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Existence and regularity of law density of a pair (diffusion, first component running maximum)

Laure Coutin*, Monique Pontier†

July 2, 2018

Abstract

Let X be a continuous d -dimensional diffusion process and M the running supremum of the first component. We show that, $\forall t > 0$, the law of the $(d+1)$ random vector (M_t, X_t) admits a density with respect to the Lebesgue measure using Malliavin's calculus. In case $d = 1$ we prove the regularity of this density.

Keywords: Running supremum process, joint law density, Malliavin calculus, regularity of the density.

A.M.S. Classification: 60J60, 60H07, 60H10.

In this paper one is interested in the joint law of the random vector (M, X) where X is a d -dimensional diffusion process and M the running supremum of its first component. When the process X is a Brownian motion the result is well known, see for instance [13]. For general Gaussian processes, the law of the maximum is studied in [2]. The cases where the process X is a martingale, or a Lévy process are deeply studied in the literature. Some specific diffusion processes as Orstein-Uhlenbeck were also investigated.

A lot of studies are devoted to the maximum of martingales, their terminal value, their maximum at terminal time. For instance look at Rogers et al. [18, 9, 3]. Cox-Obloj [7] aim is to exhibit an hedging strategy of the so-called “no touch option”, meaning that the payoff is the indicator of the set $\{\bar{S}_T < b; \underline{S}_T > a\}$, the price process S being given. They are not concerned with the law of the pair (process, its

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running maximum). A lot of papers are mainly interested in the hedging of barrier option, for instance [3].

The case of general Lévy processes is studied by Doney and Kyprianou [8]. In particular cases driven by a Brownian motion and a compound Poisson process, Roynette et al. [19] provide the Laplace transform of undershot-overshot-hitting time law. In [15, 6] the pair (the process-its running maximum) law density is proved to be a weak solution of a partial differential equation. We quote [4] which is the starting point of our study. Lagnoux et al. [14] provide the law density of such a pair, but in case of a reflected Brownian motion.

Concerning the diffusion processes, for instance the Ornstein Uhlenbeck process, the density of the running maximum law is given in [1]. Some analogous results are provided in [9] concerning random walks.

Quote Jeanblanc et al. [13] for the one-dimensional diffusion process: a PDE is obtained for the law density of the process stopped before hitting a moving barrier. In [10] a multi-dimensional diffusion (whose corresponding diffusion vector fields are commutative) joint distribution is studied at the time when a component attains its maximum on finite time interval; under regularity and ellipticity conditions the smoothness of this joint distribution is proved.

Here we look for more general (but continuous) cases where this density exists. In d -dimensional case, the law of the $(d+1)$ -random vector (M_t, X_t) is absolutely continuous, where $M_t := \sup_{s \leq t} X_s^1$. In the one-dimensional case we get the regularity of this density. Such results are used in our work in progress [5] where this density is proved to be a weak solution of a partial differential equation.

The model is as following: let a filtered probability space $(\Omega, (\mathcal{F}_t = \sigma(W_u, u \leq t))_{t \geq 0}, \mathbb{P})$ where $W := (W_u, u \geq 0)$ is a d -dimensional Brownian motion. Let X be a d -dimensional diffusion process solution to

$$dX_t = B(X_t)dt + \sum_{i=1}^d A_i(X_t)dW_t, \quad X_0 = x \in \mathbb{R}^d, \quad t > 0,$$

where $B: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $A: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ are bounded with bounded and continuous differential.

In Section 1 we recall some elements in Malliavin's calculus that we use to prove that the law of $V_t := (M_t, X_t)$ is absolutely continuous with respect to the Lebesgue measure in a general case with some standard assumptions on the coefficients A and B . In Section 2, we turn to the one-dimensional case and we prove the regularity of the density of the law of V_t under some more technical assumptions. Section 3 provides an example where the announced partial differential equation is exactly stated.

1 The law of V_t is absolutely continuous

Here it is proved that for any $t > 0$, the joint law of $V_t := (M_t, X_t)$ admits a density with respect to the Lebesgue measure. For this purpose, we use ‘‘Malliavin calculus’’ specifically Nualart’s results [16]. Let us denote $C_b^i(\mathbb{R}^d)$ the set of the functions i times differentiable, bounded, with bounded derivatives.

Theorem 1.1 *We assume that A and B satisfy*

$$(1) \quad A \text{ and } B \in C_b^1(\mathbb{R}^d)$$

and there exists a constant $c > 0$ such that

$$(2) \quad c\|v\|^2 \leq v^T A(x)A^T(x)v, \quad \forall v, x \in \mathbb{R}^d.$$

Then the joint law of $V_t = (M_t, X_t)$ admits a density with respect to the Lebesgue measure for all $t > 0$.

The next subsection recalls some useful definitions and results.

1.1 Short Malliavin calculus summary

The material of this subsection can be found in Section 1.2 of [16]. Let $\mathbb{H} = L^2([0, T], \mathbb{R}^d)$ endowed with the usual scalar product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and the associated norm $\|\cdot\|_{\mathbb{H}}$.

For all $(h, \tilde{h}) \in \mathbb{H}^2$,

$$W(h) := \int_0^T h(t)dW_t$$

is a centered Gaussian variable with variance equal to $\|h\|_{\mathbb{H}}^2$. If $\langle h, \tilde{h} \rangle_{\mathbb{H}} = 0$ then the random variables $W(h)$ and $W(\tilde{h})$ are independent.

Let \mathcal{S} denote the class of smooth random variables F defined as following:

$$(3) \quad F = f(W(h_1), \dots, W(h_n))$$

where $n \in \mathbb{N}$, $h_1, \dots, h_n \in \mathbb{H}$ and f belongs to $C_b(\mathbb{R}^n)$.

Definition 1.2 *The Malliavin derivative of a smooth variable F defined in (3) is the \mathbb{H} valued random variable given by*

$$DF = \sum_{i=1}^n \partial_i f(W(h_1), \dots, W(h_n))h_i.$$

Proposition 1.3 *The operator D is closable from $L^p(\Omega)$ into $L^p(\Omega, \mathbb{H})$ for any $p \geq 1$.*

For any $p \geq 1$, let us denote $\mathbb{D}^{1,p}$ the domain of the operator D in $L^p(\Omega)$, meaning that $\mathbb{D}^{1,p}$ is the closure of the class of smooth random variables \mathcal{S} with respect to the norm

$$\|F\|_{1,p} = (\mathbb{E}[|F|^p] + [\mathbb{E}[\|DF\|_{\mathbb{H}}^p]])^{1/p}.$$

Malliavin calculus is a powerful tool to prove the absolute continuity of random variables law. Namely Theorem 2.1.2 page 97 [16] states:

Theorem 1.4 *Let $F = (F^1, \dots, F^m)$ be a random vector satisfying the following conditions*

- (i) F^i belongs to $\mathbb{D}^{1,p}$ for $p > 1$ for all $i = 1, \dots, m$,
- (ii) the Malliavin matrix $\gamma_F := (\langle DF^i, DF^j \rangle_{\mathbb{H}})_{1 \leq i, j \leq m}$ is invertible.

