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Invariant Set Design for Constrained Discrete-Time Linear Systems with Bounded Matched Disturbance

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Abstract: Invariant set theory has been recognized as an important tool for control design of constrained systems subject to disturbances. Indeed, for a given control law, entering an invariant set guarantees recursive state and input constraint satisfaction in closed-loop. This paper focuses on discrete-time linear systems subject to bounded matched additive disturbance. The problem of the joint synthesis of control laws and associated invariant sets that are optimized with regards to the state constraints is investigated. An interpolation method is used to enlarge the controllable region.

Keywords: Invariant set, disturbance, state constraints, input constraints, nonlinear control

1. INTRODUCTION

Constrained control of dynamic systems in presence of disturbance faces two main challenges: the impact of the disturbances on the local behavior around a nominal equilibrium and the characterization of the controllable region. Both challenges have been addressed in several control design frameworks according to the tools and modelling assumptions: set theoretic methods (Blanchini (1999)), interval based approaches (Jaulin (2000)), or robust Model Predictive Control (Mayne et al. (2005)).

Set theoretic methods require a set description of the disturbances. A systematic way to assess the influence of disturbances is to compute invariant sets. Indeed, invariant sets are certificates for robust constraints satisfaction, recursive feasibility, and mitigation of the disturbances for a given control law (Mayne et al. (2000)). Such approaches have been studied in the context of model predictive control (Mayne et al. (2005)), robust time-optimal control (Mayne and Schroeder (1997)), or design of reference governors (Falcone et al. (2009)). A series of results regarding invariant set have been established for linear systems, with a linear control law, subject to additive bounded disturbance, see for example Kolmanovsky and Gilbert (1998). Of particular interest is the so-called minimal Robust Positively Invariant (mRPI) set. It is defined as the smallest invariant set for a given disturbance set (Raković et al. (2005)). It corresponds to the limit set of state trajectories for any sequence of disturbances. Characterization of the controllable region can also be addressed by set theoretic methods (Blanchini (1999); Mayne et al. (2005)). The set of interest for such controllability analysis is the largest invariant set respecting the constraints, denoted Maximal Robustly Positively Invariant (MRPI) set.

Previous work has focused on the joint synthesis of control laws and associated invariant sets tailored to state or input constraints (Corradini et al. (2014); Nguyen (2012); Tahir and Jaimoukha (2015); Raković et al. (2007); Michel et al. (2018)). In Raković et al. (2007), a characterization of families of robust control invariant sets, based on outer approximations of the mRPI set, is proposed. This characterization can be used to establish optimized invariant sets regarding the state and input constraints. The control design in Michel et al. (2018) considers bounded matched additive disturbance and adopts a sliding mode strategy. The resulting control law is the linear feedback gain minimizing the mRPI projection in a predefined direction.

In this paper we extend the linear control law and associated invariant set design strategy of Michel et al. (2018) to a larger class of state constraints, and we account for input constraints. The impact of the disturbances around the nominal equilibrium is further mitigated by relaxing the linear control structure, allowing for smaller invariant sets in the direction of the state constraints. An interpolation-based control design is then proposed to enlarge the controllable region.

The paper is organized as follows. Section 2 presents the class of system studied and important definitions. Section 3 introduces results on the design of invariant set tailored to the state constraints. Section 4 proposes an interpolation based method to enlarge the controllable region. Section 5 gives illustrative examples of the results. Finally, Section 6 draws conclusion and discusses perspectives.

Notation: For a positive integer p, define $\mathcal{I}_p = \{1, ..., p\}$. For a vector $h \in \mathbb{R}^n$, denote h_i its i^{th} element, define $||h||_\infty = \max|h_i|, i \in \mathcal{I}_n$, and $|h| = [|h_1| \; ... \; |h_n|]^\top$. For two vectors x and y, $x \leq y$ ($x < y$) denotes the element-wise (strict) inequalities between their components. Define 1_p the vector of ones of dimension p. The i^{th} power of a matrix A is denoted A^i. The i^{th} line of a matrix A is denoted A_i. Define the set of invertible matrices $\mathcal{G}_n = \{A \in \mathbb{R}^{n\times n} \mid det(A) \neq 0\}$. The set of Schur matrices of dimension n is denoted $\mathcal{C}_n$. For a matrix $A \in \mathbb{R}^{m\times n}$ and a set $\mathcal{X} \subseteq \mathbb{R}^n$, define the set $A\mathcal{X} = \{y \in \mathbb{R}^m \mid y = Ax, x \in \mathcal{X}\}$. Given two sets $\mathcal{A}, \mathcal{B}$, define $\mathcal{A} \oplus \mathcal{B} = \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ and $\mathcal{A} \ominus \mathcal{B} = \{x \mid \{x\} \oplus \mathcal{B} \subseteq \mathcal{A}\}$. The boundary of a set $\mathcal{A}$ is denoted $\partial\mathcal{A}$.

