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<b>Abstract:</b>	Let $X$ be a continuous $d$ -dimensional diffusion process and $M$ the running supremum of the first component. We show that, for any $t > 0$ , the law of the $(d+1)$ random vector $(M_t, X_t)$ admits a density with respect to the Lebesgue measure using Malliavin's calculus. In case $d=1$ we prove the regularity of this density.
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# Existence and regularity of law density of a pair (diffusion, first component running maximum)

Laure Coutin\*, Monique Pontier†

July 2, 2018

## Abstract

Let  $X$  be a continuous  $d$ -dimensional diffusion process and  $M$  the running supremum of the first component. We show that,  $\forall t > 0$ , the law of the  $(d+1)$  random vector  $(M_t, X_t)$  admits a density with respect to the Lebesgue measure using Malliavin's calculus. In case  $d = 1$  we prove the regularity of this density.

**Keywords:** Running supremum process, joint law density, Malliavin calculus, regularity of the density.

**A.M.S. Classification:** 60J60, 60H07, 60H10.

In this paper one is interested in the joint law of the random vector  $(M, X)$  where  $X$  is a  $d$ -dimensional diffusion process and  $M$  the running supremum of its first component. When the process  $X$  is a Brownian motion the result is well known, see for instance [13]. For general Gaussian processes, the law of the maximum is studied in [2]. The cases where the process  $X$  is a martingale, or a Lévy process are deeply studied in the literature. Some specific diffusion processes as Orstein-Uhlenbeck were also investigated.

A lot of studies are devoted to the maximum of martingales, their terminal value, their maximum at terminal time. For instance look at Rogers et al. [18, 9, 3]. Cox-Obloj [7] aim is to exhibit an hedging strategy of the so-called “no touch option”, meaning that the payoff is the indicator of the set  $\{\bar{S}_T < b; \underline{S}_T > a\}$ , the price process  $S$  being given. They are not concerned with the law of the pair (process, its

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running maximum). A lot of papers are mainly interested in the hedging of barrier option, for instance [3].

The case of general Lévy processes is studied by Doney and Kyprianou [8]. In particular cases driven by a Brownian motion and a compound Poisson process, Roynette et al. [19] provide the Laplace transform of undershot-overshot-hitting time law. In [15, 6] the pair (the process-its running maximum) law density is proved to be a weak solution of a partial differential equation. We quote [4] which is the starting point of our study. Lagnoux et al. [14] provide the law density of such a pair, but in case of a reflected Brownian motion.

Concerning the diffusion processes, for instance the Ornstein Uhlenbeck process, the density of the running maximum law is given in [1]. Some analogous results are provided in [9] concerning random walks.

Quote Jeanblanc et al. [13] for the one-dimensional diffusion process: a PDE is obtained for the law density of the process stopped before hitting a moving barrier. In [10] a multi-dimensional diffusion (whose corresponding diffusion vector fields are commutative) joint distribution is studied at the time when a component attains its maximum on finite time interval; under regularity and ellipticity conditions the smoothness of this joint distribution is proved.

Here we look for more general (but continuous) cases where this density exists. In  $d$ -dimensional case, the law of the  $(d+1)$ -random vector  $(M_t, X_t)$  is absolutely continuous, where  $M_t := \sup_{s \leq t} X_s^1$ . In the one-dimensional case we get the regularity of this density. Such results are used in our work in progress [5] where this density is proved to be a weak solution of a partial differential equation.

The model is as following: let a filtered probability space  $(\Omega, (\mathcal{F}_t = \sigma(W_u, u \leq t))_{t \geq 0}, \mathbb{P})$  where  $W := (W_u, u \geq 0)$  is a  $d$ -dimensional Brownian motion. Let  $X$  be a  $d$ -dimensional diffusion process solution to

$$dX_t = B(X_t)dt + \sum_{i=1}^d A_i(X_t)dW_t, \quad X_0 = x \in \mathbb{R}^d, \quad t > 0,$$

where  $B: \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $A: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  are bounded with bounded and continuous differential.

In Section 1 we recall some elements in Malliavin's calculus that we use to prove that the law of  $V_t := (M_t, X_t)$  is absolutely continuous with respect to the Lebesgue measure in a general case with some standard assumptions on the coefficients  $A$  and  $B$ . In Section 2, we turn to the one-dimensional case and we prove the regularity of the density of the law of  $V_t$  under some more technical assumptions. Section 3 provides an example where the announced partial differential equation is exactly stated.

# 1 The law of $V_t$ is absolutely continuous

Here it is proved that for any  $t > 0$ , the joint law of  $V_t := (M_t, X_t)$  admits a density with respect to the Lebesgue measure. For this purpose, we use ‘‘Malliavin calculus’’ specifically Nualart’s results [16]. Let us denote  $C_b^i(\mathbb{R}^d)$  the set of the functions  $i$  times differentiable, bounded, with bounded derivatives.

**Theorem 1.1** *We assume that  $A$  and  $B$  satisfy*

$$(1) \quad A \text{ and } B \in C_b^1(\mathbb{R}^d)$$

*and there exists a constant  $c > 0$  such that*

$$(2) \quad c\|v\|^2 \leq v^T A(x)A^T(x)v, \quad \forall v, x \in \mathbb{R}^d.$$

*Then the joint law of  $V_t = (M_t, X_t)$  admits a density with respect to the Lebesgue measure for all  $t > 0$ .*

The next subsection recalls some useful definitions and results.

## 1.1 Short Malliavin calculus summary

The material of this subsection can be found in Section 1.2 of [16]. Let  $\mathbb{H} = L^2([0, T], \mathbb{R}^d)$  endowed with the usual scalar product  $\langle \cdot, \cdot \rangle_{\mathbb{H}}$  and the associated norm  $\|\cdot\|_{\mathbb{H}}$ .

For all  $(h, \tilde{h}) \in \mathbb{H}^2$ ,

$$W(h) := \int_0^T h(t)dW_t$$

is a centered Gaussian variable with variance equal to  $\|h\|_{\mathbb{H}}^2$ . If  $\langle h, \tilde{h} \rangle_{\mathbb{H}} = 0$  then the random variables  $W(h)$  and  $W(\tilde{h})$  are independent.

Let  $\mathcal{S}$  denote the class of smooth random variables  $F$  defined as following:

$$(3) \quad F = f(W(h_1), \dots, W(h_n))$$

where  $n \in \mathbb{N}$ ,  $h_1, \dots, h_n \in \mathbb{H}$  and  $f$  belongs to  $C_b(\mathbb{R}^n)$ .

