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RECOVERING TIME-DEPENDENT SINGULAR COEFFICIENTS OF THE WAVE-EQUATION - ONE DIMENSIONAL CASE

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ABSTRACT. We consider the homogeneous wave equation in the rectangle $(0, T) \times (0, b)$, that is, in the one-dimensional space situation. The conductivity depends on the two variables t, x of time and space, and represents an unknown moving inclusion inside the background which has constant conductivity. The waves satisfy the homogeneous Dirichlet condition at $x = b$ and sufficiently smooth but unknown initial conditions at $t = 0$. We prove that the inclusion is determined by the Dirichlet-to-Neumann mapping defined on the interface $x = 0$. In fact, we show how the inclusion can be reconstructed from the detection of the singularities of the flux of special waves knowing the singularities of their trace on the interface.

1. INTRODUCTION

1.1. **The Wave Equation.** Let $\Omega =]0, b[\subset \mathbb{R}$ be a bounded open interval ($b > 0$), and consider the following initial boundary value problem

$$(1.1) \quad \begin{cases} \partial_t^2 u - \nabla_x \cdot (\gamma \nabla_x u) = 0 & \text{in } (0, T) \times \Omega, \\ u|_{x=0} = f(t) & \text{on } (0, T), \\ u|_{x=b} = 0 & \text{on } (0, T), \\ u|_{t=0} = u_0 & \text{on } \Omega, \\ \partial_t u|_{t=0} = u_1 & \text{on } \Omega, \end{cases}$$

where the conductivity $\gamma = \gamma(t, x)$ has the following form : there exist a positive constant $k \neq 1$ and a smooth function $t \mapsto a(t) \in]0, b[$ such that

$$(1.2) \quad \gamma(t, x) = \begin{cases} 1 & \text{if } x < a(t), \\ k^2 & \text{if } x \in]a(t), b[= D(t). \end{cases}$$

Moreover we assume:

$$(H1D) \quad \|\dot{a}(t)\|_\infty < \min(1, k),$$

where $\dot{a} = \frac{da}{dt}$. The inverse problem we are concern with is to obtain some informations on $a(\cdot)$ and k , by choosing carefully the data f and then measuring $\partial_x u(t, x)$ at $x = 0$.

Since the velocity of the waves in $]0, a(t)[= \Omega \setminus \overline{D(t)}$ is one, it is quite natural to consider the following functions. We set

$$(1.3) \quad \xi(t) = t - a(t),$$

$$(1.4) \quad \mu(t) = t + a(t).$$

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For simplicity, and if it is unambiguous, we write $\xi(t) = \xi$, $\mu(t) = \mu$. If needed, we extend $a(t)$ in $\mathbb{R} \setminus [0, T]$ by a smooth extension, and so we extend $D := \{\{t\} \times (a(t), b)\}$, $t \in [0, T]$, $\overline{D}^C = \{\{t\} \times (0, a(t))\}$, $t \in [0, T]$, $\partial D = \{(t, a(t))\}$, $t \in [0, T]$ too (with the same notation) by replacing $[0, T]$ by \mathbb{R} in their definition, in such a way that

$$\delta := \frac{1}{2}d(\partial D, \mathbb{R} \times \Omega) > 0, \quad |\dot{a}|_\infty < \min(1, k).$$

We put

$$t_s := \inf\{t \geq s; a(t) = t - s\}, \quad t^*(s) = 2t_s - s, \quad s \in [0, T].$$

Remark 1.1. *Since $|\dot{a}| < 1$ and $a > 0$, it becomes obvious that $\{t \geq s; a(t) = t - s\} = \{t_s\}$, and that $s \mapsto t_s$ and $t^*(\cdot)$ are smooth and increasing.*

In fact, t_0 is the necessary time delay to have the first information on $D(t)$, and t_s is the same, but with initial time at $t = s$. We denote

$$\mu_0 := \mu(t_0) = t_0 + a(t_0) = 2t_0.$$

Remark 1.2. *We obviously have $\mu(t_s) = t^*(s)$ and $\xi(t_s) = s$. Hence $\mu = t^* \circ \xi$ and $\xi^{-1}(\cdot) = t_{(\cdot)}$.*

We also define the coefficients of reflexion and transmittion, respectively,

$$(1.5) \quad \alpha(t) := \frac{1 - k + (k - \frac{1}{k})\dot{a}(t)}{1 + k + (k - \frac{1}{k})\dot{a}(t)} = \left(\frac{1 - k}{1 + k}\right) \left(\frac{1 - (1 + \frac{1}{k})\dot{a}(t)}{1 + (1 - \frac{1}{k})\dot{a}(t)}\right),$$

$$(1.6) \quad \beta(t) := \frac{2}{1 + k + (k - \frac{1}{k})\dot{a}(t)}.$$

Thanks to (H1D), the functions α and β are well-defined in $[0, T]$. We are going to deal with data and measurements as functions in the usual Sobolev space $H^s(I)$, where $s \in \mathbb{R}$ and $I \subset \mathbb{R}$ is an non empty open interval. If $s \in (0, 1)$ it can be defined by

$$H^s(I) = \left\{q \in L^2(I); \iint_{I \times I} \frac{|q(x) - q(y)|^2}{|x - y|^{1+2s}} dx dy < \infty\right\}, \quad 0 < s < 1.$$

See [2] for details. Our main result is the following one.

Theorem 1.3. *Assume that $(u_0, u_1) \in H^{r_0}(\Omega) \times H^{r_0-1}(\Omega)$ for some $r_0 \in (0, \frac{1}{2})$. Fix $f \in L^2(-\infty, T)$ such that*

- (1) $f|_{(-\infty, 0)} \in H^{r_0}(-\infty, 0)$;
- (2) $f|_{(0, t)} \in H^{r_0(1-t'/T)}((0, t))$ for $0 < t < t' \leq T$;
- (3) $f|_{(0, t')} \notin H^{r_0(1-t'/T)}((0, t'))$ for $0 \leq t < t' \leq T$.

Then, the following statements hold.

- 1) *There exists a unique solution u of (1.1) in $L^2(\Omega_T)$.*
- 2) *The quantity $\partial_x u|_{x=0}$ is defined in $H^{-1}(0, T)$ by continuous extension.*
- 3) *The distribution $g = \partial_x u|_{x=0} + f' \in H^{-1}(0, T)$ has the following form*

$$g = g_A + g_E,$$

where g_A, g_E satisfy the following properties:

- (i) $g_A(\mu) = 2\alpha(t)f'(\xi)$, $\forall \mu \in [0, T]$.
- (ii) $g_A|_{(0, \mu)} \in H^{r_0(1-\tilde{\xi}/T)-1}(0, \mu)$ for all $\mu_0 < \mu \leq T$ and all $\tilde{\xi} > \xi$.
- (iii) If $\dot{a}(t) \neq \frac{k}{1+k}$ then $g_A|_{(0, \mu)} \notin H^{r_0(1-\tilde{\xi}/T)-1}(0, \mu)$, $\forall \tilde{\xi} < \xi$.

(iv) There exists $\varepsilon > 0$ such that

$$(1.7) \quad g_E|_{(0,\mu)} \in H^{\varepsilon+r_0(1-\xi/T)-1}(0,\mu), \quad \forall \mu \in [0, T].$$

The main consequence of this is

Corollary 1.4. *Assume that $\dot{a}(t) \neq \frac{k}{1+k}$ for all t , and $(u_0, u_1) \in H^{r_0}(\Omega) \times H^{r_0-1}(\Omega)$ for some $r_0 > 0$. Let $T > 0$. We claim that:*

- 1) *We can know if $T \leq \mu_0$ or if $T > \mu_0$.*
- 2) *Assume that $T > t^*(0) = \mu_0$. Set*

$$s^* := t^{*-1}(T), \quad t_{max} := t_{s^*}.$$

Then we can recover the functions $s \mapsto t_s$, $0 \leq s \leq s^$, $t \mapsto a(t)$, $t_0 \leq t \leq t_{max}$. The constant k is the root of a second degree equation with known coefficients. If $\dot{a} \leq 0$ then this equation has no more than one positive root, and so, we are able to reconstruct k .*

Remark 1.5. *Obviously, from Corollary 1.4 and Remark 1.1, and since $t_0 = a(t_0) < b$, we can ensure the condition $T > \mu_0$ by choosing $T \geq 2b$.*

In Theorem 1.3, the existence of such a function f is ensured, thanks to the following Lemma.

