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Recovering time-dependent singular coefficients of the wave-equation - One Dimensional Case

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1 Introduction

Let \( \Omega = ]0, b[ \subset \mathbb{R}, \ b > 0 \), and consider the following initial boundary value problem

\[
\left\{ \begin{array}{ll}
L_{\gamma} u &= 0 \quad \text{in } (0,T) \times \Omega = \Omega_T, \\
u_{|x=0} &= f(t) \quad \text{on } (0,T), \\
v_{|x=b} &= 0 \quad \text{on } (0,T), \\
u_{|t=0} &= u_0 \quad \text{on } \Omega, \\
\partial_t u_{|t=0} &= u_1 \quad \text{on } \Omega,
\end{array} \right. \tag{1.1}
\]

where \( L_{\gamma} u = \partial^2_t u - \nabla_x \cdot (\gamma \nabla_x u), \gamma = \gamma(t,x) \) has the following properties:

There exist a positive constant \( k \neq 1 \) and a smooth function \( t \mapsto a(t) \in ]0, b[ \) such that

\[
\gamma(t,x) = \begin{cases} 
1 & \text{if } x < a(t), \\
k^2 & \text{if } x \in [a(t), b[D(t). 
\end{cases} \tag{1.2}
\]

We make the following assumption

\[
\text{(H1D) } ||\dot{a}(t)||_{\infty} < \min(1,k),
\]

where \( \dot{a} = \frac{da}{dt} \). The inverse problem were are concern with is to obtain some informations on \( a(\cdot) \) and \( k \), by choosing carefully the data \( f \) and then measuring \( \partial_x u(t, x) \) at \( x = 0 \).

Since the velocity of waves in \( \Omega \setminus D(t) \) is one, it is quite natural to consider the following functions. We set

\[
\xi(t) = t - a(t), \quad \mu(t) = t + a(t). \tag{1.3}
\]

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Remark 1.2. We obviously have $s$ with data and measurements as functions in the usual Sobolev space $H$. Fix $D = \{(t, a(t)), t \in [0, T]\}$, $D^C = \{(t, 0), t \in [0, T]\}$, $\partial D = \{(t, a(t)), t \in [0, T]\}$ too (with the same notation) by replacing $[0, T]$ by $\mathbb{R}$ in their definition, in such a way that

$$\delta := \frac{1}{2} \delta(\partial D, \mathbb{R} \times \Omega) > 0, \quad |\dot{a}|_\infty < \min(1, k).$$

We put

$$t_s := \inf\{t \geq s; a(t) = t - s\}, \quad t^*(s) = 2t_s - s, \quad s \in [0, T].$$

Remark 1.1. Since $|\dot{a}| < 1$ and $a > 0$, it becomes obvious that $\{t \geq s; a(t) = t - s\} = \{t_s\}$, and that $s \mapsto t_s$ and $s \mapsto t^*(\cdot)$ are smooth and increasing.

In fact, $t_0$ is the necessary time delay to have the first information on $D(t)$, and $t_s$ is the same, but with initial time at $t = s$. We set

$$\mu_0 := t_0 + a(t_0) = 2t_0.$$

Remark 1.2. We obviously have $\mu(t_s) = t^*(s)$ and $\xi(t_s) = s$. Hence $\mu = t^* \circ \xi$ and $\xi^{-1}(\cdot) = t(\cdot)$.

We also define the coefficient of reflexion/transmition by

$$\alpha(t) := \frac{1 - k + (k - \frac{1}{2})\dot{a}(t)}{1 + k + (k - \frac{1}{2})\dot{a}(t)} = \left(1 - \frac{1}{k}\right) \left(1 + \frac{1}{(1 + \frac{1}{2})\dot{a}(t)}\right), \quad (1.5)$$

$$\beta(t) := \frac{2}{1 + k + (k - \frac{1}{2})\dot{a}(t)} \quad (1.6)$$

Thanks to (H1D), the functions $\alpha$ and $\beta$ are well-defined in $[0, T]$. We shall deal with data and measurements as functions in the usual Sobolev space $H^s(I)$, where $s \in \mathbb{R}$ and $I \subset \mathbb{R}$ is an non empty open interval. If $s \in (0, 1)$ it can be defined by

$$H^s(I) = \left\{q \in L^2(I); \int_I \int_I \frac{|q(x) - q(y)|^2}{|x - y|^{1+2s}} dx dy < \infty\right\}, \quad 0 < s < 1.$$

Our main result is the following

Theorem 1.3. Assume that $(u_0, u_1) \in H^{r_0}(\Omega) \times H^{r_0-1}(\Omega)$ for some $r_0 \in (0, \frac{1}{2})$.

Fix $f \in L^2(-\infty, T)$ such that

1. $f|_{(-\infty, 0)} \in H^{r_0}(-\infty, 0)$;

2. $f|_{(0, t)} \in H^{r_0(1-t'/T)}((0, t))$ for $0 < t < t' \leq T$.
3. \( f|_{(0,t')} \notin H^{\alpha(1-t'/T)}((0,t')) \) for \( 0 \leq t < t' \leq T \).

Then, the following statements hold.
1) There exists a unique solution \( u \) of (1.1) in \( L^2(\Omega_T) \).
2) The quantity \( \partial_x u|_{x=0} \) is defined in \( H^{-1}(0,T) \) by continuous extension.
3) The distribution \( g = \partial_x u|_{x=0} + f' \in H^{-1}(0,T) \) has the following form
   \[
   g = g_A + g_E,
   \]
   where \( g_A, g_E \) satisfy the following properties:
   (i) \( g_A(\mu) = 2\alpha(t)f'(\xi), \quad \forall \mu \in [0,T] \).
   (ii) \( g_A|_{(0,\mu)} \in H^{\alpha(1-\tilde{\xi}/T)\gamma}(0,\mu) \) for all \( \mu_0 < \mu \leq T \) and all \( \tilde{\xi} > \xi \).
   (iii) If \( \dot{a}(t) \neq \frac{k}{1+x} \) then \( g_A|_{(0,\mu)} \notin H^{\alpha(1-\tilde{\xi}/T)\gamma}(0,\mu), \quad \forall \tilde{\xi} < \xi \).
   (iv) There exists \( \varepsilon > 0 \) such that
   \[
   g_E|_{(0,\mu)} \in H^{\varepsilon+\alpha(1-\tilde{\xi}/T)\gamma}(0,\mu), \quad \forall \mu \in [0,T].
   \] (1.7)

The main consequence of this is

**Corollary 1.4.** Assume that \( \dot{a}(t) \neq \frac{k}{1+x} \) for all \( t \), and \( (u_0,u_1) \in H^\gamma(\Omega) \times H^{\gamma-1}(\Omega) \) for some \( \gamma > 0 \). Let \( T > 0 \). We claim that:
1) We can know if \( T \leq \mu_0 \) or if \( T > \mu_0 \).
2) Assume that \( T > t^*(0) = \mu_0 \). Set
   \[
   s^* := t^{-1}(T), \quad t_{max} := t_{s^*}.
   \]
   Then we can recover the functions \( s \mapsto t_s \), \( 0 \leq s \leq s^* \), \( t \mapsto a(t), \) \( t_0 \leq t \leq t_{max} \).
   The constant \( k \) is the root of a second degree equation with known coefficients.
   If \( \dot{a} \leq 0 \) then this equation has no more than one positive root, and so, we are able to reconstruct \( k \).

**Remark 1.5.** Obviously, from Corollary 1.4 and Remark 1.1, and since \( t_0 = a(t_0) < b \), we can ensure the condition \( T > \mu_0 \) by choosing \( T \geq 2b \).

In Theorem 1.3, the existence of such a function \( f \) is ensured, thanks to the following

**Lemma 1.6.** For all \( R > 0 \), there exists a function \( G(t) \), \( 0 \leq t \leq 1 \), such that
1. \( G|_{(0,t)} \in H^{(1-t')/R}(0,t) \) for all \( 0 < t < t' \leq 1 \).
2. \( G|_{(0,t')} \notin H^{(1-t)/R}(0,t') \) for all \( 0 < t < t' \leq 1 \).
Remark 1.7. In Theorem 1.3, if \((u_0, u_1) \in H^{r_0}_0(\Omega) \times H^{r_0-1}(\Omega)\) for some \(r_0 \in \left(\frac{1}{2}, 1\right]\), and if \(u_0(0)\) is known, then we can fix \(f \in L^2(0,T)\) such that

1. \(f|_{[0,t]} \in H^{r_0(1-t/T)}([0,t])\) for \(0 < t \leq T\);
2. \(f|_{[0,t']} \not\in H^{r_0(1-t'/T)}([0,t'])\) for \(0 < t < t' \leq T\),

and with \(f(0) = u_0(0)\). Then, the same result holds than in Theorem 1.3, but with \(r_0 \in \left(\frac{1}{2}, 1\right]\).

