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# ON THE FACES OF THE TENSOR CONE OF SYMMETRIZABLE KAC-MOODY LIE ALGEBRAS

by

Shrawan Kumar & Nicolas Ressayre

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**Abstract.** — In this paper, we are interested in the decomposition of the tensor product of two representations of a symmetrizable Kac-Moody Lie algebra  $\mathfrak{g}$ , or more precisely in the tensor cone of  $\mathfrak{g}$ . As usual, we parametrize the integrable, highest weight (irreducible) representations of  $\mathfrak{g}$  by their highest weights. Then, the triples of such representations such that the last one is contained in the tensor product of the first two is a semigroup. This semigroup generates a rational convex cone  $\Gamma(\mathfrak{g})$  called tensor cone. If  $\mathfrak{g}$  is finite-dimensional,  $\Gamma(\mathfrak{g})$  is a polyhedral convex cone. In 2006, Belkale and the first author described this cone by an explicit finite list of inequalities. In 2010, this list of inequalities was proved to be irredundant by the second author: each such inequality corresponds to a codimension one face. In general,  $\Gamma(\mathfrak{g})$  is neither polyhedral, nor closed. Brown and the first author obtained a list of inequalities that describe  $\Gamma(\mathfrak{g})$  conjecturally. Here, we prove that each of these inequalities corresponds to a codimension one face of  $\Gamma(\mathfrak{g})$ .

## **Résumé (À propos des faces du cône tensoriel d'une algèbre de Kac-Moody)**

Dans cet article, nous nous intéressons à la décomposition du produit tensoriel de deux représentations d'une algèbre de Kac-Moody symétrisable  $\mathfrak{g}$ , et plus précisément au cône tensoriel de  $\mathfrak{g}$ . Comme d'habitude, nous paramétrons les représentations irréductibles intégrables et de plus haut poids par leur plus haut poids. Alors, les triplets de telles représentations telles que la troisième s'injecte dans le produit tensoriel des deux premières est un semi-groupe. Ces triplets engendrent un cône convexe rationnel  $\Gamma(\mathfrak{g})$  que nous appelons le *cône tensoriel*. Lorsque  $\mathfrak{g}$  est de dimension finie,  $\Gamma(\mathfrak{g})$  est un cône convexe polyédral. En 2006, Belkale et le premier auteur ont décrit ce cône par une liste finie explicite d'inégalités linéaires. En 2010, le second auteur a montré que cette liste d'inégalités n'est pas redondante : chaque inégalité correspond à une face de codimension un. En général,  $\Gamma(\mathfrak{g})$  n'est ni fermé, ni polyédral. Brown et le premier auteur ont obtenu une liste d'inégalités qui décrit conjecturalement le cône  $\Gamma(\mathfrak{g})$ . Nous montrons ici que chacune de ces inégalités correspond à une face de codimension un de  $\Gamma(\mathfrak{g})$ .

## 1. Introduction

Let  $A$  be a symmetrizable irreducible GCM (generalized Cartan matrix) of size  $l+1$ . Let  $\mathfrak{h} \supset \{\alpha_0^\vee, \dots, \alpha_l^\vee\}$  and  $\mathfrak{h}^* \supset \{\alpha_0, \dots, \alpha_l\} =: \Delta$  be a realization of  $A$  over the complex numbers  $\mathbb{C}$ . We fix an integral form  $\mathfrak{h}_{\mathbb{Z}} \subset \mathfrak{h}$  containing each  $\alpha_i^\vee$ , such that  $\mathfrak{h}_{\mathbb{Z}}^* := \text{Hom}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z})$  contains  $\Delta$  and such that  $\mathfrak{h}_{\mathbb{Z}} / \bigoplus_i \mathbb{Z}\alpha_i^\vee$  is torsion-free.

Set  $\mathfrak{h}_{\mathbb{Q}}^* = \mathfrak{h}_{\mathbb{Z}}^* \otimes \mathbb{Q} \subset \mathfrak{h}^*$ ,  $P_{+, \mathbb{Q}} := \{\lambda \in \mathfrak{h}_{\mathbb{Q}}^* : \langle \alpha_i^\vee, \lambda \rangle \geq 0 \ \forall i\}$ , and  $P_+ := \mathfrak{h}_{\mathbb{Z}}^* \cap P_{+, \mathbb{Q}}$ .