Lemma 1.6. *For all $R > 0$, there exists a function $G(t)$, $0 \leq t \leq 1$, such that*

- (1) $G|_{(0,t)} \in H^{(1-t')/R}(0,t)$ for all $0 < t < t' \leq 1$.
- (2) $G|_{(0,t')} \notin H^{(1-t)/R}(0,t')$ for all $0 < t < t' \leq 1$.

Remark 1.7. *In Theorem 1.3, if $(u_0, u_1) \in H_0^{r_0}(\Omega) \times H^{r_0-1}(\Omega)$ for some $r_0 \in (\frac{1}{2}, 1]$, and if $u_0(0)$ is known, then we can fix $f \in L^2(0, T)$ such that*

- (1) $f|_{[0,t]} \in H^{r_0(1-t/T)}([0, t])$ for $0 < t \leq T$;
- (2) $f|_{[0,t']} \notin H^{r_0(1-t'/T)}([0, t'])$ for $0 < t < t' \leq T$;
- (3) $f(0) = u_0(0)$.

Then, Theorem 1.3 with $r_0 \in (\frac{1}{2}, 1]$ is valid.

If $(u_0, u_1) \in H^{r_0}(\Omega) \times H^{r_0-1}(\Omega)$ for some $r_0 \in (\frac{1}{2}, 1]$, but if we don't know the value of $u_0(0)$, then the information is not sufficient (with our approach) to construct a function f and to make Theorem 1.3 work with $r_0 \in (\frac{1}{2}, 1]$. Hence, we are obliged to come back to the situation $(u_0, u_1) \in H^{r_1}(\Omega) \times H^{r_1-1}(\Omega)$, where $r_1 < \frac{1}{2}$.

The paper is organized as follows. In Section 2, we analyse the direct problem (1.1). In Section 3 we construct an ansatz u_A for (1.1) where f is the function of Theorem 1.3. In Section 4, we first prove Corollary 1.4, then Theorem 1.3. In particular, we analyse the error $u_E = u - u_A$.

2. STUDY OF THE DIRECT PROBLEM

2.1. Notations. We denote by $(\cdot | \cdot)$ the usual scalar product in $L^2(\Omega; dx)$, by $(\cdot)_H$ the scalar product in a Hilbert space H , by $\langle \cdot; \cdot \rangle_{H^* \times H}$ the duality product between a Hilbert space H and its dual space H^* , by $\langle \cdot; \cdot \rangle$ the duality product in $\mathcal{D}'(\Omega_T) \times \mathcal{D}(\Omega_T)$ or in $\mathcal{D}'(0, T) \times \mathcal{D}(0, T)$. We put $\mathcal{H}^1 = L^2(0, T; H_0^1(\Omega))$, $\mathcal{H}^{-1} = L^2(0, T; H^{-1}(\Omega)) = \mathcal{H}^{1*}$, $W = \{v \in \mathcal{H}^{-1}; \partial_t v \in \mathcal{H}^{-1}\}$ with obvious norms. We denote

$$E^r = H^r(\Omega) \times H^{r-1}(\Omega) \times H^r(0, T),$$

and

$$E_0^r = \begin{cases} \{(u_0, u_1, f) \in E^r; u_0(0) = f(0), u_0(b) = 0\}, & \frac{1}{2} < r \leq 1, \\ E^r, & 0 \leq r < \frac{1}{2}. \end{cases}$$

(For $r = \frac{1}{2}$ we could set E_0^r as in the case $r > \frac{1}{2}$, but the relations $u_0(0) = f(0)$ and $u_0(b) = 0$ should be modified).

We denote $\Omega_{t_1, t_2} = (t_1, t_2) \times \Omega$.

For data v_0, v_1, F , let v satisfying in some sense:

$$(2.1) \quad \begin{cases} \mathcal{L}_\gamma v = F & \text{in } \Omega_T, \\ v(t, x) = 0, & x \in \partial\Omega, \quad t \in (0, T), \\ v|_{t=0} = v_0 & \text{on } \Omega, \\ \partial_t v|_{t=0} = v_1 & \text{on } \Omega. \end{cases}$$

We formally define the following operators:

$$\begin{aligned} u &= \tilde{P}(u_0, u_1, f), \\ \partial_x u|_{x=0} + f' &= \tilde{Z}(u_0, u_1, f), \\ (u|_{t=s}, \partial_t u|_{t=s}) &= \tilde{X}(s)(u_0, u_1, f), \quad 0 \leq s \leq T, \\ v &= P(v_0, v_1, F), \\ \partial_x v|_{x=0} &= Z(v_0, v_1, F), \\ (v|_{t=s}, \partial_t v|_{t=s}) &= X(s)(v_0, v_1, F), \quad 0 \leq s \leq T, \end{aligned}$$

where u, v , are respectively solutions of (1.1), (2.1).

2.2. Main results. In this section and the one above, we state that Problems (1.1), (2.1) have a unique solution for adequate spaces.

Lemma 2.1. 1. The operator P is a continuous linear mapping from $H_0^1(\Omega) \times L^2(\Omega) \times (L^2(\Omega_T) + W)$ into $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$.
2. The operator $X(s)$ is continuous from $H_0^1(\Omega) \times L^2(\Omega) \times (L^2(\Omega_T) + W)$ into $H_0^1(\Omega) \times L^2(\Omega)$, for all $s \in [0, T]$.

Lemma 2.2. 1. The operator P continuously extends as a continuous operator from $L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1}$ into $L^2(\Omega_T)$.
2. The operator $X(s)$ continuously extends as a continuous operator from $L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1}$ into $L^2(\Omega) \times H^{-1}(\Omega)$, for all $s \in [0, T]$.

Lemma 2.3. 1. The operator \tilde{P} is a continuous linear mapping from E_0^1 into $C([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$, and continuously extends as a continuous operator from E^0 into $L^2(\Omega_T)$.
2. The operator $\tilde{X}(s)$ is continuous from E_0^1 into $H^1(\Omega) \times L^2(\Omega)$, and continuously extends as a continuous operator from E^0 into $L^2(\Omega) \times H^{-1}(\Omega)$, for all $s \in [0, T]$.

Lemma 2.4. The operator Z (respect., \tilde{Z}) is continuous from $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega_T)$ (respect., E_0^1) into $L^2(0, T)$ and continuously extends as a continuous operator from $L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1}$ (respect., E^0) into $H^{-1}(0, T)$.

Lemma 2.5. Let $t_1 \in [0, T]$. Assume that $F \in \mathcal{H}^{-1}$ has a compact support in $\mathcal{O}(t_1)$. Let $v = P(v_0, v_1, F)$. Then there exists a neighborhood \tilde{K} of $K(t_1)$ in $\overline{D^C}$ such that $v|_{\tilde{K}}$ does not depend on F , that, is, if $v_0 = v_1 = 0$, then $v|_{\tilde{K}}$ vanishes, and, in particular, $\text{supp } \partial_x v|_{x=0} \subset (\mu(t_1), T]$.

2.3. Proofs. Let us consider the family of bilinear forms $b(t)$, $t \in \mathbb{R}$, defined by

$$b(t; u, v) = \int_{\Omega} \gamma(t, x) \nabla_x u(x) \nabla_x v(x) dx, \quad \forall u, v \in H^1(\Omega).$$

Lemma 2.1 is a corollary of the following general theorem (proof in Appendix, see Part 6), which is an extension of [1, XV section 4] in which γ does not depend on the variable t .