2. PRELIMINARIES

Consider the following discrete-time linear time-invariant system

$$x^+ = Ax + B(u + w), B = \begin{bmatrix} 0_{n-m,m}^\perp & I_m \end{bmatrix}^\top \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $w \in \mathcal{W} \subseteq \mathbb{R}^m$ is an unknown bounded disturbance, and we assume that $m < n$. Note that any system $x^+ = \bar{A}x + \bar{B}(u + w)$ with rank$(\bar{B}) = m$ can be written as (1) with a linear change of coordinates. The system (1) is subject to the state and input constraints

$$x \in \mathcal{X} = \{x \in \mathbb{R}^n \mid |Hx| \leq 1_p\}, u \in \mathcal{U}, \tag{2}$$

where $H \in \mathbb{R}^{p\times n}$, $m \leq p$. The sets $\mathcal{U}$ and $\mathcal{W}$ are bounded polytopes containing the origin in their interior. This paper focuses on the stabilization of (1) to a neighborhood of the origin, characterized in terms of invariant set, along with the characterization of the controllable region.

The following definitions are based on invariant set theory (Blanchini (1999); Kolmanovsky and Gilbert (1998)).

Definition 1. The set $\mathcal{Z}$ is said to be Robustly Controlled positively Invariant (RCI) for the system (1) and constraint set $(\mathcal{X}, \mathcal{U}, \mathcal{W})$ if $\mathcal{Z} \subseteq \mathcal{X}$, and $\forall x \in \mathcal{Z}$ there exists $u \in \mathcal{U}$ such that $Ax + B(u + w) \in \mathcal{Z}, \forall w \in \mathcal{W}$.

In the following, an RCI set will refer to an RCI set for the system (1) and constraint set $(\mathcal{X}, \mathcal{U}, \mathcal{W})$.

Given a state feedback control law $\nu : \mathbb{R}^n \to \mathbb{R}^m$, we define the set $\mathcal{X}_\nu = \{x \in \mathcal{X} \mid \nu(x) \in \mathcal{U}\}$.

Definition 2. A set $\mathcal{Z} \subseteq \mathbb{R}^n$ is said Robustly Positively Invariant (RPI) for the system $x^+ = Ax + B(\nu(x) + w)$ and constraint set $(\mathcal{X}_\nu, \mathcal{W})$, if $\mathcal{Z} \subseteq \mathcal{X}_\nu$, and $\forall x \in \mathcal{Z}$, $Ax + B(\nu(x) + w) \in \mathcal{Z}, \forall w \in \mathcal{W}$.

Remark 1. Note that an RPI set for the system $x^+ = Ax + B\nu(x) + Bw$ and constraint set $(\mathcal{X}_\nu, \mathcal{W})$ is an RCI set. Likewise, from any RCI set $\mathcal{Z}$ it is possible to define a state feedback $\nu : \mathcal{Z} \to \mathcal{U}$ such that $\mathcal{Z}$ is an RPI set for the system $x^+ = Ax + B\nu(x) + Bw$ and constraint set $(\mathcal{X}_\nu, \mathcal{W})$ (see Raković et al. (2007)).

The following definitions consider a linear state feedback law $\nu(x) = Kx$, and we define the polytopic set $\mathcal{X}_K = \{x \in \mathcal{X} \mid Kx \in \mathcal{U}\}$.

Definition 3. The *Maximal Robustly Positively Invariant* (MRPI) set for the system $x^+ = (A + BK)x + Bw$ and constraint set $(\mathcal{X}_K, \mathcal{W})$ is defined as the RPI set containing all the RPI sets, denoted here $\mathcal{O}_\infty(K)$.

Definition 4. The *minimal Robust Positively Invariant* (mRPI) set for the system $x^+ = (A + BK)x + Bw$ and constraint set $(\mathcal{X}_K, \mathcal{W})$ is defined as the RPI set contained in any closed RPI set.

If $A + BK$ is Schur, the mRPI for the system $x^+ = (A + BK)x + Bw$ and constraint set $(\mathbb{R}^n, \mathcal{W})$ exists, is unique, compact and contains the origin in its interior. Moreover, it is given by the following infinite Minkowski sum

$$\mathcal{Z}_\infty(K) = \bigoplus_{i=0}^{\infty} (A + BK)^i B\mathcal{W}.$$

An RPI for the system $x^+ = (A + BK)x + Bw$ and constraint set $(\mathcal{X}_\nu, \mathcal{W})$ exists if and only if $\mathcal{Z}_\infty(K) \subseteq \mathcal{X}_K$.

Remark 2. In general, we do not have an explicit characterization of the set $\mathcal{Z}_\infty(K)$. For computational purposes, polytopic outer approximations of this set are sought (Olaru et al. (2010); Raković et al. (2005)).