Let  $\mathfrak{g} = \mathfrak{g}(A)$  be the associated Kac-Moody (KM) Lie algebra over  $\mathbb{C}$  with Cartan subalgebra  $\mathfrak{h}$ . For  $\lambda \in P_+$ ,  $L(\lambda)$  denotes the (irreducible) integrable, highest weight representation of  $\mathfrak{g}$  with highest weight  $\lambda$ . Define the (rational) *tensor cone* as

$$\Gamma(\mathfrak{g}) := \{(\lambda_1, \lambda_2, \mu) \in P_{+, \mathbb{Q}}^3 : \exists N \geq 1 \text{ such that } L(N\mu) \subset L(N\lambda_1) \otimes L(N\lambda_2)\}.$$

The aim of this paper is to describe facets (codimension one faces) of this cone. Before describing our result, we recall from [BK14] a conjectural description of  $\Gamma(\mathfrak{g})$ , due to Brown and the first author. We need some more notation.

Fix  $\{x_0, \dots, x_l\} \in \mathfrak{h}$  to be dual of the simple roots:  $\langle \alpha_j, x_i \rangle = \delta_i^j$ . Let  $Q = \bigoplus_{i=0}^l \mathbb{Z}\alpha_i$  denote the root lattice. Let  $X = G/B$  be the standard full KM-flag variety associated to  $\mathfrak{g}$ , where  $G$  is the ‘minimal’ Kac-Moody group with Lie algebra  $\mathfrak{g}$  and  $B$  is the standard Borel subgroup of  $G$ . For  $w$  in the Weyl group  $W$  of  $G$ , let  $X_w = \overline{BwB/B} \subset X$  be the corresponding Schubert variety. Let  $\{\varepsilon^w\}_{w \in W} \subset H^*(X, \mathbb{Z})$  be the (Schubert) basis dual (with respect to the standard pairing) to the basis of the singular homology of  $X$  given by the fundamental classes of  $X_w$ .

Let  $P \supset B$  be a (standard) parabolic subgroup and let  $X_P := G/P$  be the corresponding partial flag variety. Let  $W_P$  be the Weyl group of  $P$  (which is, by definition, the Weyl group of the Levi  $L$  of  $P$ ) and let  $W^P$  be the set of minimal length representatives of cosets in  $W/W_P$ . The projection map  $X \rightarrow X_P$  induces an injective homomorphism  $H^*(X_P, \mathbb{Z}) \rightarrow H^*(X, \mathbb{Z})$  and  $H^*(X_P, \mathbb{Z})$  has the Schubert basis  $\{\varepsilon_P^w\}_{w \in W^P}$  such that  $\varepsilon_P^w$  goes to  $\varepsilon^w$  for any  $w \in W^P$ . As defined by Belkale and the first author [BK06, §6] in the finite-dimensional case and extended by the first author in [Kum08] for any symmetrizable Kac-Moody case (see [BK14, §.7] for more details), there is a new deformed product  $\odot_0$  in  $H^*(X_P, \mathbb{Z})$ , which is commutative and associative. Now, we are ready to state Brown-Kumar’s conjecture [BK14].

**Conjecture 1.1.** — *Let  $\mathfrak{g}$  be any indecomposable symmetrizable Kac-Moody Lie algebra and let  $(\lambda_1, \lambda_2, \mu) \in P_+^3$ . Assume further that none of  $\lambda_j$  and  $\mu$  are  $W$ -invariant*

*and*  
 $\mu - \sum_{j=1}^2 \lambda_j \in Q$ . Then, the following are equivalent:

(a)  $(\lambda_1, \lambda_2, \mu) \in \Gamma(\mathfrak{g})$ .

(b) *For every standard maximal parabolic subgroup  $P$  in  $G$  and every choice of triples  $(w_1, w_2, v) \in (W^P)^3$  such that  $\varepsilon_P^v$  occurs with coefficient 1 in the deformed product*

$$\varepsilon_P^{w_1} \odot_0 \varepsilon_P^{w_2} \in (H^*(X_P, \mathbb{Z}), \odot_0),$$

*the following inequality holds:*

$$(I_{(w_1, w_2, v)}^P) \quad \lambda_1(w_1 x_P) + \lambda_2(w_2 x_P) - \mu(v x_P) \geq 0,$$

*where  $\alpha_{i_P}$  is the (unique) simple root not in the Levi of  $P$  and  $x_P := x_{i_P}$ .*

Note that if  $\lambda_1$  is  $W$ -invariant,  $L(\lambda_1)$  is one-dimensional and hence  $L(\lambda_1) \otimes L(\lambda_2)$  is irreducible.