Theorem 2.6. *Let $T > 0$ and $\Omega \subset \mathbb{R}^n$, $n \geq 1$, such that $H_0^1(\Omega)$ is compact in $L^2(\Omega)$. Let $\gamma(t, x) > 0$ be such that $\gamma, \gamma^{-1} \in C^0([0, T]; L^\infty(\Omega))$, $\partial_t \gamma \in L^\infty(\Omega_T)$. Let $F \in W \cup L^2(\Omega_T)$ and $v_0 \in H_0^1(\Omega)$, $v_1 \in L^2(\Omega)$. Then, there exists a unique weak solution v to (2.1), that is, $v \in C([0, T]; H_0^1(\Omega))$, $\partial_t v \in C([0, T]; L^2(\Omega))$, $v|_{t=0} = v_0$, $\partial_t v|_{t=0} = v_1$, and*

$$\frac{d}{dt}(\partial_t v|\phi) + b(t; v(t, \cdot), \phi) = \langle F(t, \cdot); \phi \rangle,$$

in the sense of $\mathcal{D}'([0, T])$, for all $\phi \in H_0^1(\Omega)$. Moreover there exists a constant C such that

$$(2.2) \quad \|\partial_t v(t, \cdot)\|_{L^2(\Omega)} + \|\partial_x v(t, \cdot)\|_{L^2(\Omega)} \leq C(\|F\|_{L^2(\Omega_t)+W} + \|v_0\|_{H_0^1(\Omega)} + \|v_1\|_{L^2(\Omega)}), \quad \forall t \in [0, T].$$

Let us show that Lemma 2.2 is a straightforward consequence of Lemma 2.1 with the operator P replaced by its adjoint P^* . Let $(v_0, v_1, F) \in L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1}$. By the principle of duality, we can write (2.1) as

$$(v|g)_{L^2(\Omega_T)} = \langle v_1, w(0) \rangle_{H^{-1} \times H_0^1} - (v_0|\partial_t w(0)) + \langle F, w \rangle_{\mathcal{H}^{-1} \times \mathcal{H}^1},$$

for all $g \in L^2(\Omega_T)$, where we put $w = P^*(0, 0, g)$. Consequently (thanks to Lax-Milgram theorem), Equation (2.1) admits a unique solution $v \in L^2(\Omega_T)$, and this shows Point 1 of Lemma 2.2. Once again, we have

$$\begin{aligned} \langle \partial_t v|_{t=T}, f_0 \rangle_{H^{-1} \times H_0^1} - (v|_{t=T}|f_1) &= \langle v_1, w(0) \rangle_{H^{-1} \times H_0^1} - (v_0|\partial_t w(0)) \\ &\quad + \langle F, w \rangle_{\mathcal{H}^{-1} \times \mathcal{H}^1}, \end{aligned}$$

for all $(f_0, f_1) \in H_0^1(\Omega) \times L^2(\Omega)$, where we put $w = P^*(f_0, f_1, 0)$. This shows that $(v|_{t=T}, \partial_t v|_{t=T}) \in L^2(\Omega) \times H^{-1}(\Omega)$. This proves Point 2 of Lemma 2.2 in the non-restrictive case $s = T$.

Let us prove Lemma 2.3. Let $\Phi(x) \in C^\infty(\mathbb{R})$ with $\Phi(0) = 1$ and with support in $[0, a_m]$, where $a_m \leq a(t)$ for all t . Let us consider $f \in H_{loc}^1(\mathbb{R})$ first. Set

$$(2.3) \quad u_{in}(t, x) = f(t - x)\Phi(x).$$

Problem (1.1) with unknown u is (at least formally) equivalent to the following one: find $v = u - u_{in}$ satisfying (2.1) with

$$(2.4) \quad v_0(x) = u_0(x) - f(-x)\Phi(x), \quad v_1(x) = u_1(x) - f'(-x)\Phi(x),$$

$$(2.5) \quad F(t, x) = -\mathcal{L}_\gamma u_{in}(t, x) = -\mathcal{L}_1 u_{in}(t, x) = -2f'(t - x)\Phi(x) + f(t - x)\Phi''(x).$$

Relation (2.5) shows that $F \in L^2(\Omega_T)$. In fact, we have $F \in W$ also, since

$$\partial_t F(t, x) = -2f''(t - x)\Phi(x) + f'(t - x)\Phi''(x),$$

and, for all $\varphi \in \mathcal{D}(\Omega_T)$,

$$\begin{aligned} \langle f''(t - x)\Phi(x), \varphi(t, x) \rangle &= \langle f''(t - x, \cdot)\Phi(x)\varphi(t, x) \rangle \\ &= \langle f'(t - x, \cdot), \partial_x(\Phi(x)\varphi(t, x)) \rangle \leq C\|\varphi\|_{\mathcal{H}^1}, \end{aligned}$$

which shows that $\partial_t F(t, x) \in \mathcal{H}^{-1}$. Similarly, we have

$$\langle f'(t-x)\Phi(x), \varphi(t, x) \rangle = \langle f(t-x), \partial_x(\Phi(x)\varphi(t, x)) \rangle \leq C\|\varphi\|_{L^2(\Omega_T)},$$

which shows that $F \in \mathcal{H}^{-1}$ if $f \in L^2_{loc}(\mathbb{R})$ only. We set

$$\begin{aligned} R: \quad & H^1_{loc}(\mathbb{R}) \rightarrow L^2(\Omega_T) \cap W \\ & f \mapsto F \text{ defined by (2.5)}, \\ S: \quad & E^1_0 \rightarrow H^1_0(\Omega) \times L^2(\Omega) \\ & (u_0, u_1, f) \mapsto (v_0, v_1) \text{ defined by (2.4)}. \end{aligned}$$

The above analysis shows that R continuously extends as a continuous operator from $L^2_{loc}(\mathbb{R})$ into \mathcal{H}^{-1} . Similarly, S continuously extends as a continuous operator from E^0 into $L^2(\Omega) \times H^{-1}(\Omega)$. Consequently, and since a solution to (1.1) can be written $u = v + u_{in}$ with $v = P(S(u_0, u_1, f), R(f))$, Point 1 of Lemma 2.3 is proved. Similarly, we prove Point 2 of Lemma 2.3, since we have $\tilde{X}(s)(u_0, u_1, f) = X(s)(S(u_0, u_1, f), Rf) + (u_{in}|_{t=s}, \partial_t u_{in}|_{t=s})$.

Let us prove Lemma 2.4. Let $(v_0, v_1, F) \in H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega_T)$. As above, for all $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi(T) = 0$, there exists a unique solution $q = q_\varphi \in L^2(\Omega_T)$ to

$$(2.6) \quad \begin{cases} \mathcal{L}_\gamma q = 0 & \text{in } \Omega_T, \\ (q(t, 0), q(t, b)) = (\varphi, 0) & \text{on } (0, T), \\ q|_{t=T} = 0 & \text{on } \Omega, \\ \partial_t q|_{t=T} = 0 & \text{on } \Omega, \end{cases}$$

since it is a particular case of Lemma 2.3 with reversal time.

Moreover, we have $q_\varphi \in C([0, T]; H^1_0(\Omega))$, $\partial_t q_\varphi \in C([0, T]; L^2(\Omega))$ with

$$(2.7) \quad \|q_\varphi|_{t=0}\|_{L^2(\Omega)} + \|\partial_t q_\varphi|_{t=0}\|_{H^{-1}(\Omega)} + \|q_\varphi\|_{L^2(\Omega_T)} \leq C\|\varphi\|_{L^2(0, T)},$$

$$(2.8) \quad \|q_\varphi|_{t=0}\|_{H^1(\Omega)} + \|\partial_t q_\varphi|_{t=0}\|_{L^2(\Omega)} + \|q_\varphi\|_{\mathcal{H}^1} + \|\partial_t q_\varphi\|_{L^2(\Omega_T)} \leq C\|\varphi\|_{H^1(0, T)}.$$

By the duality principle, and thanks to (2.7), we have in the sense of $\mathcal{D}'([0, T])$,

$$(2.9) \quad \langle \partial_x v|_{x=0}, \varphi \rangle = - \langle v_0, \partial_t q_\varphi|_{t=0} \rangle + \langle v_1, q_\varphi|_{t=0} \rangle + \langle F, q_\varphi \rangle$$

$$(2.10) \quad \leq C (\|v_0|_{t=0}\|_{H^1(\Omega)} + \|v_1\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega_T)}) \|\varphi\|_{L^2(0, T)},$$

which shows that $\partial_x v|_{x=0} \in L^2(0, T)$ and that Z is a continuous mapping from $H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega_T)$ into $L^2(0, T)$.

Now, let $(v_0, v_1, F) \in L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1}$. Then, Relation (2.9) and Estimate (2.8) imply

$$(2.11) \quad \langle \partial_x v|_{x=0}, \varphi \rangle \leq C (\|v_0|_{t=0}\|_{L^2(\Omega)} + \|v_1\|_{H^{-1}(\Omega)} + \|F\|_{\mathcal{H}^{-1}}) \|\varphi\|_{H^1(0, T)},$$

which shows that $\partial_x v|_{x=0} \in (H^1_T)^\prime \subset H^{-1}(0, T)$, the dual space of $H^1_T = \{f \in H^1(0, T); f(T) = 0\}$, and that Z continuously extends as a continuous operator from $L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1}$ into $H^{-1}(0, T)$.