The local behavior around the origin can be characterized in terms of RCI sets, or RPI sets and their associated control law. In this paper, we want to design a local control law that mitigates the impact of the disturbances on the state constraints satisfaction. Hence, our goal is to design a control law and an associated RPI set, or an RCI set, that is minimal in the direction of the state constraints. To this local control strategy we add an interpolation-based control design to enlarge the controllable region.

A measure to evaluate the minimality of invariant sets with regards to the state constraints is introduced in the following section.

3. INVARIANT SET DESIGN

The criterion for the design of RCI sets $\mathcal{Z}$ is the minimization of

$$h(\mathcal{Z}, H) = \max_{x\in\mathcal{Z}} \quad ||Hx||_\infty. \tag{3}$$

Indeed, $\mathcal{Z}$ accounts for the impact of the disturbance, and H characterizes the direction of the state constraints. However, we do not have an explicit characterization of all the RCI sets for the system (1). The proposed approach is to first minimize (3) among mRPI obtained with linear control laws, which is a choice for the computation of RPI sets using existing methods (Olaru et al. (2010); Raković et al. (2005)). The linear control structure is then relaxed to construct a decreasing sequence of RCI sets starting from this mRPI.

The computation of a feedback gain leading to an mRPI minimizing (3) is presented below and a strategy to further improve the solution by relaxing the linear control law structure is presented in Section 3.2.

3.1 Invariant set design using linear state feedback

We now briefly recall the results in Michel et al. (2018) for the minimization of (3) with (1)-(2), $p = m$ and $\mathcal{U} = \mathbb{R}^m$. The design method allows to compute the linear control law $\nu(x) = Kx$ whose mRPI set $\mathcal{Z}_\infty(K)$ minimizes (3) under the assumptions that the matrix $H = [H_{B^\perp}\ H_B]$ satisfies $H_B \in \mathcal{G}_m$. In this paper, we propose a strategy that extends the approach for the case $p > m$.

Let us consider the matrices $H_{\sigma_i} \in \mathbb{R}^{m\times n}$, $i \in \{1, ..., \binom{p}{m}\}$ as the matrices obtained from the combination of m distinct rows out of the p rows of H. Consider the partition of those matrices, $H_{\sigma_i} = [H_{\sigma_i,B^\perp}\ H_{\sigma_i,B}]$, and define the set $\mathcal{H} = \left\{H_{\sigma_i}, i \in \{1, ..., \binom{p}{m}\} \mid H_{\sigma_i,B} \in \mathcal{G}_m\right\}$. From every matrix $H_{\sigma_i} \in \mathcal{H}$ we compute the linear control law $\nu_{\sigma_i}(x) = K_{\sigma_i}x$ with the method presented in Michel et al. (2018) that minimizes

$$h(\mathcal{Z}, H_{\sigma_i}) = \max_{x\in\mathcal{Z}} \quad ||H_{\sigma_i}x||_\infty. \tag{4}$$

Define the set

$$\mathcal{K} = \left\{K_{\sigma_i}, i \in \left\{1, ..., \binom{p}{m}\right\} \mid H_{\sigma_i} \in \mathcal{H}, K_{\sigma_i}\mathcal{Z}_\infty(K_{\sigma_i}) \subseteq \mathcal{U}\right\}$$

This set contains the linear feedback gains K_{σ_i} obtained from the matrices $H_{\sigma_i} \in \mathcal{H}$, and such that $\mathcal{Z}_\infty(K_{\sigma_i})$ is an RPI set for the system (1) and constraint set $(\mathcal{X}, \mathcal{U}, \mathcal{W})$. If the set $\mathcal{K}$ is non-empty, we chose the element minimizing (3), that is

$$K = \arg\min_{K_{\sigma_i}\in\mathcal{K}} \quad \max_{x\in\mathcal{Z}_\infty(K_{\sigma_i})} \quad ||Hx||_\infty. \tag{5}$$

Using these elements we are able to state the following result.

Proposition 1. If $\mathcal{Z}_\infty(K) \subseteq \mathcal{X}$, then robust asymptotic stability of the set $\mathcal{Z}_\infty(K)$ is achieved with a region of attraction $\mathcal{O}_\infty(K)$. Additionally, the finite determination of $\mathcal{O}_\infty(K)$ is guaranteed.

The strategy presented here allows to take the state constraints into account in the design of a linear feedback gain and construct the associated mRPI in a direct manner by exploiting the matched properties of the disturbance. If the set $\mathcal{K}$ is empty, alternative design strategies based on the complete characterization of the RCI sets taking into account input constraints are to be sought (Nguyen (2012); Rakovič et al. (2007); Tahir and Jaimoukha (2015)).

In the next section we propose to improve the solution proposed in this section by relaxing the linear control structure and allowing for nonlinear control policies.

3.2 Refinement of RCI sets with nonlinear control laws

In the previous section we imposed a linear structure to the control law in the design of an RPI set. We now propose an optimization-based method to obtain a decreasing sequence of RCI sets with nonlinear control policies starting from a polytopic RCI set.