**Definition 1.2** *The Malliavin derivative of a smooth variable  $F$  defined in (3) is the  $\mathbb{H}$  valued random variable given by*

$$DF = \sum_{i=1}^n \partial_i f(W(h_1), \dots, W(h_n))h_i.$$

**Proposition 1.3** *The operator  $D$  is closable from  $L^p(\Omega)$  into  $L^p(\Omega, \mathbb{H})$  for any  $p \geq 1$ .*

For any  $p \geq 1$ , let us denote  $\mathbb{D}^{1,p}$  the domain of the operator  $D$  in  $L^p(\Omega)$ , meaning that  $\mathbb{D}^{1,p}$  is the closure of the class of smooth random variables  $\mathcal{S}$  with respect to the norm

$$\|F\|_{1,p} = (\mathbb{E}[|F|^p] + [\mathbb{E}[\|DF\|_{\mathbb{H}}^p]])^{1/p}.$$

Malliavin calculus is a powerful tool to prove the absolute continuity of random variables law. Namely Theorem 2.1.2 page 97 [16] states:

**Theorem 1.4** *Let  $F = (F^1, \dots, F^m)$  be a random vector satisfying the following conditions*

- (i)  $F^i$  belongs to  $\mathbb{D}^{1,p}$  for  $p > 1$  for all  $i = 1, \dots, m$ ,
- (ii) the Malliavin matrix  $\gamma_F := (\langle DF^i, DF^j \rangle_{\mathbb{H}})_{1 \leq i, j \leq m}$  is invertible.

*Then the law of  $F$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}^m$ .*

Using this theorem, the proof of Theorem 1.1 will be a consequence of the following:

- $X_t^i$ ,  $i = 1, \dots, d$  and  $M_t$  belongs to  $\mathbb{D}^{1,p}$   $p > 1$ , see Lemma 1.5;
- the  $(d+1) \times (d+1)$  matrix  $\gamma_V(t) := (\langle DV_t^i, DV_t^j \rangle)_{1 \leq i, j \leq d+1}$  is almost surely invertible, see Proposition 1.6.

## 1.2 Malliavin differentiability of the supremum

**Lemma 1.5** *Under Assumption (1),  $X_t^i$ ,  $i = 1, \dots, d$  and  $M_t$  belongs to  $\mathbb{D}^{1,p}$   $\forall p \geq 1$  for all  $t > 0$ .*

**Proof:** Using Theorem 2.2.1 [16], under Assumption (1),

- $X_t^i$ ,  $i = 1, d$  belong to  $\mathbb{D}^{1,\infty}$  for all  $t > 0$ ,
- $\forall t \leq T$ ,  $\forall p > 0$ ,  $\forall i = 1, \dots, d$ , there exists a constant  $C_T^p$  such that

$$(4) \quad \sup_{0 \leq r \leq t} \mathbb{E} \left( \sup_{r \leq s \leq T} |D_r X_s^i|^p \right) = C_t \leq C_T^p < \infty,$$

- the Malliavin derivative  $D_r X_t$  satisfies  $D_r X_t = 0$  for  $r > t$  almost surely and for  $r \leq t$  almost surely, using Einstein's convention (2.56 page 125 [16]):

$$(5) \quad D_r X_t^i = A^i(X_r) + \int_r^t \bar{A}_{k,\alpha}^i(s) D_r(X_s^k) dW_s^\alpha + \int_r^t \bar{B}_k^i(s) D_r(X_s^k) ds$$

where  $\bar{A}_{k,\alpha}^i(s) := \partial_k A_\alpha(X_s)$  and  $\bar{B}_k^i := \partial_k B(X_s)$  are in  $\mathbb{R}^d$ .

(i) In order to prove that  $M_t$  belongs to  $\mathbb{D}^{1,p}$  we follow the same lines as in the proof of Proposition 2.1.10 [16] with index  $p$  instead of 2. For any  $i = 1, \dots, d$ , we

establish that the  $\mathbb{H}$ -valued process  $(D.X_t^i, t \in [0, T])$  has a continuous modification and satisfies  $\mathbb{E}(\|D.X_t^i\|_H^p) < \infty$ .

We now use Appendix (A.11) in Nualart [16], as a corollary of Kolmogorov's continuity criterion: if there exist positive real numbers  $\alpha, \beta, K$  such that

$$(6) \quad E[\|D.X_{t+\tau}^i - D.X_t^i\|_H^\alpha] \leq K\tau^{1+\beta}, \quad \forall t \geq 0, \tau \geq 0$$

then  $D.X^i$  admits a continuous modification.

(ii) As a second step we prove (6). Let  $\tau > 0$ , Equation (5) yields

$$\begin{aligned} \Delta_\tau D_r(X_t^i) &:= D_r(X_{t+\tau}^i) - D_r(X_t^i) \\ &= \int_{\max(r,t)}^{\max(t+\tau,r)} \bar{B}_k^i(s) D_r(X_s^k) ds + \int_{\max(r,t)}^{\max(t+\tau,r)} \bar{A}_{k,\alpha}^i(s) D_r(X_s^k) dW_s^\alpha. \end{aligned}$$

Using the definition of  $\mathbb{H}$

$$\|\Delta_\tau D_r(X_t^i)\|_{\mathbb{H}}^2 = \int_0^T \left| \int_{\max(r,t)}^{\max(t+\tau,r)} \bar{B}_k^i(s) D_r(X_s^k) ds + \int_{\max(r,t)}^{\max(t+\tau,r)} \bar{A}_{k,\alpha}^i(s) D_r(X_s^k) dW_s^\alpha \right|^2 dr.$$

According to Jensen's inequality for  $p \geq 2$

$$\|\Delta_\tau D_r(X_t^i)\|_H^p \leq T^{\frac{p}{2}-1} \int_0^T \left| \int_{\max(r,t)}^{\max(t+\tau,r)} \bar{B}_k^i(s) D_r(X_s^k) ds + \int_{\max(r,t)}^{\max(t+\tau,r)} \bar{A}_{k,\alpha}^i(s) D_r(X_s^k) dW_s^\alpha \right|^p dr.$$

Using  $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ ,

$$\|\Delta_\tau D_r(X_t^i)\|_H^p \leq 2^{p-1} T^{\frac{p}{2}-1} \int_0^T \left[ \left| \int_t^{t+\tau} \bar{B}_k^i(s) D_r(X_s^k) ds \right|^p + \left| \int_t^{t+\tau} \bar{A}_{k,\alpha}^i(s) D_r(X_s^k) dW_s^\alpha \right|^p \right] dr.$$

The expectation of the first term is bounded using Jensen's inequality and (4) for any  $r \in [0, T]$ :

$$E \left[ \left| \int_t^{t+\tau} \bar{B}_k^i(s) D_r(X_s^k) ds \right|^p \right] \leq \|\bar{B}\|_\infty^p \tau^{p-1} \sup_r E \left[ \sup_{r \leq s \leq T} |D_r(X_s^k)|^p \right] = \|\bar{B}\|_\infty^p \tau^p C_T^p.$$

Using once again (4), Burkholder-Davis Gundy' and Jensen's inequalities, the expectation of the second term satisfies for any  $r \in [0, T]$ :

$$\begin{aligned} E \left[ \left| \int_t^{t+\tau} \bar{A}_{k,\alpha}^i(s) D_r(X_s^k) dW_s^\alpha \right|^p \right] &\leq C_p E \left[ \left( \int_t^{t+\tau} |\bar{A}_{k,\alpha}^i(s) D_r(X_s^k)|^2 ds \right)^{p/2} \right] \\ &\leq C_p \|\bar{A}\|_\infty^p \tau^{p/2-1} \int_t^{t+\tau} E \left( \sup_{r \leq s \leq T} |D_r(X_s^i)|^p \right) ds \leq C_p \|\bar{A}\|_\infty^p \tau^{p/2-1} C_T^p \tau, \end{aligned}$$

thus for any  $\tau \in [0, 1]$  there exists a constant  $D = T^{p/2} 2^{p/2-1} C_T^p (\|\bar{B}\|_\infty^p \tau^{p/2} + C_p \|\bar{A}\|_\infty^p)$  such that for any  $i = 1, \dots, d$ ,

$$E[\|D.(X_{t+\tau}^i) - D.(X_t^i)\|_H^p] \leq D\tau^{p/2}.$$

So Criterium (6) is satisfied with  $\alpha = p > 2$ ,  $\beta = p/2 - 1$ ,  $K = D$ . Kolmogorov's lemma applied to the process  $\{D.(X_t), t \in [0, T]\}$  taking it values in the Hilbert space  $\mathbb{H}$ , proves the existence of a continuous version of the process  $t \mapsto D.(X_t)$  from  $[0, T]$  to the Hilbert space  $\mathbb{H}$ .