This ends the proof of the property of Z in Lemma 2.4. Since $\partial_x u_{in}|_{x=0} = -f'$, we have $\tilde{Z}(u_0, u_1, f) = Z(S(u_0, u_1, f), Rf)$, and Point 2 of Lemma 2.4 is proved. \square

By the well-known Sobolev interpolation theory, we have also proved:

Proposition 2.7. *The operator P (respect., \tilde{P}) continuously maps $H^s(\Omega) \times H^{s-1}(\Omega) \times L^2(0, T; H^{s-1}(\Omega))$ (respect., E^s_0) into $L^2(0, T; H^s(\Omega))$, $s \in [0, 1] \setminus \{\frac{1}{2}\}$.*

The operator Z (respect., \tilde{Z}) continuously maps $H^s(\Omega) \times H^{s-1}(\Omega) \times L^2(0, T; H^{s-1}(\Omega))$ (respect., E_0^s) into $H^{s-1}(0, T)$, $s \in [0, 1] \setminus \{\frac{1}{2}\}$.

Proof of Lemma 2.5. Denote $K = K(t_1)$. Notice that $K \cap \overline{D^C} = \{(t_1, a(t_1))\}$. We assume that $v_0 = v_1 = 0$. Since $\text{supp} F \cap \overline{\Omega_{t_1}} = \emptyset$, then, thanks to Lemma 2.2 with T replaced by t_1 , v vanishes in Ω_{t_1} . Let $K' = \text{int } K$ be the interior of K . The function $v|_K \in L^2(K')$ satisfies the following equations:

$$\begin{aligned} \partial_t^2 v - \Delta_x v &= 0 \quad \text{in } K', \\ v(t, 0) &= 0, \quad t_1 < t < \mu(t_1), \\ v|_{t=t_1} &= \partial_t v|_{t=t_1} = 0 \quad \text{in } (0, a(t_1)). \end{aligned}$$

It is well-known that this implies $v|_{K'} = 0$, and so, $\text{supp } \partial_x v|_{x=0} \subset [\mu(t_1), T]$. But since the support of F does not touch ∂K , we similarly have $v|_{K_\varepsilon(t_1)} = 0$, $\text{supp } \partial_x v|_{x=0} \subset [\mu(t_1) + \delta, T]$, for some $\varepsilon > 0$ sufficiently small.

However, let us give a more straightforward and simple proof to the fact that $\text{supp } \partial_x v|_{x=0} \subset [\mu(t_1) + \delta, T]$. Fix $\delta, \varepsilon > 0$ such that $\mu(t_1) + \delta > \mu(t_1 + \varepsilon)$ and $\text{supp } F \cap K_\varepsilon(t_1) = \emptyset$. Let $t_2 \in [t_1, t_1 + \varepsilon]$, $\varphi \in H_0^1(0, \mu(t_2))$ and set $w(t, x) = \varphi(t + x)$ for $t_2 \leq t \leq \mu(t_2)$. Observe that $w = q_\varphi$ of (2.6), but with $(0, T)$ replaced by $(t_2, \mu(t_2))$. In fact, $\text{supp } w \subset K(t_2)$, and so w vanishes in $D \cap \Omega_{t_2, \mu(t_2)}$. We then have, similarly to (2.9),

$$\langle \partial_x v|_{x=0, t_2 < t < \mu(t_2)}, \varphi \rangle = - \langle v|_{t_2}, \partial_t w|_{t_2} \rangle + \langle \partial_t v|_{t_2}, w|_{t_2} \rangle + \langle F, w \rangle = 0$$

since $v|_{t_2} = \partial_t v|_{t_2} = 0$ and $\text{supp } F \cap \text{supp } w = \emptyset$. Since φ is arbitrarily chosen, this shows that $\text{supp } \partial_x v|_{x=0} \cap (t_2, \mu(t_2)) = \emptyset$, for all $t_2 \in [t_1, t_1 + \varepsilon]$. Hence, $\text{supp } \partial_x v|_{x=0} \subset [\mu(t_1 + \varepsilon), T]$. \square

3. ANSATZ

3.1. Notations. For $t \in [0, T]$ we put

$$K(t) = \{(s, x) \in [t, \mu(t)] \times \overline{\Omega}; s + x \leq \mu(t)\}, \quad \mathcal{O}(t) = \{(s, x) \in \overline{\Omega_{t, T}}; s + x > \mu(t)\}.$$

(Notice that $K(t) \subset \overline{D^C}$ and $K(t) \cap \overline{D} = \{(t, a(t))\}$.)

For $\varepsilon > 0$, $t \in [0, T]$, we put $K_\varepsilon(t) = \cup_{t \leq s \leq t + \varepsilon} K(s)$.

If $q(x)$ is sufficiently smooth in Ω , then $[q]_t := q(a(t) + 0) - q(a(t) - 0)$.

We write $g_1 \stackrel{s}{\simeq} g_2$ if g_1 or $g_2 \in H^s(0, T)$ and $g_1 - g_2 \in H^{s+\varepsilon}(0, T)$ for some $\varepsilon > 0$.

We put $C_+^j = \{f \in C^j(\mathbb{R}); f|_{\mathbb{R}^-} = 0\}$, $j \in \mathbb{N}$, which is dense in $L^2(\mathbb{R}^+) \approx \{f \in L^2(\mathbb{R}); f|_{(-\infty, 0)} = 0\}$. We consider for all $t \in [0, T]$ the formal operator $\mathcal{A}(t) = -\nabla_x(\gamma(t, \cdot)\nabla_x)$ defined from $H^1(\Omega)$ into $H^{-1}(\Omega)$ by duality:

$$\langle \mathcal{A}(t)u, w \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = (\gamma(t)\nabla_x u | \nabla_x w), \quad \forall u, w \in H^1(\Omega) \times H_0^1(\Omega).$$

Let f be a measurable function, we define the ansatz $u_A = U_A(f)$ for (1.1) as follows. Recall that $\xi(t)$ and $\mu(t)$ are defined by (1.3), (1.4), and we have

$$(3.1) \quad \xi_0 = t_0 - a(t_0) = 0,$$

$$(3.2) \quad \mu_0 = t_0 + a(t_0) = 2t_0.$$

In addition, we put, for $t \in [0, T]$,

$$(3.3) \quad \nu = t - \frac{a(t)}{k}, \quad \nu_0 = t_0 - \frac{a(t_0)}{k}.$$

Thanks to Assumption (H1D), $t \mapsto \nu(t)$ is invertible. Recall also that the coefficient of reflexion/transmission, α and β , are defined by (1.5), (1.6). Note that we have

$$(3.4) \quad \alpha(t) \frac{d\mu}{d\xi} - \beta(t) \frac{d\nu}{d\xi} = -1,$$

$$(3.5) \quad \alpha(t) + k\beta(t) = 1.$$

We also define:

$$(3.6) \quad f_2(\mu) = \alpha(t) \frac{d\mu}{d\xi} f(\xi),$$

$$(3.7) \quad f_3(\nu) = \beta(t) \frac{d\nu}{d\xi} f(\xi).$$

We put

$$u_A(t, x) = \begin{cases} f(t-x) + f_2(t+x) - f_2(t-x)\Phi_\varepsilon(x), & 0 \leq t \leq T, \quad 0 < x < a(t), \\ \Phi_\varepsilon(x-b+2\varepsilon)f_3(t-\frac{x}{k}), & 0 \leq t \leq T \quad a(t) < x < b, \end{cases}$$

where we fix $\Phi_\varepsilon \in C^\infty(\mathbb{R})$ so that $\Phi_\varepsilon(r) = 1$ if $r < \frac{1}{2}\varepsilon$, $\Phi_\varepsilon(r) = 0$ if $r > \varepsilon$, $0 < \varepsilon \leq \frac{1}{2}d(\partial D, \partial\Omega_T)$. It is clear that the linear operator $U_A : f \mapsto u_A$ is bounded from $L^2(\mathbb{R})$ into $L^2(\Omega_{\mu_T})$.

3.2. Properties of the Ansatz.

Lemma 3.1. *Let $f \in C^2(\mathbb{R})$. Then we have*

- 1) $u_A \in C^2([0, T]; H^1(\Omega))$, $u_A|_D \in C^2(\overline{D})$, $u_A|_{D^c} \in C^2(\overline{D^c})$.
- 2) *There exists a smooth function $\tau(t)$ with support in $[t_0, \mu_0]$ such that*

$$[\gamma \partial_x u_A(t)]_t = \tau(t) f(\xi(t)).$$

- 3) a) u_A vanishes near $x = b$.

b) Let $g_A = \partial_x u_A|_{x=0} + f'$. Then $g_A(\mu) = 2\alpha(t)f'(\xi)$ for $0 \leq \mu \leq T$, where t, ξ, μ are related by (1.3), (1.4), (3.3).