If \((u_0, u_1) \in H^{r_0}(\Omega) \times H^{r_0-1}(\Omega)\) for some \(r_0 \in \left(\frac{1}{2}, 1\right]\), but if we don’t know the value of \(u_0(0)\), then the information is not sufficient (with our approach) to construct \(f\) so that the result of Theorem 1.3 holds with this value \(r_0 \in \left(\frac{1}{2}, 1\right]\), and so, we are obliged to come back to the situation \((u_0, u_1) \in H^{r_1}(\Omega) \times H^{r_1-1}(\Omega)\), where \(r_1 < \frac{1}{2}\).

The paper is organized as follows. In Section 2, we analyse the direct problem (1.1). In Section 3 we construct an ansatz \(u_A\) for (1.1) where \(f\) is the function of Theorem 1.3. In Section 4, we first prove Corollary 1.4, then Theorem 1.3. In particular, we analyse the error \(u_E = u - u_A\).

2 Study of the direct problem

2.1 Notations

We denote by \((\cdot, \cdot)\) the usual scalar product in \(L^2(\Omega; dx)\), by \(\langle \cdot, \cdot \rangle_H\) the scalar product in a Hilbert space \(H\), by \(<\cdot, \cdot>_H^*\) the duality product between a Hilbert space \(H\) and its dual space \(H^*\), by \(<\cdot, \cdot>\) the duality product in \(\mathcal{D}'(\Omega_T) \times \mathcal{D}(\Omega_T)\) or in \(\mathcal{D}'(0,T) \times \mathcal{D}(0,T)\). We put \(\mathcal{H}^1 = L^2(0,T; H^1_0(\Omega))\), \(\mathcal{H}^{-1} = L^2(0,T; H^{-1}(\Omega))\) = \(\mathcal{H}^*\), \(W = \{v \in \mathcal{H}^{-1}; \partial_t v \in \mathcal{H}^{-1}\}\) with obvious norms. We denote

\[
E^r = H^r(\Omega) \times H^{r-1}(\Omega) \times H^r(0,T),
\]

and

\[
E^r_0 = \begin{cases} 
\{(u_0, u_1, f) \in E^r; u_0(0) = f(0), u_0(b) = 0\}, & \frac{1}{2} < r \leq 1, \\
E^r, & 0 \leq r < \frac{1}{2}.
\end{cases}
\]

(For \(r = \frac{1}{2}\) we could set \(E^r_0\) as in the case \(r > \frac{1}{2}\), but the relations \(u_0(0) = f(0)\) and \(u_0(b) = 0\) should be modified.)

We denote \(\Omega_{t_1, t_2} = (t_1, t_2) \times \Omega\).

For data \(v_0, v_1, F\), let \(v\) satisfying in some sense:

\[
\begin{cases} 
\mathcal{L}_\gamma v = F \text{ in } \Omega_T, \\
v(t,x) = 0, \quad x \in \partial \Omega, \quad t \in (0,T), \\
v|_{t=0} = v_0 \text{ on } \Omega, \\
\partial_t v|_{t=0} = v_1 \text{ on } \Omega.
\end{cases}
\]  

(2.1)
We formally define the following operators:

\[
\begin{align*}
    u &= \hat{P}(u_0, u_1, f), \\
    \partial_x u|_{x=0} + f' &= \hat{Z}(u_0, u_1, f), \\
    (u|_{t=s}, \partial_t u|_{t=s}) &= \hat{X}(s)(u_0, u_1, f), \quad 0 \leq s \leq T, \\
    v &= P(v_0, v_1, F), \\
    \partial_x v|_{x=0} &= Z(v_0, v_1, F), \\
    (v|_{t=s}, \partial_t v|_{t=s}) &= X(s)(v_0, v_1, F), \quad 0 \leq s \leq T,
\end{align*}
\]

where \( u, v \) are respectively solutions of (1.1), (2.1).

### 2.2 Main results

In this section and the one above, we state that Problems (1.1), (2.1) have a unique solution for adequate spaces.

**Lemma 2.1.** 1. The operator \( P \) is a continuous linear mapping from \( H^1_0(\Omega) \times L^2(\Omega) \times (L^2(\Omega_T) + W) \) into \( C([0, T]; H^1_0(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \).

2. The operator \( X(s) \) is continuous from \( H^1_0(\Omega) \times L^2(\Omega) \times (L^2(\Omega_T) + W) \) into \( H^1_0(\Omega) \times L^2(\Omega) \), for all \( s \in [0, T] \).

**Lemma 2.2.** 1. The operator \( P \) continuously extends as a continuous operator from \( L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1} \) into \( L^2(\Omega_T) \).

2. The operator \( X(s) \) continuously extends as a continuous operator from \( L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1} \) into \( L^2(\Omega) \times H^{-1}(\Omega) \), for all \( s \in [0, T] \).

**Lemma 2.3.** 1. The operator \( \hat{P} \) is a continuous linear mapping from \( E^0_0 \) into \( C([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \), and continuously extends as a continuous operator from \( E^0 \) into \( L^2(\Omega_T) \).

2. The operator \( \hat{X}(s) \) is continuous from \( E^0_0 \) into \( H^1(\Omega) \times L^2(\Omega) \), and continuously extends as a continuous operator from \( E^0 \) into \( L^2(\Omega) \times H^{-1}(\Omega) \), for all \( s \in [0, T] \).

**Lemma 2.4.** The operator \( Z \) (respect., \( \hat{Z} \)) is continuous from \( H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega_T) \) (respect., \( E^0_0 \)) into \( L^2(0,T) \) and continuously extends as a continuous operator from \( L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1} \) (respect., \( E^0 \)) into \( H^{-1}(0,T) \).

**Lemma 2.5.** Let \( t_1 \in [0, T] \). Assume that \( F \in \mathcal{H}^{-1} \) has a compact support in \( \mathcal{O}(t_1) \). Let \( v = P(v_0, v_1, F) \). Then there exists a neighborhood \( \hat{K} \) of \( K(t_1) \) in \( \overline{D^C} \) such that \( v|_{\hat{K}} \) does not depend on \( F \); that, is, if \( v_0 = v_1 = 0 \), then \( v|_{\hat{K}} \) vanishes, and, in particular, \( \text{supp } \partial_x v|_{x=0} \subset \{ \mu(t_1), T \} \).
2.3 Proofs

Let us consider the family of bilinear forms \( b(t; u, v) = \int_\Omega \gamma(t, x) \nabla_x u(x) \cdot \nabla_x v(x) \, dx \), \( \forall u, v \in H^1(\Omega) \).

Lemma 2.1 is a corollary of the following general theorem (proof in appendix), which is an extension of [1, XV section 4] where \( \gamma \) did not depend on the variable \( t \).

**Theorem 2.6.** Let \( T > 0 \) and \( \Omega \subset \mathbb{R}^n \), \( n \geq 1 \), such that \( H^1_0(\Omega) \) is compact in \( L^2(\Omega) \). Let \( \gamma(t, x) > 0 \) be such that \( \gamma, \gamma^{-1} \in C^0([0, T]; L^\infty(\Omega)) \), \( \partial_t \gamma \in L^\infty(\Omega_T) \). Let \( F \in W \cup L^2(\Omega_T) \) and \( v_0 \in H^1_0(\Omega) \), \( v_1 \in L^2(\Omega) \). Then, there exists a unique weak solution \( v \) to (2.1), that is, \( v \in C([0, T]; H^1_0(\Omega)) \), \( \partial_t v \in C([0, T]; L^2(\Omega)) \), \( v|_{t=0} = v_0 \), \( \partial_t v|_{t=0} = v_1 \), and

\[
\frac{d}{dt}(\partial_t v|\phi) + b(t; v(t, \cdot), \phi) = < F(t, \cdot), \phi >, 
\]

in the sense of \( \mathcal{D}'([0, T]) \), for all \( \phi \in H^1_0(\Omega) \). Moreover there exists a constant \( C \) such that

\[
\| \partial_t v(t, \cdot) \|_{L^2(\Omega)} + \| \partial_x v(t, \cdot) \|_{L^2(\Omega)} \leq C \left( \| F \|_{L^2(\Omega_T)} + \| v_0 \|_{H^1_0(\Omega)} + \| v_1 \|_{L^2(\Omega)} \right), \forall t \in [0, T].
\]