Finally, we conclude as Nualart's Proposition 2.1.10 proof with index  $p$  instead of 2. •

### 1.3 Invertibility of the Malliavin matrix

**Proposition 1.6** *Assume that  $B$  and  $A$  satisfy Assumptions (1) and (2) then for all  $t > 0$  the matrix  $\gamma_V(t) := (\langle DV_t^i, DV_t^j \rangle_{\mathbb{H}})_{1 \leq i, j \leq d+1}$  is almost surely invertible.*

**Proof:** The key is to introduce a new matrix which will be invertible:

$$(7) \quad \text{for all } (s, t), \quad 0 < s < t, \quad \gamma_G(s, t) := (\langle DG^i(s, t), DG^j(s, t) \rangle_{\mathbb{H}})_{1 \leq i, j \leq 2(d+1)}$$

where  $G^i(s, t) := X_t^i$ ,  $i = 1, \dots, d$  and  $G^{i+d}(s, t) = X_s^i$ ,  $i = 1, \dots, d$ .

Step 1: We introduce

- $N_{1,t} := \{\omega, \exists s \in [0, t], \quad DX_s^1 \neq DM_t \quad \text{and} \quad X_s^1 = M_t\}$ ,
- $N_{2,t} := \{\omega, \exists s \in [0, t], \quad \det(\gamma_G(s, t)) = 0\}$ ,
- $N_{3,t} := \{\omega, X_t^1 = M_t\}$ ,
- $N_t = \{\omega, \det(\gamma_V(t)) = 0\}$ .

Remark that  $N_t \subset (N_t \cap \bigcap_{i=1}^3 N_{i,t}^c) \cup \bigcup_{i=1}^3 N_{i,t}$ .

Note that  $\mathbb{P}(N_t \cap \bigcap_{i=1}^3 N_{i,t}^c) = 0$ . Indeed if  $\omega \in N_t \cap \bigcap_{i=1}^3 N_{i,t}^c$ , since  $X^1$  admits a continuous modification there exists  $s_0$  such that  $X_{s_0}^1 = M_t$ . The fact that  $\omega \in N_{3,t}^c$  implies that  $s_0 < t$ , and  $\gamma_V(t) = (\Gamma_G^{i,j}(s_0, t))_{(i,j) \in \{1, \dots, d+1\}^2}$  is a sub matrix of  $\gamma_G(s_0, t)$ . The fact that  $\gamma_V(t)$  is not invertible contradicts the fact that  $\gamma_G(s_0, t)$  is invertible. Then, it remains to prove that  $\mathbb{P}(N_{i,t}) = 0$  for  $i = 1, 2, 3$ .

Step 2: Using the same lines as the proof of Proposition 2.1.11 [16], we prove that almost surely

$$\{s : X_s^1 = M_t\} \subset \{s : DM_t = DX_s^1\}$$

meaning  $\mathbb{P}(N_{1,t}) = 0$ . We skip the details for simplicity.

Step 3: For all  $t > 0$ , almost surely for all  $s < t$ , the  $2d \times 2d$  matrix  $\gamma_G(s, t)$  is invertible, meaning that  $\forall t$ , the event  $N_{2,t}$  is negligible.

*Proof:* This matrix  $\gamma_G(s, t)$  is symmetrical and using (2.59) and (2.60) in [16] yields:

$$(8) \quad \gamma_G(s, t) = \begin{pmatrix} Y(t)C(t)Y^T(t) & Y(s)C(s)Y^T(t) \\ Y(t)C(s)Y^T(s) & Y(s)C(s)Y^T(s) \end{pmatrix}$$

where, using Einstein's convention

$$(9) \quad C^{i,j}(t) := \int_0^t Y^{-1}(u)_k^i A_l^k(X_u) A_l^{k'}(X_u) Y^{-1}(u)_{k'}^j du$$

$$Y_j^i(t) := \delta_{i,j} + \int_0^t \bar{A}_{k,l}^i(u) Y_j^k(u) dW_u^l + \int_0^t \bar{B}_k^i(u) Y^k(u) du, \quad i, j \in \{1, \dots, d\}.$$

Let us denote

$$C^{i,j}(s, t) := C^{i,j}(t) - C^{i,j}(s).$$

According to (2.58) [16] there exists a process  $Z$  such that almost surely for all  $h \in [0, T]$

$$Z(h)Y(h) = Id$$

thus for all  $t$  the matrices  $Y(t)$  are invertible.

As a consequence, for all  $(s, t) \in [0, T]^2$ , the  $(d, d)$  matrix  $Y(s, t) := Y(t)Y(s)^{-1}$  is too invertible. Then  $\gamma_G(s, t)$  (8) can be rewritten as a matrix composed with four  $(d, d)$  blocks:

$$\gamma_G(s, t) := \begin{pmatrix} Y(s, t)Y(s)[C(s) + C(s, t)]Y^T(s)Y^T(s, t) & Y(s)C(s)Y^T(s)Y^T(s, t) \\ Y(s, t)Y(s)C(s)Y^T(s) & Y(s)C(s)Y^T(s) \end{pmatrix}$$

The second line of blocks multiplied by  $Y(s, t)'$  and this one subtracted to the first line yield:

$$\det[\gamma_G(s, t)] = \det \begin{pmatrix} Y(s, t)Y(s)C(s, t)Y^T(s)Y^T(s, t) & 0 \\ Y(s, t)Y(s)C(s)Y^T(s) & Y(s)C(s)Y^T(s) \end{pmatrix}.$$

The properties of block trigonal matrix determinants prove that

$$(10) \quad \det[\gamma_G(s, t)] = |Y(s, t)Y(s)C(s, t)Y^T(s)Y^T(s, t)Y(s)C(s)Y^T(s)|$$

The processes  $Z$  are  $Y$  are diffusion processes so each of them admits a continuous modification satisfying  $Z(h)Y(h) = Id, \forall h \in [0, T]$ . Thus, almost surely the continuous process  $Z$  is invertible so satisfies almost surely for all  $0 \leq s \leq t \leq T$

$$\int_s^t \det(Z(h))^2 dh > 0.$$

Let  $\sigma(x) = \sum_{l=1}^d A_l(x)A_l(x)'$ . By definition of  $C$  (see (9))

$$C(s) = \int_0^s Y^{-1}(h)\sigma(X_h)(Y(h)^{-1})' dh, \quad C(s, t) = \int_s^t Y^{-1}(h)\sigma(X_h)(Y(h)^{-1})' dh$$