- 4) Put $F_A = \mathcal{L}_\gamma u_A$ in the sense that $F_A(t, \cdot) = \frac{d^2}{dt^2} u_A(t) + \mathcal{A}(t)u_A(t) \in H^{-1}(\Omega)$ for all t , and $F_A \in C([0, T]; H^{-1}(\Omega))$. Then, F_A can be written

$F_A(t, x) = F_1(t, x) - \tau(t)f(\xi(t))\delta_{a(t)}(x)$, where τ is smooth, and $F_1 \in C([0, T]; L^2(\Omega))$ is defined for $0 \leq t \leq T$ by

$$(3.8) \quad F_1(t, x) = \begin{cases} \Phi_2(x)f_2(t-x) + \Phi_3(x)f_2'(t-x) & 0 < x < a(t), \\ \Phi_4(x)f_3(t-\frac{x}{k}) + \Phi_5(x)f_3'(t-\frac{x}{k}), & a(t) < x < b, \end{cases}$$

where the functions Φ_j are smooth and independant of f , with compact support in $[\varepsilon/2, \varepsilon]$ for $j = 2, 3$, and in $[b-\varepsilon, b-\varepsilon/2]$ for $j = 4, 5$.

Proof. Point 1. is obvious, since we have, thanks to (3.4),

$$\begin{aligned} [u_A(t, \cdot)]_t &= f_3(\nu(t)) - f(\xi(t)) - f_2(\mu(t)) \\ &= \left(\beta(t) \frac{d\nu}{d\xi} - 1 - \alpha(t) \frac{d\mu}{d\xi} \right) f(\xi) = 0. \end{aligned}$$

Let us consider Point 2. For $0 \leq t \leq T$ we have

$$\begin{aligned} \gamma \partial_x u_A(t, a(t) - 0) &= -f'(\xi) + f_2'(\mu) = (-1 + \alpha)f'(\xi) + \frac{d(\alpha \frac{d\mu}{d\xi})}{d\mu} f(\xi), \\ \gamma \partial_x u_A(t, a(t) + 0) &= -kf_3'(\nu) = -k\beta(t)f'(\xi) - k \frac{d(\beta(t) \frac{d\nu}{d\xi})}{d\nu} f(\xi). \end{aligned}$$

Thanks to (3.5) we get

$$[\gamma \partial_x u_A(t)]_t = -\tau(t)f(\xi),$$

with

$$\tau(t) = -k \frac{d(\beta(t) \frac{d\nu}{d\xi})}{d\nu} - \frac{d(\alpha \frac{d\mu}{d\xi})}{d\mu}.$$

This ends Point 2.

Let us consider Point 3 b), since 3 a) is obvious. For $0 \leq \mu \leq T$ we have

$$\partial_x u_A(\mu, 0) = -f'(\mu) + 2f_2'(\mu) = -f'(\mu) + 2\alpha(t)f'(\xi).$$

This ends Point 3.

Let us prove Point 4. A short computation yields (3.8). Thanks to Point 2, we obtain $F_A = F_1 + \tau(t)f(\xi)$ in the required sense. This ends the proof of the lemma. \square

We define the bounded operators $U_A : C^2(\mathbb{R}) \ni f \mapsto u_A \in C^2([0, T]; H^1(\Omega))$, $T_0 : C^2(\mathbb{R}) \ni f \mapsto T_0 f \in C([0, T]; H^{-1}(\Omega))$ such that $T_0 f(t) = \tau(t)f(\xi)\delta_{a(t)}(x)$, and $T_1 : C^2(\mathbb{R}) \ni f \mapsto T_1 f = F_1 \in C([0, T]; L^2(\Omega))$, $T_A : C^2(\mathbb{R}) \ni f \mapsto T_A f = F_A \in C([0, T]; H^{-1}(\Omega))$. Notice that $T_0 f(t) \in H^{-s}(\Omega)$ for all $s > \frac{1}{2}$, $t \in [0, T]$. Obviously we have the following propositions and Lemma.

Proposition 3.2. *The operator U_A continuously extends as a bounded operator from $L^2(0, T)$ into $C([0, T]; H^{-1}(\Omega))$.*

Proposition 3.3. *The operator T_0 continuously extends as a bounded operator from $L^2(0, T)$ into $L^2(0, T; H^{-s}(\Omega))$, $\forall s > \frac{1}{2}$.*

Lemma 3.4. *1) The operator T_A is continuous from $C^2(\mathbb{R})$ into $L^2(0, T; H^{-1}(\Omega))$ and, for all $s \in [0, \frac{1}{2})$, it extends as a continuous operator from $H^s(0, T)$ into $L^2(0, T; H^{s-1}(\Omega))$.*

2) The operator $G_A : f \mapsto \partial_x U_A(f)|_{x=0} + f'$ is continuous from $C^2(\mathbb{R})$ into $C^0([0, T])$, and, for all $s \in [0, \frac{1}{2})$, it extends as a continuous operator from $H^s(0, T)$ into $H^{s-1}(0, T)$.

3) Let f be such as in Theorem 1.3, then $g_A := G_A f$ satisfies (ii) and (iii) of Theorem 1.3.

Proof of Lemma 3.4. Point 1). Thanks to Lemma 3.3, it is sufficient to prove this with T_A replaced by T_1 . Thanks to the interpolation theory, it is sufficient to prove that T_1 is a bounded operator from $L^2(0, T)$ into $L^2(0, T; H^{-1}(\Omega))$ and from $H_0^1(0, T)$ into $L^2(\Omega_T)$, that is obvious. Hence Point 1) holds. Point 2) is obvious for the same reason. Point 3) is obvious, since $\alpha(t) \neq 0$ for all t . \square

3.3. Modification of F_1 . The regularity of F_1 is not sufficient for us, we replace it by the following one, $F_{\varepsilon, \tilde{\mu}}$, which is equivalent to F_1 in the sense of Lemma 2.5.

Let $\tilde{\mu} \in [0, T]$, put $\tilde{t} = \mu^{-1}(\tilde{\mu})$, $\tilde{\xi} = \xi(\tilde{t})$, $\tilde{\nu} = \nu(\tilde{t})$, and consider a smooth function $\phi(\cdot; \varepsilon, \tilde{\mu})$ defined in \mathbb{R}^2 such that $\phi(t, x; \varepsilon, \tilde{\mu}) = 1$ for $(t, x) \in \overline{\Omega_{\tilde{t}}} \cup K_{\varepsilon/2}(\tilde{t})$, $\phi(t, x; \varepsilon, \tilde{\mu}) = 0$ for $t \geq \tilde{t} + \varepsilon$ and $(t, x) \notin K_\varepsilon(\tilde{t})$. For $s \in [0, \frac{1}{2})$, $f \in H^s(\mathbb{R})$ and $F_1 = T_1(f)$ we put

$$F_{\varepsilon, \tilde{\mu}}(t, x) = F_1(t, x)\phi(t, x; \varepsilon, \tilde{\mu}).$$

We have the two following properties.

Lemma 3.5. *For $\varepsilon < \delta$, the support of $F_1 - F_{\varepsilon, \tilde{\mu}}$ is contained in $\mathcal{O}(\tilde{t})$.*

Proof. Since $F_1 - F_{\varepsilon, \tilde{\mu}} = (1 - \phi(\cdot; \varepsilon, \tilde{\mu}))F_1$, the support of $F_1 - F_{\varepsilon, \tilde{\mu}}$ is contained in $\text{supp}(1 - \phi(\cdot; \varepsilon, \tilde{\mu})) \cap \text{supp} F_1$. But $\text{supp}(1 - \phi(\cdot; \varepsilon, \tilde{\mu})) \subset \overline{\Omega_T} \setminus (\Omega_{\tilde{t}} \cup \text{int}(K_{\varepsilon/2}(\tilde{t})))$. Then the proof is done if we show that $(\tilde{t}, a(\tilde{t})) \notin \text{supp}(F_1 - F_{\varepsilon, \tilde{\mu}})$. But, thanks to (3.8), the support of F_1 is localized in $\{x \leq \varepsilon\} \cup \{x \geq b - \varepsilon\}$ that does not touch ∂D . \square

Lemma 3.6. *Let f be as in Theorem 1.3. There exists $c > 0$ and $\varepsilon_0 > 0$, independent of f , such that, for all $\varepsilon \in (0, \varepsilon_0)$, $\tilde{\mu} \in [0, T]$, $F_{\varepsilon, \tilde{\mu}} \in C([0, T]; H^{r_0(1-\tilde{\xi}/T)+c\varepsilon-1}(\Omega))$.*

To prove it, we use the following well-known property.

