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To cite this version:
Messaouda Ahmane, Isma Bouchemakh, Eric Sopena. On the Broadcast Independence Number of Locally Uniform 2-Lobsters. 2019. hal-02010990

HAL Id: hal-02010990
https://hal.science/hal-02010990
Preprint submitted on 7 Feb 2019

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On the Broadcast Independence Number
of Locally Uniform 2-Lobsters

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February 7, 2019

Abstract

Let $G$ be a simple undirected graph. A broadcast on $G$ is a function $f : V(G) \to \mathbb{N}$ such that $f(v) \leq e_G(v)$ holds for every vertex $v$ of $G$, where $e_G(v)$ denotes the eccentricity of $v$ in $G$, that is, the maximum distance from $v$ to any other vertex of $G$. The cost of $f$ is the value $\text{cost}(f) = \sum_{v \in V(G)} f(v)$. A broadcast $f$ on $G$ is independent if for every two distinct vertices $u$ and $v$ in $G$, $d_G(u,v) > \max\{f(u), f(v)\}$, where $d_G(u,v)$ denotes the distance between $u$ and $v$ in $G$. The broadcast independence number of $G$ is then defined as the maximum cost of an independent broadcast on $G$.

A caterpillar is a tree such that, after the removal of all leaf vertices, the remaining graph is a non-empty path. A lobster is a tree such that, after the removal of all leaf vertices, the remaining graph is a caterpillar. In [M. Ahmane, I. Bouchemakh and E. Sopena. On the Broadcast Independence Number of Caterpillars. Discrete Applied Mathematics, in press (2018)], we studied independent broadcasts of caterpillars. In this paper, carrying on with this line of research, we consider independent broadcasts of lobsters and give an explicit formula for the broadcast independence number of a family of lobsters called locally uniform 2-lobsters.

Keywords: Independence; Broadcast independence; Lobster.

MSC 2010: 05C12, 05C69.

1 Introduction

All the graphs we consider in this paper are simple and loopless undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph $G$, respectively.

For any two vertices $u$ and $v$ of $G$, the distance $d_G(u,v)$ between $u$ and $v$ in $G$ is the length (number of edges) of a shortest path joining $u$ and $v$. The eccentricity $e_G(v)$ of a vertex $v$ in $G$ is the maximum distance from $v$ to any other vertex of $G$. The minimum eccentricity in $G$ is the radius $\text{rad}(G)$ of $G$, while the maximum eccentricity in $G$ is the diameter $\text{diam}(G)$ of $G$.

A function $f : V(G) \to \{0, \ldots, \text{diam}(G)\}$ is a broadcast on $G$ if for every vertex $v$ of $G$, $f(v) \leq e_G(v)$. The value $f(v)$ is called the $f$-value of $v$. Given a broadcast $f$ on $G$, an $f$-broadcast vertex is a vertex $v$ with $f(v) > 0$. The set of all $f$-broadcast vertices is denoted $V(f)^+$. If $u \in V(f)^+$ is a broadcast vertex, $v \in V(G)$ and $d_G(u,v) \leq f(u)$, we say that $u$ $f$-dominates $v$. In particular, every $f$-broadcast vertex $f$-dominates itself. The cost $\text{cost}(f)$ of a broadcast $f$ on $G$ is given by

$$\text{cost}(f) = \sum_{v \in V(G)} f(v) = \sum_{v \in V(f)^+} f(v).$$

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A broadcast \( f \) on \( G \) is a dominating broadcast if every vertex of \( G \) is \( f \)-dominated by some vertex of \( V^+_f \). The minimum cost of a dominating broadcast on \( G \) is the broadcast domination number of \( G \), denoted \( \gamma_b(G) \). A broadcast \( f \) on \( G \) is an independent broadcast if every \( f \)-broadcast vertex is \( f \)-dominated only by itself. The maximum cost of an independent broadcast on \( G \) is the broadcast independence number of \( G \), denoted \( \beta_b(G) \). An independent broadcast on \( G \) with cost \( \beta \) is an independent \( \beta \)-broadcast. An independent \( \beta_b(G) \)-broadcast on \( G \) is an optimal independent broadcast. Note here that any optimal independent broadcast is necessarily a dominating broadcast.

The notions of broadcast domination and broadcast independence were introduced by D.J. Erwin in his Ph.D. thesis [18] under the name of cost domination and cost independence, respectively. During the last decade, broadcast domination has been investigated by several authors (see e.g. [3, 4, 8, 10, 12, 17, 19, 21, 23, 31]), while independent broadcast domination has attracted much less attention (see [2, 11]), until the recent work of Bessy and Rautenbach. In [6], these authors prove that \( \beta_b(G) \leq 4\alpha(G) \) for every graph \( G \), where \( \alpha(G) \) denotes the independence number of \( G \), that is, the maximum cardinality of an independent set in \( G \). In [6], they prove that \( \beta_b(G) < 2\alpha(G) \) whenever \( G \) has girth at least 6 and minimum degree at least 3, or girth at least 4 and minimum degree at least 5. Answering questions posed in [22] and [17], they prove in [5] that deciding whether \( \beta_b(G) \geq k \) for a given planar graph with maximum degree four and a given positive integer \( k \) is an NP-complete problem, and, using an approach based on dynamic programming, they prove that determining the value of \( \beta_b(T) \) for a tree \( T \) of order \( n \) can be done in time \( O(n^4) \).

Our goal, initiated in [2], is to give explicit formulas for \( \beta_b(T) \), whenever \( T \) belongs to some particular subclass of trees, that can be computed in (hopefully) linear time. Recall that a caterpillar is a tree such that deleting all its pendant vertices leaves a simple path, called the spine of the caterpillar. A lobster is then a tree such that deleting all its pendant vertices leaves a caterpillar. The spine of such a lobster is the spine of the so-obtained caterpillar. A vertex belonging to the spine of a caterpillar, or of a lobster, is called a spine-vertex and an internal spine-vertex if it is not an end vertex of the spine. The length of a lobster \( L \) is the length (number of edges) of its spine.

Note that if \( L \) is a lobster of length 0, then the unique spine-vertex of \( L \) must be of degree at least 2, since otherwise, deleting all leaves of \( L \) would leave a single edge, which is not a caterpillar. Hence, \( \text{diam}(L) = k + 4 \) for every lobster \( L \) of length \( k \).

In [2], we gave an explicit formula for the broadcast independence number of caterpillars having no two consecutive internal spine vertices of degree 2. The aim of this paper is to pursue the study of independent broadcasts of trees by considering the case of locally uniform 2-lobsters.

Let \( G \) be a graph and \( A \subset V(G) \), \( |A| \geq 2 \), be a set of pairwise antipodal vertices in \( G \), that is, at distance \( \text{diam}(G) \) from each other. The function \( f \) defined by \( f(u) = \text{diam}(G) - 1 \) for every vertex \( u \in A \), and \( f(v) = 0 \) for every vertex \( v \not\in A \), is clearly an independent \( |A|(|\text{diam}(G) - 1)| \)-broadcast on \( G \).

Observation 1 (Dunbar et al. [17]) For every graph \( G \) of order at least 2 and every set \( A \subset V(G) \), \( |A| \geq 2 \), of pairwise antipodal vertices in \( G \), \( \beta_b(G) \geq |A|(|\text{diam}(G) - 1| \geq 2(\text{diam}(G) - 1)) \).

In this paper, we determine the broadcast independence number of locally uniform 2-lobsters. The paper is organised as follows. We introduce in the next section the main definitions and a few preliminary results. We then consider in Section 3 the case of locally uniform 2-lobsters and prove our main result, which gives an explicit formula for the broadcast independence number of such lobsters. We then propose some concluding remarks in Section 4.

2 Preliminaries

Let \( G \) be a graph and \( H \) be a subgraph of \( G \). Since \( d_H(u,v) \geq d_G(u,v) \) for every two vertices \( u, v \in V(H) \), every independent broadcast \( f \) on \( G \) satisfying \( f(u) \leq e_H(u) \) for every vertex \( u \in V(H) \) is an independent broadcast on \( H \). Hence we have:
Observation 2 If $H$ is a subgraph of $G$ and $f$ is an independent broadcast on $G$ satisfying $f(u) \leq e_{H}(u)$ for every vertex $u \in V(H)$, then the restriction $f_{H}$ of $f$ to $V(H)$ is an independent broadcast on $H$.

For any independent broadcast $f$ on a graph $G$, and any subgraph $H$ of $G$, we denote by $f^{*}(H)$ the $f$-value of $H$ defined as

$$f^{*}(H) = \sum_{v \in V(H)} f(v).$$

Observe that $f^{*}(G) = \text{cost}(f)$.

The following lemma shows that, for any graph $G$ of order at least 3, if $v$ is a vertex of $G$ having at least one pendent neighbour, then no independent broadcast $f$ on $G$ with $f(v) > 0$ can be optimal.

**Lemma 3** Let $G$ be a graph of order at least 3 and $v$ be a vertex of $G$ having a pendent neighbour $u$. If $f$ is an independent broadcast on $G$ with $f(v) > 0$, then there exists an independent broadcast $f'$ on $G$ with $\text{cost}(f') > \text{cost}(f)$.

**Proof.** The mapping $f'$ defined by $f'(u) = f(v) + 1$, $f'(v) = 0$ and $f'(w) = f(w)$ for every vertex $w \in V(G) \setminus \{u, v\}$ is clearly an independent broadcast on $G$ with $\text{cost}(f') > \text{cost}(f)$. \hfill $\square$

The following lemma was given in [2]. However, we include its proof here for the sake of completeness.

**Lemma 4** Let $T$ be a tree of order at least 3, and $T'$ be a subtree of $T$ of order at least 2, with root $r$. Let $f$ be an optimal independent broadcast on $T$. If $r$ is an $f$-broadcast vertex, then $T'$ contains at least one other $f$-broadcast vertex. In particular, this implies that if $T'$ is a subtree of height $h$, that is, $e_{T'}(r) = h$, then $f(r) < h$.

**Proof.** Suppose to the contrary that $f(r) > 0$ and $f(u) = 0$ for every vertex $u \in V(T') \setminus \{r\}$. Let $t' = e_{T'}(r)$ and $\overline{t'} = e_{T-(T'-r)}(r)$.

If $f(r) < t'$, the independent broadcast $f'$ given by $f'(v) = f(r)$ for some vertex $v$ in $T'$ with $d_{T'}(r, v) = t'$ and $f'(u) = f(u)$ for every vertex $u \in V(T') \setminus \{v\}$ is such that $\text{cost}(f') = \text{cost}(f) + f(r)$, contradicting the optimality of $f$.

If $f(r) \geq \overline{t'}$, then $r$ is the unique $f$-broadcast vertex, which implies $\text{cost}(f) < 2(\text{diam}(T) - 1)$, again contradicting the optimality of $f$ by Observation [1].

Hence $\overline{t'} > f(r) \geq t'$. Let now $v$ be any neighbour of $r$ in $T'$. Since $\overline{t'} > f(r) \geq t'$, we have $e_{T'}(v) = e_{T'}(r) + 1 = \overline{t'} + 1 > f(r) + 1$. The function $f'$ defined by $f'(r) = 0$, $f'(v) = f(r) + 1$ and $f'(u) = f(u)$ for every vertex $u \in V(T) \setminus \{r, v\}$ is therefore an independent broadcast on $T$ with $\text{cost}(f') = \text{cost}(f) + 1$, contradicting the optimality of $f$.

This completes the proof. \hfill $\square$

In order to formally define locally uniform lobsters, and then locally uniform 2-lobsters, we introduce some notation.

**Notation 5** ($S_1$, $S_2$) A tree $T$ rooted at a vertex $r$ is of type $S_1$ if every leaf of $T$ is at distance 1 from $r$, which means that $T$ is a star with center $r$. A tree $T$ rooted at a vertex $r$ is of type $S_2$ if every leaf of $T$ is at distance 2 from $r$.