Let us show that Lemma 2.2 is a straightforward consequence of Lemma 2.1 with the operator \( P \) replaced by its adjoint \( P^\ast \). Let \( (v_0, v_1, F) \in L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1} \). By the principle of duality, we can write (2.1) as

\[
(v|g)_{L^2(\Omega_T)} = < v_1, w(0) >_{H^{-1} \times H^1_0} - (v_0|\partial_tw(0)) + < F, w >_{H^{-1} \times H^1} 
\]

for all \( g \in L^2(\Omega_T) \), where we put \( w = P^\ast(0, 0, g) \). Consequently (thanks to Lax-Milgram theorem), Equation (2.1) admits a unique solution \( v \in L^2(\Omega_T) \), and this shows Point 1 of Lemma 2.2. Once again, we have

\[
< \partial_t v|_{t=T}, f_0 >_{H^{-1} \times H^1_0} - (v|_{t=T} f_1) = < v_1, w(0) >_{H^{-1} \times H^1_0} - (v_0|\partial_tw(0)) + < F, w >_{H^{-1} \times H^1} 
\]

for all \( (f_0, f_1) \in H^1_0(\Omega) \times L^2(\Omega) \), where we put \( w = P^\ast(f_0, f_1, 0) \). This shows that \( (v|_{t=T}, \partial_t v|_{t=T}) \in L^2(\Omega) \times H^{-1}(\Omega) \). This proves Point 2 of Lemma 2.2 in the non-restrictive case \( s = T \).

Let us prove Lemma 2.3. Let \( \Phi(x) \in C^\infty(\mathbb{R}) \) with \( \Phi(0) = 1 \) and with support in \([0, a_m]\), where \( a_m \leq a(t) \) for all \( t \). Let us consider \( f \in H^1_{loc}(\mathbb{R}) \) first. Set

\[
u_{in}(t, x) = f(t - x)\Phi(x).
\]

(2.3)
Problem (1.1) with unknown $u$ is (at least formally) equivalent to the following one: find $v = u - u_{in}$ satisfying (2.1) with

$$v_0(x) = u_0(x) - f(-x)\Phi(x), \quad v_1(x) = u_1(x) - f'(-x)\Phi(x), \quad (2.4)$$

$$F(t,x) = -\mathcal{L}_u u_{in}(t,x) = -\mathcal{L}_1 u_{in}(t,x) = -2f'(t-x)\Phi(x) + f(t-x)\Phi'(x).$$

Relation (2.5) shows that $F \in L^2(\Omega_T)$. In fact, we have $F \in W$ also, since

$$\partial_t F(t,x) = -2f''(t-x)\Phi(x) + f'(t-x)\Phi''(x),$$

and, for all $\varphi \in D(\Omega_T)$,

$$< f''(t-x)\Phi(x), \varphi(t,x) > = < f''(t-x), \Phi(x)\varphi(t,x) >$$

$$= < f'(t-x), \partial_x(\Phi(x)\varphi(t,x)) > \leq C\|\varphi\|_{H^1},$$

which shows that $\partial_t F(t,x) \in H^{-1}$. Similarly, we have

$$< f'(t-x)\Phi(x), \varphi(t,x) > = f(t-x), \partial_x(\Phi(x)\varphi(t,x)) \leq C\|\varphi\|_{L^2(\Omega_T)},$$

which shows that $F \in H^{-1}$ if $f \in L^2_{loc}(\mathbb{R})$ only. We set

$$R: \quad H^1_{loc}(\mathbb{R}) \quad \rightarrow \quad \begin{cases} L^2(\Omega_T) \cap W \\ F \text{ defined by (2.5)} \end{cases}$$

$$S: \quad E^1 \quad \rightarrow \quad \begin{cases} H^1_0(\Omega) \times L^2(\Omega) \\ (u_0, u_1, f) \mapsto (v_0, v_1) \text{ defined by (2.4).} \end{cases}$$

The above analysis shows that $R$ continuously extends as a continuous operator from $L^2_{loc}(\mathbb{R})$ into $H^{-1}$. Similarly, $S$ continuously extends as a continuous operator from $E^0$ into $L^2(\Omega) \times H^{-1}(\Omega)$. Consequently, and since a solution to (1.1) can be written $u = v + u_{in}$ with $v = P(S(u_0, u_1, f), R(f))$, Point 1 of Lemma 2.3 is proved. Similarly, we prove Point 2 of Lemma 2.3, since we have

$$\hat{X}(s)(u_0, u_1, f) = X(s)(S(u_0, u_1, f), R(f)) + (u_{in}|_{t=s}, \partial_t u_{in}|_{t=s}).$$

Let us prove Lemma 2.4. Let $(v_0, v_1, F) \in H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega_T)$.

As above, for all $\varphi \in D(\mathbb{R})$ such that $\varphi(T) = 0$, there exists a unique solution $q = q_\varphi \in L^2(\Omega_T)$ to

$$\begin{cases} \mathcal{L}_q q = 0 \quad \text{in} \quad \Omega_T, \\
q|_{t=T} = 0 \quad \text{on} \quad (0,T), \\
\partial_t q|_{t=T} = 0 \quad \text{on} \quad \Omega, \end{cases} \quad (2.6)$$

since it is a particular case of Lemma 2.3 with reversal time.

Moreover, we have $q_\varphi \in C([0,T]; H^1_0(\Omega)), \partial_t q_\varphi \in C([0,T]; L^2(\Omega))$ with

$$\|q_\varphi\|_{L^2(\Omega)} + \|\partial_t q_\varphi\|_{L^2(\Omega_T)} \leq C\|\varphi\|_{L^2(\Omega)},$$

$$\|q_\varphi\|_{H^1(\Omega)} + \|\partial_t q_\varphi\|_{L^2(\Omega)} + \|q_\varphi\|_{H^1} + \|\partial_t q_\varphi\|_{L^2(\Omega_T)} \leq C\|\varphi\|_{H^1(\Omega)}. \quad (2.7)$$
By the duality principle, and thanks to (2.7), we have in the sense of $\mathcal{D}'([0,T])$,
\[
< \partial_x v|_{x=0}, \varphi > = -< v_0, \partial_t q_\varphi |_{t=0} > + < v_1, q_\varphi |_{t=0} > + < F, q_\varphi > (2.9)
\]
\[
\leq C \left( \| v_0 \|_{L^2(\Omega)} + \| v_1 \|_{L^2(\Omega)} + \| F \|_{L^2(\Omega_T)} \right) \| \varphi \|_{(2.1)(\Omega)},
\]
which shows that $\partial_x v|_{x=0} \in L^2(0,T)$ and that $Z$ is a continuous mapping from
$H^1_t(\Omega) \times L^2(\Omega) \times L^2(\Omega_T)$ into $L^2(0,T)$.

Now, let $(v_0, v_1, F) \in L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1}$. Then, Relation (2.9) and
Estimate (2.8) imply
\[
< \partial_x v|_{x=0}, \varphi > \leq C \left( \| v_0 \|_{L^2(\Omega)} + \| v_1 \|_{H^{-1}(\Omega)} + \| F \|_{\mathcal{H}^{-1}} \right) \| \varphi \|_{(2.1)(\Omega)},
\]
which shows that $\partial_x v|_{x=0} \in (H^1_t)^\prime \subset H^{-1}(0,T)$, the dual space of $H^1_t = \{ f \in H^1(0,T); f(T) = 0 \}$, and that $Z$ continuously extends as a continuous operator
from $L^2(\Omega) \times H^{-1}(\Omega) \times \mathcal{H}^{-1}$ into $H^{-1}(0,T)$.

This ends the proof of the property of $Z$ in Lemma 2.4. Since $\partial_x u_\text{in}|_{x=0} = -f'$,
we have $\tilde{Z}(u_0, u_1, f) = Z(S(u_0, u_1, f), Rf)$, and Point 2 of Lemma 2.4 is proved.

By the well-known Sobolev interpolation theory, we have also proved:

**Proposition 2.7.** The operator $P$ (respect., $\tilde{P}$) continuously maps $H^s(\Omega) \times H^{s-1}(\Omega) \times L^2(0,T;H^{-1}(\Omega))$ (respect., $E_0^s$) into $L^2(0,T;H^s(\Omega))$, $s \in [0,1]\left(\frac{1}{2}\right)$.
The operator $Z$ (respect., $\tilde{Z}$) continuously maps $H^s(\Omega) \times H^{s-1}(\Omega) \times L^2(0,T;H^s(\Omega))$ (respect., $E_0^s$) into $H^{-1}(0,T)$, $s \in [0,1]\left(\frac{1}{2}\right)$.