Proposition 3.7. *Let $g \in H^s(\mathbb{R})$ for some $s \in [-1, 0]$. Let $r \in \mathbb{R}^*$ and $G(t, x) = g(t + rx)$, $(t, x) \in \Omega_T$. Then $G \in C([0, T]; H^s(\Omega))$.*

Let us prove Lemma 3.6. Observe that, by definition of $\phi(\cdot; \varepsilon, \tilde{\mu})$, and thanks to (3.8), the support of $F_{\varepsilon, \tilde{\mu}}|_{\Omega_{\tilde{\mu}}}$ is a subset of the set

$$E(\varepsilon, \tilde{\mu}) = K_\varepsilon(\tilde{t}) \cup (\overline{\Omega_{\tilde{t}+\varepsilon}} \cap \overline{D^C}) \cup (\overline{\Omega_{\tilde{t}+\varepsilon}} \cap \overline{D} \cap \{b - \varepsilon \leq x \leq b\}).$$

Firstly, let $(t, x) \in K_\varepsilon(\tilde{t}) \cup (\overline{\Omega_{\tilde{t}+\varepsilon}} \cap \overline{D^C})$. Then we have $t - x \leq \tilde{t} + \varepsilon$, and so

$$\xi(\mu^{-1}(t - x)) < \xi(\mu^{-1}(\tilde{t} + \varepsilon)) < \xi(\mu^{-1}(\tilde{\mu} - \delta + \varepsilon)),$$

since the functions ξ and μ^{-1} are smooth and non decreasing, and $\delta < a(\tilde{t}) = \tilde{\mu} - \tilde{t}$. So, for ε sufficiently small and some $c > 0$ (values that are independent of t, x), we have

$$(3.9) \quad \xi(\mu^{-1}(t - x)) < \tilde{\xi} - c\varepsilon, \quad (t, x) \in K_\varepsilon(\tilde{t}) \cup (\overline{\Omega_{\tilde{t}+\varepsilon}} \cap \overline{D^C}).$$

Secondly, let $(t, x) \in \overline{\Omega_{\tilde{t}+\varepsilon}} \cap \overline{D} \cap \{b - \varepsilon \leq x \leq b\}$. Then $t - \frac{x}{k} \leq \nu(t) - \frac{\delta - \varepsilon}{k}$ and so, for ε sufficiently small and some $c > 0$,

$$\xi(\nu^{-1}(t - \frac{x}{k})) \leq \xi(\nu^{-1}(\nu(t) - \frac{\delta - \varepsilon}{k})) < \tilde{\xi} - C\varepsilon.$$

We thus have

$$(3.10) \quad \xi(\nu^{-1}(t - \frac{x}{k})) < \tilde{\xi} - C\varepsilon, \quad (t, x) \in \overline{\Omega_{\tilde{t}+\varepsilon}} \cap \overline{D} \cap \{b - \varepsilon \leq x \leq b\}.$$

Since F_1 is expressed in terms of $f'_2(t - x)$, $f_2(t - x)$ in D^C , and in terms of $f'_3(t - \frac{x}{k})$, $f_3(t - \frac{x}{k})$ in D , and since the support of $F_{\varepsilon, \tilde{\mu}}$ is contained in $E(\varepsilon, \tilde{\mu})$, then, thanks to (3.9), (3.10), we see that $F_{\varepsilon, \tilde{\mu}}$ can be expressed in terms of $f|_{(-\infty, r)}$ and $f'|_{(-\infty, r)}$, $r = \tilde{\xi} - c\varepsilon$ only. Hence, thanks to Proposition 3.7, the conclusion follows. \square

4. PROOF OF THE MAIN RESULTS

4.1. Proof of Corollary 1.4.

Firstly, notice that $\alpha(t) \neq 0 \iff \dot{\alpha}(t) \neq \frac{k}{1+k}$.

1) If $T \leq \mu_0$ then $g = 0$ in $(0, T)$, and if $T > \mu_0$ then $g \neq 0$ since $g|_{(\mu_0, T)} \notin H^{r_0(1-s^*/T)-1}(\mu_0, T)$. Hence, the knowledge of g provides $T \leq \mu_0$ or $T > \mu_0$.

2)

- Let $\mu \in [\mu_0, T]$. Thanks to Theorem 1.3, we can construct

$$\xi = \inf\{r > 0; g|_{(0, \mu)} \in H^{r_0(1-r/T)-1}(0, \mu)\},$$

and so the invertible function $\mu \mapsto \xi$ from $[\mu_0, T]$ into $[0, s^*]$. (This implies that s^* is recovered too). Putting $t = \frac{1}{2}(\mu + \xi)$, we recover t_{s^*} which is t for $\mu = T$, and also the functions $t \mapsto \xi = \xi(t)$, $t \mapsto \mu(t)$, $t \mapsto$

$a(t) = \frac{1}{2}(\mu(t) - \xi(t))$, for $t \in [t_0, t_{s^*}]$. We then construct the functions $t_{(\cdot)} = (\xi(\cdot))^{-1}$, $t^*(\cdot) = 2t_{(\cdot)} - \text{id}$.

- Thanks to the above point and to (i) of Theorem 1.3, the smooth function $\alpha(\cdot)$ can be recover as the unique one such that $\mu \mapsto g(\mu) - \alpha(t)f'(\xi)$ belongs to $H^{\varepsilon+r_0(1-\xi/T)}(0, \mu)$ for some $\varepsilon > 0$ and all $\mu \in (0, T)$. Then, k is root of the following equation:

$$(4.1) \quad (\alpha + 1 + \dot{a}(\alpha - 1))k^2 + (\alpha - 1)k + \dot{a}(1 - \alpha) = 0.$$

Denote by k_1, k_2 the roots, such that $k_1 \leq k_2$. We show that $k_1 \leq 0$. A short computation shows that

$$(\alpha + 1 + \dot{a}(\alpha - 1)) = \frac{2}{D} \left(\frac{(1 - \dot{a})^2}{1 + \dot{a}} \right) > 0, \quad D = k(1 + \dot{a}) + 1 - \dot{a}/k > 0.$$

We have

$$(4.2) \quad k_1 k_2 = \frac{\dot{a}(1 - \alpha)}{\alpha + 1 + \dot{a}(\alpha - 1)} = \dot{a}(k_1 + k_2).$$

If $\dot{a} \leq 0$ then, the second equality in (4.2) implies that it is impossible to have $0 < k_1 \leq k_2$.

??????????,

□

Remark 4.1. Theorem 1.3 allows us to recover $t^*(\cdot) = \mu \circ \xi^{-1}$ as:

$$t^*(s) := \sup\{t > s; g|_{[s,t]} \in H^{r_0(1-t/T)-1}([0, t])\},$$

and shows that

$$t^*(s) = \sup\{t > s; g_A|_{[s,t]} \in H^{r_0(1-t/T)-1}([0, t])\}.$$

4.2. Analysis of the error. Let (u_0, u_1, f) , r_0 be as in Theorem 1.3. Put $u = \tilde{P}(u_0, u_1, f)$, $g = \tilde{Z}(u_0, u_1, f)$, $u_A = U_A(f)$ and

$$u_E = u - u_A, \quad F_A = T_A f, \quad g_A = \partial_x u_A|_{x=0}, \quad g_E = g - g_A = \partial_x u_E|_{x=0},$$

where u_A is defined in Section 3. Let us prove the estimate (1.7) (see (iv) of Theorem 1.3). For the sake of clarity, we replace μ , t , ξ , respectively by $\tilde{\mu}$, $\tilde{t} = \mu^{-1}(\tilde{\mu})$, $\tilde{\xi} = \xi(\tilde{t})$. Put $u_{E,0} = u_0 - u_A(0)$, $u_{E,1} = u_1 - \partial_t u_A|_{t=0}$. In view of Subsection 3, the function u_E satisfies

$$(4.3) \quad \begin{cases} \mathcal{L}_\gamma u_E &= -F_A \quad \text{in } \Omega_{\tilde{\mu}}, \\ u_E|_{x=0,b} &= 0 \quad \text{on } (0, \tilde{\mu}), \\ u_E|_{t=0} &= u_{E,1} \quad \text{on } \Omega, \\ \partial_t u_E|_{t=0} &= u_{E,1} \quad \text{on } \Omega. \end{cases}$$