Let $L$ be a lobster with spine $v_0 \ldots v_k$, $k \geq 0$. The subtree of $v_i$, $0 \leq i \leq k$, denoted $S_i$, is the maximal subtree of $L$ rooted at $v_i$ that contains no spine-vertex except $v_i$. A spine-subtree of $L$ is a subtree of some $v_i$, $0 \leq i \leq k$. A branch of a spine-subtree $S_i$ is a maximal subtree of $S_i$ containing $v_i$ and exactly one neighbour of $v_i$. Therefore, if $v_i$ has degree $d$ in $S_i$, then $S_i$ has $d$ distinct branches.

A locally uniform lobster is then defined as follows.
**Definition 6 (Locally uniform lobster)** A lobster $L$ is *locally uniform* if every spine-subtree of $L$ is of type either $S_1$ or $S_2$. In other words, all branches of any spine-vertex have the same depth.

The following observation directly follows from this definition.

**Observation 7** If $L$ is a locally uniform lobster with spine $v_0 \ldots v_k$, $k \geq 0$, then both spine-subtrees $S_0$ and $S_k$ are of type $S_2$.

Indeed, if $S_0$ or $S_k$ is of type $S_1$, then $v_0$ or $v_k$ is a leaf of the caterpillar obtained by deleting all leaves of $L$, which implies that $v_0 \ldots v_k$ is not the spine of $L$, a contradiction.

Observe that Lemma 4 implies in particular the following result for locally uniform lobsters.

**Corollary 8** If $L$ is a locally uniform lobster with spine $v_0 \ldots v_k$, $k \geq 0$, and $f$ is an optimal independent broadcast on $L$, then the two following conditions hold.

1. If $v$ is a vertex having a pendant neighbour, then $f(v) = 0$.
2. For every $i$, $0 \leq i \leq k$, $f(v_i) = 0$ if $S_i$ is of type $S_1$, and $f(v_i) \leq 1$ if $S_i$ is of type $S_2$.

Moreover, the following lemma says that for every optimal independent broadcast on a locally uniform lobster with spine $v_0 \ldots v_k$, $k \geq 0$, both the spine-subtrees $S_0$ and $S_k$ contain an $f$-broadcast vertex.

**Lemma 9** If $L$ is a locally uniform lobster with spine $v_0 \ldots v_k$, $k \geq 0$, and $f$ is an optimal independent broadcast on $L$, then $f^*(S_0) > 0$ and $f^*(S_k) > 0$.

**Proof.** It is enough to prove the result for $S_0$. Assume to the contrary that $f^*(S_0) = 0$, and let $v$ be a vertex of $L$ that $f$-dominates the leaves of $S_0$. Since $f^*(S_0) = 0$, we necessarily have $f(v) \geq 4$ which implies that $v$ is unique. By Corollary 8, $v$ must be a leaf of $L$. Let $\ell$ be any leaf of $S_0$.

Let $S$ denote the spine-subtree containing $v$. If $S$ is of type $S_1$ and $f(v) + d_L(\ell, v) > \text{diam}(L) + 1$, or $S$ is of type $S_2$ and $f(v) + d_L(\ell, v) > \text{diam}(L) + 3$, then $v$ is the unique $f$-broadcast vertex of $L$, which contradicts the optimality of $f$ by Observation 4. We now define the mapping $f'$ on $V(L)$ given by $f'(u) = f(u)$ for every vertex $u \notin \{v, \ell\}$, $f'(v) = 0$, and $f'(\ell) = f(v) + d_L(\ell, v) - 2$. The mapping $f'$ is clearly an independent broadcast on $L$ and, since $d_L(v, \ell) \geq 4$, we get $\text{cost}(f') > \text{cost}(f)$, contradicting the optimality of $f$. \qed

We now define 2-lobsters and locally uniform 2-lobsters.

**Definition 10 (2-lobster)** A lobster $L$ is a *2-lobster* if every spine-subtree of $L$ has at least two branches.

**Definition 11 (Locally uniform 2-lobster)** A *locally uniform 2-lobster* is a 2-lobster which is locally uniform (see Figure 1).

Due to their special structure, we can improve the lower bound on the broadcast independence number of locally uniform 2-lobsters of length $k \geq 1$. 

![Figure 1: A sample locally uniform 2-lobster.](image)
Observation 12  For every locally uniform 2-lobster $L$ of length $k \geq 1$, 
$$\beta_0(L) \geq 2(k - 1) + 12 = 2(\text{diam}(L) - 1) + 4.$$ 

To see that, consider the function $f$ on $V(L)$ defined as follows. For each branch of $S_0$ and $S_k$, pick one leaf and set its $f$-broadcast value to 3, and, for each branch of every $S_i$, $1 \leq i \leq k - 1$, if $k > 1$, pick one leaf and set its $f$-broadcast value to 1. The mapping $f$ is clearly an independent broadcast on $L$ and, since both $S_0$ and $S_k$ are of type $S_2$ and every spine-subtree of $L$ has at least two branches, we get $\text{cost}(f) = 2(k - 1) + 12$.

3  Independent broadcasts of locally uniform 2-lobsters

In this section we determine the broadcast independence number of locally uniform 2-lobsters. Recall that by Observation 12, $\beta_0(L) > 2(\text{diam}(L) - 1)$ for every locally uniform 2-lobster $L$ of length $k \geq 1$ (the special case of locally uniform 2-lobsters of length 0 will be considered separately, in Lemma 21).

We first introduce some notation and define different types of spine-subtrees in Subsection 3.1. We then define the value $\beta^*(L)$ for every locally uniform 2-lobster $L$ in Subsection 3.2 prove that every such lobster $L$ admits an independent $\beta^*(L)$-broadcast in Subsection 3.3 and that it cannot admit any independent broadcast with cost strictly greater than $\beta^*(L)$ in Subsection 3.4. This allows us to finally state our main result in Subsection 3.5. 

3.1  Different types of spine-subtrees

Let $L$ be a locally uniform 2-lobster with spine $v_0 \ldots v_k$, $k \geq 0$. Two spine-subtrees $S_i$ and $S_{i+1}$, $0 \leq i \leq k - 1$, are called neighbouring spine-subtrees. Moreover, we say that $S_i$ precedes $S_{i+1}$, and that $S_{i+1}$ follows $S_i$. A sequence of $p$ spine-subtrees, $p \geq 2$, is a sequence of consecutive spine-subtrees of the form $S_1 \ldots S_{i+p-1}$ for some $i, 0 \leq i \leq k - p + 1$.

We will say that two independent broadcasts $f_1$ and $f_2$ on a locally uniform 2-lobster $L$ are similar if their values on each spine-subtree of $L$ are equal, that is, $f_1(S_i) = f_2(S_i)$ for every $i$, $0 \leq i \leq k$. Observe that any two independent broadcasts have the same cost.

A 1-leaf of $L$ is a pendent vertex of $L$ adjacent to a spine-vertex. A pendent vertex which is not a 1-leaf is a 2-leaf (recall that every pendent vertex is at distance at most 2 from a spine-vertex). An only-leaf is a leaf whose neighbour has only one leaf neighbour. Therefore, an only-leaf in a locally uniform 2-lobster is necessarily a 2-leaf, and is then called a 2-only-leaf. Two leaves having the same neighbour are said to be sister-leaves.

Notation 13  $(\lambda_1, \lambda_2, \lambda^*_2)$ For every $i, 0 \leq i \leq k$, we denote by $\lambda_1(S_i)$, $\lambda_2(S_i)$ and $\lambda^*_2(S_i)$, the number of 1-leaves, of 2-leaves, and of 2-only-leaves of $S_i$, respectively. Moreover, we extend these three functions to the whole lobster $L$, by letting

$$\lambda_1(L) = \sum_{i=0}^{i=k} \lambda_1(S_i), \quad \lambda_2(L) = \sum_{i=0}^{i=k} \lambda_2(S_i), \quad \text{and} \quad \lambda^*_2(L) = \sum_{i=0}^{i=k} \lambda^*_2(S_i).$$

Let $v_i$ be a spine-vertex of $L$ with $t$ non-spine neighbours, denoted $w^1_i, \ldots, w^t_i$. For every $j$, $1 \leq j \leq t$, the branch $B^j_i$ of $v_i$ is the maximal spine-subtree of $S_i$, rooted at $v_i$, containing the edge $v_iw^j_i$ but no edge $v_iw^j_i$ with $j' \neq j$. We then define two types of branches.

Notation 14  $(\mathcal{H}_1, \mathcal{H}_2, \alpha_1, \alpha_2, \alpha^*_2)$ A branch is of type $\mathcal{H}_1$ if it does not contain any 2-leaf, and of type $\mathcal{H}_2$ if it does not contain any 1-leaf. For every spine-subtree $S_i$, $0 \leq i \leq k$, we denote by $\alpha_1(S_i)$ and $\alpha_2(S_i)$ the number of branches of $S_i$ of type $\mathcal{H}_1$ and of type $\mathcal{H}_2$, respectively. Moreover, we denote by $\alpha^*_2(S_i)$ the number of branches of $S_i$ of type $\mathcal{H}_2$ having at most two 2-leaves.
Since all branches of any spine-subtree of a locally uniform 2-lobster are of the same type, we get $\alpha_1(S_i) \geq 2$, $\alpha_1(S_i) = \alpha_2^*(S_i) = 0$, if $S_i$ is of type $S_1$, and $\alpha_1(S_i) = 0$, $\alpha_2(S_i) \geq 2$, $\alpha_2^*(S_i) \geq 0$, if $S_i$ is of type $S_2$.

**Notation 15** ($b_i$) For every $i$, $0 \leq i \leq k$, we denote by $b_i$ the number of branches of the spine-subtree $S_i$.

Observe that $b_i = \text{deg}_L(v_i) - 2$ if $1 \leq i \leq k - 1$, and $b_i = \text{deg}_L(v_i) - 1$ if $i \in \{0, k\}$.

In order to define various types of spine-subtrees, we will use the following notation.

**Notation 16 (Operators on types of spine-subtrees)** Let $X$, $Y$ and $Z$ be any types of spine-subtrees. We then define the following types.

- $\vec{X}$
  - A spine-subtree $S$ is of type $\vec{X}$ if $S$ is not of type $X$.
- $X|Y$
  - A spine-subtree $S$ is of type $X|Y$ if $S$ is of type $X$ or $Y$.
- $X, Y$
  - A sequence of two spine-subtrees $SS'$ is of type $X \cdot Y$, or simply $X Y$, if $S$ is of type $X$ and $S'$ is of type $Y$.
- $X[P_1, \ldots, P_p]$.
  - For any properties $P_1, \ldots, P_p$, $p \geq 1$, a spine-subtree $S$ is of type $X[P_1, \ldots, P_p]$ if $S$ is a spine-subtree of type $X$ satisfying properties $P_1, \ldots, P_p$. For instance, a spine-subtree $S$ is of type $S_2[\lambda_2 \geq 5, \alpha_2^* \leq 3]$ if $S$ is a spine-subtree of type $S_2$ with at least five leaves, having at most three branches with at most two leaves. Similarly, a branch of type $Y[P_1, \ldots, P_p]$ is a branch of type $Y$ satisfying properties $P_1, \ldots, P_p$. For instance, a branch of type $H_2[\lambda_2 = 3]$ is a branch of type $H_2$ having three 2-leaves.
- $\langle X \rangle Y, \langle X \rangle Y(Z)$.
  - A spine-subtree $S$ is of type $\langle X \rangle Y$ (resp. $\langle X \rangle Z$) if $S$ is a spine-subtree of type $Y$ and the spine-subtree $S'$ preceding $S$ (resp. following $S$) is of type $X$ (resp. $Z$). A spine-subtree $S$ is then of type $\langle X \rangle Y(Z)$ if $S$ is of type $\langle X \rangle Y$ and of type $\langle Y \rangle (Z)$.
- $\varnothing$.
  - Slightly abusing the notation, we use the symbol $\varnothing$ to denote an “empty spine-subtree”, so that, for instance, a spine-subtree $S$ is of type $\langle \varnothing \rangle Y$ (resp. $\varnothing(Z)$) if $S$ is a spine-subtree of type $Y$ and the spine-subtree $S'$ preceding $S$ (resp. following $S$) is of type $\varnothing$ (resp. $Z$). A spine-subtree $S$ is of type $\langle \varnothing \rangle Y(Z)$ if $S$ is of type $\langle \varnothing \rangle Y$ and of type $\langle Y \rangle (Z)$.
- $\{X_1 \ldots X_p\}^+, \{X_1 \ldots X_p\}^*$.
  - For any types of spine-subtrees $X_1, \ldots, X_p$, $p \geq 1$, a sequence of spine-subtrees $S_{i_1}, \ldots, S_{i_p}$, $0 \leq i \leq \ell - p j$, $0 \leq j \leq \lfloor k/p \rfloor$, is of type $\{X_1 \ldots X_p\}^+$, if every spine-subtree $S_{i_l}, i \leq \ell \leq i + p j$ is of type $X_{i_l-i+1} (\mod p)$, and none of the sequences $S_{i-p}, \ldots, S_{i}, S_{i+p}$ and $S_{i_1}, S_{i+p}, \ldots, S_{i+p+p}$ is of type $\{X_1 \ldots X_p\}^+$ (the sequence is thus maximal). Moreover, we will denote by $\{X_1 \ldots X_p\}^*$ the type $\varnothing[\{X_1 \ldots X_p\}^+]$.