Let Ω_0 be a polytopic RCI set. We consider the control law $\nu_0 : \Omega_0 \to \mathcal{U}$:

$$\begin{aligned}\nu_0(x) = \arg\ &\underset{u}{\text{minimize}}\ \alpha\\ &\text{subject to}\ Ax + Bu \in \Omega_0 \ominus B\mathcal{W} && (6)\\ &\qquad H(Ax + Bu) \in \alpha H(\Omega_0 \ominus B\mathcal{W})\\ &\qquad u \in \mathcal{U} && (7)\\ &\qquad 0 \le \alpha \le 1 && (8)\end{aligned}$$

which, thanks to (6), (7), and (8), satisfies

$$\forall x \in \Omega_0, Ax + B\nu_0(x) \in \Omega_0 \ominus B\mathcal{W}, \nu_0(x) \in \mathcal{U}.$$

The above definition of ν_0 seeks to minimize the scaling factor α in the direction of the state constraints defined by the matrix H as in (2). From the definition of an RCI set, the feasible domain of the above optimization problem is guaranteed to be non-empty. Note that the above optimization problem is convex (convex constraints and linear cost function).

We then define the set Ω_1 as

$$\Omega_1 = \text{ConvexHull}\{Av + B\nu_0(v), v \in \mathcal{V}(\Omega_0)\} \oplus B\mathcal{W}.$$

By construction, the set Ω_1 is polytopic, satisfies $\Omega_1 \subseteq \Omega_0$, and is an RPI set for the system $x^+ = Ax + B\nu_0(x) + Bw$ and constraint set $(\mathcal{X}_{\nu_0}, \mathcal{W})$.

Remark 3. Note that the image set, namely

$$\{z \in \mathbb{R}^n \mid z = Ax + B\nu_0(x) + Bw, x \in \Omega_0, w \in \mathcal{W}\}$$

might not be convex.

Likewise, we can define a sequence of polytopic RCI sets $\Omega_i, i \in \mathbb{N}$, as

$$\Omega_{i+1} = \text{ConvexHull}\{Av + B\nu_i(v), v \in \mathcal{V}(\Omega_i)\} \oplus B\mathcal{W}, i \in \mathbb{N},$$

where

$$\begin{aligned}\nu_i(x) = \arg\ &\underset{u}{\text{minimize}}\ \alpha\\ &\text{subject to}\ Ax + Bu \in \Omega_i \ominus B\mathcal{W}\\ &\qquad H(Ax + Bu) \in \alpha H(\Omega_i \ominus B\mathcal{W})\\ &\qquad u \in \mathcal{U}\\ &\qquad 0 \le \alpha \le 1.\end{aligned}$$

By construction, we have $\forall i \in \mathbb{N}, \forall j \in \mathbb{N}$, $\Omega_{i+j} \subseteq \Omega_i$, thus the objective (3) decreases (not strictly) as i increases.

Let $k \in \mathbb{N}$, and let $\Omega = \Omega_k$. Consider the following control law $\nu : \Omega_0 \to \mathcal{U}$:

$$\nu(x) = \nu_k(x), x \in \Omega_k, \tag{9a}$$
$$\nu(x) = \nu_i(x), x \in \Omega_i \backslash \Omega_{i+1}, i \in \{0, ..., k-1\}, \tag{9b}$$
$$\nu(x) = \nu_0(x), x \in \Omega_0. \tag{9c}$$

For all $x \in \Omega_0$, finite time convergence to Ω is guaranteed. Indeed, if $x \in \Omega_i, i \in \mathcal{I}_{k-1}$, $x^+ = Ax + B\nu(x) + Bw \in \Omega_{i+1}, \forall w \in \mathcal{W}$.

We have presented a method to construct, starting from a polytopic RCI set, a decreasing sequence of polytopic RCI sets that aims at minimizing (3).

In the following section we define the control law $\nu(x)$ outside of the set Ω_0 using interpolation-based methods to extend the controllable region.

4. ENLARGEMENT OF THE CONTROLLABLE REGION

The previous section focused on the behavior of the system in a neighborhood around the origin. The proposed control law is defined locally. We now propose a method to enlarge the controllable region using interpolation-based control.

Let K be the stabilizing gain as given in (5), Ω_0 be a polytopic outer approximation of the mRPI $\mathcal{Z}_\infty(K)$, and

$\Omega = \Omega_k$ for a given $k \in \mathbb{N}$, the k^{th} element of the decreasing polytopic RCI sets sequence starting from Ω_0 as defined in Section 3.2.

A first step to enlarge the basin of attraction is to define the control law on the set $\mathcal{O}_\infty(K)\backslash\Omega_0$ as

$$\nu(x) = Kx, x \in \mathcal{O}_\infty(K)\backslash\Omega_0. \tag{10}$$

Indeed, any element of $\mathcal{O}_\infty(K)\backslash\Omega_0$ is robustly steered to Ω_0 with the linear control law $u = Kx$.