So we have $u_E = P(u_{E,0}, u_{E,1}, -F_A)$. Recall that, thanks to Lemma ??, we have $T_0(f) \in L^2(0, \tilde{\mu}; H^{-s}(\Omega))$, for all $s > \frac{1}{2}$. Thanks to Proposition 2.7, we have

$$(4.4) \quad Z(0, 0, T_0(f))|_{(0, \tilde{\mu})} \in H^{-s}(0, \tilde{\mu}), \quad \forall s > \frac{1}{2}.$$

Let us prove that $u_{E,0} \in H^{r_0}(\Omega)$, $u_{E,1} \in H^{r_0-1}(\Omega)$. Observe that $u_A(0)(x) = (f(-x) + f_2(x) + f_2(-x)\Phi_\varepsilon(x))\chi_{x < a(0)} + f_3(-x/k)\Phi_\varepsilon(x - b + 2\varepsilon)\chi_{x > a(0)}$. For $x < a(0) = t_0$ we have

$$\xi(\mu^{-1}(x)) < \xi(\mu^{-1}(t_0)) < \xi(\mu^{-1}(\mu_0)) = \xi(t_0) = 0,$$

and, similarly, $\xi(\mu^{-1}(-x)) \leq \xi(\mu^{-1}(0)) < 0$. For $x > a(0)$ we have

$$\xi(\nu^{-1}(-x/k)) < \xi(\nu^{-1}(-t_0/k)) < \xi(\nu^{-1}(\nu_0)) = 0.$$

Hence, $u_A(0)$ can be expressed in terms of $f(\xi)$ for $\xi < 0$. Since $f|_{(-\infty, 0]} \in H^{r_0}(-\infty, 0)$, then $u_A(0) \in H^{r_0}(\Omega)$. Thanks to the assumption on u_0 , we then have $u_{E,0} \in H^{r_0}(\Omega)$. Similarly, we have $u_{E,1} \in H^{r_0-1}(\Omega)$. Thanks to (3.6), the regularity of $f_2|_{(0, \tilde{\mu})}$ is given by those of $f|_{(0, \tilde{\xi})}$, that is, $f_2|_{(0, \tilde{\mu})} \in H^{r_0(1-\xi'/T)}((0, \tilde{\mu}))$, for all $\xi' > \tilde{\xi}$. Thus, thanks to Proposition 2.7, we have

$$(4.5) \quad Z(u_{E,0}, u_{E,1}, 0)|_{(0, \tilde{\mu})} \in H^{r_0-1}(0, \tilde{\mu}).$$

Thanks to Lemma 2.5 with t_1 replaced by \tilde{t} and T by $\tilde{\mu}$, and to Lemma 3.5, we have

$$(4.6) \quad Z(0, 0, -F_1)|_{(0, \tilde{\mu})} = Z(0, 0, -F_{\varepsilon, \tilde{\mu}})|_{(0, \tilde{\mu})}.$$

Thanks to Lemma 3.6, if $\varepsilon > 0$ is sufficiently small, we have

$$F_{\varepsilon, \tilde{\mu}}|_{\Omega_{\tilde{\mu}}} \in L^2([0, \tilde{\mu}]; H^{r_0(1-\tilde{\xi}/T)+c\varepsilon-1}(\Omega)),$$

and so, thanks to (4.6) and by applying Proposition 2.7, we obtain

$$(4.7) \quad Z(0, 0, -F_1)|_{(0, \tilde{\mu})} \in H^{r_0(1-\tilde{\xi}/T)+\varepsilon-1}(0, \tilde{\mu}),$$

for some $\varepsilon > 0$ (independent of $\tilde{\mu}$).

Thanks to (4.4), (4.5) (4.7), and since $g_E = Z(u_{E,0}, u_{E,1}, 0) + Z(0, 0, T(0)f) + Z(0, 0, -F_1)$, the proof of (1.7) is done. □

5. APPENDIX: THE FUNCTION G

Let $I = (0, 1)$ and let $\{a_n\}_{n \in \mathbb{N}^*}$ be a dense sequence in \bar{I} . We set

$$f_n(x) = ((x - a_n)_+)^{1/2 - a_n},$$

$$G(x) = \sum_{n \in \mathbb{N}^*} \frac{1}{2^n} f_n(x), \quad x \in I,$$

where $z_+ = \max(0, z)$ for $z \in \mathbb{R}$. The function G is increasing.

For $0 < s < 1$ we set the following Sobolev space:

$$H^s(I) = \left\{ q \in L^2(I); \int \int_{I \times I} \frac{|q(x) - q(y)|^2}{|x - y|^{1+2s}} dx dy < \infty \right\}.$$

Lemma 5.1. *Let $b \in (0, 1]$, $r > -\frac{1}{2}$, $s \in (0, 1)$, $a \in [0, b)$. Set $f(x) = ((x - a)_+)^r$, $I_b = (0, b)$. We have $f \in H^s(I_b)$ if, and only if, $r > s - 1/2$. In such a case, we have*

$$(5.1) \quad \int \int_{I \times I} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2s}} dx dy \leq C_s \left(\frac{1}{2r+1} + \frac{r^2}{2r-2s+1} \right) (b-a)^{2r-2s+1},$$

for some $C_s > 0$.

Proof. Firstly, assume $b = 1$. We then have

$$J := \int \int_{I_1 \times I_1} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2s}} dx dy = 2 \int_0^1 dy \left(\int_0^y \frac{|f(x) - f(y)|^2}{|x - y|^{1+2s}} dx \right)$$

$$= 2(K_1 + K_2),$$

$$K_1 := \int_a^1 dy \left(\int_0^a \frac{(y-a)^{2r}}{(y-x)^{1+2s}} dx \right),$$

$$K_2 := \int_a^1 dy \left(\int_a^y \frac{((y-a)^r - (x-a)^r)^2}{(y-x)^{1+2s}} dx \right).$$

We thus have

$$K_1 = \frac{1}{2s} \int_a^1 (y-a)^{2r} \left[\frac{1}{(y-x)^{2s}} \right]_0^a dy = \frac{1}{2s} \int_a^1 \left((y-a)^{2r-2s} - \frac{(y-a)^{2r}}{y^{2s}} \right) dy.$$

If $a = 0$, then $K_1 = 0$. If $a > 0$, then $K_1 < \infty$ if, and only if, $2r > 2s - 1$. In such a case, we have

$$(5.2) \quad K_1 \leq \frac{1}{2s(2r-2s+1)} (1-a)^{2r-2s+1}.$$

Assume $2r > 2s - 1$. We have

$$K_2 = \int_0^{1-a} dy \left(\int_0^y \frac{(y^r - x^r)^2}{(y-x)^{1+2s}} dx \right) = \int_0^{1-a} y^{2r-2s} dy \left(\int_0^1 \frac{(1-t)^2}{(1-t)^{1+2s}} dt \right)$$

$$= \frac{C(r, s)}{2r-2s+1} (1-a)^{2r-2s+1},$$

where

$$C(r, s) = \int_0^1 \frac{(1-t)^2}{(1-t)^{1+2s}} dt = \int_0^{1/2} \frac{(1-t)^2}{(1-t)^{1+2s}} dt + \int_{1/2}^1 \frac{(1-t)^2}{(1-t)^{1+2s}} dt$$

$$(5.3) \quad \leq C_s \left(\frac{1}{2r+1} + \frac{r^2}{2r-2s+1} \right).$$

Since $C(r, s) > 0$, then $K_2 = +\infty$ if $2r \leq 2s - 1$. Hence, the sum $K_1 + K_2$ converges iff $2r > 2s - 1$.

Assume $2r > 2s - 1$. Thanks to (5.2) and (5.3), we obtain (5.1).

Assume $b \in (0, 1)$. The proof of this case is easily obtained from that of the previous case by setting $a = a'b$, $x = x'b$, $y = y'b$. \square

Lemma 5.2. *For $0 < s < 1$ and $b \in (0, 1]$, we have $G \in H^s(0, b)$ if $s < 1 - b$ and $G \notin H^s(0, b)$ if $s > 1 - b$.*

Proof. For $x, y \in I$, we have, thanks to the Schwarz inequality, (5.4)

$$|G(x) - G(y)|^2 \leq \left(\sum_{n \geq 1} \frac{1}{2^n} \right) \left(\sum_{n \geq 1} \frac{1}{2^n} |f_n(x) - f_n(y)|^2 \right) = \sum_{n \geq 1} \frac{1}{2^n} |f_n(x) - f_n(y)|^2.$$

Let $I_b = (0, b)$, $A_b = \{n \in \mathbb{N}^*; a_n \geq b\}$, $B_b = \mathbb{N}^* \setminus A_b = \{n; a_n < b\}$.