Then the law of F is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^m .

Using this theorem, the proof of Theorem 1.1 will be a consequence of the following:

- X_t^i , $i = 1, \dots, d$ and M_t belongs to $\mathbb{D}^{1,p}$ $p > 1$, see Lemma 1.5;
- the $(d+1) \times (d+1)$ matrix $\gamma_V(t) := (\langle DV_t^i, DV_t^j \rangle)_{1 \leq i, j \leq d+1}$ is almost surely invertible, see Proposition 1.6.

1.2 Malliavin differentiability of the supremum

Lemma 1.5 *Under Assumption (1), X_t^i , $i = 1, \dots, d$ and M_t belongs to $\mathbb{D}^{1,p}$ $\forall p \geq 1$ for all $t > 0$.*

Proof: Using Theorem 2.2.1 [16], under Assumption (1),

- X_t^i , $i = 1, d$ belong to $\mathbb{D}^{1,\infty}$ for all $t > 0$,
- $\forall t \leq T$, $\forall p > 0$, $\forall i = 1, \dots, d$, there exists a constant C_T^p such that

$$(4) \quad \sup_{0 \leq r \leq t} \mathbb{E} \left(\sup_{r \leq s \leq T} |D_r X_s^i|^p \right) = C_t \leq C_T^p < \infty,$$

- the Malliavin derivative $D_r X_t$ satisfies $D_r X_t = 0$ for $r > t$ almost surely and for $r \leq t$ almost surely, using Einstein's convention (2.56 page 125 [16]):

$$(5) \quad D_r X_t^i = A^i(X_r) + \int_r^t \bar{A}_{k,\alpha}^i(s) D_r(X_s^k) dW_s^\alpha + \int_r^t \bar{B}_k^i(s) D_r(X_s^k) ds$$

where $\bar{A}_{k,\alpha}^i(s) := \partial_k A_\alpha^i(X_s)$ and $\bar{B}_k^i := \partial_k B^i(X_s)$ are in \mathbb{R}^d .

(i) In order to prove that M_t belongs to $\mathbb{D}^{1,p}$ we follow the same lines as in the proof of Proposition 2.1.10 [16] with index p instead of 2. For any $i = 1, \dots, d$, we

establish that the \mathbb{H} -valued process $(D.X_t^i, t \in [0, T])$ has a continuous modification and satisfies $\mathbb{E}(\|D.X_t^i\|_H^p) < \infty$.

We now use Appendix (A.11) in Nualart [16], as a corollary of Kolmogorov's continuity criterion: if there exist positive real numbers α, β, K such that

$$(6) \quad E[\|D.X_{t+\tau}^i - D.X_t^i\|_H^\alpha] \leq K\tau^{1+\beta}, \quad \forall t \geq 0, \tau \geq 0$$

then $D.X^i$ admits a continuous modification.

(ii) As a second step we prove (6). Let $\tau > 0$, Equation (5) yields

$$\begin{aligned} \Delta_\tau D_r(X_t^i) &:= D_r(X_{t+\tau}^i) - D_r(X_t^i) \\ &= \int_{\max(r,t)}^{\max(t+\tau,r)} \bar{B}_k^i(s) D_r(X_s^k) ds + \int_{\max(r,t)}^{\max(t+\tau,r)} \bar{A}_{k,\alpha}^i(s) D_r(X_s^k) dW_s^\alpha. \end{aligned}$$

Using the definition of \mathbb{H}

$$\|\Delta_\tau D_r(X_t^i)\|_{\mathbb{H}}^2 = \int_0^T \left| \int_{\max(r,t)}^{\max(t+\tau,r)} \bar{B}_k^i(s) D_r(X_s^k) ds + \int_{\max(r,t)}^{\max(t+\tau,r)} \bar{A}_{k,\alpha}^i(s) D_r(X_s^k) dW_s^\alpha \right|^2 dr.$$

According to Jensen's inequality for $p \geq 2$

$$\|\Delta_\tau D_r(X_t^i)\|_H^p \leq T^{\frac{p}{2}-1} \int_0^T \left| \int_{\max(r,t)}^{\max(t+\tau,r)} \bar{B}_k^i(s) D_r(X_s^k) ds + \int_{\max(r,t)}^{\max(t+\tau,r)} \bar{A}_{k,\alpha}^i(s) D_r(X_s^k) dW_s^\alpha \right|^p dr.$$

Using $(a+b)^p \leq 2^{p-1}(a^p + b^p)$,

$$\|\Delta_\tau D_r(X_t^i)\|_H^p \leq 2^{p-1} T^{\frac{p}{2}-1} \int_0^T \left[\left| \int_t^{t+\tau} \bar{B}_k^i(s) D_r(X_s^k) ds \right|^p + \left| \int_t^{t+\tau} \bar{A}_{k,\alpha}^i(s) D_r(X_s^k) dW_s^\alpha \right|^p \right] dr.$$

The expectation of the first term is bounded using Jensen's inequality and (4) for any $r \in [0, T]$:

$$E \left[\left| \int_t^{t+\tau} \bar{B}_k^i(s) D_r(X_s^k) ds \right|^p \right] \leq \|\bar{B}\|_\infty^p \tau^{p-1} \sup_r E \left[\sup_{r \leq s \leq T} |D_r(X_s^k)|^p \right] = \|\bar{B}\|_\infty^p \tau^p C_T^p.$$

Using once again (4), Burkholder-Davis Gundy' and Jensen's inequalities, the expectation of the second term satisfies for any $r \in [0, T]$:

$$\begin{aligned} E \left[\left| \int_t^{t+\tau} \bar{A}_{k,\alpha}^i(s) D_r(X_s^k) dW_s^\alpha \right|^p \right] &\leq C_p E \left[\left(\int_t^{t+\tau} |\bar{A}_{k,\alpha}^i(s) D_r(X_s^k)|^2 ds \right)^{p/2} \right] \\ &\leq C_p \|\bar{A}\|_\infty^p \tau^{p/2-1} \int_t^{t+\tau} E \left(\sup_{r \leq s \leq T} |D_r(X_s^i)|^p \right) ds \leq C_p \|\bar{A}\|_\infty^p \tau^{p/2-1} C_T^p \tau, \end{aligned}$$

thus for any $\tau \in [0, 1]$ there exists a constant $D = T^{p/2} 2^{p/2-1} C_T^p (\|\bar{B}\|_\infty^p \tau^{p/2} + C_p \|\bar{A}\|_\infty^p)$ such that for any $i = 1, \dots, d$,

$$E[\|D.(X_{t+\tau}^i) - D.(X_t^i)\|_H^p] \leq D\tau^{p/2}.$$

So Criterium (6) is satisfied with $\alpha = p > 2$, $\beta = p/2 - 1$, $K = D$. Kolmogorov's lemma applied to the process $\{D.(X_t), t \in [0, T]\}$ taking it values in the Hilbert space \mathbb{H} , proves the existence of a continuous version of the process $t \mapsto D.(X_t)$ from $[0, T]$ to the Hilbert space \mathbb{H} .