The definition of the control law outside of $\mathcal{O}_\infty(K)$ proposed here relies on the existence of $L-1$ stabilizing control laws $u_j(x), j \in \{2, ..., L\}$ and associated RPI sets $\mathcal{X}_j \subseteq \mathcal{X}, j \in \{2, ..., L\}$. That is $\forall j \in \{2, ..., L\}, \forall x \in \mathcal{X}_j$,

$$u_j(x) \in \mathcal{U}, Ax + B(u_j(x) + w) \in \mathcal{X}_j, \forall w \in \mathcal{W}.$$

We denote $u_1(x)$ the control law given by (9) and (10), and $\mathcal{X}_1 = \mathcal{O}_\infty(K)$.

Remark 4. The sets $\mathcal{X}_j, j = \{2, ..., L\}$ are assumed to be convex and compact. This assumption is not restrictive provided $Conv\{\mathcal{X}_j\}$ is an admissible convex RPI set.

Several methods to design the pairs $u_j(x), \mathcal{X}_j$ exist in the literature. Among those, we can cite

- an LMI-based method to compute the invariant ellipsoid $E(x_0)$, and the associated feedback gain $u = K_{x_0}x$, that contains the most important extension on a direction defined by a reference point (Nguyen (2012)). This is of particular interest to enlarge the basin of attraction in specific directions.
- Tube-based Model Predictive Control (Mayne et al. (2006)).

Let us now define $\mathcal{X}_{ch} = Conv\left(\{\mathcal{X}_j, j \in \mathcal{I}_L\}\right)$.

Any point $x \in \mathcal{X}_{ch}$ can be written $x = \sum_{j=1}^{L} \lambda_j x_j$, with $\sum_{j=1}^{L} \lambda_j = 1, \lambda_j \geq 0, x_j \in \mathcal{X}_j, \forall j \in \mathcal{I}_L$.

Remark 5. The above expression of x is not unique.

By denoting $\hat{x}_j = \lambda_j x_j$, we have $x = \sum_{j=1}^{L} \hat{x}_j$ and $\hat{x}_j \in \lambda_j \mathcal{X}_j$. To perform a selection among the feasible λ_j and $\hat{x}_j$ at each time-step, we minimize online the following linear cost function subject to convex constraints

$$\begin{aligned} \underset{\lambda_j, \hat{x}_j, j \in \mathcal{I}_L}{\text{minimize}} \quad & -\lambda_1 \\ \text{subject to} \quad & x = \sum_{j=1}^{L} \hat{x}_j \\ & \hat{x}_j \in \lambda_j \mathcal{X}_j, \forall j \in \mathcal{I}_L \\ & \lambda_j \geq 0, \forall j \in \mathcal{I}_L \\ & \sum_{j=1}^{L} \lambda_j = 1 \end{aligned} \tag{11}$$

Let us denote $(\hat{x}_j^*(x), \lambda_j^*(x)), j \in \mathcal{I}_L$ the solution of (11) for the state x, and define for all $j \in \mathcal{I}_L$,

$$x_j^*(x) = \frac{\hat{x}_j^*(x)}{\lambda_j^*(x)}, \text{if} \quad \lambda_j^* \neq 0,$$

$$x_j^*(x) = 0, \text{if} \quad \lambda_j^* = 0.$$

The dependency on x will be dropped for clarity purpose. We use this selection to define the control law $\nu : \mathcal{X}_{ch} \to \mathcal{U}$,

$$\nu(x) = \sum_{j=1}^{L} \lambda_j^* u_j(x_j^*). \tag{12}$$

Note that $\forall x \in \mathcal{X}_1, \nu(x) = u_1(x)$. We have the following result regarding the above control law.

Proposition 2. The set $\mathcal{X}_{ch}$ is an RPI set for the system $x^+ = Ax + B\nu(x) + Bw$ and constraint set $(\mathcal{X}_\nu, \mathcal{W})$.

Proof. Let $x \in \mathcal{X}_{ch}$ and $w \in \mathcal{W}$. We have $\forall j \in \mathcal{I}_L$,

$$u_j(x_j^*) \in \mathcal{U}, x_j^{*+} = Ax_j^* + Bu_j(x_j^*) + Bw \in \mathcal{X}_j.$$

Moreover, x^+ satisfies

$$x^+ = \sum_{j=1}^{L} \lambda_j^* \left(Ax_j^* + Bu_j(x_j^*) + Bw\right) = \sum_{j=1}^{L} \lambda_j^* x_j^{*+}. \tag{13}$$

Since the set $\mathcal{U}$ is convex, and by definition of the set $\mathcal{X}_{ch}$, we have $\nu(x) \in \mathcal{U}, x^+ \in \mathcal{X}_{ch}$. $\square$

In view of the stability analysis for the closed-loop system, we introduce the following positive definite function:

$$V(x) = 1 - \lambda_1^*(x).$$

Based on the constraints on the interpolation factors we have, $\forall x \in \mathcal{X}_{ch}, 0 \leq V(x) \leq 1$, and $V(x) = 0$ if and only if $x \in \mathcal{X}_1$.