**Proof of Lemma 2.5.** Denote $K = K(t_1)$. Notice that $K \cap \overline{D^c} = \{(t_1, a(t_1))\}$. We assume that $v_0 = v_1 = 0$. Since $\text{supp} F \cap \overline{\Omega_{t_1}} = \emptyset$, then, thanks to Lemma 2.2 with $T$ replaced by $t_1$, $v$ vanishes in $\Omega_{t_1}$. Let $K' = \text{int} K$ the interior of $K$.
The function $v|_{K} \in L^2(K')$ satisfies the following equations:
\[
\partial_t^2 v - \Delta_x v = 0 \quad \text{in} \quad K',
\]
\[
v(t,0) = 0, \quad t_1 < t < \mu(t_1),
\]
\[
v|_{t=t_1} = \partial_t v|_{t=t_1} = 0 \quad \text{in} \quad (0,a(t_1)).
\]
It is well-known that this implies $v|_{K'} = 0$, and so, $\text{supp} \partial_x v|_{x=0} \subset [\mu(t_1), T]$. But since the support of $F$ does not touch $\partial K$, we similarly have $v|_{K_{\mu(t_1)}} = 0$, $\text{supp} \partial_x v|_{x=0} \subset [\mu(t_1) + \delta, T]$, for some $\delta > 0$ sufficiently small.

However, let us give a more straightforward and simple proof to the fact that
$\text{supp} \partial_x v|_{x=0} \subset [\mu(t_1) + \delta, T]$. Fix $\delta, \epsilon > 0$ such that $\mu(t_1) + \delta > \mu(t_1 + \epsilon)$ and $\text{supp} F \cap K_{\epsilon}(t_1) = \emptyset$. Let $t_2 \in (t_1, t_1 + \epsilon)$, $\varphi \in H^1_0(0, \mu(t_2))$ and set $w(t,x) = \varphi(t + x)$ for $t_2 \leq t \leq \mu(t_2)$. Observe that $w = \varphi$ of (2.6), but with $(0,T)$ replaced by

\[
8
\]
have, similarly to (2.9),
\[ < \partial_x v|_{x=0,t_2 < t < \mu(t_2)}, \varphi > = - < v|_{t_2}, \partial_t w|_{t_2} > + < \partial_t v|_{t_2}, w|_{t_2} > + < F, w > = 0 \]
since \( v|_{t_2} = \partial_t v|_{t_2} = 0 \) and \( \text{supp} \ F \cap \text{supp} \ w = \emptyset \). Since \( \varphi \) is arbitrarily chosen, this shows that \( \text{supp} \ \partial_x v|_{x=0} \cap (t_2, \mu(t_2)) = \emptyset \), for all \( t_2 \in [t_1, t_1 + \varepsilon] \). Hence, \( \text{supp} \ \partial_x v|_{x=0} \subset [\mu(t_1 + \varepsilon), T] \). \( \square \)

3 Ansatz

3.1 Notations

For \( t \in [0, T] \) we put
\[ K(t) = \{(s, x) \in [t, \mu(t)] \times \overline{\Omega}; \ s+x \leq \mu(t)\}, \quad \mathcal{O}(t) = \{(s, x) \in \overline{\Omega_t}; s+x > \mu(t)\}. \]
(Notice that \( K(t) \subset \overline{D^c} \) and \( K(t) \cap \overline{D} = \{(t, a(t))\}).

For \( \varepsilon > 0 \), \( t \in [0, T] \), we put \( K_\varepsilon(t) = \bigcup_{t \leq s \leq t+\varepsilon} K(s) \).

If \( q(x) \) is sufficiently smooth in \( \Omega \), then \( [q]_t := q(a(t) + 0) - q(a(t) - 0) \).

We write \( g_1 \preceq g_2 \) if \( g_1 \) or \( g_2 \in H^s(0, T) \) and \( g_1 - g_2 \in H^{s+\varepsilon}(0, T) \) for some \( \varepsilon > 0 \).

We put \( C^j_+ = \{f \in C^j(\mathbb{R}); \ f|_{\mathbb{R}^-} = 0\}, \ j \in \mathbb{N} \), which is dense in \( L^2(\mathbb{R}) \approx \{f \in L^2(\mathbb{R}); \ f|_{(-\infty, 0]} = 0\} \). We consider for all \( t \in [0, T] \) the formal operator \( A(t) = -\nabla_x (\gamma(t, \cdot) \nabla_x) \) defined from \( H^1(\Omega) \) into \( H^{-1}(\Omega) \) by duality:
\[ < A(t)u, w >_{H^{-1}(\Omega) \times H^1(\Omega)} = (\gamma(t) \nabla_x u | \nabla_x w), \quad \forall u, w \in H^1(\Omega) \times H^1_0(\Omega). \]

Let \( f \) be a measurable function, we define the ansatz \( u_A = U_A(f) \) for (1.1) as follows. Recall that \( \xi(t) \) and \( \mu(t) \) are defined by (1.3), (1.4), and we have
\[ \xi_0 = t_0 - a(t_0) = 0, \quad \mu_0 = t_0 + a(t_0) = 2t_0. \] (3.1) (3.2)

In addition, we put, for \( t \in [0, T] \),
\[ \nu = t - \frac{a(t)}{k}, \quad \nu_0 = t_0 - \frac{a(t_0)}{k}. \] (3.3)

Thanks to Assumption (H1D), \( t \mapsto \nu(t) \) is invertible. Recall also that the coefficient of reflexion/transmission, \( \alpha \) and \( \beta \), are defined by (1.5), (1.6). Note that we have
\[ \alpha(t) \frac{d\mu}{d\xi} - \beta(t) \frac{d\nu}{d\xi} = -1, \quad \alpha(t) + k\beta(t) = 1. \] (3.4) (3.5)
We also define:

\[ f_2(\mu) = \alpha(t) \frac{d\mu}{d\xi} f(\xi), \] (3.6)

\[ f_3(\nu) = \beta(t) \frac{d\nu}{d\xi} f(\xi). \] (3.7)

We put

\[ u_A(t, x) = \begin{cases} f(t - x) + f_2(t + x) - f_2(t - x)\Phi_\varepsilon(x), & 0 \leq t \leq T, \quad 0 < x < a(t), \\ \Phi_\varepsilon(x - b + 2\varepsilon)f_3(t - \frac{\xi}{r}), & 0 \leq t \leq T, \quad a(t) < x < b, \end{cases} \]

where we fix \( \Phi_\varepsilon \in C^\infty(\mathbb{R}) \) so that \( \Phi_\varepsilon(r) = 1 \) if \( r < \frac{1}{2}\varepsilon \), \( \Phi_\varepsilon(r) = 0 \) if \( r > \varepsilon \), \( 0 < \varepsilon \leq \frac{1}{2}(\partial D, \partial \Omega_T) \). It is clear that the linear operator \( U_A : f \mapsto u_A \) is bounded from \( L^2(\mathbb{R}) \) into \( L^2(\Omega_{\mu_T}) \).

### 3.2 Properties of the Ansatz

**Lemma 3.1.** Let \( f \in C^2(\mathbb{R}) \). Then we have

1) \( u_A \in C^2([0, T]; H^1(\Omega)), u_A|_D \in C^2(\mathcal{D}), u_A|_{D^c} \in C^2(\mathcal{D}^c) \).

2) There exists a smooth function \( \tau(t) \) with support in \([t_0, \mu_0]\) such that

\[ [\gamma \partial_x u_A(t)]_t = \tau(t) f(\xi(t)). \]

3) a) \( u_A \) vanishes near \( x = b \).

b) Let \( g_A = \partial_x u_A|_{x=0} + f' \). Then \( g_A(\mu) = 2\alpha(t)f'(\xi) \) for \( 0 \leq \mu \leq T \), where \( t, \xi, \mu \) are related by (3.3).