Finally, we conclude as Nualart's Proposition 2.1.10 proof with index p instead of 2. •

1.3 Invertibility of the Malliavin matrix

Proposition 1.6 *Assume that B and A satisfy Assumptions (1) and (2) then for all $t > 0$ the matrix $\gamma_V(t) := (\langle DV_t^i, DV_t^j \rangle_{\mathbb{H}})_{1 \leq i, j \leq d+1}$ is almost surely invertible.*

Proof: The key is to introduce a new matrix which will be invertible:

$$(7) \quad \text{for all } (s, t), \quad 0 < s < t, \quad \gamma_G(s, t) := (\langle DG^i(s, t), DG^j(s, t) \rangle_{\mathbb{H}})_{1 \leq i, j \leq 2(d+1)}$$

where $G^i(s, t) := X_t^i$, $i = 1, \dots, d$ and $G^{i+d}(s, t) = X_s^i$, $i = 1, \dots, d$.

Step 1: We introduce

- $N_{1,t} := \{\omega, \exists s \in [0, t], \quad DX_s^1 \neq DM_t \quad \text{and} \quad X_s^1 = M_t\}$,
- $N_{2,t} := \{\omega, \exists s \in [0, t], \quad \det(\gamma_G(s, t)) = 0\}$,
- $N_{3,t} := \{\omega, X_t^1 = M_t\}$,
- $N_t = \{\omega, \det(\gamma_V(t)) = 0\}$.

Remark that $N_t \subset (N_t \cap \cap_{i=1}^3 N_{i,t}^c) \cup \cup_{i=1}^3 N_{i,t}$.

Note that $\mathbb{P}(N_t \cap \cap_{i=1}^3 N_{i,t}^c) = 0$. Indeed if $\omega \in N_t \cap \cap_{i=1}^3 N_{i,t}^c$, since X^1 admits a continuous modification there exists s_0 such that $X_{s_0}^1 = M_t$. The fact that $\omega \in N_{3,t}^c$ implies that $s_0 < t$, and $\gamma_V(t) = (\Gamma_G^{i,j}(s_0, t))_{(i,j) \in \{1, \dots, d+1\}^2}$ is a sub matrix of $\gamma_G(s_0, t)$. The fact that $\gamma_V(t)$ is not invertible contradicts the fact that $\gamma_G(s_0, t)$ is invertible. Then, it remains to prove that $\mathbb{P}(N_{i,t}) = 0$ for $i = 1, 2, 3$.

Step 2: Using the same lines as the proof of Proposition 2.1.11 [16], we prove that almost surely

$$\{s : X_s^1 = M_t\} \subset \{s : DM_t = DX_s^1\}$$

meaning $\mathbb{P}(N_{1,t}) = 0$. We skip the details for simplicity.

Step 3: For all $t > 0$, almost surely for all $s < t$, the $2d \times 2d$ matrix $\gamma_G(s, t)$ is invertible, meaning that $\forall t$, the event $N_{2,t}$ is negligible.

Proof: This matrix $\gamma_G(s, t)$ is symmetrical and using (2.59) and (2.60) in [16] yields:

$$(8) \quad \gamma_G(s, t) = \begin{pmatrix} Y(t)C(t)Y^T(t) & Y(s)C(s)Y^T(t) \\ Y(t)C(s)Y^T(s) & Y(s)C(s)Y^T(s) \end{pmatrix}$$

where, using Einstein's convention

$$(9) \quad C^{i,j}(t) := \int_0^t Y^{-1}(u)_k^i A_l^k(X_u) A_l^{k'}(X_u) Y^{-1}(u)_{k'}^j du$$

$$Y_j^i(t) := \delta_{i,j} + \int_0^t \bar{A}_{k,l}^i(u) Y_j^k(u) dW_u^l + \int_0^t \bar{B}_k^i(u) Y^k(u) du, \quad i, j \in \{1, \dots, d\}.$$

Let us denote

$$C^{i,j}(s, t) := C^{i,j}(t) - C^{i,j}(s).$$

According to (2.58) [16] there exists a process Z such that almost surely for all $h \in [0, T]$

$$Z(h)Y(h) = Id$$

thus for all t the matrices $Y(t)$ are invertible.

As a consequence, for all $(s, t) \in [0, T]^2$, the (d, d) matrix $Y(s, t) := Y(t)Y(s)^{-1}$ is too invertible. Then $\gamma_G(s, t)$ (8) can be rewritten as a matrix composed with four (d, d) blocks:

$$\gamma_G(s, t) := \begin{pmatrix} Y(s, t)Y(s)[C(s) + C(s, t)]Y^T(s)Y^T(s, t) & Y(s)C(s)Y^T(s)Y^T(s, t) \\ Y(s, t)Y(s)C(s)Y^T(s) & Y(s)C(s)Y^T(s) \end{pmatrix}$$

The second line of blocks multiplied by $Y(s, t)'$ and this one subtracted to the first line yield:

$$\det[\gamma_G(s, t)] = \det \begin{pmatrix} Y(s, t)Y(s)C(s, t)Y^T(s)Y^T(s, t) & 0 \\ Y(s, t)Y(s)C(s)Y^T(s) & Y(s)C(s)Y^T(s) \end{pmatrix}.$$

The properties of block trigonal matrix determinants prove that

$$(10) \quad \det[\gamma_G(s, t)] = |Y(s, t)Y(s)C(s, t)Y^T(s)Y^T(s, t)Y(s)C(s)Y^T(s)|$$

The processes Z are Y are diffusion processes so each of them admits a continuous modification satisfying $Z(h)Y(h) = Id, \forall h \in [0, T]$. Thus, almost surely the continuous process Z is invertible so satisfies almost surely for all $0 \leq s \leq t \leq T$

$$\int_s^t \det(Z(h))^2 dh > 0.$$

Let $\sigma(x) = \sum_{l=1}^d A_l(x)A_l(x)'$. By definition of C (see (9))

$$C(s) = \int_0^s Y^{-1}(h)\sigma(X_h)(Y(h)^{-1})' dh, \quad C(s, t) = \int_s^t Y^{-1}(h)\sigma(X_h)(Y(h)^{-1})' dh$$

We now follow the proof of Theorem 2.3.1 page 127 [16]: for $v \in \mathbb{R}^d$, using the uniform ellipticity (Assumption (2)):

$$v'\sigma(X_s)v \geq c|v|^2, \quad \forall s > 0, \quad \forall v \in \mathbb{R}^d.$$