In the case where  $\mathfrak{g}$  is a semisimple Lie algebra, Conjecture 1.1 was proved by Belkale and the first author in [BK06]. The following result is due to the second author.

**Theorem 1.2** ([Res21]). — *In the case where  $\mathfrak{g}$  is affine untwisted, Conjecture 1.1 holds.*

The conjecture in the general symmetrizable case is still open. But it is conceivable that the inductive proof in the case of affine  $\mathfrak{g}$  obtained by the second author might be amenable to handle the general symmetrizable case.

Let us come back to the case where  $\mathfrak{g}$  is semisimple. Then,  $\Gamma(\mathfrak{g})$  is a closed convex polyhedral cone, and Conjecture 1.1 (Belkale-Kumar's theorem) describes  $\Gamma(\mathfrak{g})$  in  $(\mathfrak{h}_{\mathbb{Q}}^*)^3$  by (finitely many) explicit inequalities. (Recall that a rational cone  $\mathcal{C}$  is called *convex* if for  $x, y \in \mathcal{C}$  and  $0 < \alpha < 1, \alpha \in \mathbb{Q}, \alpha x + (1-\alpha)y \in \mathcal{C}$ .) In the case of  $\mathfrak{g} = \mathfrak{sl}_n$ , a larger set of inequalities describing  $\Gamma(\mathfrak{g})$  was conjectured by Horn [Hor62] and proved by Klyachko [Kly98] (combining the saturation result of Knutson-Tao [KT99]). A larger set of inequalities describing  $\Gamma(\mathfrak{g})$  for any semisimple  $\mathfrak{g}$  was known earlier (see [BS00]). The irredundancy of the above set of inequalities  $I_{(w_1, w_2, v)}^P$  was proved by Knutson-Tao-Woodward in type A [KTW04] and by the second author in general [Res10]. (See [Kum14, §1] for more details on the history.) The irredundancy assertion is the statement that each inequality  $I_{(w_1, w_2, v)}^P$  in Conjecture 1.1 corresponds to a face of  $\Gamma(\mathfrak{g})$  of codimension one. The aim of this paper is to extend this result to any symmetrizable Kac-Moody Lie algebra. We, in fact, prove the following (stronger) result for any (not necessarily maximal) standard parabolic subgroup  $P$ .

**Theorem 1.3.** — *Let  $\mathfrak{g}$  be any indecomposable symmetrizable Kac-Moody Lie algebra. Let  $P$  be a standard parabolic subgroup in  $G$  and let  $(w_1, w_2, v) \in (W^P)^3$  be a triple such that  $\varepsilon_P^v$  occurs with coefficient 1 in the deformed product*

$$\varepsilon_P^{w_1} \odot_0 \varepsilon_P^{w_2} \in (H^*(X_P, \mathbb{Z}), \odot_0).$$

*Then, the set of  $(\lambda_1, \lambda_2, \mu) \in \Gamma(\mathfrak{g})$  such that for all  $\alpha_j \notin \Delta(P)$ ,*

$$(I_{(w_1, w_2, v)}^j) \quad \lambda_1(w_1 x_j) + \lambda_2(w_2 x_j) - \mu(v x_j) = 0$$

*has codimension  $\sharp(\Delta \setminus \Delta(P))$  in  $\Gamma(\mathfrak{g})$ , where  $\Delta(P) \subset \Delta$  is the set of simple roots of the Levi subgroup  $L$  of  $P$ .*

Let  $\mathcal{C}$  denote the cone determined by the inequalities in Conjecture 1.1. For  $P$  maximal, Theorem 1.3 implies that if one removes any of the inequalities  $I_{(w_1, w_2, v)}^P$ , the cone thus obtained is strictly larger than  $\mathcal{C}$ .

Theorem 1.3 implies that  $\mathcal{C}$  is locally polyhedral. This property of  $\mathcal{C}$  plays an important role in the inductive proof of Theorem 1 from [Res21]. (Note that in [Res21], the local polyhedrality is proved in a totally different way.) As a consequence, one can hopefully think about Theorem 1.3 as a first step towards a proof of Conjecture 1.1.

Combining Theorems 1.2 and 1.3, we get the following.

**Corollary 1.4.** — *For any untwisted affine Kac-Moody Lie algebra  $\mathfrak{g}$ , the inequalities  $I_{(w_1, w_2, v)}^P$  in Conjecture 1.1 give an irredundant and complete set of inequalities determining the cone  $\Gamma(\mathfrak{g})$ .*

To prove Theorem 1.3 we will use (geometric) Theorem 1.5 below. Let us introduce some more notation.