Proposition 3. The control law (12) ensures that the closed-loop system is robustly stable in the sense of Lyapunov (*non-increase along the system trajectories*) for all initial conditions $x \in \mathcal{X}_{ch}$.

Proof. From (13), $\forall w \in \mathcal{W}$, the pairs $(\lambda_j^* x_j^{*+}, \lambda_j^*), j \in \mathcal{I}_L$ satisfy the constraints of the optimization problem (11) for the state x^+.

Hence, $\forall x \in \mathcal{X}_{ch}, \forall w \in \mathcal{W}, V(x^+) \leq 1 - \lambda_1^*(x) = V(x)$ $\square$

Let us introduce the following definition.

Definition 5. (Robust Contractivity) A set $\mathcal{S}$ is α robustly contractive for the closed-loop system $x^+ = f(x, w), w \in \mathcal{W}$ if there exists $0 \leq \alpha < 1$ such that

$$\forall x \in \mathcal{S}, \forall w \in \mathcal{W}, x^+ \in \alpha\mathcal{S}.$$

The following assumption will be considered in view of the convergence analysis.

Assumption 1. The sets $\mathcal{X}_j, j \in \mathcal{I}_L$ are respectively α_j robustly contractive for the closed-loop systems $x^+ = Ax + B(u_j(x) + w), w \in \mathcal{W}$.

Let us define $\alpha = \max(\{\alpha_j, j \in \mathcal{I}_L\})$ in view of the following result.

Proposition 4. The set $\mathcal{X}_{ch}$ is α robustly contractive for the closed-loop system $x^+ = Ax + B\nu(x) + Bw, w \in \mathcal{W}$ with $\nu(x)$ defined in (12) and under the conditions of the Assumption 1.

Proof. Let $x \in \mathcal{X}_{ch}$. From Assumption 1, we have, $\forall j \in \mathcal{I}_L, \forall w \in \mathcal{W}, x_j^{*+} = Ax_j^* + B(u_j(x_j^*) + w) \in \alpha_j \mathcal{X}_j \subseteq \alpha \mathcal{X}_j$. Hence, $x^+ = \sum_{j=1}^{L} \lambda_j^* x_j^{*+} \in \alpha \mathcal{X}_{ch}$. $\square$

Proposition 5. The control law (12) ensures that the set $\mathcal{X}_1$ is robustly asymptotically stable for all initial conditions $x \in \mathcal{X}_{ch}$.

Proof. First, let us prove that if $V(x) = 1$ then $x \in \partial\mathcal{X}_{ch}$.

Let us assume that $x \notin \partial \mathcal{X}_{ch}$. Then, $\exists \epsilon > 0$ such that $(1+\epsilon)x \in \mathcal{X}_{ch}$. Moreover, $\mathcal{X}_1$ has a non-empty interior, hence $\exists \delta > 0$ such that $\delta x \in \mathcal{X}_1$.

Considering $\gamma = min(\epsilon, \delta)$, we have concomitantly

$$(1+\gamma)x \in \mathcal{X}_{ch}, \gamma x \in \mathcal{X}_1.$$

If $\gamma \geq 1$, then $\delta \geq 1$ and it follows $x \in \mathcal{X}_1$, which leads to $V(x) = 0$. Else, if $\gamma < 1$ and we can rewrite x as

$$x = (1-\gamma)((1+\gamma)x) + \gamma(\gamma x),$$

where $(1+\gamma)x \in \mathcal{X}_{ch}$, $\gamma x \in \mathcal{X}_1$, $0 < (1-\gamma) < 1$ and $0 < \gamma < 1$. Hence we get $V(x) \leq 1-\gamma < 1$. We conclude that if $V(x) = 1$ then $x \in \partial \mathcal{X}_{ch}$.

Let $w \in \mathcal{W}$. To prove robust asymptotic stability of the set $\mathcal{X}_1$, we consider three cases regarding the value of $V(x)$.

Case 1: $V(x) = 1$. From Proposition 4, we have $x^+ \in \alpha \mathcal{X}_{ch}$. Thus, $V(x^+) < 1 = V(x)$.