With $v = (Y(h)^{-1})'u$ we get

$$u'Y(h)^{-1}\sigma(X_h)(Y(h)^{-1})'u \geq cu'Y(h)^{-1}(Y(h)^{-1})'u$$

and

$$u'C(s)u = \int_0^s u'Y(h)^{-1}\sigma(X(h))(Y(h)^{-1})'udh \geq c \int_0^s u'Y(h)^{-1}(Y(h)^{-1})'udh = c|u|^2 \int_0^s \det(Z(h))^2 dh.$$

Similarly

$$u'C(s, t)u = \int_s^t u'Y(h)^{-1}\sigma(X(h))(Y(h)^{-1})'udh \geq c|u|^2 \int_s^t \det(Z(h))^2 dh.$$

Thus almost surely for all $s \in]0, t[$, $C(s)$ and $C(s, t)$ are invertible. Since $Y(s, t)$, $Y(s)$ are too invertible as a consequence, from (10) the matrix $\gamma_G(s, t)$ is invertible. The process $t \rightarrow D.(X_t)$ takes its values in \mathbb{H} and admits a continuous modification (see (6) above) and the set of invertible matrices is an open set then,

$$\mathbb{P}(\{\omega, \exists s \in]0, t[, \det(\gamma_G(s, t)) = 0\}) = \mathbb{P}(N_{2,t}) = 0. \quad \square$$

Step 4: Under Assumptions (1) and (2), time t being fixed, almost surely $M_t > X_t^1$ meaning the event $N_{3,t}$ is negligible.

Proof: For sake of completeness we prove this result. Actually it is Proposition 18 [10] but with stronger assumptions than ours.

The set $\{M_t = X_t^1\}$ is detailed as follows:

$$(11) \quad \{\omega, M_t(\omega) = X_t^1(\omega)\} = \Omega_1 \cup \Omega_2 \text{ where} \\ \Omega_1 = \{\omega, \exists s < t \mid \forall u \in [s, t], X_u^1(\omega) = X_t^1(\omega)\}, \Omega_2 = \{\omega \mid \forall u < t, X_u^1(\omega) < X_t^1(\omega)\}.$$

Using Assumptions (1) and (2), $A^{-1}B$ is bounded, thus an equivalent change of equivalent probability measure can be operated using Girsanov Theorem: the probability measure \mathbb{P}_0 is defined as

$$(12) \quad \frac{d\mathbb{P}_0}{d\mathbb{P}|_{\mathcal{F}_t}} = L_t, \quad L_t := \exp\left(-\int_0^t (BA^{-1})^i(X_s)dW_s^i - \frac{1}{2}\int_0^t \|(BA^{-1}(X_s))\|^2 ds\right).$$

Then X^1 is a $(\mathcal{F}, \mathbb{P}_0)$ martingale:

$$(13) \quad X_t^1 = X_0^1 + \int_0^t \sum_j A^{1,j}(X_s)d\tilde{W}_s^j$$

where \tilde{W} is a $(\mathcal{F}, \mathbb{P}_0)$ d -dimensional Brownian motion. The bracket of X^1 , actually independent of the probability measure in continuous case, is

$$\langle X^1, X^1 \rangle_t = \int_0^t \sum_j (A^{1,j}(X_s))^2 ds.$$

Assumption (2) applied to $v = (1, 0, \dots, 0)$ implies that for any x , $\sum_j A^{1,j}(x)A^{1,j}(x) \geq c > 0$. This has two consequences:

- For all rational numbers $q < q'$ in $[0, T]$

$$\langle X^1, X^1 \rangle_{q'} - \langle X^1, X^1 \rangle_q > c(q' - q) > 0.$$

According to Proposition 1.13 page 119 [17], for all rational numbers $q < q'$ in $[0, T]$ X^1 is not constant on the interval $[q, q']$. Let us remark

$$\{\omega, \exists s > t \mid \forall u \in [s, t], X_u^1(\omega) = X_t^1(\omega)\} \subset \cup_{q < q' < T, q, q' \in \mathbb{Q}} \{\omega \mid u \in [q, q'], X_u^1(\omega) = X_q^1(\omega)\}$$

thus

$$\mathbb{P}_0(\{\omega \mid \exists s > t, \forall u \in [s, t], X_u^1(\omega) = X_t^1(\omega)\}) = 0.$$

The probability measures \mathbb{P}_0 and \mathbb{P} are equivalent so

$$(14) \quad \mathbb{P}(\{\omega, \exists s > t \mid \forall u \in [s, t], X_u^1(\omega) = X_t^1(\omega)\}) = 0.$$

• Using Dambis-Dubins-Schwarz' Theorem (Theorem 1.6 Chapter V [17]), and once again that $\sum_j (A^{1,j}(X_s))^2 \geq c$, then

$$\langle X^1, X^1 \rangle_\infty = \int_0^\infty \sum_j (A^{1,j}(X_s))^2 ds = +\infty.$$

So there exists a \mathbb{P}_0 Brownian motion B such that

$$X^1(t) = B_{\langle X^1, X^1 \rangle_t}, \quad \forall t > 0.$$

Here is followed step by step the proof of Theorem 2.7 Chapter I [17] (Lévy's modulus of continuity), but without absolute value: for all $N \in \mathbb{R}$

$$\mathbb{P}_0 \left(\limsup_{\varepsilon \rightarrow 0} \left\{ \sup_{\substack{0 \leq t_1, t_2 \leq N \\ t_1 - t_2 < \varepsilon}} \frac{B_{t_1} - B_{t_2}}{h(\varepsilon)} = 1 \right\} \right) = 1$$

where $h(s) = \sqrt{2s \log(1/s)}$, $s \in [0, 1]$.

This is equivalent to

$$\mathbb{P}_0 \left(\liminf_{\varepsilon \rightarrow 0} \left\{ \sup_{\substack{0 \leq t_1, t_2 \leq N \\ t_1 - t_2 < \varepsilon}} \frac{B_{t_1} - B_{t_2}}{h(\varepsilon)} \neq 1 \right\} \right) = 0.$$

We remark that

$$\begin{aligned} \Omega_2 &= \{\omega, \forall u \in [0, t], X^1(u) < X^1(t)\} = \{\omega, \forall u \in [0, t], B_{\langle X^1, X^1 \rangle_u} < B_{\langle X^1, X^1 \rangle_t}\} \\ &\subset \cup_N \cup_n \cap_{k \geq n} \left\{ \sup_{\substack{0 \leq t_1, t_2 \leq N \\ t_1 - t_2 < 1/k}} \frac{B_{t_1} - B_{t_2}}{h(1/k)} \leq 0 \right\} \\ &\subset \cup_N \liminf_{1/n \rightarrow 0} \left\{ \sup_{\substack{0 \leq t_1, t_2 \leq N \\ t_1 - t_2 < 1/n}} \frac{B_{t_1} - B_{t_2}}{h(1/n)} \neq 1 \right\} \end{aligned}$$

thus $\mathbb{P}_0(\Omega_2) = 0$. The probability measures \mathbb{P} and \mathbb{P}_0 are equivalent, so

$$(15) \quad \mathbb{P}(\Omega_2) = \mathbb{P}(\{\omega, \forall s \in [0, t], X_s^1(\omega) < X_t^1(\omega)\}) = 0.$$

Finally (11), (14), (15) prove that

$$(16) \quad \mathbb{P}(\{M_t = X_t^1\}) = \mathbb{P}(N_{3,t}) = 0.$$

□

These four steps conclude the proof of Proposition 1.6, so Assumption (ii) in Theorem 1.4 is satisfied and Theorem 1.1 is proved.