4) Put \( F_A = L_A u_A \) in the sense that \( F_A(t, \cdot) = \frac{d^2}{dx^2}u_A(t) + A(t)u_A(t) \in H^{-1}(\Omega) \) for all \( t \), and \( F_A \in C([0, T]; H^{-1}(\Omega)) \). Then, \( F_A \) can be written

\[ F_A(t, x) = F_1(t, x) - \tau(t)f(\xi(t))\delta_{x,t}(x), \]

where \( \tau \) is smooth, and \( F_1 \in C([0, T]; L^2(\Omega)) \) is defined for \( 0 \leq t \leq T \) by

\[ F_1(t, x) = \begin{cases} \Phi_2(x)f_2(t - x) + \Phi_3(x)f_3(t - x) & 0 < x < a(t), \\ \Phi_4(x)f_3(t - \frac{\xi}{r}) + \Phi_5(x)f_3(t - \frac{\xi}{r}) & a(t) < x < b, \end{cases} \] (3.8)

where the functions \( \Phi_j \) are smooth and independant of \( f \), with compact support in \([\varepsilon/2, \varepsilon]\) for \( j = 2, 3 \), and in \([b - \varepsilon, b - \varepsilon/2]\) for \( j = 4, 5 \).

**Proof.** Point 1. is obvious, since we have, thanks to (3.4),

\[ [u_A(t, \cdot)]_t = f_3(\nu(t)) - f(\xi(t)) - f_2(\mu(t)) = \left( \beta(t) \frac{d\nu}{d\xi} - 1 - \alpha(t) \frac{d\mu}{d\xi} \right) f(\xi) = 0. \]

Let us consider Point 2. For \( 0 \leq t \leq T \) we have

\[ \gamma \partial_x u_A(t, a(t) - 0) = -f'(\xi) + f_2'(\mu) = (-1 + \alpha)f'(\xi) + \frac{d(\alpha \frac{df}{d\mu})}{d\mu} f(\xi) \]

\[ \gamma \partial_x u_A(t, a(t) + 0) = -kf_3'(\nu) = -k\beta(t)f'(\xi) - k \frac{d(\beta(t) \frac{df}{d\nu})}{d\nu} f(\xi). \]
Let us prove Point 4. A short computation yields

$$[\gamma \partial_x u_A(t)]_t = -\tau(t)f(\xi),$$

with

$$\tau(t) = -k \frac{d(\beta(t) \frac{d\nu}{d\mu})}{d\nu} - \frac{d(\alpha(t) \frac{d\nu}{d\mu})}{d\mu}.$$ 

This ends Point 2.

Let us consider Point 3 b), since 3 a) is obvious. For $0 \leq \mu \leq T$ we have

$$\partial_x u_A(\mu, 0) = -f'(\mu) + 2f'_2(\mu) = -f'(\mu) + 2\alpha(t)f'(\xi).$$

This ends Point 3.

Let us prove Point 4. A short computation yields (3.8). Thanks to Point 2, we obtain $F_A = F_1 + \tau(t)f(\xi)$ in the required sense. This ends the proof of the lemma.

We define the bounded operators $U_A : C^2(\mathbb{R}) \ni f \mapsto u_A \in C^2([0, T]; H^1(\Omega))$, $T_0 : C^2(\mathbb{R}) \ni f \mapsto T_0 f \in C([0, T]; H^{-1}(\Omega))$ such that $T_0 f(t) = \tau(t)f(\xi)\delta_{\alpha(t)}(x)$, and $T_1 : C^2(\mathbb{R}) \ni f \mapsto T_1 f = F_1 \in C([0, T]; L^2(\Omega))$, $T_A : C^2(\mathbb{R}) \ni f \mapsto T_A f = F_A \in C([0, T]; H^{-1}(\Omega))$. Notice that $T_0 f(t) \in H^{-s}(\Omega)$ for all $s > \frac{1}{2}$, $t \in [0, T]$.

Obviously we have the following propositions and Lemma.

**Proposition 3.2.** The operator $U_A$ continuously extends as a bounded operator from $L^2(0, T)$ into $C([0, T]; H^{-1}(\Omega))$.

**Proposition 3.3.** The operator $T_0$ continuously extends as a bounded operator from $L^2(0, T)$ into $L^2(0, T; H^{-s}(\Omega))$, $\forall s > \frac{1}{2}$.

**Lemma 3.4.** 1) The operator $T_A$ is continuous from $C^2(\mathbb{R})$ into $L^2(0, T; H^{-1}(\Omega))$ and, for all $s \in [0, \frac{1}{2})$, it extends as a continuous operator from $H^s(0, T)$ into $L^2(0, T; H^{s-1}(\Omega))$.

2) The operator $G_A : f \mapsto \partial_x U_A(f)|_{x=0} + f'$ is continuous from $C^2(\mathbb{R})$ into $C^0([0, T])$, and, for all $s \in [0, \frac{1}{2})$, it extends as a continuous operator from $H^s(0, T)$ into $H^{s-1}(0, T)$.

3) Let $f$ such as in Theorem 1.3, then $g_A := G_A f$ satisfies (ii) and (iii) of Theorem 1.3.

Proof of Lemma 3.4. Point 1). Thanks to Lemma 3.3, it is sufficient to prove this with $T_A$ replaced by $T_1$. Thanks to the interpolation theory, it is sufficient to prove that $T_1$ is a bounded operator from $L^2(0, T)$ into $L^2(0, T; H^{-1}(\Omega))$ and from $H^1_0(0, T)$ into $L^2(\Omega_T)$, that is obvious. Hence Point 1) holds. Point 2) is obvious for the same reason. Point 3) is obvious, since $\alpha(t) \neq 0$ for all $t$. \qed
3.3 Modification of $F_1$

The regularity of $F_1$ is not sufficient for us, we replace it by the following one, $F_{\varepsilon, \mu}$, which is equivalent to $F_1$ in the sense of Lemma 2.5.

Let $\mu \in [0, T]$, put $\tilde{t} = \mu^{-1}(\tilde{\mu})$, $\tilde{\xi} = \xi(\tilde{t})$, $\tilde{\nu} = \nu(\tilde{t})$, and consider a smooth function $\phi(\cdot; \varepsilon, \mu)$ defined in $\mathbb{R}^2$ such that $\phi(t, x; \varepsilon, \mu) = 1$ for $(t, x) \in \overline{\Omega_t} \cup K_{\varepsilon/2}(\tilde{t})$, $\phi(t, x; \varepsilon, \mu) = 0$ for $t \geq \tilde{t} + \varepsilon$ and $(t, x) \notin K_\varepsilon(\tilde{t})$. For $s \in [0, \frac{1}{2}]$, $f \in H^s(\mathbb{R})$ and $F_1 = T_1(f)$ we put

$$ F_{\varepsilon, \mu}(t, x) = F_1(t, x) \phi(t, x; \varepsilon, \mu). $$

We have the two following properties.

**Lemma 3.5.** For $\varepsilon < \delta$, the support of $F_1 - F_{\varepsilon, \mu}$ is contained in $\mathcal{O}(\tilde{t})$.

**Proof.** Since $F_1 - F_{\varepsilon, \mu} = (1 - \phi(\cdot; \varepsilon, \mu))F_1$, the support of $F_1 - F_{\varepsilon, \mu}$ is contained in $\text{supp}(1 - \phi(\cdot; \varepsilon, \mu)) \cap \text{supp} F_1$. But $\text{supp}(1 - \phi(\cdot; \varepsilon, \mu)) \subset \overline{\Omega_t} \cup \text{int}(K_{\varepsilon/2}(\tilde{t}))$. Then the proof is done if we show that $(\tilde{t}, a(\tilde{t})) \notin \text{supp}(F_1 - F_{\varepsilon, \mu})$. But, thanks to (3.8), the support of $F_1$ is localized in $\{x \leq \varepsilon\} \cup \{x \geq b - \varepsilon\}$ that does not touch $\partial D$. \hfill \qed

**Lemma 3.6.** Let $f$ as in Theorem 1.3. There exists $c > 0$ and $\varepsilon_0 > 0$, independent of $f$, such that, for all $\varepsilon \in (0, \varepsilon_0)$, $\mu \in [0, T]$, $F_{\varepsilon, \mu} \in C([-1, 0]; H^{0(1-\xi/T)+\varepsilon^{-1}}(\Omega))$.

To prove it, we use the following well-known property.

**Proposition 3.7.** Let $g \in H^s(\mathbb{R})$ for some $s \in [-1, 0]$. Let $r \in \mathbb{R}^+$ and $G(t, x) = g(t + rx)$, $(t, x) \in \Omega_T$. Then $G \in C([-1, 0]; H^s(\Omega))$.