Case 2: $0 < V(x) < 1$. We have $\lambda_1^*(x) > 0$. Given that the set $\mathcal{X}_1$ is α_1 robustly contractive, $x_1^{*+} \in \alpha_1 \mathcal{X}_1$. Moreover, $\mathcal{X}_1$ has an non-empty interior, hence $\exists \epsilon > 0$ such that $x_1^{*+} + \epsilon(x^+ - x_1^{*+}) \in \mathcal{X}_1$. Note that if $\epsilon \geq 1$, $x^+ \in \mathcal{X}_1$ and thus $V(x^+) = 0$. Else, we denote

$$z_1 = x_1^{*+} + \epsilon(x^+ - x_1^{*+}) \in \mathcal{X}_1. \quad (14)$$

This leads to $x_1^{*+} = \dfrac{z_1 - \epsilon x^+}{1-\epsilon}$. We can rewrite x^+ as$x^+ = \lambda_1^* \dfrac{z_1 - \epsilon x^+}{1-\epsilon} + \sum_{j=2}^{L} \lambda_j^* x_j^{*+}$. Hence, $x^+(1 + \lambda_1^* \dfrac{\epsilon}{1-\epsilon}) = \lambda_1^* \dfrac{z_1}{1-\epsilon} + \sum_{j=2}^{L} \lambda_j^* x_j^{*+}$.This is equivalent to

$$x^+ = \frac{\lambda_1^* z_1}{1-\epsilon(1-\lambda_1^*)} + \sum_{j=2}^{L} \frac{\lambda_j^*}{(1+\lambda_1^* \frac{\epsilon}{1-\epsilon})} x_j^{*+}.$$

From $0 < \epsilon < 1$, we have $\lambda_1^* > \dfrac{\lambda_1^*}{1-\epsilon(1-\lambda_1^*)} > 0$. Moreover, we can show that

$$\frac{\lambda_1^*}{1-\epsilon(1-\lambda_1^*)} + \sum_{j=2}^{L} \frac{\lambda_j^*}{(1+\lambda_1^* \frac{\epsilon}{1-\epsilon})} = 1$$

and $\forall j \in \mathcal{I}_L, \dfrac{\lambda_j^*}{(1+\lambda_1^* \frac{\epsilon}{1-\epsilon})} \geq 0$. This proves

$$V(x^+) \leq 1 - \frac{\lambda_1^*}{1-\epsilon(1-\lambda_1^*)} < 1 - \lambda_1^* = V(x).$$

Case 3: $V(x) = 0$. The control action is given by $\nu(x) = u_1(x)$ and it guarantees $x^+ \in \mathcal{X}_1$. Hence, $V(x^+) = 0$.

This proves robust asymptotic stability of the set $\mathcal{X}_1$ for any initial condition $x \in \mathcal{X}_{ch}$. □

We have proposed a design strategy to enlarge the controllability of the local control law defined in the previous section. Convergence properties have been established with regards to different assumptions on the sets considered for the interpolation-based control strategy.

5. SIMULATION RESULTS

The results presented in Section 3 and 4 are now illustrated. Consider the following system, constraints and disturbance set

$$x+ = Ax + B(u+w), A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\mathcal{X} = \left\{x \in \mathbb{R}^2 \mid |Hx| \leq 1_3\right\}, H = \begin{bmatrix} 0.375 & 0.25 \\ 0.1786 & 0.357 \\ -0.25 & 0.25 \end{bmatrix},$$

$$\mathcal{U} = \{u \in \mathbb{R} \mid |u| \leq 1.5\}, \mathcal{W} = \{w \in \mathbb{R} \mid |w| \leq 0.5\}.$$

In this exemple, $p = 3$, $n = 2$, and $m = 1$. The 1×3 submatrices of H verifying $H \in \mathcal{H}$ are $H_i, i = 1, 2, 3$. We use the results in Michel et al. (2018) to obtain the associated gains

$$K_1 = [-0.9 \ -1.9], K_2 = [-0.5 \ -1.5], \quad K_3 = [-1.5 \ -2.5].$$

In this scenario, it is possible to have an explicit representation of the sets $\mathcal{Z}_\infty(K_i)$ given the eigenstructure of the matrices $A + BK_i$ and the dimensions of the problem. We represent these sets and the state constraints in Figure 1. Note that $\mathcal{Z}_\infty(K_i) \subseteq \mathcal{X}$, for all $i \in \{1, 2, 3\}$. We have

$$K_1\mathcal{Z}_\infty(K_1) = [-1.2 \ ; \ 1.2], K_2\mathcal{Z}_\infty(K_2) = [-1 \ ; \ 1],$$
$$K_3\mathcal{Z}_\infty(K_3) = [-3.5 \ ; \ 3.5].$$

For $\mathcal{Z}_\infty(K_i)$ to be an RPI set, it has to satisfy $\mathcal{Z}_\infty(K_i) \subseteq \mathcal{X}$ and $K_i\mathcal{Z}_\infty(K_i) \subseteq \mathcal{U}$. Hence, the gain K_3 is not admissible. Moreover, we have

$$h(\mathcal{Z}_\infty(K_1), H) = 0.44, h(\mathcal{Z}_\infty(K_2), H) = 0.5.$$

The set $\mathcal{Z}_\infty(K_1)$ minimizes (3). Due to the dimension of the problem and the proposed approach, the refinement proposed Section 3 does not reduce the invariant set $(\Omega_1 = \Omega_0 = \mathcal{Z}_\infty(K_1))$. The MRPI of the feedback gains K_1 and K_2 are presented in Figure 2.