2 Case $d = 1$, regularity of the density

In the one-dimensional case we study the regularity of the density of the law of the vector V_t .

We recall the model, the following stochastic differential equation (see [6])

$$(17) \quad dX_t = B(X_t)dt + A(X_t)dW_t, \quad X_0 = x, \quad t \in [0, T],$$

where W is a Brownian motion, A and B satisfy Assumptions (1) and (2). Moreover Assumption (2) yields that $A^{-1}B$ is bounded. We recall the notations $V := (M, X)$ where $M_t = \sup_{s \leq t} X_s$, $t \in \mathbb{R}$.

From Theorem 1.1. in Section 1, the law of the pair $V_t = (M_t, X_t)$ admits a density. With Malliavin calculus tools, we are not able to prove the regularity of the density on the diagonal $\{(m, x) : m = x\}$. So we turn to the regularity of the density on the open subset $\{(m, x), m > \max(0, x)\}$.

Theorem 2.1 *Under Assumptions (2), $B \in C_b^4(\mathbb{R})$, $A \in C_b^5(\mathbb{R})$, $\forall t > 0$, the density of the vector V_t restricted to the open set $\Delta := \{(m, x), m > \max(0, x)\}$ belongs to $C^{2,2}(\Delta)$.*

2.1 Reduction to a drifted Brownian motion

Since we are in a one-dimensional setting, we use a Lamperti transformation in order to reduce the problem to the case of a diffusion with additive noise. A priori $dX_t = B(X_t)dt + A(X_t)dW_t$. We look for an increasing function $\varphi \in C_b^2$ such that the coefficient of dW would be 1. Itô formula yields

$$d\varphi(X_t) = \varphi'(X_t)B(X_t)dt + \frac{1}{2}\varphi''(X_t)A^2(X_t)dt + \varphi'(X_t)A(X_t)dW_t.$$

A sufficient condition is to choose φ such that $\varphi' = \frac{1}{A}$. As is A , φ' is bounded above and below uniformly. Then $Y = \varphi(X)$ satisfies

$$(18) \quad dY_t = \left[\frac{B}{A} \circ \varphi^{-1}(Y_t) - \frac{1}{2}A' \circ \varphi^{-1}(Y_t) \right] dt + dW_t.$$

Remark that $\tilde{B} = \frac{B}{A} \circ \varphi^{-1} - \frac{1}{2}A' \circ \varphi^{-1} \in C_b^1$ as a consequence of Assumptions of Theorem 2.1. Moreover φ' being positive, φ is increasing and $Y_t^* = \varphi(X_t^*)$. From Theorem 1.1, the law of the pair (Y_t^*, Y_t) admits a density with respect to Lebesgue measure and the link between V_t and (Y_t^*, Y_t) law densities is done by the following lemma.

Lemma 2.2 *We assume that A and B satisfy Assumptions of Theorem 2.1. Then the density law of (M_t, X_t) , $p_V(\cdot, \cdot, t)$, satisfies*

$$p_V(b, a; t) = \frac{p_{Y^*, Y}(\varphi(b), \varphi(a); t)}{A(b)A(a)}$$

where φ is defined by $\varphi'(x) = \frac{1}{A(x)}$ and $p_{Y^*, Y}(\cdot, \cdot, t)$ is the pair (Y_t^*, Y_t) law density.

Proof: It is enough to identify the density law of the pair $V_t = (M_t, X_t)$ using, for any bounded measurable F , the following

$$E[F(M_t, X_t)] = E[F(\varphi^{-1}(Y_t^*), \varphi^{-1}(Y_t))] = \int F(\varphi^{-1}(\beta), \varphi^{-1}(\alpha)) p_{Y^*, Y}(\beta, \alpha; t) d\beta d\alpha.$$

We operate the change of variables $b = \varphi^{-1}(\beta)$, $a = \varphi^{-1}(\alpha)$, so $d\beta = \varphi'(b)db = \frac{1}{A(b)}db$ and $d\alpha = \varphi'(a)da = \frac{1}{A(a)}da$ get the result. \bullet

With such a lemma, the regularity of the density p will be the one of $p_{Y^*, Y}, \varphi, A$. Theorem 2.1 is a consequence of the following theorem where B is indeed $\frac{B}{A} \circ \varphi^{-1} - \frac{1}{2}A' \circ \varphi^{-1}$.

Theorem 2.3 *Assume that B belongs to $C_b^4(\mathbb{R})$, then for all $t > 0$ the law of (M_t, X_t) has a $C^{2,2}$ density with respect to the Lebesgue measure on $\Delta = \{(b, a), b > a, b > 0\}$ where $M_t = \sup_{s \leq t} X_s$ and X_t is the solution of*

$$dX_t = B(X_t)dt + dW_t.$$

2.2 Change of probability measure

We now consider the model with $A = 1$ meaning

$$dX_t = B(X_t)dt + dW_t.$$

Recall the equivalent change of probability measure $Q = L\mathbb{P}$ where L is defined as $dL_t = -L_t B(X_t)dW_t$. Abusing of notation, remark that under Q , here we use W instead of X which is a Q -Brownian motion. Denoting $Z = L^{-1}$, $\mathbb{P} = ZQ$,

$$(19) \quad Z_t = \exp\left[\int_0^t B(W_s)dW_s - \frac{1}{2} \int_0^t B^2(W_s)ds\right].$$

Since B is bounded, $Z_t \in L^p$ for any p .

Under Q the law of V_t is the one of (W_t^*, W_t) where $W_t^* := \sup_{s \leq t} W_s$.