Fix a standard parabolic subgroup  $P$  of  $G$ . For  $w \in W^P$ , we set

$$\Delta^-(w) = \{\alpha \in \Delta : \ell(s_\alpha w) = \ell(w) - 1\},$$

and

$$\Delta^+(w) = \{\alpha \in \Delta : \ell(s_\alpha w) = \ell(w) + 1 \text{ and } s_\alpha w \in W^P\},$$

where  $s_\alpha$  is the (simple) reflection corresponding to the (simple) root  $\alpha$ . It is easy to see that for any  $\alpha \in \Delta^-(w)$ ,  $s_\alpha w \in W^P$ .

Let  $B^-$  denote the Borel subgroup of  $G$  opposite to  $B$ . Consider the flag ind-variety  $\mathcal{X} := (G/B^-)^2 \times G/B$  and  $\text{Pic}^G(\mathcal{X})$  the group of  $G$ -linearized line bundles on  $\mathcal{X}$ . For  $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ , denote the line bundle  $\mathcal{L}^-(\lambda) := G \times^{B^-} \mathbb{C}_\lambda$  over  $G/B^-$  (resp.  $\mathcal{L}(\lambda) := G \times^B \mathbb{C}_{-\lambda}$  over  $G/B$ ) associated to the principal  $B^-$ -bundle  $G \rightarrow G/B^-$  (resp. the  $B$ -bundle  $G \rightarrow G/B$ ) via the one-dimensional representation  $\mathbb{C}_\lambda$  of  $B^-$  (resp. the representation  $\mathbb{C}_{-\lambda}$  of  $B$  given by the character  $e^{-\lambda}$ ).

Fix  $(\lambda_1, \lambda_2, \mu) \in P_+^3$ . By an analogue of the Borel-Weil theorem for any Kac-Moody group  $G$  (cf. [Kum02, Corollary 8.3.12]), the  $G$ -linearized line bundle  $\mathcal{L} := \mathcal{L}^-(\lambda_1) \boxtimes \mathcal{L}^-(\lambda_2) \boxtimes \mathcal{L}(\mu)$  on  $\mathcal{X}$  is such that the dimension of the space  $H^0(\mathcal{X}, \mathcal{L})^G$  of  $G$ -invariant sections is the multiplicity of  $L(\mu)$  in  $L(\lambda_1) \otimes L(\lambda_2)$  (cf. [BK14, Proof of Theorem 3.2]). From this we see that  $\Gamma(\mathfrak{g})$  is a convex subset of  $P_{+, \mathbb{Q}}^3$ .

Fix  $(w_1, w_2, v) \in (W^P)^3$  as in Theorem 1.3 and let  $L \supset T$  denote the standard Levi subgroup of  $P$ , where  $T$  is the standard maximal torus of  $G$  with Lie algebra  $\mathfrak{h}$ . The base point  $B/B$  in  $G/B$  is denoted by  $\underline{o}$ . Similarly,  $\underline{o}^- = B^-/B^-$ . Set

$$x_0 = (w_1^{-1} \underline{o}^-, w_2^{-1} \underline{o}^-, v^{-1} \underline{o}) \in \mathcal{X}.$$

For  $\alpha \in \Delta^+(w_1)$ , we set

$$x_{\alpha,1} = (w_1^{-1} s_\alpha \underline{o}^-, w_2^{-1} \underline{o}^-, v^{-1} \underline{o}) \in \mathcal{X}.$$

Similarly, we define  $x_{\alpha,2}$  associated to  $\alpha \in \Delta^+(w_2)$ . For  $\alpha \in \Delta^-(v)$ , we set

$$x_{\alpha,3} = (w_1^{-1} \underline{o}^-, w_2^{-1} \underline{o}^-, v^{-1} s_\alpha \underline{o}) \in \mathcal{X}.$$

For any  $(\alpha, i)$  as above, we denote by  $\ell_{\alpha,i}$  the unique  $T$ -stable curve in  $\mathcal{X}$  containing  $x_0$  and  $x_{\alpha,i}$ ; then  $\ell_{\alpha,i} \simeq \mathbb{P}^1$  and  $x_0$  and  $x_{\alpha,i}$  are the two  $T$ -fixed points in  $\ell_{\alpha,i}$ . Explicitly,

$$\ell_{\alpha,1} = (w_1^{-1} P_\alpha^- \underline{o}^-, w_2^{-1} \underline{o}^-, v^{-1} \underline{o}) \subset \mathcal{X},$$

where  $P_\alpha^-$  is the minimal (opposite) parabolic subgroup containing  $B^-$  and  $s_\alpha$ . Similarly,  $\ell_{\alpha,2}$  and  $\ell_{\alpha,3}$  can be described explicitly.