For all $n \in B_b$, thanks to Lemma 5.1, we have $f_n \in H^{1-b}(0, 1)$, since $1/2 - a_n > (1 - b) - 1/2$. For all $n \in A_b$, we have $f_n \in H^{1-b}(I_b)$, since $f_n|_{I_b} = 0$.

Let $0 < s < 1 - b$. By using (5.4), and (5.1), we have

$$\begin{aligned} J_{b,s} &:= \int \int_{I_b \times I_b} \frac{|G(x) - G(y)|^2}{|x - y|^{1+2s}} dx dy \leq \sum_{n \in B_b} \frac{1}{2^n} \int \int_{I_b \times I_b} \frac{|f_n(x) - f_n(y)|^2}{|x - y|^{1+2s}} dx dy \\ &\leq C_s \sum_{n \in B_b} \frac{1}{2^n} \left(\frac{1}{1 - a_n} + \frac{1}{1 - a_n - s} \right) (b - a_n)^{2(1-s-a_n)} \\ &\leq C_s \sum_{n \in B_b} \frac{1}{2^n} \left(\frac{1}{1 - b} + \frac{1}{1 - b - s} \right) (b - a_n)^{2(1-s-a_n)} < \infty \end{aligned}$$

since $(b - a_n)^{2(1-s-a_n)} \leq 1$ for all $n \in B_b$, $0 < s < 1 - b$.

Let $s \in (1 - b, 1)$. For all $n \in \mathbb{N}^*$ and $x > y$ we have $G(x) - G(y) \geq f_n(x) - f_n(y)$. Fix $n \in A_{1-s} \cap B_b$, that is, $1 - s \leq a_n < b$. Thanks to Lemma 5.1, we have $f_n \notin H^s(I_b)$, and then

$$J_{b,s} \geq \frac{1}{2^n} \int \int_{I_b \times I_b} \frac{|f_n(x) - f_n(y)|^2}{|x - y|^{1+2s}} dx dy = \infty.$$

This ends the proof. \square

6. APPENDIX: PROOF OF THEOREM 2.6

Let $F \in L^2(\Omega_T)$, $v_0 \in H_0^1(\Omega)$, $v_1 \in L^2(\Omega)$. Denote $M^1 := \{v \in C([0, T]; H_0^1(\Omega)), \partial_t v \in C([0, T]; L^2(\Omega))\}$, $M_0^1 := \{v \in M^1; v|_{t=0} = 0, \partial_t v|_{t=0} = 0\}$.

6.1. Energy estimate. Put

$$E(t)(v) = \frac{1}{2} \int_{\Omega} |\partial_t v|^2 + \frac{1}{2} \int_{\Omega} \gamma(t, \cdot) |\partial_x v|^2, \quad v \in M^1.$$

We claim that, for all $v \in M^1$ such that $L_\gamma v =: f \in L^2(\Omega_T) + W$, the following (standart) estimate, which implies (2.2), holds:

$$(6.1) \quad E(t)(v) \leq C \left(\|f\|_{L^2(0,t;\Omega)}^2 + E(0)(v) \right), \quad \forall t \in [0, T],$$

for some constant C .

Proof. It is sufficient to show (6.1) for $t = T$. Assume that $f \in L^2(\Omega_T)$. Put $\rho =$

$\sup_Q \frac{|\dot{\gamma}|}{\gamma}$ and $\Pi_0 \in C^1([0, T]; (0, +\infty))$ such that $\delta^{-1}\Pi_0 \leq -\Pi'_0$ for some $\delta \in (0, \frac{1}{\rho})$.
 (For example, $\Pi_0 = e^{-\frac{t}{\delta}}$.) Put

$$Q(v) = \int_0^T E(t)(v) \Pi_0 dt, \quad C_0(f) = \int_Q f^2 \Pi_0.$$

We formally have, thanks to the Schwarz inequality,

$$\begin{aligned} \delta^{-1}Q(v) &\leq - \int_0^T E(t)(v) \Pi'_0 dt = [-E(t)(v) \Pi_0(t)]_0^T - \int_0^T \frac{dE(t)(v)}{dt} \Pi_0 dt \\ &\leq E(0)(v) \Pi_0(0) - E(T)(v) \Pi_0(T) - \frac{1}{2} \int_Q \Pi_0 \dot{\gamma} |\partial_t v|^2 - \int_Q \Pi_0 f \partial_t v \\ &\leq E(0)(v) \Pi_0(0) - E(T)(v) \Pi_0(T) + \rho Q(v) + \sqrt{2C_0(f)} \sqrt{Q(v)}, \end{aligned}$$

Hence, we obtain

$$(\delta^{-1} - \rho)Q(v) + E(T)(v) \Pi_0(T) \leq E(0)(v) \Pi_0(0) + \sqrt{2C_0(f)} \sqrt{Q(v)},$$

and so,

$$(6.2) \quad Q(v) + E(T)(v) \Pi_0(T) \leq C(C_0(f) + E(0)(v) \Pi_0(0)).$$

Then (6.2) follows.

6.2. Uniqueness. Consequently, if $v \in M_0^1$ satisfies (2.1) with $F = 0$, then $E(t)(v) = 0$ for all t , and so $v = 0$. This shows that Problem (2.1) admits at most one solution in M^1 .

6.3. Existence. Let $(\lambda_j, e_j)_{1 \leq j}$ be the family of spectral values of the positive operator $-\Delta_x$ in $H_0^1(\Omega)$, i.e such that $(e_i, e_j)_{L^2(\Omega)} = \delta_{ij}$, $-\Delta e_j = \lambda_j e_j$, and $\lambda_j \nearrow +\infty$. The data v_0, v_1, F are then written $v_0 = \sum_{j=1}^{\infty} v_{0,j} e_j$, $v_1 = \sum_{j=1}^{\infty} v_{1,j} e_j$, $F(t, \cdot) = \sum_{j=1}^{\infty} F_j(t) e_j$, with

$$\sum_{j=1}^{\infty} \{ \lambda_j |v_{0,j}|^2 + |v_{1,j}|^2 + \int_0^T |F_j(t)|^2 dt \} < \infty.$$

Let $N \in \mathbb{N}^*$, and put $E_N = \text{span}\{e_1, \dots, e_N\}$, $V_{k,N} = (v_{k,1}, \dots, v_{k,N})$, $k = 0, 1$, $F_N = \sum_{j=1}^N F_j(t) e_j$, $B_N(t) = (b_{i,j}(t))_{1 \leq i,j \leq N}$ with $b_{i,j}(t) = (\nabla e_i, \nabla e_j)_{L^2(\Omega; \gamma(t, \cdot))}$, and consider the following vectorial differential equation: find $V_N(t) = (v_1(t), \dots, v_N(t))$ such that

$$\frac{d^2}{dt^2} V_N(t) + V_N(t) B_N(t) = F_N(t), \quad 0 \leq t \leq T,$$

with the initial condition $V_N(0) = V_{0,N}$, $\frac{d}{dt} V_N(0) = V_{1,N}$. Since $B_N(\cdot)$ is continuous, the theorem of Cauchy-Lipschitz implies existence and uniqueness for $V_N(t)$. Note that $B_N(t)$ is positive since, for all $U = (u_1, \dots, u_N)$, setting $u(x) = \sum_{j=1}^N u_j e_j(x)$, we have

$$U B_N(t) {}^t U = \int_{\Omega} |\nabla_x u|^2 \gamma(t, x) dx \geq C \|\nabla_x u\|_{L^2(\Omega)}^2 = C \sum_{j=1}^N \lambda_j |u_j|^2,$$

where C is a constant such that $0 < C \leq \gamma$ in Q . Let $v_N(t) = \sum_{j=1}^N v_j(t) e_j(x)$. Then, a standart energy estimate for $E_N(t)(v_N) = \frac{1}{2} (\dot{V}_N^2(t) + V_N(t) B_N(t) V_N(t))$, as above, implies that there exists a positive constant C such that

$$\|\dot{v}_N(t)\|_{L^2(\Omega)} + \|\partial_x v_N(t)\|_{L^2(\Omega)} \leq C(\|v_0\|_{H^1(\Omega)} + \|v_1\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}), \quad 0 \leq t \leq T.$$

Passing to the limit $N \rightarrow +\infty$, we can conclude by standard arguments that $(v_N)_N$ converges to a function $v \in C([0, T]; H_0^1(\Omega))$ satisfying (2.1).

The proof of Theorem 2.6 is done in the case $F \in L^2(\Omega_T)$. The case $F \in W$ is similar.

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