Lemma 2.4 *As soon as $B \in C_b^4$, then $Z_t \in \mathbb{D}^{4,p}$ for any $p > 1$, where $\mathbb{D}^{4,p}$ is the set of random variables 4 times differentiable in the Malliavin sense with derivatives in L^p .*

Proof: Using the definition of Z_t we only have to prove $\forall p > 1$

$$\int_0^t B(W_s)dW_s - \frac{1}{2} \int_0^t B^2(W_s)ds \in \mathbb{D}^{4,p}.$$

Obviously by linearity $\int_0^t B^2(W_s)ds \in \mathbb{D}^{4,p}$ since actually $D_{s_1, \dots, s_i}^i \int_0^t B^2(W_s)ds = \int_{\max_j s_j}^t (B^2)^{(i)}(W_s)ds$. Using the Malliavin derivative $D_r[\int_0^t B(W_s)dW_s] = B(W_r) + \int_r^t B'(W_s)dW_s$ and recursively $D^i Z_t$, $i = 1, \dots, 4$, are well defined.

Finally the lemma is proved using the boundness of B and its derivatives. •

From now on, using a change of time unit, it is enough to consider the case $t = 1$.

2.3 Local non degeneracy of $F := V_1 = (W_1^*, W_1)$

Let us recall Definition 2.1.2 in Nualart [16].

Definition 2.5 *A random vector $F = (F_1, \dots, F_m)$ whose components are in $\mathbb{D}^{1,2}$ is locally non degenerate in an open set $\Delta \subset \mathbb{R}^m$ if there exist elements $u_\Delta^j \in \mathbb{D}^\infty(\mathbb{H})$, $j = 1, \dots, m$ and an $m \times m$ random matrix $\gamma_\Delta = (\gamma_\Delta^{i,j})$ such that $\gamma_\Delta^{i,j} \in \mathbb{D}^\infty$, $|\det \gamma_\Delta|^{-1} \in L^q(\Omega)$ for all $q \geq 1$, and $\langle DF^i, u_\Delta^j \rangle = \gamma_\Delta^{i,j}$ on $\{F \in \Delta\}$ for any $i, j = 1, \dots, m$.*

Proposition 2.6 *Let $\eta > 0$. The random vector $F := V_1 = (W_1^*, W_1)$ is locally non degenerate in any open set $\Delta_\eta = \{(b, a), b - a > \eta, b > \eta\}$.*

Proof: The key is to prove that the random vector $F := V_1 = (W_1^*, W_1)$ satisfies Definition (2.5).

Step 1: The F components belong to $\mathbb{D}^{1,2}$

Indeed, $D_t W_1 = \mathbf{1}_{[0,1]}(t)$, and $D_t W_1^* = \mathbf{1}_{[0,\tau]}(t)$ where $\tau := \inf\{s, W_s = W_1^*\}$.

Step 2

Let the process \tilde{W} be defined as $\tilde{W}_t := W_{1-t} - W_1$.

Similarly to [16] Proposition 2.1.12 page 112 $\forall p > 2$, let $s \in [0, 1]$ and γ such that $2p\gamma \in (1, p - 1)$ we introduce:

$$\begin{aligned} Y^1(s) &= \int_{[0,s]^2} \frac{(W_u - W_{u'})^{2p}}{(u - u')^{1+2p\gamma}} du du' \\ Y^2(s) &= \int_{[0,s]^2} \frac{(\tilde{W}_u - \tilde{W}_{u'})^{2p}}{(u - u')^{1+2p\gamma}} du du' \end{aligned}$$

Following this proposition proof, it can be proved that:

$$(20) \quad \exists R(\eta, p, \gamma) \text{ such that } Y^1(s) \leq R \Rightarrow \sup_{z \in [0,s]} W_z \leq \eta;$$

$$\text{similarly } Y^2(s) \leq R \Rightarrow \sup_{z \in [0,s]} \tilde{W}_z \leq \eta.$$

Let $\psi : \mathbb{R}^+ \rightarrow [0, 1]$ be infinitely differentiable and such that $\psi(x) = 1$ when $x \in [0, R/2]$, $\psi(x) = 0$ when $x \geq R$. Denote

$$(21) \quad u_\eta^1(s) = \psi(Y^1(s)), \quad u_\eta^2(s) = \psi(Y^2(1-s))$$

and

$$(22) \quad \Gamma_\eta = \begin{pmatrix} \int_0^1 \psi(Y^1(s)) ds & \int_0^1 \psi(Y^1(s)) ds \\ 0 & \int_0^1 \psi(Y^2(1-s)) ds \end{pmatrix}.$$

Step 3

The processes $u_\eta^i \in \mathbb{D}^\infty(H)$ $i = 1, 2$, by the construction of Y^i and the definition of the function ψ . As a consequence, the random matrix Γ_η belongs to \mathbb{D}^∞ .

Step 4 Note that $(\det \Gamma_\eta)^{-1} = [\int_0^1 \psi(Y^1(s)) ds \int_0^1 \psi(Y^2(1-s)) ds]^{-1}$. We will prove that $\forall q \geq 1$,

$$(23) \quad \left(\int_0^1 \psi(Y^1(s)) ds \right)^{-1} \text{ and } \left(\int_0^1 \psi(Y^2(1-s)) ds \right)^{-1} \in L^q.$$

Indeed using the trick of Proposition 2.1.12 [16] page 114 $\int_0^1 \psi(Y^1(s)) ds \geq \int_0^1 \mathbf{1}_{\{Y^1(s) < R/2\}} ds = \text{Leb}\{s : Y^1(s) < R/2\} = (Y^1)^{-1}(R/2) > 0$, since the non decreasing function Y^1 is invertible; so for any q , $\left(\int_0^1 \psi(Y^1(s)) ds \right)^{-q} \leq [(Y^1)^{-1}(R/2)]^{-q}$. Recall that

$$\mathbb{E} \left([(Y^1)^{-1}(R/2)]^{-q} \right) = \int_0^\infty \mathbb{P}\{(Y^1)^{-1}(R/2) < x^{-1/q}\} dx.$$

On the other hand for any y , $\mathbb{P}\{(Y^1)^{-1}(R/2) < y\} = \mathbb{P}\{R/2 < Y^1(y)\} \leq (\frac{2}{R})^q \mathbb{E}[(Y^1)^q(y)]$.
 We compute $\mathbb{E}[(Y^1)^q(y)]$ as

$$\mathbb{E} \left(\left[\int_{[0,y]^2} \frac{(W_u - W_{u'})^{2p}}{(u - u')^{1+2p\gamma}} dud u' \right]^q \right).$$

Using Jensen's inequality,

$$\mathbb{E}[(Y^1)^q(y)] \leq y^{2q-2} \left[\int_{[0,y]^2} \frac{\mathbb{E}[(W_u - W_{u'})^{2pq}]}{(u - u')^{q(1+2p\gamma)}} dud u' \right]$$

Since W is a Brownian motion, $\mathbb{E}[(W_u - W_{u'})^{2pq}] \leq C|u - u'|^{pq}$ and using $2p\gamma < p - 1$,

$$y^{2q-2} \int_{[0,y]^2} \frac{\mathbb{E}[(W_u - W_{u'})^{2pq}]}{(u - u')^{q(1+2p\gamma)}} dud u' \leq C y^{q(1+p-2\gamma p)}.$$