We now follow the proof of Theorem 2.3.1 page 127 [16]: for  $v \in \mathbb{R}^d$ , using the uniform ellipticity (Assumption (2)):

$$v'\sigma(X_s)v \geq c|v|^2, \quad \forall s > 0, \quad \forall v \in \mathbb{R}^d.$$

With  $v = (Y(h)^{-1})'u$  we get

$$u'Y(h)^{-1}\sigma(X_h)(Y(h)^{-1})'u \geq cu'Y(h)^{-1}(Y(h)^{-1})'u$$

and

$$u'C(s)u = \int_0^s u'Y(h)^{-1}\sigma(X(h))(Y(h)^{-1})'udh \geq c \int_0^s u'Y(h)^{-1}(Y(h)^{-1})'udh = c|u|^2 \int_0^s \det(Z(h))^2 dh.$$

Similarly

$$u'C(s,t)u = \int_s^t u'Y(h)^{-1}\sigma(X(h))(Y(h)^{-1})'udh \geq c|u|^2 \int_s^t \det(Z(h))^2 dh.$$

Thus almost surely for all  $s \in ]0, t[$ ,  $C(s)$  and  $C(s, t)$  are invertible. Since  $Y(s, t)$ ,  $Y(s)$  are too invertible as a consequence, from (10) the matrix  $\gamma_G(s, t)$  is invertible. The process  $t \rightarrow D.(X_t)$  takes its values in  $\mathbb{H}$  and admits a continuous modification (see (6) above) and the set of invertible matrices is an open set then,

$$\mathbb{P}(\{\omega, \exists s \in ]0, t[, \det(\gamma_G(s, t)) = 0\}) = \mathbb{P}(N_{2,t}) = 0. \quad \square$$

Step 4: Under Assumptions (1) and (2), time  $t$  being fixed, almost surely  $M_t > X_t^1$  meaning the event  $N_{3,t}$  is negligible.

*Proof:* For sake of completeness we prove this result. Actually it is Proposition 18 [10] but with stronger assumptions than ours.

The set  $\{M_t = X_t^1\}$  is detailed as follows:

$$(11) \quad \{\omega, M_t(\omega) = X_t^1(\omega)\} = \Omega_1 \cup \Omega_2 \text{ where} \\ \Omega_1 = \{\omega, \exists s < t \mid \forall u \in [s, t], X_u^1(\omega) = X_t^1(\omega)\}, \Omega_2 = \{\omega \mid \forall u < t, X_u^1(\omega) < X_t^1(\omega)\}.$$

Using Assumptions (1) and (2),  $A^{-1}B$  is bounded, thus an equivalent change of equivalent probability measure can be operated using Girsanov Theorem: the probability measure  $\mathbb{P}_0$  is defined as

$$(12) \quad \frac{d\mathbb{P}_0}{d\mathbb{P}|_{\mathcal{F}_t}} = L_t, \quad L_t := \exp\left(-\int_0^t (BA^{-1})^i(X_s)dW_s^i - \frac{1}{2}\int_0^t \|(BA^{-1}(X_s))\|^2 ds\right).$$

Then  $X^1$  is a  $(\mathcal{F}, \mathbb{P}_0)$  martingale:

$$(13) \quad X_t^1 = X_0^1 + \int_0^t \sum_j A^{1,j}(X_s)d\tilde{W}_s^j$$

where  $\tilde{W}$  is a  $(\mathcal{F}, \mathbb{P}_0)$   $d$ -dimensional Brownian motion. The bracket of  $X^1$ , actually independent of the probability measure in continuous case, is

$$\langle X^1, X^1 \rangle_t = \int_0^t \sum_j (A^{1,j}(X_s))^2 ds.$$

Assumption (2) applied to  $v = (1, 0, \dots, 0)$  implies that for any  $x$ ,  $\sum_j A^{1,j}(x)A^{1,j}(x) \geq c > 0$ . This has two consequences:

- For all rational numbers  $q < q'$  in  $[0, T]$

$$\langle X^1, X^1 \rangle_{q'} - \langle X^1, X^1 \rangle_q > c(q' - q) > 0.$$

According to Proposition 1.13 page 119 [17], for all rational numbers  $q < q'$  in  $[0, T]$   $X^1$  is not constant on the interval  $[q, q']$ . Let us remark

$$\{\omega, \exists s > t \mid \forall u \in [s, t], X_u^1(\omega) = X_t^1(\omega)\} \subset \cup_{q < q' < T, q, q' \in \mathbb{Q}} \{\omega \mid u \in [q, q'], X_u^1(\omega) = X_q^1(\omega)\}$$

thus

$$\mathbb{P}_0(\{\omega \mid \exists s > t, \forall u \in [s, t], X_u^1(\omega) = X_t^1(\omega)\}) = 0.$$

The probability measures  $\mathbb{P}_0$  and  $\mathbb{P}$  are equivalent so

$$(14) \quad \mathbb{P}(\{\omega, \exists s > t \mid \forall u \in [s, t], X_u^1(\omega) = X_t^1(\omega)\}) = 0.$$

• Using Dambis-Dubins-Schwarz' Theorem (Theorem 1.6 Chapter V [17]), and once again that  $\sum_j (A^{1,j}(X_s))^2 \geq c$ , then

$$\langle X^1, X^1 \rangle_\infty = \int_0^\infty \sum_j (A^{1,j}(X_s))^2 ds = +\infty.$$

So there exists a  $\mathbb{P}_0$  Brownian motion  $B$  such that

$$X^1(t) = B_{\langle X^1, X^1 \rangle_t}, \quad \forall t > 0.$$

Here is followed step by step the proof of Theorem 2.7 Chapter I [17] (Lévy's modulus of continuity), but without absolute value: for all  $N \in \mathbb{R}$

$$\mathbb{P}_0 \left( \limsup_{\varepsilon \rightarrow 0} \left\{ \sup_{\substack{0 \leq t_1, t_2 \leq N \\ t_1 - t_2 < \varepsilon}} \frac{B_{t_1} - B_{t_2}}{h(\varepsilon)} = 1 \right\} \right) = 1$$

where  $h(s) = \sqrt{2s \log(1/s)}$ ,  $s \in [0, 1]$ .