Let us prove Lemma 3.6. Observe that, by definition of $\phi(\cdot; \varepsilon, \mu)$, and thanks to (3.8), the support of $F_{\varepsilon, \mu}|_{\Omega_k}$ is a subset of the set $E(\varepsilon, \mu) = K_\varepsilon(\tilde{t}) \cup (\overline{\Omega_{t+\varepsilon}} \cap C) \cup (\overline{\Omega_{t+\varepsilon}} \cap D) \cap \{b - \varepsilon \leq x \leq b\}$.

Firstly, let $(t, x) \in K_\varepsilon(\tilde{t}) \cup (\overline{\Omega_{t+\varepsilon}} \cap C)$. Then we have $t - x \leq \tilde{t} + \varepsilon$, and so

$$ \xi(\mu^{-1}(t - x)) < \xi(\mu^{-1}(\tilde{t} + \varepsilon)) < \xi(\mu^{-1}(\tilde{\mu} + \varepsilon)). $$

since the functions $\xi$ and $\mu^{-1}$ are smooth and non-decreasing, and $\delta < a(\tilde{t}) = \tilde{\mu} - \tilde{t}$. So, for $\varepsilon$ sufficiently small and some $c > 0$ (values that are independent of $t, x$), we have

$$ \xi(\mu^{-1}(t - x)) < \tilde{\xi} - c \varepsilon, \quad (t, x) \in K_\varepsilon(\tilde{t}) \cup (\overline{\Omega_{t+\varepsilon}} \cap \overline{C}). \quad (3.9) $$

Secondly, let $(t, x) \in \overline{\Omega_{t+\varepsilon}} \cap D \cap \{b - \varepsilon \leq x \leq b\}$. Then $t - \frac{x}{k} \leq \nu(t) - \frac{\delta - \varepsilon}{k}$ and so, for $\varepsilon$ sufficiently small and some $c > 0$,

$$ \xi(\nu^{-1}(t - \frac{x}{k})) \leq \xi(\nu^{-1}(\nu(t) - \frac{\delta - \varepsilon}{k})) < \tilde{\xi} - C \varepsilon. $$
We thus have
\[ \xi(\nu^{-1}(t - \frac{x}{k})) < \tilde{\xi} - C\varepsilon, \quad (t, x) \in \overline{\Omega_{\nu+\varepsilon}} \cap \overline{D} \cap \{b - \varepsilon \leq x \leq b\}. \] (3.10)

Since \( F_1 \) is expressed in terms of \( f_2'(t - x) \), \( f_2(t - x) \) in \( D^C \), and in terms of \( f_3'(t - \tilde{\xi}) \), \( f_3(t - \tilde{\xi}) \) in \( D \), and since the support of \( F_{\varepsilon, \tilde{\mu}} \) is contained in \( E(\varepsilon, \tilde{\mu}) \), then, thanks to (3.9), (3.10), we see that \( F_{\varepsilon, \tilde{\mu}} \) can be expressed in terms of \( f|_{(-\infty, r)} \) and \( f'|_{(-\infty, r)} \), \( r = \tilde{\xi} - c\varepsilon \) only. Hence, thanks to Proposition 3.7, the conclusion follows. \( \square \)

4 Proof of the main results

4.1 Proof of Corollary 1.4

Firstly, notice that \( \alpha(t) \neq 0 \iff \dot{a}(t) \neq \frac{k}{1 + \dot{a}}. \)

1) If \( T \leq \mu_0 \) then \( g = 0 \) in \((0, T)\), and if \( T > \mu_0 \) then \( g \neq 0 \) since \( g|_{(\mu_0, T)} \notin H^{r_0(1-s^*/T)-1}(\mu_0, T) \). Hence, the knowledge of \( g \) provides \( T \leq \mu_0 \) or \( T > \mu_0 \).

2)

- Let \( \mu \in [\mu_0, T] \). Thanks to Theorem 1.3, we can construct
  \[ \xi = \inf\{r > 0; g|_{(0, \mu)} \in H^{r_0(1-r/T)-1}(0, \mu)\}, \]

  and so the invertible function \( \mu \mapsto \xi \) from \([\mu_0, T]\) into \([0, s^*]. \) (This implies that \( s^* \) is recovered too). Putting \( t = \frac{1}{2}(\mu + \xi) \), we recover \( t_{s^*} \) which is \( t \) for \( \mu = T \), and also the functions \( t \mapsto \xi = \xi(t), \) \( t \mapsto \mu(t), \) \( t \mapsto a(t) = \frac{1}{2}(\mu(t) - \xi(t)), \) for \( t \in [t_0, t_{s^*}]. \) We then construct the functions \( t_{s^*} = (\xi(\cdot))^{-1}, t_{s^*}(\cdot) = 2t(\cdot) - \text{id}. \)

- Thanks to the above point and to (i) of Theorem 1.3, the smooth function \( \alpha(\cdot) \) can be recover as the unique one such that \( \mu \mapsto g(\mu) - \alpha(t)f'(\xi) \) belongs to \( H^{s+r_0(1-\xi/T)}(0, \mu) \) for some \( \varepsilon > 0 \) and all \( \mu \in (0, T). \) Then, \( k \) is root of the following equation:
  \[ (\alpha + 1 + \dot{a}(\alpha - 1))k^2 + (\alpha - 1)k + \dot{a}(1 - \alpha) = 0. \] (4.1)

Denote by \( k_1, k_2 \) the roots, such that \( k_1 \leq k_2. \) We show that \( k_1 \leq 0. \) A short computation shows that
  \[ (\alpha + 1 + \dot{a}(\alpha - 1)) = \frac{2}{D} \left( \frac{(1 - \dot{a})^2}{1 + \dot{a}} \right) > 0, \quad D = k(1 + \dot{a}) + 1 - \dot{a}/k > 0. \]
We have
\[ k_1 k_2 = \frac{\dot{\alpha}(1 - \alpha)}{\alpha + 1 + \ddot{\alpha}(\alpha - 1)} = \ddot{\alpha}(k_1 + k_2). \tag{4.2} \]

If \( \dot{\alpha} \leq 0 \) then, the second equality in (4.2) implies that it is impossible to have \( 0 < k_1 \leq k_2 \).

\[ \text{Remark 4.1.} \text{ Theorem 1.3 allows us to recover } t^*(\cdot) = \mu \circ \xi^{-1} \text{ as:} \]
\[ t^*(s) := \sup\{t > s; g|_{x,t} \in H^{r_0(1-t/T)-1}([0,t])\}, \]
and shows that
\[ t^*(s) = \sup\{t > s; g_A|_{x,t} \in H^{r_0(1-t/T)-1}([0,t])\}. \]

4.2 Analysis of the error

Let \((u_0, u_1, f), r_0\) as in Theorem 1.3. Put \( u = \hat{P}(u_0, u_1, f), g = \bar{Z}(u_0, u_1, f), \)
\( u_A = U_A(f) \) and
\[ u_E = u - u_A, \quad F_A = T_A f, \quad g_A = \partial_x u_A|_{x=0}, \quad g_E = g - g_A = \partial_x u_E|_{x=0}, \]
where \( u_A \) is defined in Section 3. Let us prove the estimate (1.7) (see (iv) of Theorem 1.3). For the sake of clarity, we replace \( \mu, t, \xi \), respectively by \( \hat{\mu}, \tilde{t} = \mu^{-1}(\hat{\mu}), \xi = \xi(\tilde{t}) \). Put \( u_{E,0} = u_0 - u_A(0), u_{E,1} = u_1 - \partial_t u_A|_{t=0} \). In view of Subsection 3, the function \( u_E \) satisfies
\[
\begin{align*}
   & \mathcal{L}_\gamma u_E = -F_A \quad \text{in } \Omega_{\tilde{\mu}}, \\
   & u_E|_{x=0,b} = 0 \quad \text{on } (0, \tilde{\mu}), \\
   & u_E|_{t=0} = u_{E,1} \quad \text{on } \Omega, \\
   & \partial_t u_E|_{t=0} = u_{E,1} \quad \text{on } \Omega.
\end{align*}
\tag{4.3}
\]
So we have \( u_E = P(u_{E,0}, u_{E,1}, -F_A) \). Recall that, thanks to Lemma ??, we have \( T_0(f) \in L^2(0, \tilde{\mu}; H^{-s}(\Omega)), \) for all \( s > \frac{1}{2} \). Thanks to Proposition 2.7, we have
\[ Z(0,0,T_0(f))|_{(0,\tilde{\mu})} \in H^{-s}(0, \tilde{\mu}), \quad \forall s > \frac{1}{2}. \tag{4.4} \]
Let us prove that \( u_{E,0} \in H^{r_0}(\Omega), u_{E,1} \in H^{r_0-1}(\Omega) \). Observe that \( u_A(0)(x) = (f(-x) + f_2(x) + f_3(-x)\Phi(x))x_{x<a(0)} + f_3(-x/k)\Phi(x - b + 2\varepsilon)X_{x>a(0)}. \) For \( x < a(0) = t_0 \) we have
\[ \xi(\mu^{-1}(x)) < \xi(\mu^{-1}(t_0)) < \xi(\mu^{-1}(t_0)) = \xi(t_0) = 0, \]
for
and, similarly, \(\xi(\mu^{-1}(-x)) \leq \xi(\mu^{-1}(0)) < 0\). For \(x > a(0)\) we have

\[\xi(\nu^{-1}(-x/k)) < \xi(\nu^{-1}(-t_0/k)) < \xi(\nu^{-1}(\nu_0)) = 0.\]