The method proposed in Section 4 to enlarge the controllable region $\mathcal{O}_\infty(K_1) = \mathcal{X}_1$ of the controller $u_1(x) = K_1 x$ has been tested in simulation. The control laws $u_j, j = 2, 3, 4$ and the associated RPI sets $\mathcal{X}_j, j = \{2, 3, 4\}$ have been obtained using Tube-Based MPC as presented in Mayne et al. (2005) with

$$K_{tube,1} = K_2, K_{tube,2} = [-0.61 \ -1.61],$$
$$K_{tube,3} = [-0.58 \ -1.55],$$

and a prediction horizon $N = 10$. The weight matrices Q, R, and P as defined in Mayne et al. (2005) do not impact the region of attraction of the tube-based MPC controllers. We also consider $\mathcal{X}_5 = \mathcal{O}_\infty(K_2)$ and $u_5(x) = K_2 x$.

The set $\mathcal{X}_{ch} = \text{Conv}(\mathcal{X}_i, i = 1, ..., 5)$ can be seen in Figure 3. Trajectories emanating from $\partial \mathcal{X}_{ch}$ and converging to $\mathcal{Z}_\infty(K_1)$ are represented in Figure 4.

6. CONCLUSION

We have proposed a control law for discrete-time linear systems subject to matched disturbances and input and state constraints. The proposed method allows to mitigate the impact of the disturbances in the direction of the state constraints in a neighborhood of the origin. The effect of the disturbances is assessed through invariant set. A first approach that imposes a linear structure to the control law is proposed. The control structure is then relaxed to further mitigate the impact of the disturbance on state constraints satisfaction, leading to smaller invariant sets

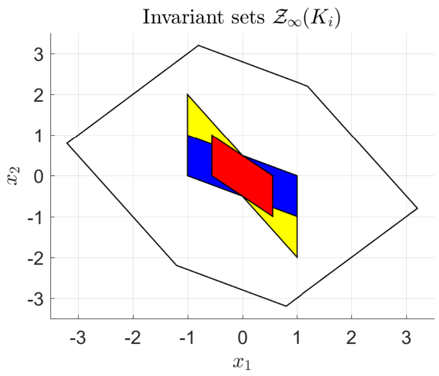

Fig. 1. $\mathcal{Z}_\infty(K_i)$ for K_1 (red), K_2 (blue), and K_3 (yellow), and the state constraints $\mathcal{X}$ (black).

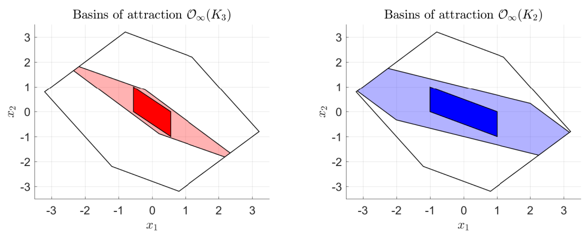

Fig. 2. The MRPI $\mathcal{O}_\infty(K_i)$ and the mRPI $\mathcal{Z}_\infty(K_i)$ for K_1 (left), and K_2 (right).

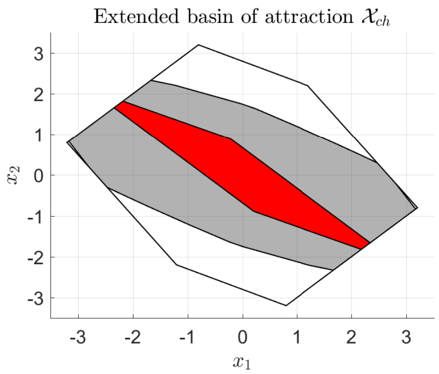

Fig. 3. The enlarged controllable region $\mathcal{X}_{ch}$ (grey) and the initial MRPI $\mathcal{O}_\infty(K_1)$ (red).

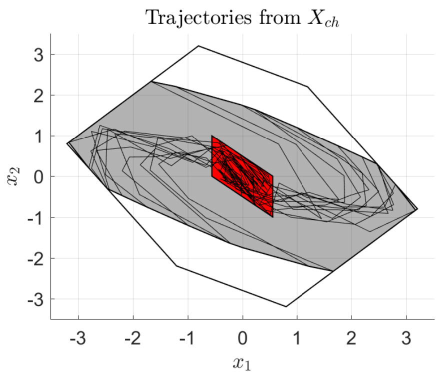

Fig. 4. The enlarged controllable region $\mathcal{X}_{ch}$ (grey), the mRPI $\mathcal{Z}_\infty(K_1)$ (red), and trajectories initialized on several vertices of $\mathcal{X}_{ch}$.

in the constraints direction. To this local control strategy we add an interpolation-based control design to enlarge the controllable region.

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