Our aim now is twofold. We will first construct, for any locally uniform 2-lobster $L$, an independent broadcast $f^*$ on $L$ with cost($f^*$) = $\beta^*(L)$, for some value $\beta^*(L)$, and then prove that the value $\beta^*(L)$ is the optimal cost of an independent broadcast on $L$.

The independent broadcast $f^*$ will be constructed in four steps, that is, we will construct a sequence of independent broadcasts $f_1, \ldots, f_4$, with cost($f_i$) $\leq$ cost($f_{i+1}$) for every $i$, $1 \leq i \leq 3$, and then set $f^* = f_4$. Each step will consist in modifying the broadcast values of some vertices, according to the type of the spine-subtree, or of the sequence of spine-subtrees, they belong to.

We now introduce the specific types of spine-subtrees, or types of sequences of spine-subtrees, that will be used. All these types are illustrated in Figure 2 (do not consider the depicted broadcast values yet, they will be discussed later, in Claim 2).
Definition 17 ($A$, $B$, $C_a$, $C_b$, $C_c$, $C$, $D$)

We define the following types of spine-subtrees.

- $A = S_2[\alpha_2^* = 0, \alpha_2 \geq 2]$. A spine-subtree of type $A$ is a spine-subtree of type $S_2$ with at least two branches of type $H_2$, all of them having at least three leaves.
- $B = \langle S_1 \rangle S_1[\lambda_1 = 2] | S_1[\lambda_1 = 2] \langle S_1 \rangle$. A spine-subtree of type $B$ is a spine-subtree of type $S_1$ with two leaves, having at least one neighbouring spine-subtree of type $S_1$.
- $C_a = S_1[\lambda_1 \geq 3]$. A spine-subtree of type $C_a$ is a spine-subtree of type $S_1$ with at least three leaves.
- $C_b = S_2[\alpha_2^* = 1, \alpha_2 \geq 2]$. A spine-subtree of type $C_b$ is a spine-subtree of type $S_2$ having at least two branches of type $H_2$ with exactly one of them having at most two leaves.
- $C_c = \langle S_2 \rangle S_1[\lambda_1 = 2] \langle S_2 \rangle$. A spine-subtree of type $C_c$ is a spine-subtree of type $S_1$ with two leaves having two neighbouring spine-subtrees of type $S_2$.
- $C = C_a | C_b | C_c$.
- $D = S_2[\alpha_2^* \geq 2]$. A spine-subtree of type $D$ is a spine-subtree of type $S_2$ with at least two branches having at most two leaves.

The following observation directly follows from the previous definition, considering the neighbouring requirements, and will be useful later.

Observation 18 A spine-subtree of type $B$ or $C_a$ cannot have a spine-subtree of type $C_c$ as a neighbouring spine-subtree.

We now claim that the set of types $\{A, B, C_a, C_b, C_c, D\}$ induces a partition of the spine-subtrees of any locally uniform 2-lobster (with possibly empty parts).
Proposition 19 Let $\mathcal{T} = \{A, B, C_a, C_b, C_c, D\}$, and $L$ be any locally uniform 2-lobster. Every spine-subtree of $L$ belongs to exactly one type in $\mathcal{T}$.

Proof. Clearly, the types in $\mathcal{T}$ are pairwise disjoint, that is, no spine-subtree of $L$ can belong to two types from this set (see Figure 2).

We now prove that every spine-subtree of a locally uniform 2-lobster belongs to exactly one type in this set. Indeed, consider any such spine-subtree $S$.

1. If $S$ is of type $S_1$, then either $S$ has at least three leaves (type $C_a$), or two leaves and a neighbouring spine-subtree of type $S_1$ (type $B$), or two leaves and no neighbouring spine-subtree of type $S_1$ (type $C_c$).

2. Suppose now that $S$ is of type $S_2$. Since $S$ has at least two branches, we get that either each of these branches have at least three leaves (type $A$), or $S$ has exactly one branch with at most two leaves (type $C_b$), or $S$ has at least two branches with at most two leaves (type $D$).

This completes the proof. □

3.2 Definition of $\beta^*(L)$

We are now able to define the value $\beta^*(L)$ for any locally uniform 2-lobster $L$, which will be proven to be the optimal cost of an independent broadcast on $L$. The value $\beta^*(L)$ will be expressed as a formula involving the number of 1-leaves, 2-leaves and 2-only-leaves, and the number of spine-subtrees, or sequences of spine-subtrees, of types defined in the previous subsection, appearing in $L$.

Finally, recall that $\lambda_1(L)$, $\lambda_2(L)$ and $\lambda_3^*(L)$ denote the number of 1-leaves, of 2-leaves and of 2-only-leaves in $L$, respectively. We are now able to define $\beta^*(L)$.

Definition 20 ($\beta^*(L)$) Let $L$ be a locally uniform 2-lobster. We then let

$$\beta^*(L) = \nu_1(L) + \nu_2(L) + \nu_3(L) + \nu_4(L),$$

where

- $\nu_1(L) = \lambda_1(L) + \lambda_2(L) + \lambda_3^*(L)$ is the total number of leaves in $L$, where each 2-only-leaf is counted twice.
- $\nu_2(L)$ is the number of branches in $L$ with at most two 2-leaves, that belong to a spine-subtree of type $S_2$ (that is, of depth 2).
- $\nu_3(L)$ is the number of spine-subtrees of type $C_c$ in $L$.
- $\nu_4(L)$ is the sum, taken over all sequences of spine-subtrees $S$ in $L$ of type

$$\langle \overline{C_c} (\emptyset | A | B | C_a) \rangle . \langle (C | A) \rangle . \langle (\emptyset | A | B | C_a) \overline{C} \rangle,$$

of the value

$$\ell(S) + 1 \over 2 - \#_{C_b, C_c}(S),$$

where $\ell(S)$ denotes the number of spine-subtrees in $S$, and $\#_{C_b, C_c}(S)$ the number of spine-subtrees of type $C_b$ or $C_c$ in $S$.

3.3 Lower bound

We will now prove that every locally uniform 2-lobster admits an independent broadcast $f$ with $\text{cost}(f) = \beta^*(L)$. We consider the case of locally uniform 2-lobsters of length 0 separately.

Lemma 21 If $L$ is a locally uniform 2-lobster of length $k = 0$, then there exists an independent broadcast $f$ on $L$ with $\text{cost}(f) = \beta^*(L)$, thus implying $\beta_b(L) \geq \beta^*(L)$. 
Proof. Recall that since \( k = 0 \), \( L = S_0 \) is necessarily of type \( S_2 \) (Observation 7) and has at least two branches, so that \( \text{diam}(L) = 4 \). We construct an independent broadcast \( f \) on \( L \) as follows, by considering each branch separately. Let \( B \) be any branch of \( L \). If \( B \) has at most two leaves, then we set \( f(\ell) = 3 \) for one leaf \( \ell \) of \( B \), and \( f(\ell') = 0 \) for every sister-leaf \( \ell' \) of \( \ell \), if any. If \( B \) has at least three leaves, then we set \( f(\ell) = 1 \) for every leaf \( \ell \) of \( B \). Finally, if \( S_0 \) is of type \( A \), then we set \( f(v_0) = 1 \) (in that case, \( \nu_4(L) = 1 \)).

We then have \( \text{cost}(f) = \nu_1(L) + \nu_2(L) + \nu_4(L) \), and thus, since \( \nu_3(L) = 0 \), \( \text{cost}(f) = \beta^*(L) \). \( \square \)

Lemma 22 Every locally uniform 2-lobster \( L \) of length \( k \geq 1 \) admits an independent broadcast \( f \) with \( \text{cost}(f) = \beta^*(L) \), thus implying \( \beta_b(L) \geq \beta^*(L) \).

Proof. We will construct a sequence of four independent broadcasts \( f_1, \ldots, f_4 \) on \( L \), step by step, such that \( \text{cost}(f_4) = \beta^*(L) \). Each independent broadcast \( f_i \), \( 2 \leq i \leq 4 \), is obtained by possibly modifying the independent broadcast \( f_{i-1} \), and is such that \( \text{cost}(f_i) \geq \text{cost}(f_{i-1}) \). Moreover, for each independent broadcast \( f_i \), \( 1 \leq i \leq 3 \), we will have \( f_i(v_j) = 0 \) for every spine-vertex \( v_j \), \( 0 \leq j \leq k \), while we may have \( f_3(v_j) = 1 \).

These modifications are illustrated in Figures 3 where dashed edges represent optional edges. These figures should help the reader to see that each mapping \( f_i \) is a valid independent broadcast on \( L \).

Step 1. Let \( f_1 \) be the mapping defined by \( f_1(u) = 2 \) if \( u \) is an only-leaf, \( f_1(u) = 1 \) if \( u \) is a leaf which is not an only-leaf and \( f_1(u) = 0 \) otherwise (see Figure 3a).

Clearly, \( f_1 \) is an independent broadcast on \( L \) with

\[
\text{cost}(f_1) = \lambda_1(L) + \lambda_2(L) + \lambda_2^*(L) = \nu_1(L).
\]

Step 2. We modify \( f_1 \) as follows, to obtain \( f_2 \). For every branch \( B_i^j \) of type \( H_2[\lambda_2 \leq 2] \), \( 0 \leq i \leq k \), \( 1 \leq j \leq s_i \), such that \( S_i \) is a spine-subtree of type \( S_2 \), we let \( f_2(\ell) = 3 \) for one leaf \( \ell \) of \( B_i^j \), and \( f_2(\ell') = 0 \) for the sister-leaf \( \ell' \) of \( \ell \), if any (see Figure 3b).

Again, \( f_2 \) is an independent broadcast on \( L \) with

\[
\text{cost}(f_2) = \text{cost}(f_1) + \nu_2(L).
\]

Step 3. We modify \( f_2 \) as follows, to obtain \( f_3 \). For every spine-subtree \( S \) of type \( C_c \), we let \( f_3(\ell) = 3 \) for one leaf \( \ell \) of \( S \), and \( f_3(\ell') = 0 \) for the sister-leaf \( \ell' \) of \( \ell \), if any (see Figure 3c). Note here that setting \( f_3(\ell) = 3 \) is allowed, since \( L \) does not contain any vertex \( x \) with \( f_3(x) > 0 \) at distance at most \( 3 \) from the leaves of \( S \), and \( f_3(\ell'') \leq 3 \) for every 2-leaf \( \ell'' \) of \( L \).

Again, \( f_3 \) is an independent broadcast on \( L \), and, since the broadcast value of every considered spine-subtree \( S \) has been increased by 1, we have

\[
\text{cost}(f_3) = \text{cost}(f_2) + \nu_3(L).
\]

Before describing the last step, we prove a claim on the \( f_3 \)-values and introduce some terminology.