This is equivalent to

$$\mathbb{P}_0 \left( \liminf_{\varepsilon \rightarrow 0} \left\{ \sup_{\substack{0 \leq t_1, t_2 \leq N \\ t_1 - t_2 < \varepsilon}} \frac{B_{t_1} - B_{t_2}}{h(\varepsilon)} \neq 1 \right\} \right) = 0.$$

We remark that

$$\begin{aligned} \Omega_2 &= \{\omega, \forall u \in [0, t], X^1(u) < X^1(t)\} = \{\omega, \forall u \in [0, t], B_{\langle X^1, X^1 \rangle_u} < B_{\langle X^1, X^1 \rangle_t}\} \\ &\subset \cup_N \cup_n \cap_{k \geq n} \left\{ \sup_{\substack{0 \leq t_1, t_2 \leq N \\ t_1 - t_2 < 1/k}} \frac{B_{t_1} - B_{t_2}}{h(1/k)} \leq 0 \right\} \\ &\subset \cup_N \liminf_{1/n \rightarrow 0} \left\{ \sup_{\substack{0 \leq t_1, t_2 \leq N \\ t_1 - t_2 < 1/n}} \frac{B_{t_1} - B_{t_2}}{h(1/n)} \neq 1 \right\} \end{aligned}$$

thus  $\mathbb{P}_0(\Omega_2) = 0$ . The probability measures  $\mathbb{P}$  and  $\mathbb{P}_0$  are equivalent, so

$$(15) \quad \mathbb{P}(\Omega_2) = \mathbb{P}(\{\omega, \forall s \in [0, t], X_s^1(\omega) < X_t^1(\omega)\}) = 0.$$

Finally (11), (14), (15) prove that

$$(16) \quad \mathbb{P}(\{M_t = X_t^1\}) = \mathbb{P}(N_{3,t}) = 0.$$

□

These four steps conclude the proof of Proposition 1.6, so Assumption (ii) in Theorem 1.4 is satisfied and Theorem 1.1 is proved.

## 2 Case $d = 1$ , regularity of the density

In the one-dimensional case we study the regularity of the density of the law of the vector  $V_t$ .

We recall the model, the following stochastic differential equation (see [6])

$$(17) \quad dX_t = B(X_t)dt + A(X_t)dW_t, \quad X_0 = x, \quad t \in [0, T],$$

where  $W$  is a Brownian motion,  $A$  and  $B$  satisfy Assumptions (1) and (2). Moreover Assumption (2) yields that  $A^{-1}B$  is bounded. We recall the notations  $V := (M, X)$  where  $M_t = \sup_{s \leq t} X_s$ ,  $t \in \mathbb{R}$ .

From Theorem 1.1. in Section 1, the law of the pair  $V_t = (M_t, X_t)$  admits a density. With Malliavin calculus tools, we are not able to prove the regularity of the density on the diagonal  $\{(m, x) : m = x\}$ . So we turn to the regularity of the density on the open subset  $\{(m, x), m > \max(0, x)\}$ .

**Theorem 2.1** *Under Assumptions (2),  $B \in C_b^4(\mathbb{R})$ ,  $A \in C_b^5(\mathbb{R})$ ,  $\forall t > 0$ , the density of the vector  $V_t$  restricted to the open set  $\Delta := \{(m, x), m > \max(0, x)\}$  belongs to  $C^{2,2}(\Delta)$ .*

## 2.1 Reduction to a drifted Brownian motion

Since we are in a one-dimensional setting, we use a Lamperti transformation in order to reduce the problem to the case of a diffusion with additive noise. A priori  $dX_t = B(X_t)dt + A(X_t)dW_t$ . We look for an increasing function  $\varphi \in C_b^2$  such that the coefficient of  $dW$  would be 1. Itô formula yields

$$d\varphi(X_t) = \varphi'(X_t)B(X_t)dt + \frac{1}{2}\varphi''(X_t)A^2(X_t)dt + \varphi'(X_t)A(X_t)dW_t.$$

A sufficient condition is to choose  $\varphi$  such that  $\varphi' = \frac{1}{A}$ . As is  $A$ ,  $\varphi'$  is bounded above and below uniformly. Then  $Y = \varphi(X)$  satisfies

$$(18) \quad dY_t = \left[ \frac{B}{A} \circ \varphi^{-1}(Y_t) - \frac{1}{2}A' \circ \varphi^{-1}(Y_t) \right] dt + dW_t.$$

Remark that  $\tilde{B} = \frac{B}{A} \circ \varphi^{-1} - \frac{1}{2}A' \circ \varphi^{-1} \in C_b^1$  as a consequence of Assumptions of Theorem 2.1. Moreover  $\varphi'$  being positive,  $\varphi$  is increasing and  $Y_t^* = \varphi(X_t^*)$ . From Theorem 1.1, the law of the pair  $(Y_t^*, Y_t)$  admits a density with respect to Lebesgue measure and the link between  $V_t$  and  $(Y_t^*, Y_t)$  law densities is done by the following lemma.

**Lemma 2.2** *We assume that  $A$  and  $B$  satisfy Assumptions of Theorem 2.1. Then the density law of  $(M_t, X_t)$ ,  $p_V(\cdot, \cdot, t)$ , satisfies*

$$p_V(b, a; t) = \frac{p_{Y^*, Y}(\varphi(b), \varphi(a); t)}{A(b)A(a)}$$

where  $\varphi$  is defined by  $\varphi'(x) = \frac{1}{A(x)}$  and  $p_{Y^*, Y}(\cdot, \cdot, t)$  is the pair  $(Y_t^*, Y_t)$  law density.

**Proof:** It is enough to identify the density law of the pair  $V_t = (M_t, X_t)$  using, for any bounded measurable  $F$ , the following

$$E[F(M_t, X_t)] = E[F(\varphi^{-1}(Y_t^*), \varphi^{-1}(Y_t))] = \int F(\varphi^{-1}(\beta), \varphi^{-1}(\alpha)) p_{Y^*, Y}(\beta, \alpha; t) d\beta d\alpha.$$

We operate the change of variables  $b = \varphi^{-1}(\beta)$ ,  $a = \varphi^{-1}(\alpha)$ , so  $d\beta = \varphi'(b)db = \frac{1}{A(b)}db$  and  $d\alpha = \varphi'(a)da = \frac{1}{A(a)}da$  get the result.  $\bullet$

With such a lemma, the regularity of the density  $p$  will be the one of  $p_{Y^*, Y}, \varphi, A$ . Theorem 2.1 is a consequence of the following theorem where  $B$  is indeed  $\frac{B}{A} \circ \varphi^{-1} - \frac{1}{2}A' \circ \varphi^{-1}$ .

**Theorem 2.3** *Assume that  $B$  belongs to  $C_b^4(\mathbb{R})$ , then for all  $t > 0$  the law of  $(M_t, X_t)$  has a  $C^{2,2}$  density with respect to the Lebesgue measure on  $\Delta = \{(b, a), b > a, b > 0\}$  where  $M_t = \sup_{s \leq t} X_s$  and  $X_t$  is the solution of*

$$dX_t = B(X_t)dt + dW_t.$$

## 2.2 Change of probability measure

We now consider the model with  $A = 1$  meaning

$$dX_t = B(X_t)dt + dW_t.$$

Recall the equivalent change of probability measure  $Q = L\mathbb{P}$  where  $L$  is defined as  $dL_t = -L_t B(X_t)dW_t$ . Abusing of notation, remark that under  $Q$ , here we use  $W$  instead of  $X$  which is a  $Q$ -Brownian motion. Denoting  $Z = L^{-1}$ ,  $\mathbb{P} = ZQ$ ,

$$(19) \quad Z_t = \exp\left[\int_0^t B(W_s)dW_s - \frac{1}{2} \int_0^t B^2(W_s)ds\right].$$

Since  $B$  is bounded,  $Z_t \in L^p$  for any  $p$ .

Under  $Q$  the law of  $V_t$  is the one of  $(W_t^*, W_t)$  where  $W_t^* := \sup_{s \leq t} W_s$ .