Hence, \(u_A(0)\) can be expressed in terms of \(f(\xi)\) for \(\xi < 0\). Since \(f|_{(-\infty, 0]} \in H^{\tau_0}(-\infty, 0)\), then \(u_A(0) \in H^{\tau_0}(\Omega)\). Thanks to the assumption on \(u_0\), we then have \(u_{E,0} \in H^{\tau_0}(\Omega)\). Similarly, we have \(u_{E,1} \in H^{\tau_0-1}(\Omega)\). Thanks to (3.6), the regularity of \(f_2|_{(0, \tilde{\mu})}\) is given by those of \(f|_{(0, \tilde{\xi})}\), that is, \(f_2|_{(0, \tilde{\mu})} \in H^{\tau_0(1-\tilde{\xi}'/T)}((0, \tilde{\mu}))\), for all \(\xi' > \tilde{\xi}\). Thus, thanks to Proposition 2.7, we have

\[Z(0, 0, -F_1)|_{(0, \tilde{\mu})} \in H^{\tau_0-1}(0, \tilde{\mu}).\]  

(4.5)

Thanks to Lemma 2.5 with \(t_1\) replaced by \(\tilde{t}\) and \(T\) by \(\tilde{\mu}\), and to Lemma 3.5, we have

\[Z(0, 0, -F_1)|_{(0, \tilde{\mu})} = Z(0, 0, -F_{\varepsilon, \tilde{\mu}})|_{(0, \tilde{\mu})}.\]  

(4.6)

Thanks to Lemma 3.6, if \(\varepsilon > 0\) is sufficiently small, we have

\[F_{\varepsilon, \tilde{\mu}}|_{[0, \tilde{\mu}]} \in L^2([0, \tilde{\mu}]; H^{\tau_0(1-\tilde{\xi}'/T)+\varepsilon-1}(\Omega)),\]

and so, thanks to (4.6) and by applying Proposition 2.7, we obtain

\[Z(0, 0, -F_1)|_{(0, \tilde{\mu})} \in H^{\tau_0(1-\tilde{\xi}'/T)+\varepsilon-1}(0, \tilde{\mu}),\]  

(4.7)

for some \(\varepsilon > 0\) (independent of \(\tilde{\mu}\)).

Thanks to (4.4), (4.5) (4.7), and since \(g_E = Z(u_{E,0}, u_{E,1}, 0) + Z(0, 0, T(0)f) + Z(0, 0, -F_1)\), the proof of (1.7) is done. 

\(\square\)
5 Appendix: the function $G$

Let $I = (0, 1)$ and a dense sequence $\{a_n\}_{n \in \mathbb{N}}$ in $I$. We set

$$f_n(x) = ((x - a_n)_{+})^{1/2 - a_n},$$

$$G(x) = \sum_{n \in \mathbb{N}^*} \frac{1}{2^n} f_n(x), \quad x \in I,$$

where $z_{+} = \max(0, z)$ for $z \in \mathbb{R}$. The function $G$ is increasing.

For $0 < s < 1$ we set the following Sobolev space:

$$H^s(I) = \left\{ q \in L^2(I); \int \int_{I \times I} \frac{|q(x) - q(y)|^2}{|x - y|^{1+2s}} \, dx \, dy < \infty \right\}.$$

**Lemma 5.1.** Let $b \in (0, 1], r > -\frac{1}{2}$, $s \in (0, 1)$, $a \in [0, b)$. Set $f(x) = ((x - a)_{+})^r$, $I_b = (0, b)$. We have $f \in H^s(I_b)$ if, and only if, $r > s - 1/2$. In such a case, we have

$$\int \int_{I \times I} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2s}} \, dx \, dy \leq C_s \left( \frac{1}{2r + 1} + \frac{y^2}{2r - 2s + 1} \right) (b - a)^{2r - 2s + 1},$$

(5.1)

for some $C_s > 0$.

**Proof.** Firstly, let $b = 1$. We have

$$J := \int \int_{I \times I} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2s}} \, dx \, dy = 2 \int_0^1 dy \left( \int_0^y \frac{|f(x) - f(y)|^2}{|x - y|^{1+2s}} \, dx \right) = 2(K_1 + K_2),$$

$$K_1 := \int_a^1 dy \left( \int_0^a \frac{(y - a)^{2r}}{|y - x|^{1+2s}} \, dx \right),$$

$$K_2 := \int_a^1 dy \left( \int_a^y \frac{(y - a)^r - (x - a)^r)^2}{(y - x)^{1+2s}} \, dx \right).$$

We have

$$K_1 = \frac{1}{2s} \int_a^1 (y - a)^{2r} \left[ \frac{1}{(y - x)^{2s}} \right]^a_0 \, dy = \frac{1}{2s} \int_a^1 \left( (y - a)^{2r - 2s} - \frac{(y - a)^{2r}}{y^{2s}} \right) \, dy.$$

If $a = 0$, then $K_1 = 0$. If $a > 0$, then $K_1 < \infty$ if, and only if, $2r > 2s - 1$. In such a case, we have

$$K_1 \leq \frac{1}{2s(2r - 2s + 1)} (1 - a)^{2r - 2s + 1}.$$

(5.2)
Let $2r > 2s - 1$. We have
\[
K_2 = \int_0^{1-a} dy \left( \int_0^y \frac{(y^r - x^r)^2}{(y-x)^{1+2s}} \, dx \right) = \int_0^{1-a} y^{2r-2s} dy \left( \int_0^1 \frac{(1-t^r)^2}{(1-t)^{1+2s}} \, dt \right) \\
= \frac{C(r,s)}{2r - 2s + 1}(1-a)^{2r-2s+1},
\]
where
\[
C(r,s) = \int_0^1 \frac{(1-t^r)^2}{(1-t)^{1+2s}} \, dt = \int_0^{1/2} \frac{(1-t^r)^2}{(1-t)^{1+2s}} \, dt + \int_{1/2}^1 \frac{(1-t^r)^2}{(1-t)^{1+2s}} \, dt
\leq C_s \left( \frac{1}{2r+1} + \frac{r^2}{2r - 2s + 1} \right).
\]
Since $C(r,s) > 0$, then $K_2 = +\infty$ if $2r \leq 2s - 1$. Hence, the sum $K_1 + K_2$ converges if $2r > 2s - 1$. If $2r > 2s - 1$, thanks to (5.2) and (5.3), we obtain (5.1).

Secondly, the case $b \in (0,1)$ is easily proved by setting $a = a'b, x = x'b, y = y'b$.

\[\square\]

**Lemma 5.2.** For $0 < s < 1$ and $b \in (0,1)$, we have $G \in H^s(0,b)$ if $s < 1 - b$ and $G \not\in H^s(0,b)$ if $s > 1 - b$.

Proof. For $x, y \in I$, we have, thanks to the Schwarz inequality,
\[
|G(x) - G(y)|^2 \leq \left( \sum_{n \geq 1} \frac{1}{2^n} \right) \left( \sum_{n \geq 1} \frac{1}{2^n} |f_n(x) - f_n(y)|^2 \right) = \sum_{n \geq 1} \frac{1}{2^n} |f_n(x) - f_n(y)|^2.
\]
(5.4)

Let $I_b = (0,b)$, $A_b = \{n \in \mathbb{N}^*; a_n \geq b\}$, $B_b = \mathbb{N}^* \setminus A_r = \{n; a_n < b\}$.

For all $n \in B_b$, thanks to Lemma 5.1, we have $f_n \in H^{1-b}(0,1)$, since $1/2 - a_n > (1 - b) - 1/2$. For all $n \in A_b$, we have $f_n \in H^{1-b}(I_b)$, since $f_n|_{I_b} = 0$.