Claim 23 After step 3, the \( f_3 \)-values of the vertices of \( L \), depending on the type of the spine-subtree they belong to, are those values depicted in Figure 3.
(a) The independent broadcast $f_1$ on a sample locally uniform 2-lobster

(b) From $f_1$ to $f_2$

(c) From $f_2$ to $f_3$

(d) From $f_3$ to $f_4$, for a sequence of type $\langle \emptyset \rangle \mathcal{A} \mathcal{C}_c. \mathcal{A}. \mathcal{A}. \mathcal{A}(\mathcal{B}, \mathcal{B})$ (cost increases by 2)

Figure 3: Proof of Lemma 22 from $f_1$ to $f_4$. 
Proof. Let \( S \) be any spine-subtree of \( L \). If \( S \) is of type \( S_1 \), then either \( S \) has at least three leaves (type \( C_3 \)), or two leaves and a neighbouring spine-subtree of type \( S_1 \) (type \( B \)), and has thus not been modified in steps 2 or 3 in both cases, or two leaves and no neighbouring spine-subtree of type \( S_1 \) (type \( C_2 \)), and has thus been modified in step 3. In all these cases, the \( f_3 \)-values of the vertices in \( S \) are those values depicted in Figure 2(b), (c) or (e).

Suppose now that \( S \) is of type \( S_2 \). In that case, either each of these branches have at least three leaves (type \( A \)), and has thus not been modified in steps 2 or 3, or \( S \) has exactly one branch with at most two leaves (type \( C_b \)), or \( S \) has at least two branches with at most two leaves (type \( D \)), and has thus been modified in step 2. In all these cases, the \( f_3 \)-values of the vertices in \( S \) are those values depicted in Figure 2(a), (d) or (f).

This completes the proof of the claim. \( \square \)

A spine-subtree \( S \) exceeds by \( e \), for some integer \( e \geq 1 \), if \( S \) contains a 1-leaf with broadcast value \( e + 1 \), or a 2-leaf with broadcast value \( e + 2 \). Therefore, if a spine-subtree \( S_i, 0 \leq i \leq k \), of a locally uniform 2-lobster \( L \) exceeds by \( e \geq 1 \), then none of the spine vertices \( v_{i−e}, . . . , v_i, . . . , v_{i+e} \) can be a broadcast vertex.

We can then partition the set \( T = \{ A, B, C_a, C_b, C_c, D \} \) in three parts \( E_0, E_1 \) and \( E_2 \), corresponding to the types of spine-subtrees exceeding by 0, 1 and 2, respectively, after Step 3 (as given in Claim 23). In order to be complete, we will also say that the “empty subtree”, of type \( \emptyset \), does not exceed. Therefore, we have

\[
E_0 = \{ \emptyset, A, B, C_a \}, \quad E_1 = \{ C_b, D \} \quad \text{and} \quad E_2 = \{ C_c \}.
\]

Moreover, we denote by \( \overline{E_i} \) the complement of \( E_i \) for every \( i, 0 \leq i \leq 2 \), that is, \( \overline{E_i} = (T \cup \{ \emptyset \}) \setminus E_i \).

Let \( S \) be a spine-subtree of type \( A \). By increasing \( S \) by one, we mean giving the broadcast value 1 to the root of \( S \) (observe that only leaves of \( S \) are \( f_3 \)-broadcast vertices, and that \( f_3(\ell) = 1 \) for every such leaf \( \ell \)).

Let now \( S \) be a spine-subtree of type \( C_b \) or \( C_c \). By decreasing \( S \) by one, we mean the following:

- If \( S \) is of type \( C_b \), then we give the broadcast value 2 to one leaf of the (unique) branch of type \( H_{2|\lambda_2 \leq 2} \), and the broadcast value 0 to its sister-leaf, if any (by Claim 23 \( f_3(\ell) = 3 \) for one leaf \( \ell \) of \( S \), and \( f_3(\ell') = 0 \) for the sister-leaf \( \ell' \) of \( \ell \), if any).
- If \( S \) is of type \( C_c \), then we give the broadcast value 1 to each of the two leaves of \( S \) (by Claim 23 \( f_3(\ell) = 3 \) for one leaf \( \ell \) of \( S \), and \( f_3(\ell') = 0 \) for the sister-leaf \( \ell' \) of \( \ell \)).

Observe that after having being decreased by one, a spine-subtree of type \( C_b \) or \( C_c \) does no longer exceed.

We are now able to describe the fourth step of the proof. The key idea of this last step is to increase by one some spine-subtrees of type \( A \), and decrease by one some spine-subtrees of type \( C_b \) or \( C_c \), provided that this results in a strict increasing of the cost of the current independent broadcast on \( L \).

Step 4. We modify \( f_3 \) as follows, to obtain \( f_4 \). For every sequence of spine-subtrees \( A_0X_1A_1 . . . X_pA_p, p \geq 0 \), of type

\[
T_4 = (\overline{E_2}, E_0) A, (\{ C|A \}, A) R (\overline{E_0}, \overline{E_2}),
\]

we decrease by one each spine-subtree \( X_i \) of type \( C_b \) or \( C_c \), \( 1 \leq i \leq p \), and increase by one each spine-subtree \( A_j, 0 \leq j \leq p \) (see Figure 3(d)). Note that this can be done since none of the spine-subtrees \( X_i, 1 \leq i \leq p \), exceeds and no spine-subtree outside the sequence can prevent us from doing so on the extremal spine-subtrees \( A_0 \) and \( A_p \).

The broadcast value of the whole sequence is thus increased by \( p + 1 \), minus the number of spine-subtrees of type \( C_b \) or \( C_c \). Since the number of spine-subtrees of type \( C_b \) or \( C_c \) is at most \( p \), this broadcast value always increases. Therefore, doing the above modification for every sequence of spine-subtrees of type \( T_4 \), the so-obtained independent broadcast \( f_4 \) satisfies

\[
\text{cost}(f_4) = \text{cost}(f_3) + \nu_4(L).
\]
We finally get $\text{cost}(f_4) = \beta^*(L)$, as required. This completes the proof. \hfill $\square$

### 3.4 Upper bound

We first prove that, for every locally uniform 2-lobster \( L \), we can choose an optimal independent broadcast on \( L \) that satisfies some given properties.

The next lemma shows that if \( f \) is an optimal independent broadcast on a locally uniform 2-lobster \( L \) of length \( k \geq 1 \), then there exists an optimal independent broadcast \( \tilde{f} \) on \( L \) such that the \( f \)-values of the vertices in the spine-subtrees \( S_0 \) and \( S_k \) are at most 3.

**Lemma 24** If \( L \) is a locally uniform 2-lobster of length \( k \geq 1 \), and \( f \) is an optimal independent broadcast on \( L \), then there exists an optimal independent broadcast \( \tilde{f} \) on \( L \) such that \( \tilde{f}(\ell) \leq 3 \) for every leaf \( \ell \) of \( S_0 \) and \( S_k \).

**Proof.** Recall first that, by Observation\[\text{a}\] both spine-subtrees \( S_0 \) and \( S_k \) must be of type \( S_2 \). Also note that, by symmetry, it is enough to prove the result for \( S_0 \). If \( S_0 \) has at least two broadcast leaves, then the broadcast value of each of them is at most 3, since every two such leaves are at distance at most 4 from each other. We thus only need to consider the case when \( S_0 \) has a unique broadcast leaf. Moreover, we can assume that the broadcast value of this leaf is at least 7, since otherwise, by setting the broadcast value of any two leaves at distance 4 from each other to 3, we would get either a broadcast satisfying the requirement of the lemma, or a contradiction with the optimality of the broadcast. Therefore, we get that the result holds if \( k \leq 3 \) since, in that case, \( \text{diam}(L) \leq 7 \), which implies \( f(v) \leq 6 \) for every vertex \( v \) of \( L \) since \( f \) is maximal.

The proof now is by contradiction. Let \( L \) be a counter-example to the lemma, of length \( k \geq 4 \), and \( f \) be an optimal independent broadcast on \( L \) which minimizes the value of \( f(\ell) = \alpha \), where \( \ell \) is the (unique) \( f \)-broadcast leaf of \( S_0 \). We thus have \( \alpha \geq 7 \).

Observe that at least one vertex at distance \( \alpha + 1 \) from \( \ell \) must be an \( f \)-broadcast vertex, since otherwise we could increase the value of \( f(\ell) \) by 1, contradicting the optimality of \( f \). Let \( x \) denote any such vertex. The spine-subtrees \( S_1, \ldots, S_{\alpha-4} \) do not contain any \( f \)-broadcast vertex (since every such vertex is \( f \)-dominated by \( \ell \)), and \( x \) is either a 2-leaf of \( S_{\alpha-3} \), a 1-leaf of \( S_{\alpha-2} \), or the spine-vertex \( v_{\alpha-1} \).

We consider four cases, depending on whether these vertices are \( f \)-broadcast vertices or not. For each of these cases, we assume that none of the previous cases occurs.

1. **\( v_{\alpha-1} \) is an \( f \)-broadcast vertex.**
   In this case, by Corollary\[\text{b}\] we know that \( f(x) = 1 \). Consider the spine-subtree \( S_{\alpha-3} \). If \( S_{\alpha-3} \) is of type \( S_1 \), then all its vertices are \( f \)-dominated by \( \ell \), and the mapping \( g \) defined by \( g(\ell) = \alpha - 1 \), \( g(\ell_{\alpha-3}) = 2 \) for one leaf \( \ell_{\alpha-3} \) of \( S_{\alpha-3} \) and \( g(v) = f(v) \) for every other vertex \( v \) of \( L \) is clearly an independent broadcast on \( L \) with \( \text{cost}(g) = \text{cost}(f) - 1 + 2 = \text{cost}(f) + 1 \), which contradicts the optimality of \( f \) (see Figure\[\text{c}\]). Now, if \( S_{\alpha-3} \) is of type \( S_2 \), then \( f(\ell_{\alpha-3}) \leq 3 \) for every leaf \( \ell_{\alpha-3} \) of \( S_{\alpha-3} \). Therefore, the mapping \( g \) defined by \( g(\ell) = \alpha - 2 \), \( g(\ell_{\alpha-4}) = 2 \) for one leaf \( \ell_{\alpha-4} \) of \( S_{\alpha-4} \) and \( g(v) = f(v) \) for every other vertex \( v \) of \( L \) is clearly an independent broadcast on \( L \), with \( \text{cost}(g) = \text{cost}(f) - 2 + 2 = \text{cost}(f) \), which contradicts the minimality of \( \alpha \) (see Figure\[\text{d}\]), where \( S_{\alpha-4} \) is supposed to be of type \( S_1 \), the case \( S_1 \) of type \( S_2 \) being similar). Therefore, \( v_{\alpha-1} \) cannot be an \( f \)-broadcast vertex.

2. **Both a 2-leaf \( \ell_{\alpha-3} \) of \( S_{\alpha-3} \) and a 1-leaf \( \ell_{\alpha-2} \) of \( S_{\alpha-2} \) are \( f \)-broadcast vertices.**
   In this case, we necessarily have \( f(\ell_{\alpha-3}) \leq 3 \) and \( f(\ell_{\alpha-2}) \leq 3 \). Therefore, the mapping \( g \) defined by \( g(\ell) = \alpha - 2 \), \( g(\ell_{\alpha-4}) = 3 \) for one leaf \( \ell_{\alpha-4} \) of \( S_{\alpha-4} \) and \( g(v) = f(v) \) for every other vertex \( v \) of \( L \) is clearly an independent broadcast on \( L \) with \( \text{cost}(g) = \text{cost}(f) - 2 + 3 = \text{cost}(f) + 1 \), which contradicts the optimality of \( f \) (see Figure\[\text{e}\]), where again \( S_{\alpha-4} \) is supposed to be of type \( S_1 \), the case \( S_1 \) of type \( S_2 \) being similar).
(a) Case 1, $S_{\alpha-2}$ is of type $S_1$

(b) Case 1, $S_{\alpha-2}$ is of type $S_2$

(c) Case 2

Figure 4: Independent broadcasts for the proof of Lemma [24] case 1 and case 2 (only one branch per spine-subtree is depicted).
Figure 5: Independent broadcasts for the proof of Lemma 24, case 3 (only one branch per spine-subtree is depicted).
3. A 1-leaf $\ell_{\alpha-2}$ of $S_{\alpha-2}$ is an $f$-broadcast vertex.