Choosing $y = x^{-1/q}$,

$$\mathbb{P}\{(Y^1)^{-1}(R/2) < x^{-1/q}\} \leq C_1 x^{-(1+p-2\gamma p)}.$$

Using this bound for $x \geq 1$, and $2\gamma p - p - 1 > -2$,

$$\mathbb{E} \left([(Y^1)^{-1}(R/2)]^{-q} \right) = \int_0^\infty \mathbb{P}\{(Y^1)^{-1}(R/2) < x^{-1/q}\} dx < \infty.$$

Recall that \tilde{W} is a Brownian motion (time reversal), so both processes $(Y_s^i, 0 \leq s \leq 1), i = 1, 2$, have the same law so the two factors in $\det \Gamma_\eta$ have the same law. So, for any q , $(\int_0^1 \psi(Y^2(1-s)) ds)^{-1}$ also belongs to L^q . This achieves the proof of (23) and as a conclusion, $(\det \Gamma_\eta)^{-1} \in L^q$ for any q .

Step 5

According to Definition 2.5 we now prove that the following random matrix

$$(24) \quad \begin{pmatrix} \langle DW_1^*, u_\eta^1 \rangle & \langle DW_1, u_\eta^1 \rangle \\ \langle DW_1^*, u_\eta^2 \rangle & \langle DW_1, u_\eta^2 \rangle \end{pmatrix}$$

coincides with Γ_η on the set $\{V_1 \in \Delta_\eta\}$.

(i) By definition of the time τ , $W_\tau = W_1^* > \eta$, this contradicts (20) so $Y^1(s) \geq R$ thus $\psi(Y^1(s)) = 0$ as soon as $s \geq \tau$; $\langle DW_1^*, u_\eta^1 \rangle = \int_0^\tau \psi(Y^1(s)) ds = \int_0^1 \psi(Y^1(s)) ds$.

(ii) Concerning $\langle DW_1^*, u_\eta^2 \rangle = \int_0^\tau u_\eta^2(s) ds$, it is null:

On the set $\{W_1^* > \eta, W_1^* - W_1 > \eta\}$, using $W_1^* = W_\tau$, so $W_\tau - W_1 > \eta$, we get $W_\tau - W_1 = Z_{1-\tau} > \eta$ so $\eta \leq Z_{1-\tau} \leq \sup_{0 \leq u \leq 1-s} Z_u$ for $s \leq \tau$. Once again, this contradicts (20) so $Y^2(1-s) > R$, and the definition of ψ yields $u_\eta^2(s) = 0, \forall s \in [0, \tau]$

(iii) Concerning $\langle DW_1, u_\eta^i \rangle$, since $DW_1 = 1$ on $[0, 1]$, $\langle DW_1, u_\eta^2 \rangle = \int_0^1 \psi(Y^2(1-s)) ds$ and $\langle DW_1, u_\eta^1 \rangle = \int_0^1 \psi(Y^1(s)) ds$. •

2.4 Proof of Theorem 2.3

Let $v_0 = (x_0^*, x_0) \in \Delta_\eta := \{(x^*, x), x^* > \eta, x^* - x > \eta\}$ and $\delta < \frac{1}{2}d(v_0, \Delta_\eta^c)$ and $\delta < \delta' < d(v_0, \Delta_\eta^c)$. Recall the notations $u_\eta^i, i = 1, 2, \Gamma_\eta$ (21) and (22).

(i) Let $\varphi \in C_\eta^\infty(\mathbb{R}^m)$ meaning that $\text{support}(\phi) \subset \Delta_\eta$ and denote $F = (W_1^*, W_1)$, then using the definition of \mathbb{P} and \mathbb{Q}

$$(25) \quad \mathbb{E}(\partial_{1,2}^2 \varphi(F)) = \mathbb{E}_Q(\partial_{1,2}^2 \varphi(F) Z_1)$$

where Z is defined in (19).

Since F is locally non degenerate (Proposition 2.6), we apply Proposition 2.1.4 (2.25) [16] to the multi-index $\alpha = (1, 2)$ and $G = Z_1$. Since we use Proposition 2.1.4 [16] with index $\alpha = (1, 2)$, we only need Z_1 belonging to $\cap_{p>1} \mathbb{D}_p^2$. Later, we will use twice these indices, so then we will need Z_1 belonging to $\cap_{p>1} \mathbb{D}_p^4$, see Lemma 2.4.

Using identity (2.25) of Proposition 2.1.4 [16] to the multi-index $\alpha = (1, 2)$ and $G = Z_1$ in the right hand of (25) we obtain

$$(26) \quad \mathbb{E}(\partial_{1,2}^2 \varphi(F)) = \mathbb{E}_Q(\varphi(F) H_{1,2}(Z_1)),$$

where, following Formulas (2.26) and (2.27) of Proposition 2.1.4 [16], expressed in our framework:

$$H_{(i)}(Z_1) = \sum_{j=1}^2 \delta(Z_1(\Gamma_\eta^{-1})^{i,j} u_\eta^j), = \delta(Z_1(\Gamma_\eta^{-1} u_\eta)_i) = Z_1 \delta((\Gamma_\eta^{-1} u_\eta)_i) - \int_0^1 D_s Z_1(\Gamma_\eta^{-1} u_\eta(s))_i ds, \quad i = 1, 2$$

and for $\beta = (1, 2)$

$$H_{1,2}(Z_1) = H_{(2)}(H_{(1)}(Z_1)).$$

We detail this definition:

$$(27) \quad H_{1,2}(Z_1) = \delta[H_{(1)}(Z_1(\Gamma_\eta^{-1} u_\eta)_2)] = \delta[\delta[Z_1(\Gamma_\eta^{-1} u_\eta)_1](\Gamma_\eta^{-1} u_\eta)_2].$$

Since

$$(28) \quad \varphi(W_1^*, W_1) = \int_{-\infty}^{W_1^*} \int_{-\infty}^{W_1} \partial_{1,2}^2 \varphi(x^*, x) dx^* dx$$

with Fubini in (26) we obtain

$$(29) \quad \mathbb{E}[\partial_{1,2}^2 \varphi(F)] = \int_{\mathbb{R}^2} \partial_{1,2}^2 \varphi(x^*, x) E_Q[\mathbf{1}_{W_1 > x, W_1^* > x^*} H_{1,2}(Z_1)] dx^* dx.$$

Then Equations (27) and (29) imply that the law of F has a density with respect to the Lebesgue measure given by

$$p_V(x^*, x; 1) = E_Q[\mathbf{1}_{W_1 > x, W_1^* > x^*} \delta[\delta[Z_1(\Gamma_\eta^{-1} u_\eta)_1](\Gamma_\eta^{-1} u_\eta)_2]].$$