**Lemma 2.4** *As soon as  $B \in C_b^4$ , then  $Z_t \in \mathbb{D}^{4,p}$  for any  $p > 1$ , where  $\mathbb{D}^{4,p}$  is the set of random variables 4 times differentiable in the Malliavin sense with derivatives in  $L^p$ .*

**Proof:** Using the definition of  $Z_t$  we only have to prove  $\forall p > 1$

$$\int_0^t B(W_s)dW_s - \frac{1}{2} \int_0^t B^2(W_s)ds \in \mathbb{D}^{4,p}.$$

Obviously by linearity  $\int_0^t B^2(W_s)ds \in \mathbb{D}^{4,p}$  since actually  $D_{s_1, \dots, s_i}^i \int_0^t B^2(W_s)ds = \int_{\max_j s_j}^t (B^2)^{(i)}(W_s)ds$ . Using the Malliavin derivative  $D_r[\int_0^t B(W_s)dW_s] = B(W_r) + \int_r^t B'(W_s)dW_s$  and recursively  $D^i Z_t$ ,  $i = 1, \dots, 4$ , are well defined.

Finally the lemma is proved using the boundness of  $B$  and its derivatives. •

From now on, using a change of time unit, it is enough to consider the case  $t = 1$ .

## 2.3 Local non degeneracy of $F := V_1 = (W_1^*, W_1)$

Let us recall Definition 2.1.2 in Nualart [16].

**Definition 2.5** *A random vector  $F = (F_1, \dots, F_m)$  whose components are in  $\mathbb{D}^{1,2}$  is locally non degenerate in an open set  $\Delta \subset \mathbb{R}^m$  if there exist elements  $u_\Delta^j \in \mathbb{D}^\infty(\mathbb{H})$ ,  $j = 1, \dots, m$  and an  $m \times m$  random matrix  $\gamma_\Delta = (\gamma_\Delta^{i,j})$  such that  $\gamma_\Delta^{i,j} \in \mathbb{D}^\infty$ ,  $|\det \gamma_\Delta|^{-1} \in L^q(\Omega)$  for all  $q \geq 1$ , and  $\langle DF^i, u_\Delta^j \rangle = \gamma_\Delta^{i,j}$  on  $\{F \in \Delta\}$  for any  $i, j = 1, \dots, m$ .*

**Proposition 2.6** *Let  $\eta > 0$ . The random vector  $F := V_1 = (W_1^*, W_1)$  is locally non degenerate in any open set  $\Delta_\eta = \{(b, a), b - a > \eta, b > \eta\}$ .*

**Proof:** The key is to prove that the random vector  $F := V_1 = (W_1^*, W_1)$  satisfies Definition (2.5).

Step 1: The  $F$  components belong to  $\mathbb{D}^{1,2}$

Indeed,  $D_t W_1 = \mathbf{1}_{[0,1]}(t)$ , and  $D_t W_1^* = \mathbf{1}_{[0,\tau]}(t)$  where  $\tau := \inf\{s, W_s = W_1^*\}$ .

Step 2

Let the process  $\tilde{W}$  be defined as  $\tilde{W}_t := W_{1-t} - W_1$ .

Similarly to [16] Proposition 2.1.12 page 112  $\forall p > 2$ , let  $s \in [0, 1]$  and  $\gamma$  such that  $2p\gamma \in (1, p - 1)$  we introduce:

$$\begin{aligned} Y^1(s) &= \int_{[0,s]^2} \frac{(W_u - W_{u'})^{2p}}{(u - u')^{1+2p\gamma}} du du' \\ Y^2(s) &= \int_{[0,s]^2} \frac{(\tilde{W}_u - \tilde{W}_{u'})^{2p}}{(u - u')^{1+2p\gamma}} du du' \end{aligned}$$

Following this proposition proof, it can be proved that:

$$(20) \quad \exists R(\eta, p, \gamma) \text{ such that } Y^1(s) \leq R \Rightarrow \sup_{z \in [0,s]} W_z \leq \eta;$$

$$\text{similarly } Y^2(s) \leq R \Rightarrow \sup_{z \in [0,s]} \tilde{W}_z \leq \eta.$$

Let  $\psi : \mathbb{R}^+ \rightarrow [0, 1]$  be infinitely differentiable and such that  $\psi(x) = 1$  when  $x \in [0, R/2]$ ,  $\psi(x) = 0$  when  $x \geq R$ . Denote

$$(21) \quad u_\eta^1(s) = \psi(Y^1(s)), \quad u_\eta^2(s) = \psi(Y^2(1-s))$$

and

$$(22) \quad \Gamma_\eta = \begin{pmatrix} \int_0^1 \psi(Y^1(s)) ds & \int_0^1 \psi(Y^1(s)) ds \\ 0 & \int_0^1 \psi(Y^2(1-s)) ds \end{pmatrix}.$$

Step 3

The processes  $u_\eta^i \in \mathbb{D}^\infty(H)$   $i = 1, 2$ , by the construction of  $Y^i$  and the definition of the function  $\psi$ . As a consequence, the random matrix  $\Gamma_\eta$  belongs to  $\mathbb{D}^\infty$ .

Step 4 Note that  $(\det \Gamma_\eta)^{-1} = [\int_0^1 \psi(Y^1(s)) ds \int_0^1 \psi(Y^2(1-s)) ds]^{-1}$ . We will prove that  $\forall q \geq 1$ ,

$$(23) \quad \left( \int_0^1 \psi(Y^1(s)) ds \right)^{-1} \text{ and } \left( \int_0^1 \psi(Y^2(1-s)) ds \right)^{-1} \in L^q.$$

Indeed using the trick of Proposition 2.1.12 [16] page 114  $\int_0^1 \psi(Y^1(s)) ds \geq \int_0^1 \mathbf{1}_{\{Y^1(s) < R/2\}} ds = \text{Leb}\{s : Y^1(s) < R/2\} = (Y^1)^{-1}(R/2) > 0$ , since the non decreasing function  $Y^1$  is invertible; so for any  $q$ ,  $\left( \int_0^1 \psi(Y^1(s)) ds \right)^{-q} \leq [(Y^1)^{-1}(R/2)]^{-q}$ . Recall that

$$\mathbb{E} \left( [(Y^1)^{-1}(R/2)]^{-q} \right) = \int_0^\infty \mathbb{P}\{(Y^1)^{-1}(R/2) < x^{-1/q}\} dx.$$

On the other hand for any  $y$ ,  $\mathbb{P}\{(Y^1)^{-1}(R/2) < y\} = \mathbb{P}\{R/2 < Y^1(y)\} \leq (\frac{2}{R})^q \mathbb{E}[(Y^1)^q(y)]$ . We compute  $\mathbb{E}[(Y^1)^q(y)]$  as

$$\mathbb{E} \left( \left[ \int_{[0,y]^2} \frac{(W_u - W_{u'})^{2p}}{(u - u')^{1+2p\gamma}} dud u' \right]^q \right).$$

Using Jensen's inequality,

$$\mathbb{E}[(Y^1)^q(y)] \leq y^{2q-2} \left[ \int_{[0,y]^2} \frac{\mathbb{E}[(W_u - W_{u'})^{2pq}]}{(u - u')^{q(1+2p\gamma)}} dud u' \right]$$