We let $\beta = f(\ell_{\alpha-2})$. We necessarily have $\beta \leq \alpha$. If $\beta = 1$, the mapping $g$ defined by $g(\ell) = \alpha - 1$, $g(\ell_{\alpha-3}) = 2$ for one leaf $\ell_{\alpha-3}$ of $S_{\alpha-3}$ and $g(v) = f(v)$ for every other vertex $v$ of $L$, is clearly an independent broadcast on $L$ with $\text{cost}(g) = \text{cost}(f) + 1$, which contradicts the optimality of $f$ (see Figure 5(a), where $S_{\alpha-3}$ is supposed to be of type $S_1$, the case $S_{\alpha-3}$ of type $S_2$ being similar).

Suppose now $\beta \geq 2$ and let $g_{\alpha-2}$ be the mapping defined by $g_{\alpha-2}(\ell) = 3$, $g_{\alpha-2}(\ell_j) = 2$ for one leaf $\ell_j$ of each spine-subtree $S_j$, $1 \leq j \leq \alpha - 3$, $g_{\alpha-2}(\ell_{\alpha-2}) = 2$ and $g(v) = f(v)$ for every other vertex $v$ of $L$. The mapping $g_{\alpha-2}$ is clearly an independent broadcast on $L$, with

$$\text{cost}(g_{\alpha-2}) = \text{cost}(f) - \alpha - \beta + 3 + 2(\alpha - 2) = \text{cost}(f) + (\alpha - \beta) - 1$$

(see Figure 5(b), where $S_1, \ldots, S_{\alpha-3}$ are supposed to be of type $S_1$, all other cases being similar). Therefore, the mapping $g_{\alpha-2}$ contradicts either the optimality of $f$ or the minimality of $\alpha$ if $\beta < \alpha$.

We can thus assume $\beta = \alpha$. Suppose first that the spine-subtree $S_{\alpha-1}$ contains a leaf $\ell_{\alpha-1}$ which is $f$-dominated only by $\ell_{\alpha-2}$. (Observe that this is in particular the case if $S_{\alpha-1}$ is of type $S_2$.) In that case, we consider the mapping $g_{\alpha-1}$ whose definition is similar to the above definition of $g_{\alpha-2}$, by simply replacing $\alpha - 2$ by $\alpha - 1$. The mapping $g_{\alpha-1}$ is clearly an independent broadcast on $L$, with

$$\text{cost}(g_{\alpha-1}) = \text{cost}(f) - 2\alpha + 3 + 2(\alpha - 1) = \text{cost}(f) + 1,$$

which contradicts the optimality of $f$ (see Figure 5(c), where $S_1, \ldots, S_{\alpha-3}$ and $S_{\alpha-1}$ are supposed to be of type $S_1$, all other cases being similar).

Therefore, $S_{\alpha-1}$ is of type $S_1$ and each of its 1-leaves is $f$-dominated by $\ell_{\alpha-2}$ and (at least) one other vertex $x$. Moreover, we necessarily have $f(x) = f(\ell_{\alpha-2}) = \alpha \geq 7$, which implies the uniqueness of $x$, since a 2-leaf of $S_{2a-4}$ and a 1-leaf of $S_{2a-3}$ are at distance 4 from each other, and $r_{2a-2}$ cannot be an $f$-broadcast vertex by Corollary 8. We then consider the mapping $g'_{\alpha-1}$ defined by $g'_{\alpha-1}(x) = \alpha - 1$ and $g'_{\alpha-1}(v) = g_{\alpha-1}(v)$ for every other vertex $v$ of $L$. Again, the mapping $g'_{\alpha-1}$ is clearly an independent broadcast on $L$, with

$$\text{cost}(g'_{\alpha-1}) = \text{cost}(g_{\alpha-1}) - 1 = \text{cost}(f),$$

which contradicts the minimality of $\alpha$ (see Figure 5(d), where $x$ is a 1-leaf of $S_{2\alpha-3}$, the case when $x$ is a 2-leaf of $S_{2\alpha-4}$ being similar).

4. A 2-leaf $\ell_{\alpha-3}$ of $S_{\alpha-3}$ is an $f$-broadcast vertex.

Note first that if $\alpha - 3 = k$, then the optimality of $f$ implies $f(\ell_{\alpha-3}) = \alpha$, so that $\text{cost}(f) = 2\alpha = 2(\text{diam}(L) - 1)$, in contradiction with Observation 12. We thus have $\alpha - 3 < k$.

We let $\beta = f(\ell_{\alpha-3})$. We necessarily have $\beta \leq \alpha$.

If $\beta \leq 2$, the mapping $g$ defined by $g(\ell) = \alpha - 1$, $g(\ell_{\alpha-3}) = 3$, and $g(v) = f(v)$ for every other vertex $v$ of $L$, is clearly an independent broadcast on $L$ with $\text{cost}(g) = \text{cost}(f) - 1 - \beta + 3 = \text{cost}(f) - \beta + 2 \geq \text{cost}(f)$, which contradicts the minimality of $\alpha$ or the optimality of $f$ (see Figure 5(a)).

Suppose now $\beta \geq 3$ and let $g_{\alpha-3}$ be the mapping defined by $g_{\alpha-3}(\ell) = 3$, $g_{\alpha-3}(\ell_j) = 2$ for one leaf $\ell_j$ of each spine-subtree $S_j$, $1 \leq j \leq \alpha - 4$, $g_{\alpha-3}(\ell_{\alpha-3}) = 3$ and $g(v) = f(v)$ for every other vertex $v$ of $L$. The mapping $g_{\alpha-3}$ is clearly an independent broadcast on $L$, with

$$\text{cost}(g_{\alpha-3}) = \text{cost}(f) - \alpha - \beta + 3 + 2(\alpha - 4) + 3 = \text{cost}(f) + (\alpha - \beta) - 2$$

(see Figure 5(b), where $S_1, \ldots, S_{\alpha-4}$ are supposed to be of type $S_1$, all other cases being similar). Therefore, the mapping $g_{\alpha-3}$ contradicts either the optimality of $f$ or the minimality of $\alpha$ if $\beta < \alpha - 1$.  

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We can thus assume $\alpha - 1 \leq \beta \leq \alpha$, so that $\beta \geq 6$. Suppose that the spine-subtree $S_{\alpha-2}$ contains a leaf $\ell_{\alpha-2}$ which is $f$-dominated only by $\ell_{\alpha-3}$. In that case, we consider the mapping $g_{\alpha-2}$ defined by $g_{\alpha-2}(\ell) = 3$, $g_{\alpha-2}(\ell_j) = 2$ for one leaf $\ell_j$ of each spine-subtree $S_j$, $1 \leq j \leq \alpha - 4$, $g_{\alpha-2}(\ell_{\alpha-3}) = g_{\alpha-2}(\ell_{\alpha-2}) = 3$, and $g(\nu) = f(\nu)$ for every other vertex $\nu$ of $L$ (see Figure 6(c), where $S_1, \ldots, S_{\alpha-4}$ and $S_{\alpha-2}$ are supposed to be of type $S_1$, all other cases being similar). The mapping $g_{\alpha-2}$ is clearly an independent broadcast on $L$, with

$$\text{cost}(g_{\alpha-2}) = \text{cost}(f) - \alpha - \beta + 3 + 2(\alpha - 4) + 3 + 3 = \text{cost}(f) + (\alpha - \beta) + 1 > \text{cost}(f),$$

which contradicts the optimality of $f$.

Therefore, each leaf of $S_{\alpha-2}$ is $f$-dominated by $\ell_{\alpha-3}$ and (at least) one other vertex $x$. Moreover, we necessarily have $f(\ell_{\alpha-3}) - 1 \leq f(x) \leq f(\ell_{\alpha-3})$ if $S_{\alpha-2}$ is of type $S_1$, or $f(x) = f(\ell_{\alpha-3})$ if $S_{\alpha-2}$ is of type $S_2$. Hence, $f(x) \geq f(\ell_{\alpha-3}) - 1 = \beta - 1 \geq 5$, which implies the uniqueness of $x$, since a 2-leaf of $S_{2\alpha-6}$ and a 1-leaf of $S_{2\alpha-5}$ are at distance 4 from each other, and $v_{2\alpha-4}$ cannot be an $f$-broadcast vertex by Corollary 8.

We consider two subcases, depending on whether $x$ is a 1-leaf of $S_{2\alpha-5}$ or a 2-leaf of $S_{2\alpha-6}$.

(a) $x$ is a 1-leaf of $S_{2\alpha-5}$.

In this case, we consider the mapping $g_x$ defined by $g_x(\ell) = g_x(\ell_{\alpha-3}) = 3$, $g_x(x) = 2$, $g_x(\ell_j) = 2$ for one leaf $\ell_j$ for each of the spine-subtrees $S_j$, $j \in \{1, \ldots, 2\alpha - 6\} \setminus \{\alpha - 3\},$
and $g_x(v) = f(v)$ for every other vertex $v$ of $L$ (see Figure 7(a)). Again, the mapping $g_x$ is clearly an independent broadcast on $L$, with

$$\text{cost}(g_x) = \text{cost}(f) - \alpha + 3 + 2(\alpha - 4) - \beta + 3 + 2(\alpha - 3) - f(x) + 2$$

$$= \text{cost}(f) + 3\alpha - \beta - f(x) - 6.$$

We thus get a contradiction on the minimality of $\alpha$ or the optimality of $f$ since $\alpha \geq 7$ and $f(x) \leq \beta \leq \alpha$.

(b) $x$ is a 2-leaf of $S_{2a-6}$.

In this case, we consider the mapping $g_x$ defined by $g_x(\ell) = g_x(\ell_{a-3}) = g_x(x) = 3$, $g_x(\ell_j) = 2$ for one leaf $\ell_j$ for each of the spine-subtrees $S_j$, $j \in \{1, \ldots, 2a-7\} \setminus \{a-3\}$, and $g_x(v) = f(v)$ for every other vertex $v$ of $L$ (see Figure 7(b)). Again, the mapping $g_x$ is clearly an independent broadcast on $L$, with

$$\text{cost}(g_x) = \text{cost}(f) - \alpha + 3 + 2(\alpha - 4) - \beta + 3 + 2(\alpha - 4) - f(x) + 3$$

$$= \text{cost}(f) + 3\alpha - \beta - f(x) - 7.$$

We thus get a contradiction on the minimality of $\alpha$ or the optimality of $f$ since $\alpha \geq 7$ and $f(x) \leq \beta \leq \alpha$.

We thus obtain a contradiction in each case, which implies that no counter-example to the lemma exists. This completes the proof.

Let $f$ be any independent broadcast on a locally uniform 2-lobster $L$ and $S_i$ be any spine-subtree of $L$. Recall that $f^*(S_i)$ denotes the broadcast value of $S_i$, that is, the sum of the broadcast values of the vertices of $S_i$. The next lemma shows that if $f$ is an optimal independent broadcast on a locally uniform 2-lobster $L$ of length $k \geq 1$, then there exists an optimal independent broadcast $\tilde{f}$ on $L$ such that every spine-subtree of $L$ contains an $\tilde{f}$-broadcast vertex.

**Lemma 25** If $L$ is a locally uniform 2-lobster of length $k \geq 1$, and $f$ is an optimal independent broadcast on $L$, then there exists an optimal independent broadcast $\tilde{f}$ on $L$ such that $f^*(S_i) > 0$ for every $i$, $0 \leq i \leq k$. 

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Figure 7: Independent broadcasts for the proof of Lemma 24, case 4 continued (only one branch per spine-subtree is depicted).
Proof. Since the result directly follows from Lemma 9 if \( k = 1 \), we only need to consider the case \( k > 1 \). Assume to the contrary that there does not exist any such independent broadcast \( \tilde{f} \), and let \( f \) be an optimal independent broadcast on \( L \) that maximises the number of \( f \)-broadcast leaves. Let \( i \) be the smallest index such that \( f^*(S_i) = 0 \), and \( \ell_i \) be any leaf of \( S_i \).