(ii) We now turn to the differentiability of this density. Let $\alpha = (1, 2)$ and $\beta = 1$, or 2 or $(i, j), i, j \leq 2$, using the definition of \mathbb{P} and Q

$$(30) \quad \mathbb{E}(\partial_{\alpha, \beta}^{2+|\beta|} \varphi(F)) = \mathbb{E}_Q[\partial_{\alpha, \beta}^{2+|\beta|} \varphi(F) Z_1].$$

Then using Proposition 2.1.4 [16] for $\alpha + \beta$

$$(31) \quad \mathbb{E}(\partial_{\beta}^{|\beta|} (\partial_{1,2}^2 \varphi(F))) = \mathbb{E}_Q[\varphi(F) H_{\beta}(H_{\alpha}(Z_1))]$$

Using representation (28) of φ

$$(32) \quad \mathbb{E}[(\partial_{\alpha, \beta}^{2+|\beta|} \varphi(F))] = \int_{\mathbb{R}^2} \partial_{1,2}^2 \varphi(x^*, x) \mathbb{E}_Q[\mathbf{1}_{W_1^* > x^*, W_1 > x} H_{\beta}(H_{\alpha}(Z_1))] dx dx^*.$$

Let ϕ be a regular function with support in the ball $B(v_0, \delta) \subset \Delta_{\eta}$ and $\xi = \partial_{\alpha}^2 \varphi$ so $\partial_{\alpha, \beta}^{2+|\beta|} \varphi = \partial_{\beta}^{|\beta|} \xi$; we apply (32):

$$(33) \quad \mathbb{E}[(\partial_{\beta}^{|\beta|} \xi(F))] = \int_{\mathbb{R}^2} \xi(x^*, x) \mathbb{E}_Q[\mathbf{1}_{W_1^* > x^*, W_1 > x} H_{\beta}(H_{\alpha}(Z_1))] dx dx^*.$$

The left hand above is identified as $\int_{\mathbb{R}^2} \partial_{\beta}^{|\beta|} \xi(x^*, x) p_V(x^*, x; 1) dx^* dx$ but this one, using an integration by part, is also $\pm \int_{\mathbb{R}^2} \xi(x^*, x) \partial_{\beta}^{|\beta|} p_V(x^*, x; 1) dx^* dx$. Thus we can identified on any ball $B_{\delta}(v)$, using the notation $|\beta|$ is the length of β :

$$\partial_{\beta}^{|\beta|} p_V(x^*, x; 1) = (-1)^{|\beta|} \mathbb{E}_Q[\mathbf{1}_{W_1^* > x^*, W_1 > x} H_{\beta}(H_{\alpha}(Z_1))].$$

•

3 Monotonous image of the Brownian motion

Here is provided a simple example which is completely solvable and connected with [5] where is studied the density of the law as a weak solution to a partial differential equation.

• Firstly one recalls the standard result concerning the Brownian motion. The density of the law of (W_t^*, W_t) is well known, $\forall t > 0$, see [13] Section 3.2, and is defined on \mathbb{R}^2 by

$$(34) \quad p_W(b, a; t) := 2 \frac{(2b - a)}{\sqrt{2\pi t^3}} e^{-\frac{(2b-a)^2}{2t}} \mathbf{1}_{\sup(0, a), \infty[}(b), t > 0.$$

It could be checked that Corollary 1 [6] provides for all $t > 0$, a partial differentiable equation and boundary conditions for the density of the law of $V_1 := (W_1^*, W_1)$:

$$(35) \quad \begin{aligned} \partial_t p_W(b, a; t) &= \frac{1}{2} \partial_{22}^2 p_W(b, a; t), \quad \forall b > \max(0, a), \quad t > 0 \\ \partial_1 p_W(b, b; t) + 2\partial_2 p_W(b, b; t) &= 0, \quad \forall b > 0, \quad t > 0, \\ \partial_t p_{W^*}(b, t) + \frac{1}{2}(\partial_1 p_W(b, b, t) + \partial_2 p_W(b, b, t)) &= 0. \end{aligned}$$

• Another case is easily deduced: the particular case of a one-dimensional monotonous image of the Brownian motion, as following:

$$dX_t = \sigma(X_t)dW_t + 1/2\sigma'(X_t)\sigma(X_t)dt, \quad X_0 = x_0$$

with $\sigma \in C^1$ and $\inf_{x \in \mathbb{R}} \sigma(x) \geq c > 0$. Let φ be the solution to the differential equation

$$\frac{d}{dx}\varphi(x) = \sigma(\varphi(x)), \quad \varphi(0) = x_0$$

then $X = \varphi(W)$. Moreover since φ is continuous and increasing, $M = \varphi(W^*)$ where W^* is the running maximum of the Brownian motion.

Thus, using p_W defined in (34), the law of the pair $V_t = (M_t, X_t) = (\varphi(W_t^*), \varphi(W_t))$ admits the density on \mathbb{R}^2 : $p_V(m, x; t) := \frac{1}{\sigma(m)\sigma(x)}p_W(\varphi^{-1}(m); \varphi^{-1}(x); t)$ and similarly $p_M(m; t) := \frac{1}{\sigma(m)}p_{W^*}(\varphi^{-1}(m); t)$.

Let \mathcal{L} denote the infinitesimal generator of X , then its adjoint is

$$\mathcal{L}^*F(x) := \frac{1}{2}[\sigma'(x)^2 + \sigma\sigma''(x)]F(x) + \frac{3}{2}\sigma\sigma'(x)F'(x) + \frac{1}{2}\sigma(x)^2F''(x).$$

Standard change of variables, starting from the partial differentiable system (35), provides for all $t > 0$, a partial differentiable system for the density of the law of $V_t = (M_t, X_t)$,

$$\begin{aligned} \partial_t p_V(m, x; t) &= \mathcal{L}^*(p(\cdot; t))(m, x), \\ (36) \quad 3\sigma'(m)p_V(m, m; t) + \sigma(m)[\partial_1 + 2\partial_2](p_V)(m, m; t) &= 0, \quad \forall m > 0, \\ \partial_t p_M(m, t) + \sigma'(m)p_V(m, m; t) + \frac{1}{2}\sigma(m)(\partial_1 + \partial_2)(p_V)(m, m; t) &= 0. \end{aligned}$$

Remark 3.1 *This example is the “Example 4” in [10], but there only the law of M_t is studied instead of the pair as we do.*

4 Conclusion

We emphasize here that such results on the law of the couple (M, X) could be used when we are interested in barrier options, since the law of time activation (or extinction) is strongly related to that of the couple (M, X) . Moreover, for concrete applications, the tools built in [11, 12] would be used.

Finally, in a work in progress [5], results as (36) could be generalised to a diffusion process X under some convenient hypotheses as (1) and (2).

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