Since  $W$  is a Brownian motion,  $\mathbb{E}[(W_u - W_{u'})^{2pq}] \leq C|u - u'|^{pq}$  and using  $2p\gamma < p - 1$ ,

$$y^{2q-2} \int_{[0,y]^2} \frac{\mathbb{E}[(W_u - W_{u'})^{2pq}]}{(u - u')^{q(1+2p\gamma)}} dud u' \leq C y^{q(1+p-2\gamma p)}.$$

Choosing  $y = x^{-1/q}$ ,

$$\mathbb{P}\{(Y^1)^{-1}(R/2) < x^{-1/q}\} \leq C_1 x^{-(1+p-2\gamma p)}.$$

Using this bound for  $x \geq 1$ , and  $2\gamma p - p - 1 > -2$ ,

$$\mathbb{E} \left( [(Y^1)^{-1}(R/2)]^{-q} \right) = \int_0^\infty \mathbb{P}\{(Y^1)^{-1}(R/2) < x^{-1/q}\} dx < \infty.$$

Recall that  $\tilde{W}$  is a Brownian motion (time reversal), so both processes  $(Y_s^i, 0 \leq s \leq 1), i = 1, 2$ , have the same law so the two factors in  $\det \Gamma_\eta$  have the same law. So, for any  $q$ ,  $(\int_0^1 \psi(Y^2(1-s)) ds)^{-1}$  also belongs to  $L^q$ . This achieves the proof of (23) and as a conclusion,  $(\det \Gamma_\eta)^{-1} \in L^q$  for any  $q$ .

### Step 5

According to Definition 2.5 we now prove that the following random matrix

$$(24) \quad \begin{pmatrix} \langle DW_1^*, u_\eta^1 \rangle & \langle DW_1, u_\eta^1 \rangle \\ \langle DW_1^*, u_\eta^2 \rangle & \langle DW_1, u_\eta^2 \rangle \end{pmatrix}$$

coincides with  $\Gamma_\eta$  on the set  $\{V_1 \in \Delta_\eta\}$ .

(i) By definition of the time  $\tau$ ,  $W_\tau = W_1^* > \eta$ , this contradicts (20) so  $Y^1(s) \geq R$  thus  $\psi(Y^1(s)) = 0$  as soon as  $s \geq \tau$ ;  $\langle DW_1^*, u_\eta^1 \rangle = \int_0^\tau \psi(Y^1(s)) ds = \int_0^1 \psi(Y^1(s)) ds$ .

(ii) Concerning  $\langle DW_1^*, u_\eta^2 \rangle = \int_0^\tau u_\eta^2(s) ds$ , it is null:

On the set  $\{W_1^* > \eta, W_1^* - W_1 > \eta\}$ , using  $W_1^* = W_\tau$ , so  $W_\tau - W_1 > \eta$ , we get  $W_\tau - W_1 = Z_{1-\tau} > \eta$  so  $\eta \leq Z_{1-\tau} \leq \sup_{0 \leq u \leq 1-s} Z_u$  for  $s \leq \tau$ . Once again, this contradicts (20) so  $Y^2(1-s) > R$ , and the definition of  $\psi$  yields  $u_\eta^2(s) = 0, \forall s \in [0, \tau]$

(iii) Concerning  $\langle DW_1, u_\eta^i \rangle$ , since  $DW_1 = 1$  on  $[0, 1]$ ,  $\langle DW_1, u_\eta^2 \rangle = \int_0^1 \psi(Y^2(1-s)) ds$  and  $\langle DW_1, u_\eta^1 \rangle = \int_0^1 \psi(Y^1(s)) ds$ . •

## 2.4 Proof of Theorem 2.3

Let  $v_0 = (x_0^*, x_0) \in \Delta_\eta := \{(x^*, x), x^* > \eta, x^* - x > \eta\}$  and  $\delta < \frac{1}{2}d(v_0, \Delta_\eta^c)$  and  $\delta < \delta' < d(v_0, \Delta_\eta^c)$ . Recall the notations  $u_\eta^i, i = 1, 2, \Gamma_\eta$  (21) and (22).

(i) Let  $\varphi \in C_\eta^\infty(\mathbb{R}^m)$  meaning that  $\text{support}(\phi) \subset \Delta_\eta$  and denote  $F = (W_1^*, W_1)$ , then using the definition of  $\mathbb{P}$  and  $\mathbb{Q}$

$$(25) \quad \mathbb{E}(\partial_{1,2}^2 \varphi(F)) = \mathbb{E}_Q(\partial_{1,2}^2 \varphi(F) Z_1)$$

where  $Z$  is defined in (19).

Since  $F$  is locally non degenerate (Proposition 2.6), we apply Proposition 2.1.4 (2.25) [16] to the multi-index  $\alpha = (1, 2)$  and  $G = Z_1$ . Since we use Proposition 2.1.4 [16] with index  $\alpha = (1, 2)$ , we only need  $Z_1$  belonging to  $\cap_{p>1} \mathbb{D}_p^2$ . Later, we will use twice these indices, so then we will need  $Z_1$  belonging to  $\cap_{p>1} \mathbb{D}_p^4$ , see Lemma 2.4.

Using identity (2.25) of Proposition 2.1.4 [16] to the multi-index  $\alpha = (1, 2)$  and  $G = Z_1$  in the right hand of (25) we obtain

$$(26) \quad \mathbb{E}(\partial_{1,2}^2 \varphi(F)) = \mathbb{E}_Q(\varphi(F) H_{1,2}(Z_1)),$$

where, following Formulas (2.26) and (2.27) of Proposition 2.1.4 [16], expressed in our framework:

$$H_{(i)}(Z_1) = \sum_{j=1}^2 \delta(Z_1(\Gamma_\eta^{-1})^{i,j} u_\eta^j), = \delta(Z_1(\Gamma_\eta^{-1} u_\eta)_i) = Z_1 \delta((\Gamma_\eta^{-1} u_\eta)_i) - \int_0^1 D_s Z_1(\Gamma_\eta^{-1} u_\eta(s))_i ds, \quad i = 1, 2$$

and for  $\beta = (1, 2)$

$$H_{1,2}(Z_1) = H_{(2)}(H_{(1)}(Z_1)).$$

We detail this definition:

$$(27) \quad H_{1,2}(Z_1) = \delta[H_{(1)}(Z_1(\Gamma_\eta^{-1} u_\eta)_2)] = \delta[\delta[Z_1(\Gamma_\eta^{-1} u_\eta)_1](\Gamma_\eta^{-1} u_\eta)_2].$$

Since

$$(28) \quad \varphi(W_1^*, W_1) = \int_{-\infty}^{W_1^*} \int_{-\infty}^{W_1} \partial_{1,2}^2 \varphi(x^*, x) dx^* dx$$

with Fubini in (26) we obtain

$$(29) \quad \mathbb{E}[\partial_{1,2}^2 \varphi(F)] = \int_{\mathbb{R}^2} \partial_{1,2}^2 \varphi(x^*, x) E_Q[\mathbf{1}_{W_1 > x, W_1^* > x^*} H_{1,2}(Z_1)] dx^* dx.$$

Then Equations (27) and (29) imply that the law of  $F$  has a density with respect to the Lebesgue measure given by

$$p_V(x^*, x; 1) = E_Q[\mathbf{1}_{W_1 > x, W_1^* > x^*} \delta[\delta[Z_1(\Gamma_\eta^{-1} u_\eta)_1](\Gamma_\eta^{-1} u_\eta)_2]].$$