Suppose first that \( S_i \) is of type \( S_2 \), so that \( \ell_i \) is a 2-leaf. From the choice of \( S_i \), we know that \( f^*(S_{i-1}) > 0 \). Moreover, we claim that \( S_{i-1} \) is of type \( S_2 \) and that, for every leaf \( \ell_{i-1} \) of \( S_{i-1} \), \( f(\ell_{i-1}) \leq 4 \). Indeed, this follows from Lemma 24 if \( i = 1 \). If \( i \geq 2 \), \( S_{i-1} \) cannot be of type \( S_1 \) since otherwise any vertex that \( f \)-dominates \( \ell_i \) would also \( f \)-dominate all the leaves of \( S_{i-1} \) (such a dominating vertex must belong to some \( S_j \) with \( j > i \)). Thus \( S_{i-1} \) is of type \( S_2 \) and the fact that \( f^*(S_{i-2}) > 0 \) implies \( f(\ell_{i-1}) \leq 4 \) for every leaf \( \ell_{i-1} \) of \( S_{i-1} \). Therefore, \( \ell_i \) is necessarily \( f \)-dominated by a vertex \( y \in S_j \), for some \( j > i \). Consider the mapping \( g \) defined by \( g(y) = f(y) - 1 \), \( g(\ell_i) = 1 \), and \( g(v) = f(v) \) for every other vertex \( v \) of \( L \). The mapping \( g \) is an independent broadcast on \( L \) with \( \text{cost}(g) = \text{cost}(f) \), which contradicts the maximality of the number of \( f \)-broadcast leaves.

Suppose now that \( S_i \) is of type \( S_1 \), so that \( \ell_i \) is a 1-leaf, and that \( \ell_i \) is \( f \)-dominated by a unique vertex \( y \). If \( y \in S_{i-1} \) then the mapping \( g \) defined by \( g(y) = f(y) - 1 \), \( g(\ell_i) = 1 \), and \( g(v) = f(v) \) for every other vertex \( v \) of \( L \), is an independent broadcast on \( L \), with \( \text{cost}(g) = \text{cost}(f) \), which contradicts the maximality of the number of \( f \)-broadcast leaves. If \( y \notin S_{i-1} \), then \( \ell_i \) is \( f \)-dominated by some \( j > i \). Without loss of generality, \( y \in S_j \), which contradicts the maximality of the number of \( f \)-broadcast leaves.

Suppose finally that \( S_i \) is of type \( S_1 \), and that \( \ell_i \) is \( f \)-dominated by two distinct vertices \( y_1 \) and \( y_2 \), with \( y_1 \in S_{i-1} \) and \( y_2 \in S_{i-2} \). Note that we necessarily have, without loss of generality, \( i_1 = i - 1 \) and \( i < i_2 \). We claim that \( f(y_1) \geq 3 \) and \( f(y_2) \geq 3 \). Indeed, if \( f(y_1) = 2 \), then \( y_1 = v_{i-1} \), which contradicts Corollary 8. The case \( f(y_2) = 2 \) is similar. Moreover, we clearly have either \( f(y_1) = 3 \) and \( f(y_2) = d_L(y_2, \ell_i) \), if \( S_{i-1} \) is of type \( S_1 \), or \( f(y_1) = 4 \) and \( d_L(y_2, \ell_i) \leq f(y_2) \leq d_L(y_2, \ell_i) + 1 \), if \( S_{i-1} \) is of type \( S_2 \). We consider two cases, depending on the value of \( f(y_2) \).

1. \( f(y_2) = d_L(y_2, \ell_i) \).
   In that case, the mapping \( g \) defined by \( g(y_1) = f(y_1) - 1 \), \( g(y_2) = f(y_2) - 1 \), \( g(\ell_i) = 2 \), and \( g(v) = f(v) \) for every other vertex \( v \) of \( L \), is an independent broadcast on \( L \), with \( \text{cost}(g) = \text{cost}(f) \), which contradicts the maximality of the number of \( f \)-broadcast leaves.

2. \( f(y_2) = d_L(y_2, \ell_i) + 1 \).
   In that case, we necessarily have that \( S_{i-1} \) is of type \( S_2 \) and \( f(y_1) = 4 \) on one hand, and \( f(y_2) \geq 4 \) on the other hand, which implies that \( f^*(S_{i-1}) = 0 \), if \( i_2 < k \).
   If \( y_2 \) is not a 1-leaf of \( S_{i+1} \), then the mapping \( g \) defined by \( g(y_1) = 3 \), \( g(y_2) = d_L(y_2, \ell_i) - 1 \), \( g(\ell_i) = 2 + f(y_2) - d_L(y_2, \ell_i) \), and \( g(v) = f(v) \) for every other vertex \( v \) of \( L \), is an independent broadcast on \( L \), with \( \text{cost}(g) = \text{cost}(f) \), which contradicts the maximality of the number of \( f \)-broadcast leaves.

Otherwise, that is, \( y_2 \) is a 1-leaf of \( S_{i+1} \) and \( f(y_2) = 4 \), we cannot give to \( \ell_i \) the broadcast value \( 2 + f(y_2) - d_L(y_2, \ell_i) = 2 + 4 - 3 = 3 \), since \( \ell_i \) would then dominate \( y_2 \). Observe that since \( S_{i+1} \) is of type \( S_1 \), we have \( i + 1 < k \), so that \( S_{i+2} \) exists. If \( S_{i+2} \) is of type \( S_2 \), then the leaves of \( S_{i+2} \) are necessarily \( f \)-dominated only by \( y_2 \). Let \( \ell_{i+2} \) be any leaf of \( S_{i+2} \). In that case, the mapping \( g \) defined by \( g(y_1) = 3 \), \( g(\ell_i) = 2 \), \( g(y_2) = 2 \), \( g(\ell_{i+2}) = 1 \), and \( g(v) = f(v) \) for every other vertex \( v \) of \( L \), is an independent broadcast on \( L \), with \( \text{cost}(g) = \text{cost}(f) \), which contradicts the maximality of the number of \( f \)-broadcast leaves.

Suppose finally that \( S_{i+2} \) is of type \( S_1 \), and let \( \ell_{i+2} \) denote any 1-leaf of \( S_{i+2} \). Note that \( y_2 \) \( f \)-dominates \( \ell_{i+2} \). If \( \ell_{i+2} \) is \( f \)-dominated only by \( y_2 \), then the mapping \( g \) defined by \( g(y_1) = 3 \), \( g(\ell_i) = 2 \), \( g(y_2) = 2 \), \( g(\ell_{i+2}) = 2 \), and \( g(v) = f(v) \) for every other vertex \( v \) of \( L \), is an independent broadcast on \( L \), with \( \text{cost}(g) > \text{cost}(f) \), which contradicts the optimality of \( f \).
Otherwise, let \( z \) be the other vertex of \( L \) which \( f \)-dominates \( \ell_{i+2} \). Then, the mapping \( g \) defined by \( g(y_1) = 3, g(z) = 2, g(y_2) = 2, g(\ell_{i+2}) = 2, g(z) = g(z) - 1 \) and \( g(v) = f(v) \) for every other vertex \( v \) of \( L \), is an independent broadcast on \( L \), with \( \text{cost}(g) = \text{cost}(f) \), which contradicts the maximality of the number of \( f \)-broadcast leaves.

This completes the proof. \[ \square \]

From Lemma 25 and Corollary 8 we get the following corollary.

**Corollary 26** If \( L \) is a locally uniform 2-lobster of length \( k \geq 1 \), and \( f \) is an optimal independent broadcast on \( L \), then there exists an optimal independent broadcast \( \tilde{f} \) on \( L \) such that, for every spine-subtree \( S_i \) of \( L \), \( 0 \leq i \leq k \), and every vertex \( x \) of \( S_i \), \( \tilde{f}(x) \leq 1 \) if \( x = v_i \), \( \tilde{f}(x) \leq 3 \) if \( x \) is a 1-leaf of \( S_i \), and \( \tilde{f}(x) \leq 4 \) if \( x \) is a 2-leaf of \( S_i \).

**Proof.** Let \( \tilde{f} \) be an optimal independent broadcast on \( L \) such that \( \tilde{f}^*(S_i) > 0 \) for every \( i, 0 \leq i \leq k \). The existence of \( \tilde{f} \) is guaranteed by Lemma 25. If \( x = v_i \), then \( \tilde{f}(x) \leq 1 \) follows from Corollary 8. Otherwise, assuming that the claimed bound on \( \tilde{f}(x) \) is not satisfied would imply \( \tilde{f}^*(S) = 0 \) for a neighbouring spine-subtree \( S \) of \( S_i \), in contradiction with our assumption on \( \tilde{f} \). \[ \square \]

The next two lemmas show that if \( f \) is an optimal independent broadcast on a locally uniform 2-lobster \( L \) of length \( k \geq 1 \), then there exists an optimal independent broadcast \( \tilde{f} \) on \( L \) such that the \( \tilde{f} \)-value of every spine-subtree \( S \) of \( L \) is bounded from above by a value depending on the type of \( S \).

Recall that \( T_4 \) denotes the type of sequence used in step 4 in the proof of Lemma 22 that is

\[ T_4 = (\mathcal{E}_2, \mathcal{E}_0) \ . \ (\mathcal{A} \ . \ (\mathcal{C} \ . \ (\mathcal{A} \ . \ (\mathcal{E}_0, \mathcal{E}_2) \right). \]

In the following, when we say that a spine-subtree \( S_i \) appears as an \( \mathcal{A} \)-spine-subtree (resp. as an \( \mathcal{C} \)-spine-subtree) in a sequence of type \( T_4 \), we mean that \( S_i = A_j \) (resp. \( S_i = X_j \)) for some \( j, \) \( 0 \leq j \leq p \) (resp. \( 1 \leq j \leq p \)), in the corresponding sequence \( A_0X_1A_1 \ldots X_pA_p \).

**Lemma 27** If \( L \) is a locally uniform 2-lobster of length \( k \geq 1 \), and \( f \) is an optimal independent broadcast on \( L \), then there exists an optimal independent broadcast \( \tilde{f} \) on \( L \) such that, for every spine-subtree \( S_i \) of \( L \), \( 0 \leq i \leq k \), \( \tilde{f} \) satisfies the following properties.

1. \( \tilde{f}^*(S_i) > 0 \).
2. If \( \tilde{f}^*(S_i) = \lambda_1(S_i) \), or \( \tilde{f}^*(S_i) = \lambda_2(S_i) \), then \( \tilde{f}(\ell) = 1 \) for every leaf of \( S_i \).
3. If \( S_i \) is of type \( S_1 \), then
   (a) \( \tilde{f}^*(S_i) \leq \lambda_1(S_i) \) if \( S_i \) is of type \( \mathcal{B} \) or \( \mathcal{C}_a \),
   (b) \( \tilde{f}^*(S_i) \leq 3 \) if \( S_i \) is of type \( \mathcal{C}_c \),
   (c) \( \tilde{f}^*(S_i) \leq 2 \) if \( S_i \) is of type \( \mathcal{C}_c \) and \( S_i \) belongs to a sequence of type \( T_4 \),
4. If \( S_i \) is of type \( S_2 \), then
   (a) \( \tilde{f}^*(S_i) \leq \lambda_2(S_i) + 1 \) if \( S_i \) is of type \( \mathcal{A} \),
   (b) \( \tilde{f}^*(S_i) \leq \lambda_2(S_i) + \lambda_3(S_i) + \alpha_2(S_i) \) if \( S_i \) is of type \( \mathcal{C}_b \) or \( \mathcal{D} \),
   (c) \( \tilde{f}^*(S_i) \leq \lambda_2(S_i) + \lambda_3(S_i) + \alpha_2(S_i) - 1 \) if \( S_i \) is of type \( \mathcal{C}_b \) and \( S_i \) belongs to a sequence of type \( T_4 \),
5. If \( S_i \) is not of type \( \mathcal{A} \), then \( \tilde{f}(v_i) = 0 \).