Let $0 < s < 1 - b$. By using (5.4), and (5.1), we have
\[
J_{b,s} := \int_{I_b \times I_b} \frac{|G(x) - G(y)|^2}{|x-y|^{1+2s}} \, dx \, dy \leq \sum_{n \in B_b} \frac{1}{2^n} \int_{I_b \times I_b} \frac{|f_n(x) - f_n(y)|^2}{|x-y|^{1+2s}} \, dx \, dy
\leq C_s \left( \sum_{n \in B_b} \frac{1}{2^n} \left( \frac{1}{1-a_n} + \frac{1}{1-a_n - s} \right) (b-a_n)^{2(1-s-a_n)} \right)
\leq C_s \left( \sum_{n \in B_b} \frac{1}{2^n} \left( \frac{1}{1-b} + \frac{1}{1-b - s} \right) (b-a_n)^{2(1-s-a_n)} \right) < \infty
\]

since $(b-a_n)^{2(1-s-a_n)} \leq 1$ for all $n \in B_b$, $0 < s < 1 - b$.

Let $s \in (1 - b, 1)$. For all $n \in \mathbb{N}^*$ and $x > y$ we have $G(x) - G(y) \geq f_n(x) - f_n(y)$. Fix $n \in A_{1-s} \cap B_b$, that is, $1 - s \leq a_n < b$. Thanks to Lemma 5.1, we have $f_n \not\in H^s(I_b)$, and then
\[
J_{b,s} \geq \frac{1}{2^n} \int_{I_b \times I_b} \frac{|f_n(x) - f_n(y)|^2}{|x-y|^{1+2s}} \, dx \, dy = \infty.
\]

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This ends the proof.

6 Proof of Theorem 2.6

Let \( F \in L^2(\Omega_T) \), \( v_0 \in H^1_0(\Omega) \), \( v_1 \in L^2(\Omega) \). Denote \( M^1 := \{ v \in C([0, T]; H^1_0(\Omega)) \}, \partial_t v \in C([0, T]; L^2(\Omega)) \} \), \( M^0_1 = \{ v \in M; v|_{t=0} = 0, \partial_t v|_{t=0} = 0 \} \).

6.1 Energy estimate.

Put
\[
E(t)(v) = \frac{1}{2} \int_\Omega |\partial_t v|^2 + \frac{1}{2} \int_\Omega \gamma(t, \cdot)|\partial_x v|^2, \quad v \in M^1.
\]

We claim that, for all \( v \in M^1 \) such that \( L_\gamma v =: f \in L^2(\Omega_T) + W \), the following (standart) estimate, which implies (2.2), holds.

\[
E(t)(v) \leq C \left( \| f \|_{L^2(0, T; \Omega)}^2 + E(0)(v) \right), \quad \forall t \in [0, T],
\]

for some constant \( C \).

Proof. It is sufficient to show (6.1) for \( t = T \). Assume that \( f \in L^2(\Omega_T) \). Put \( \rho = \sup_{Q} \frac{|\dot{\gamma}|}{\gamma} \) and \( \Pi_0 \in C^1([0, T]; (0, +\infty)) \) such that \( \delta^{-1} \Pi_0 \leq -\Pi_0' \) for some \( \delta \in (0, \frac{1}{\rho}) \). (For example, \( \Pi_0 = e^{-\frac{t}{\delta}} \)). Put
\[
Q(v) = \int_0^T E(t)(v) \Pi_0 dt, \quad C_0(f) = \int_Q f^2 \Pi_0.
\]

We formally have, thanks te the Schwarz inequality,

\[
\begin{align*}
\delta^{-1} Q(v) &\leq - \int_0^T E(t)(v) \Pi_0 dt = [-E(t)(v) \Pi_0(t)]_0^T - \int_0^T \frac{dE(t)(v)}{dt} \Pi_0 dt \\
&\leq E(0)(v) \Pi_0(0) - E(T)(v) \Pi_0(T) - \frac{1}{2} \int_Q \Pi_0 |\partial_t v|^2 - \int_Q \Pi_0 f \partial_t v \\
&\leq E(0)(v) \Pi_0(0) - E(T)(v) \Pi_0(T) + \rho Q(v) + \sqrt{2C_0(f)} \sqrt{Q(v)},
\end{align*}
\]

Hence, we obtain

\[
(\delta^{-1} - \rho) Q(v) + E(T)(v) \Pi_0(T) \leq E(0)(v) \Pi_0(0) + \sqrt{2C_0(f)} \sqrt{Q(v)},
\]

and so,

\[
Q(v) + E(T)(v) \Pi_0(T) \leq C(C_0(f) + E(0)(v) \Pi_0(0)). \tag{6.2}
\]

Then (6.2) follows.
6.2 Uniqueness

Consequently, if \( v \in M^1_0 \) satisfies (2.1) with \( F = 0 \), then \( E(t)(v) \equiv 0 \) for all \( t \), and so \( v \equiv 0 \). This shows that Problem (2.1) admits at most one solution in \( M^1 \).

6.3 Existence

Let \( (\lambda_j, e_j)_{1 \leq j} \) be the family of spectral values of the positive operator \(-\Delta_x\) in \( H^1_0(\Omega) \), i.e. such that \( (e_i, e_j)_{L^2(\Omega)} = \delta_{ij} \), \(-\Delta e_j = \lambda_j e_j\), and \( \lambda_j \to +\infty \). The data \( v_0, v_1, F \) are then written \( v_0 = \sum_{j=1}^{\infty} v_{0,j} e_j \), \( v_1 = \sum_{j=1}^{\infty} v_{1,j} e_j \), \( F(t, \cdot) = \sum_{j=1}^{\infty} F_j(t)e_j \), with

\[
\sum_{j=1}^{\infty} (\lambda_j |v_{0,j}|^2 + |v_{1,j}|^2 + \int_0^T |F_j(t)|^2 \, dt) < \infty.
\]

Let \( N \in \mathbb{N}^* \), and put \( E_N = \text{span}\{e_1, \ldots, e_N\} \), \( V_{k,N} = (v_{k,1}, \ldots, v_{k,N}) \), \( k = 0, 1 \), \( F_N = \sum_{j=1}^{N} F_j(t)e_j \), \( B_N(t) = (b_{i,j}(t))_{1 \leq i,j \leq N} \) with \( b_{i,j}(t) = (\nabla e_i, \nabla e_j)_{L^2(\Omega; \gamma(t) \cdot dx)} \), and consider the following vectorial differential equation: find \( V_N(t) = (v_1(t), \ldots, v_N(t)) \) such that

\[
\frac{d^2}{dt^2} V_N(t) + V_N(t) B_N(t) = F_N(t), \quad 0 \leq t \leq T,
\]

with the initial condition \( V_N(0) = V_{0,N}, \frac{d}{dt} V_N(0) = V_{1,N} \). Since \( B_N(\cdot) \) is continuous, the theorem of Cauchy-Lipschitz implies existence and uniqueness for \( V_N(t) \). Note that \( B_N(t) \) is positive since, for all \( U = (u_1, \ldots, u_N) \), setting \( u(x) = \sum_{j=1}^{N} u_j e_j(x) \), we have

\[
UB_N(t) U = \int_{\Omega} |\nabla_x u|^2 \gamma(t, x) dx \geq C||\nabla_x u||_{L^2(\Omega)}^2 = C \sum_{j=1}^{N} \lambda_j |u_j|^2,
\]

where \( C \) is a constant such that \( 0 < C \leq \gamma \) in \( Q \). Let \( v_N(t) = \sum_{j=1}^{N} v_j(t)e_j(x) \). Then, a standard energy estimate for \( E_N(t)(v_N) = \frac{1}{2}(V_N^2(t) + V_N(t)B_N(t)V_N(t)) \), as above, implies that there exists a positive constant \( C \) such that

\[
||v_N(t)||_{L^2(\Omega)} + ||\partial_x v_N(t)||_{L^2(\Omega)} \leq C(||v_0||_{H^1(\Omega)} + ||v_1||_{H^1(\Omega)} + ||F||_{L^2(\Omega)}), \quad 0 \leq t \leq T.
\]

Passing to the limit \( N \to +\infty \), we can conclude by standard arguments that \( (v_N)_N \) converges to a function \( v \in C([0, T]; H^1_0(\Omega)) \) satisfying (2.1).

The proof of Theorem 2.6 in done in the case \( F \in L^2(\Omega_T) \). The case \( F \in W \) is similar.
References