(ii) We now turn to the differentiability of this density. Let  $\alpha = (1, 2)$  and  $\beta = 1$ , or 2 or  $(i, j), i, j \leq 2$ , using the definition of  $\mathbb{P}$  and  $Q$

$$(30) \quad \mathbb{E}(\partial_{\alpha, \beta}^{2+|\beta|} \varphi(F)) = \mathbb{E}_Q[\partial_{\alpha, \beta}^{2+|\beta|} \varphi(F) Z_1].$$

Then using Proposition 2.1.4 [16] for  $\alpha + \beta$

$$(31) \quad \mathbb{E}(\partial_{\beta}^{|\beta|} (\partial_{1,2}^2 \varphi(F))) = \mathbb{E}_Q[\varphi(F) H_{\beta}(H_{\alpha}(Z_1))]$$

Using representation (28) of  $\varphi$

$$(32) \quad \mathbb{E}[(\partial_{\alpha, \beta}^{2+|\beta|} \varphi(F))] = \int_{\mathbb{R}^2} \partial_{1,2}^2 \varphi(x^*, x) \mathbb{E}_Q[\mathbf{1}_{W_1^* > x^*, W_1 > x} H_{\beta}(H_{\alpha}(Z_1))] dx dx^*.$$

Let  $\phi$  be a regular function with support in the ball  $B(v_0, \delta) \subset \Delta_{\eta}$  and  $\xi = \partial_{\alpha}^2 \varphi$  so  $\partial_{\alpha, \beta}^{2+|\beta|} \varphi = \partial_{\beta}^{|\beta|} \xi$ ; we apply (32):

$$(33) \quad \mathbb{E}[(\partial_{\beta}^{|\beta|} \xi(F))] = \int_{\mathbb{R}^2} \xi(x^*, x) \mathbb{E}_Q[\mathbf{1}_{W_1^* > x^*, W_1 > x} H_{\beta}(H_{\alpha}(Z_1))] dx dx^*.$$

The left hand above is identified as  $\int_{\mathbb{R}^2} \partial_{\beta}^{|\beta|} \xi(x^*, x) p_V(x^*, x; 1) dx^* dx$  but this one, using an integration by part, is also  $\pm \int_{\mathbb{R}^2} \xi(x^*, x) \partial_{\beta}^{|\beta|} p_V(x^*, x; 1) dx^* dx$ . Thus we can identified on any ball  $B_{\delta}(v)$ , using the notation  $|\beta|$  is the length of  $\beta$ :

$$\partial_{\beta}^{|\beta|} p_V(x^*, x; 1) = (-1)^{|\beta|} \mathbb{E}_Q[\mathbf{1}_{W_1^* > x^*, W_1 > x} H_{\beta}(H_{\alpha}(Z_1))].$$

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### 3 Monotonous image of the Brownian motion

Here is provided a simple example which is completely solvable and connected with [5] where is studied the density of the law as a weak solution to a partial differential equation.

• Firstly one recalls the standard result concerning the Brownian motion. The density of the law of  $(W_t^*, W_t)$  is well known,  $\forall t > 0$ , see [13] Section 3.2, and is defined on  $\mathbb{R}^2$  by

$$(34) \quad p_W(b, a; t) := 2 \frac{(2b - a)}{\sqrt{2\pi t^3}} e^{-\frac{(2b-a)^2}{2t}} \mathbf{1}_{\sup(0, a), \infty[}(b), t > 0.$$

It could be checked that Corollary 1 [6] provides for all  $t > 0$ , a partial differentiable equation and boundary conditions for the density of the law of  $V_1 := (W_1^*, W_1)$ :

$$(35) \quad \begin{aligned} \partial_t p_W(b, a; t) &= \frac{1}{2} \partial_{22}^2 p_W(b, a; t), \quad \forall b > \max(0, a), \quad t > 0 \\ \partial_1 p_W(b, b; t) + 2\partial_2 p_W(b, b; t) &= 0, \quad \forall b > 0, \quad t > 0, \\ \partial_t p_{W^*}(b, t) + \frac{1}{2}(\partial_1 p_W(b, b, t) + \partial_2 p_W(b, b, t)) &= 0. \end{aligned}$$

• Another case is easily deduced: the particular case of a one-dimensional monotonous image of the Brownian motion, as following:

$$dX_t = \sigma(X_t)dW_t + 1/2\sigma'(X_t)\sigma(X_t)dt, \quad X_0 = x_0$$

with  $\sigma \in C^1$  and  $\inf_{x \in \mathbb{R}} \sigma(x) \geq c > 0$ . Let  $\varphi$  be the solution to the differential equation

$$\frac{d}{dx}\varphi(x) = \sigma(\varphi(x)), \quad \varphi(0) = x_0$$

then  $X = \varphi(W)$ . Moreover since  $\varphi$  is continuous and increasing,  $M = \varphi(W^*)$  where  $W^*$  is the running maximum of the Brownian motion.

Thus, using  $p_W$  defined in (34), the law of the pair  $V_t = (M_t, X_t) = (\varphi(W_t^*), \varphi(W_t))$  admits the density on  $\mathbb{R}^2$ :  $p_V(m, x; t) := \frac{1}{\sigma(m)\sigma(x)}p_W(\varphi^{-1}(m); \varphi^{-1}(x); t)$  and similarly  $p_M(m; t) := \frac{1}{\sigma(m)}p_{W^*}(\varphi^{-1}(m); t)$ .

Let  $\mathcal{L}$  denote the infinitesimal generator of  $X$ , then its adjoint is

$$\mathcal{L}^*F(x) := \frac{1}{2}[\sigma'(x)^2 + \sigma\sigma''(x)]F(x) + \frac{3}{2}\sigma\sigma'(x)F'(x) + \frac{1}{2}\sigma(x)^2F''(x).$$

Standard change of variables, starting from the partial differentiable system (35), provides for all  $t > 0$ , a partial differentiable system for the density of the law of  $V_t = (M_t, X_t)$ ,

$$\begin{aligned} \partial_t p_V(m, x; t) &= \mathcal{L}^*(p(\cdot; t))(m, x), \\ (36) \quad 3\sigma'(m)p_V(m, m; t) + \sigma(m)[\partial_1 + 2\partial_2](p_V)(m, m; t) &= 0, \quad \forall m > 0, \\ \partial_t p_M(m, t) + \sigma'(m)p_V(m, m; t) + \frac{1}{2}\sigma(m)(\partial_1 + \partial_2)(p_V)(m, m; t) &= 0. \end{aligned}$$

**Remark 3.1** *This example is the “Example 4” in [10], but there only the law of  $M_t$  is studied instead of the pair as we do.*

## 4 Conclusion

We emphasize here that such results on the law of the couple  $(M, X)$  could be used when we are interested in barrier options, since the law of time activation (or extinction) is strongly related to that of the couple  $(M, X)$ . Moreover, for concrete applications, the tools built in [11, 12] would be used.

Finally, in a work in progress [5], results as (36) could be generalised to a diffusion process  $X$  under some convenient hypotheses as (1) and (2).

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