**Proof.** Thanks to Lemma 25 we know that we can choose an independent broadcast \( \tilde{f} \) on \( L \) which satisfies Item 1. By Corollary 26 we get that the \( \tilde{f} \)-value of every 1-leaf is at most 3, and that the \( \tilde{f} \)-value of every 2-leaf is at most 4. This observation will be implicitly used all along the proof.
Note also that if \( \tilde{f}^\ast(S_i) = \lambda_1(S_i) \), or \( \tilde{f}^\ast(S_i) = \lambda_2(S_i) \), then we can obviously modify \( \tilde{f} \), in order to satisfy Item 2 without modifying its cost.

We now prove that \( \tilde{f} \) can be chosen in such a way that it satisfies all the other items of the lemma. Let \( S_i \) be any spine-subtree of \( L \).

Note first that the above observation already proves Item 3\( A \). Moreover, observe that we necessarily have \( \tilde{f}^\ast(S_i) \leq \lambda_1(S_i) \) if \( S_i \) is of type \( B \) or \( C_\alpha \), since in each of these cases, the only way to attain this value is to have one leaf \( \ell \) of \( S_i \) with \( \tilde{f}(\ell) = \tilde{f}^\ast(S_i) \), which would imply that a neighbouring spine-vertex of \( S_i \) has no \( \tilde{f} \)-broadcast vertex, in contradiction with Item 1. This proves Item 3\( A \).

Suppose now that \( S_i \) is of type \( A \). If the broadcast value of a 2-leaf of \( S_i \) is 2, then its at least two sister-leaves cannot be \( f \)-broadcast vertices since this would contradict the optimality of \( \tilde{f} \) (by giving the broadcast value 1 to each 2-leaf of a branch \( B \) of \( S_i \), we get \( \tilde{f}^\ast(B) \geq \lambda_2(B) \)). Therefore, the greatest possible value of \( \tilde{f}^\ast(S_i) \) is obtained when the spine-vertex \( v_i \) and all the 2-leaves of \( S_i \) have \( \tilde{f} \)-value 1. This gives \( \tilde{f}^\ast(S_i) \leq \lambda_2(S_i) + 1 \), which proves Item 4\( A \).

Suppose now that \( S_i \) is of type \( C_b \) or \( D \). Observe first that, if \( \tilde{f}(v_i) = 1 \), then the broadcast value of every leaf of \( S_i \) is 1. The optimality of \( \tilde{f} \) then implies that \( S_i \) has a unique branch \( B \) with two leaves and no branch with a unique leaf, since otherwise we could set to 0 the broadcast value of \( v_i \) and to 3 the broadcast value of one leaf of every such branch, and thus increase the cost of \( \tilde{f} \). (Note also that, for the same cost, we can set \( \tilde{f}(v_i) = 0 \) and set to 0 and 3 the broadcast value of the two leaves of this branch. This remark will be useful in the next paragraph.) In that case, we thus have \( \tilde{f}^\ast(S_i) = \lambda_2(S_i) + 1 \leq \lambda_2(S_i) + \lambda_2^2(S_i) + \alpha_2(S_i) \), which proves Item 4\( A \).

Suppose now \( \tilde{f}(v_i) = 0 \) and let \( B \) be any branch of \( S_i \). The optimality of \( \tilde{f} \) then implies the following. If \( B \) has one or two 2-leaves, the \( \tilde{f} \)-value of one of these leaves is 3 (otherwise, we would have \( \tilde{f}^\ast(B) \leq 2 \)). If \( B \) has at least three leaves, the largest possible value of \( \tilde{f}^\ast(B) \) is \( \lambda_2(B) \), since as soon as a 2-leaf has a broadcast value at least 2, none of its sister-leaves can be a broadcast vertex. (Note that if \( B \) has three 2-leaves, then either one of them has \( \tilde{f} \)-value 3, or, for the same cost, each of them has \( \tilde{f} \)-value 1.) Therefore, \( \tilde{f}^\ast(S_i) \leq \lambda_2(S_i) + \lambda_2^2(S_i) + \alpha_2(S_i) \), which proves Item 4\( A \).

Suppose now that \( S_i \) is of type \( C_b \) or \( C_c \), and belongs to some sequence of type \( T_4 \). In such a sequence, each spine-subtree of type \( C \) is associated with one of its neighbouring spine-subtrees of type \( A \), in such a way that no spine-subtree of type \( A \) is associated with two distinct spine-subtrees of type \( C \). Let \( S_i' \) denote the spine-subtree associated with \( S_i \) (we have \( S_i' \notin \{S_{i-1}, S_{i+1}\} \)). On the one hand, from the above discussion about spine-subtrees of type \( A \), we know that their largest possible broadcast value can be attained only if their spine-vertex has broadcast value 1. On the other hand, from the above discussion about spine-subtrees of type \( C_b \) or \( C_c \), we know that their largest possible broadcast value can be attained only if one leaf \( \ell \) of the unique branch \( B \) of \( S_i \) having at most two, or exactly two leaves, has a broadcast value of 3. Therefore, \( S_i \) and \( S_i' \) cannot get their largest possible broadcast value both at the same time. We thus need either to remove the broadcast value of the spine-vertex of \( S_i' \), or to give the broadcast value 2 to \( \ell \) if \( \ell \) is a 2-only-leaf, or 1 to each 2-leaf of \( B \) otherwise. This second choice proves that the optimal independent broadcast \( \tilde{f} \) can be chosen in order to satisfy Items 4\( B \) and 4\( C \).

It remains to prove Item 5. If \( S_i \) is of type \( S_1 \), the result follows from Lemma 3. We thus only need to consider the case when \( S_i \) is of type \( C_b \) or \( D \). Suppose that \( \tilde{f}(v_i) > 1 \) by Corollary 3. If \( S_i \) is of type \( C_b \), then we can set \( \tilde{f}(v_i) = 0 \), \( \tilde{f}(\ell) = 3 \) for a 2-leaf \( \ell \) of \( S_i \) belonging to the unique branch of \( S_i \) having at most two leaves, and \( \tilde{f}(\ell') = 0 \) for the sister-leaf \( \ell' \) of \( \ell \), if any. Such a modification does not decrease the cost of \( \tilde{f} \) and we are done. If \( S_i \) is of type \( D \), then we cannot have \( \tilde{f}(v_i) = 1 \), since this would contradict the optimality of \( \tilde{f} \), as the previous modification can be done on the at least two branches of \( S_i \) having at most two 2-leaves.

This completes the proof. \( \square \)

**Lemma 28** If \( L \) is a locally uniform 2-lobster of length \( k \geq 1 \), and \( f \) is an optimal independent broadcast on \( L \), then there exists an optimal independent broadcast \( \tilde{f} \) on \( L \) such that, for every spine-subtree \( S_i \) of \( L \), \( 0 \leq i \leq k \), \( \tilde{f} \) satisfies the following properties.
1. \( \tilde{f} \) satisfies all the items of Lemma \([27]\).

2. \( \tilde{f}^*(S_i) \leq \lambda_2(S_i) \) if \( S_i \) is of type \( \mathcal{A} \) and \( S_i \) does not appear as an \( \mathcal{A} \)-spine-subtree in a sequence of type \( T_4 \).

**Proof.** Thanks to Lemma \([27]\), we know that we can choose an independent broadcast on \( L \) which satisfies Item \([1]\). So consider such a broadcast \( \tilde{f} \) on \( L \). Recall that by Item \([5]\) of Lemma \([27]\), \( \tilde{f}(v_i) = 0 \) for every spine-subtree \( S_i \) which is not of type \( \mathcal{A} \). Moreover, by Item \([13]\) of Lemma \([27]\), if \( S_i \) is a spine-subtree of \( L \) of type \( \mathcal{A} \), then \( \tilde{f}^*(S_i) \leq \lambda_2(S_i) + 1 \), and, as observed in the proof of that lemma, the only way to attain this value is to give a broadcast value of 1 to the spine-vertex and to all the 2-leaves of \( S_i \).

Suppose that there exists in \( L \) a spine-subtree \( S_i \) of type \( \mathcal{A} \), that does not appear as an \( \mathcal{A} \)-spine-subtree in any sequence of type \( T_4 \), and such that \( \tilde{f}(v_i) = 1 \), which implies \( \tilde{f}(\ell) = 1 \) for every leaf \( \ell \) of \( S_i \) since \( \tilde{f} \) is optimal. Such a spine-subtree will be called a bad spine-subtree. Moreover, suppose that \( S_i \) is the leftmost such bad spine-subtree of \( L \). We claim that the broadcast \( \tilde{f} \) can be modified, without decreasing its cost, in such a way that either the number of bad spine-subtrees in \( L \) strictly decreases, or this number is still the same but the index of the leftmost bad spine-subtree in \( L \) strictly increases, which will prove Item \([2]\).

All along this proof, we will modify the independent broadcast \( \tilde{f} \) on some spine-subtrees, according to their type. By applying the standard modification of \( \tilde{f} \) on a spine-subtree \( S_j \) of \( L \), we mean the following.

- If \( S_j \) is a bad spine-subtree, then we set \( \tilde{f}(v_j) = 0 \).
- If \( S_j \) is of type \( \mathcal{A} \) and is not a bad spine-subtree, then we set \( \tilde{f}(v_j) = 1 \).
- If \( S_j \) is of type \( \mathcal{C}_0 \), then we set \( \tilde{f}(\ell) = 3 \) for a 2-leaf \( \ell \) of the unique branch of \( S_j \) having at most two leaves.
- If \( S_j \) is of type \( \mathcal{C}_c \), then we set \( \tilde{f}(\ell) = 3 \) for a 1-leaf \( \ell \) of \( S_j \), and \( \tilde{f}(\ell') = 0 \) for the sister-leaf of \( \ell \).
- If \( S_j \) is of type \( \mathcal{D} \), then we set \( \tilde{f}(\ell) = 3 \) for one 2-leaf \( \ell \) of each branch of \( S_j \) having at most two 2-leaves, and \( \tilde{f}(\ell') = 0 \) for its sister-leaf \( \ell' \), if any.

We first consider the case when \( S_i \) belongs to some sequence of type \( T_4 \), but not as an \( \mathcal{A} \)-spine-subtree, which implies that the length of this sequence is at least 3. Let \( A_0X_1A_1\ldots X_pA_p, p \geq 1 \), denote the corresponding sequence. Every spine-subtree corresponding to some \( X_j \), \( 1 \leq j \leq p \), is surrounded by two spine-subtrees of type \( \mathcal{A} \). If any such spine-subtree \( S_{a_i} \) (corresponding to some \( X_j \)) is of type \( \mathcal{A} \), then having \( \tilde{f}(s_{a_i}) = 1 \) would imply that none of \( s_{a_i-1} \) and \( s_{a_i+1} \) is an \( \tilde{f} \)-broadcast vertex. We can thus apply the standard modification of \( \tilde{f} \) on all the spine-subtrees \( A_0, X_1, A_1, \ldots, X_p, A_p \), without decreasing the cost of \( \tilde{f} \).

We suppose now that \( S_i \) does not belong to any sequence of type \( T_4 \). The following claims will be useful in the sequel.

**Claim 29** If a bad spine-subtree \( S_j \), not belonging to any sequence of type \( T_4 \), has a neighbouring spine-subtree of type \( \mathcal{D} \), then we can modify \( \tilde{f} \), without decreasing its cost, in such a way that the number of bad spine-subtrees strictly decreases.

**Proof.** Suppose that \( S_{i+1} \) exists and is of type \( \mathcal{D} \) (the case when \( S_{i-1} \) exists and is of type \( \mathcal{D} \) is similar). If \( S_{i+2} \) does not exist, or if \( S_{i+2} \) exists and \( \tilde{f}(v_{i+2}) = 0 \), then we can apply the standard modification of \( \tilde{f} \) on \( S_i \) and \( S_{i+1} \), without decreasing the cost of \( \tilde{f} \). Finally, if \( S_{i+2} \) exists and \( \tilde{f}(v_{i+2}) = 1 \), we get that \( S_{i+2} \) is also a bad spine-subtree of \( L \). In that case, we can apply the standard modification of \( \tilde{f} \) on \( S_i, S_{i+1} \) and \( S_{i+2} \), without decreasing the cost of \( \tilde{f} \) since \( S_{i+1} \) has at least two branches with at most two leaves. \( \square \)
Claim 30 Let $S_i$ be the leftmost bad spine-subtree in $L$, that does not belong to any sequence of type $T_4$. If $S_{i+1}$ is of type $A$ and $S_{i+2}$ is of type $B$, $C_a$ or $C_c$, then we can modify $\tilde{f}$, without decreasing its cost, so that the index of the leftmost bad spine-subtree in $L$ strictly increases.

Proof. Since $\tilde{f}(v_i) = 1$, we get that $\tilde{f}^*(S_{i+2}) = \lambda_1(S_{i+2})$ (the $\tilde{f}$-broadcast value of every leaf in $S_{i+2}$ is 1), so that we can apply the standard modification of $\tilde{f}$ on $S_i$ and $S_{i+1}$ (recall that $\tilde{f}(\ell) = 1$ for every leaf of $S_{i+2}$ by Item 2 of Lemma 27), without decreasing the cost of $\tilde{f}$, but strictly increasing the index of the leftmost bad spine-subtree in $L$. $\square$

Claim 31 If $S_i$ is the leftmost bad spine-subtree in $L$, that does not belong to any sequence of type $T_4$, then we can assume that $S_{i+1}$ is of type $C_b$ or $C_c$.

Proof. If $S_{i-1}$ is of type $C_c$, then, since $\tilde{f}(v_i) = 1$, we get that $\tilde{f}(\ell) = 1$ for every 1-leaf of $S_{i-1}$.

Similarly, if $S_{i-1}$ is of type $C_b$, then $\tilde{f}(\ell) \leq 2$ for every 2-leaf of $S_{i-1}$. In both cases, we can thus apply the standard modification of $\tilde{f}$ on $S_i$ and $S_{i-1}$, without decreasing the cost of $\tilde{f}$.

Thanks to Claim 29, we can thus assume that $S_{i-1}$ either does not exist or is of type $A$, $C_a$ or $B$ (these two latter cases imply that $S_{i-2}$ cannot be of type $C_c$, and is thus necessarily of type $B$ or $C_a$). If $S_{i-1}$ is of type $A$ and $S_{i-2}$ is of type $C_c$, then, similarly as above, we can apply the standard modification of $\tilde{f}$ on $S_i$ and $S_{i-2}$, without decreasing the cost of $\tilde{f}$.

In all the remaining cases, $S_i$ is of type $\langle E_2, E_0 \rangle A$, so that we necessarily have that either $S_{i+1}$ is of type $C_b$ or $C_c$, or $S_{i+1}$ is of type $A$ and $S_{i+2}$ is of type $C_c$, since otherwise $S_i$ would be a sequence of type $T_4$. But the case when $S_{i+1}$ is of type $A$ and $S_{i+2}$ is of type $C_c$ is covered by Claim 30.

This concludes the proof. $\square$

Thanks to Claims 29 and 31, we can assume that $S_{i-1}$ is not of type $D$, and that $S_{i+1}$ is of type either $C_b$ or $C_c$. We consider these two cases separately when $S_{i+2}$ is not a bad spine-subtree, or together otherwise.

1. $S_{i+1}$ is of type $C_b$ and $S_{i+2}$ is not a bad spine-subtree.

In that case, we can apply the standard modification of $\tilde{f}$ on $S_i$ and $S_{i+1}$ and we are done.

2. $S_{i+1}$ is of type $C_c$ and $S_{i+2}$ is not a bad spine-subtree.

In this case, $S_{i+2}$ necessarily exists and is of type $S_2$. If $S_{i+3}$ is not a bad spine-subtree, then we can apply the standard modification of $\tilde{f}$ on $S_i$, $S_{i+1}$, $S_{i+2}$ and $S_{i+3}$, without decreasing the cost of $\tilde{f}$.

(a) $S_{i+3}$ is a bad spine-subtree and $S_{i+2}$ is of type $C_b$.

In that case, we get that $\tilde{f}(\ell) \leq 2$ for every 2-leaf $\ell$ of $S_{i+2}$, so that we can apply the standard modification of $\tilde{f}$ on $S_i$, $S_{i+1}$, $S_{i+2}$ and $S_{i+3}$, without decreasing the cost of $\tilde{f}$.

(b) $S_{i+3}$ is a bad spine-subtree and $S_{i+2}$ is of type $A$.

In that case, $S_{i+4}$ must exist and must be of type $C_c$, since otherwise the sequence $S_i S_{i+1} S_{i+2}$ would be of type $T_4$. Observe then that $S_{i+3}$ is somehow “in the same situation as $S_i$”.

Let $L' = S_1 S_2 S_3 S_4 \ldots S_s$, $s \geq 5$, be the maximal subsequence of $L$, starting at $S_i$ (that is, $S'_1 = S_i$), whose type is a prefix of $(A, C_c, A)^*$ (considered as a word), and such that $S'_j$ is a bad spine-subtree if $j \equiv 1 \pmod{3}$. We then have three cases, according to the value of $(s \mod{3})$.

i. If $s \equiv 1 \pmod{3}$, which means that $S_{i+s}$ is not of type $C_c$, we get that $S_i \ldots S_{i+s-1}$ is a sequence of type $T_4$, a contradiction.

ii. If $s \equiv 0 \pmod{3}$, which means that $S_{i+s}$ is not a bad spine-subtree, we can apply the standard modification of $\tilde{f}$ on all the spine-subtrees $S'_j$ of $L'$ with $j \not\equiv 0 \pmod{3}$, without decreasing the cost of $\tilde{f}$.

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We are now able to prove the main result of our paper.

3.5 Main result

We claim that for every spine-subtree

Assume \( k \) in its proof reaches the upper bounds on the broadcast values stated in Lemma 27. We can thus or 28, which will prove the theorem.

1. If \( S_i \) is of type \( C_b \), then \( f^*(S_i) \) has been set to \( \lambda_1(S_i) \) in Step 1, and never modified in the following steps.

2. If \( S_i \) is of type \( C_b \), then \( f^*(S_i) \) has been set to \( \lambda_2(S_i) + \lambda_3(S_i) + \alpha_2(S_i) \) in Steps 1 and 2. Moreover, if \( S_i \) belongs to some sequence of type \( T_4 \), then \( f^*(S_i) \) has been decreased by 1 in Step 4.

3. \( S_{i+1} \) is of type \( C_b \) or \( C_c \), and \( S_{i+2} \) is a bad spine-subtree.

In this case, we get that either \( S_{i+3} \) is of type \( C_b \) or \( C_c \), or \( S_{i+3} \) is of type \( A \) and \( S_{i+4} \) is of type \( C_c \), since otherwise \( S_i S_{i+1} S_{i+2} \) would be a sequence of type \( T_4 \). Therefore, \( S_{i+2} \) is “in the same situation as \( S_i \)”.

Let \( L' = S_1'S_2'S_3' \ldots S_s' \), \( s \geq 4 \), be the maximal subsequence of \( L \), starting at \( S_i \) (that is, \( S_1' = S_i \)), whose type is a prefix of \( (A.(C_b(C_c))^* \) (considered as a word), and such that \( S_j' \) is a bad spine-subtree if \( j \equiv 1 \pmod{2} \). We then have two cases, according to the value of \( s \pmod{2} \).

(a) If \( s \equiv 1 \pmod{2} \), which means that \( S_{i+s} \) is not of type \( C_b \) nor \( C_c \), we get that \( S_{i+s} \) is of type \( A \) and \( S_{i+s+1} \) is of type \( C_c \), since otherwise \( L' \) would be a sequence of type \( T_4 \).

In that case, since \( S_{i+s-1} \) is a bad spine-subtree, we get that \( f^*(S_{i+s+1}) = 2 \), so that we can apply the standard modification of \( f \) on the spine-subtrees \( S_i, \ldots, S_{i+s} \), without decreasing the cost of \( f \).

(b) If \( s \equiv 0 \pmod{2} \), which means that \( S_{i+s} \) is not a bad spine-subtree, we consider two cases.

If \( S_{i+s-1} \) is of type \( C_b \), then we can apply the standard modification of \( f \) on the spine-subtrees \( S_i, \ldots, S_{i+s-1} \), without decreasing the cost of \( f \).

If \( S_{i+s-1} \) is of type \( C_c \), we get that \( S_{i+s} \) is of type either \( A \) (but not a bad spine-subtree), \( C_b \) or \( D \). If \( S_{i+s+1} \) is not a bad spine-subtree, then, again, we can apply the standard modification of \( f \) on the spine-subtrees \( S_i, \ldots, S_{i+s-1} \), without decreasing the cost of \( f \).

If \( S_{i+s+1} \) is a bad spine-subtree, we get a new subsequence \( L' \), whose length has been increased by 1, so that we now have \( s \equiv 1 \pmod{2} \) and the previous case applies.

This completes the proof. \( \square \)

3.5 Main result

We are now able to prove the main result of our paper.

**Theorem 32** For every locally uniform 2-lobster \( L \) of length \( k \geq 0 \), \( \beta_b(L) = \beta^*(L) \).

**Proof.** If \( k = 0 \), the result follows from Lemma 21, observing that the independent broadcast built in its proof reaches the upper bounds on the broadcast values stated in Lemma 27. We can thus assume \( k \geq 1 \). By Lemma 22, we know that there exists an independent broadcast \( f \) on \( L \) with cost \( f(L) = \beta^*(L) \). Let \( f \) be the independent broadcast on \( L \) constructed in the proof of Lemma 22.

We claim that for every spine-subtree \( S_i \) of \( L \), \( f^*(S_i) \) equals the upper bound given in Lemmas 27 or 28, which will prove the theorem.

1. If \( S_i \) is of type \( B \) or \( C_a \), then \( f^*(S_i) \) has been set to \( \lambda_1(S_i) \) in Step 1, and never modified in the following steps.

2. If \( S_i \) is of type \( C_b \), then \( f^*(S_i) \) has been set to \( \lambda_2(S_i) + \lambda_3(S_i) + \alpha_2(S_i) \) in Steps 1 and 2. Moreover, if \( S_i \) belongs to some sequence of type \( T_4 \), then \( f^*(S_i) \) has been decreased by 1 in Step 4.
3. If $S_i$ is of type $C$, then $f^*(S_i)$ has been set to 3 in Steps 1 and 3. Moreover, if $S_i$ belongs to some sequence of type $T_4$, then $f^*(S_i)$ has been decreased by 1 in Step 4.

4. If $S_i$ is of type $D$, then $f^*(S_i)$ has been set to $\lambda_2(S_i) + \lambda_4(S_i) + \alpha_2(S_i)$ in Steps 1 and 2, and never modified in the following steps.

5. Finally, if $S_i$ is of type $A$, then $f^*(S_i)$ has been set to $\lambda_1(S_i)$ in Step 1. Moreover, if $S_i$ belongs to some sequence of type $T_4$, then $f^*(S_i)$ has been increased by 1 in Step 4.

This concludes the proof. 

4 Concluding remarks

In this paper, we have given an explicit formula for the broadcast independence number of a subclass of lobsters, called locally uniform 2-lobsters. Moreover, it is easily seen that computing the value $\beta^*(L)$ for a locally uniform 2-lobster $L$ of length $k$ can be done in linear time (simply processing the spine-subtrees $S_0, \ldots, S_k$ in that order), which improves the result of Bessy and Rautenbach [5] for this particular subclass of trees.

A natural question, as a first step, would be to extend our result to the whole subclass of locally uniform lobsters. In fact, we were able to give an explicit formula for every such lobster not containing any spine-subtree of type $Z$, that is, having exactly one branch and three 2-leaves (see [1]). However, the proof is then quite involved and we thus decided to only consider in this paper the restricted class of locally uniform 2-lobsters. Determining when the optimal broadcast value of a spine-subtree of type $Z$ is 3 or 4 appears to be not so easy.

The more general question of giving an explicit formula for the broadcast independence number of the whole class of lobsters is certainly more challenging.

References