Consider now

$$C = Lw_1^{-1} \underline{o}^- \times Lw_2^{-1} \underline{o}^- \times Lv^{-1} \underline{o},$$

acted on by  $L$  diagonally.

**Theorem 1.5.** — *Let  $P$  and  $(w_1, w_2, v) \in (W^P)^3$  be as in Theorem 1.3. Fix  $(\lambda_1, \lambda_2, \mu) \in (\mathfrak{h}_{\mathbb{Z}}^*)^3$  such that*

$$\forall \alpha_j \notin \Delta(P), \quad \lambda_1(w_1 x_j) + \lambda_2(w_2 x_j) - \mu(v x_j) = 0.$$

*Let  $\mathcal{L} := \mathcal{L}^-(\lambda_1) \boxtimes \mathcal{L}^-(\lambda_2) \boxtimes \mathcal{L}(\mu)$  denote the associated line bundle on  $\mathcal{X}$ . We assume that, for any  $i = 1, 2$  and  $\alpha \in \Delta^+(w_i)$ , the restriction of  $\mathcal{L}$  to  $\ell_{\alpha,i}$  is nonnegative. Similarly, we assume that for any  $\alpha \in \Delta^-(v)$  the restriction of  $\mathcal{L}$  to  $\ell_{\alpha,3}$  is nonnegative.*

*Then, the restriction map induces an isomorphism:*

$$H^0(\mathcal{X}, \mathcal{L})^G \simeq H^0(C, \mathcal{L})^L.$$

To prove Theorem 1.3, we have to produce line bundles  $\mathcal{L}$  on  $\mathcal{X}$  having nonzero  $G$ -invariant sections and satisfying the equalities  $(I_{(w_1, w_2, v)}^j)$ . To do this we start with a line bundle  $\mathcal{M}$  on  $\mathcal{X}$  whose restriction  $\mathcal{M}|_C$  admits an  $L$ -invariant section  $\sigma$ . Now, we want to extend  $\sigma$  to a regular  $G$ -invariant section on  $\mathcal{X}$ . The first step is to extend  $\sigma$  to a rational  $G$ -invariant section. Even though this rational section can have poles, we are able to kill them by adding an explicit line bundle  $\mathcal{L}'$  to  $\mathcal{M}$ . An informed reader will notice that the strategy is similar to the one used by the second author in [Res10]. Nevertheless, there are numerous difficulties because of infinite-dimensional phenomena. For example, we have no abstract construction of line bundles arising from divisors; the order of a pole along a divisor is not so easy to define (and even if it is defined, such an order could be infinite) etc. In this paper, we overcome these difficulties by making various constructions more explicit which extend to our infinite-dimensional situation.

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## 2. Zariski's main theorem

We recall a consequence of the Zariski's main theorem for our later use.

**Proposition 2.1.** — *Let  $f : Y \rightarrow Z$  be a proper birational morphism between two quasiprojective irreducible varieties. We assume that we have an open subset  $\tilde{Y}$  of  $Y$  such that  $f(Y \setminus \tilde{Y})$  has codimension at least two in  $Z$  and that  $Z$  is normal. Let  $\mathcal{L}$  be a line bundle over  $Z$ .*

*Then,  $f^* : H^0(Z, \mathcal{L}) \rightarrow H^0(Y, f^*(\mathcal{L}))$  and the restriction map  $r : H^0(Y, f^*(\mathcal{L})) \rightarrow H^0(\tilde{Y}, f^*(\mathcal{L}))$  are both isomorphisms.*

*Proof.* — To prove that  $f^*$  is an isomorphism, use the proof of Zariski's main theorem as in [Har77, Chap. III, Corollary 11.4].

To prove that  $r$  is an isomorphism, consider the following commutative diagram:

$$\begin{array}{ccc}
H^0(Z, \mathcal{L}) & \xrightarrow[\sim]{\beta} & H^0(Z \setminus f(Y \setminus \tilde{Y}), \mathcal{L}) \\
f^* \downarrow \wr & & \wr \downarrow f^* \\
H^0(Y, f^* \mathcal{L}) & \xrightarrow[r_1]{\sim} & H^0(Y \setminus f^{-1}(f(Y \setminus \tilde{Y})), f^* \mathcal{L}) \\
& \searrow r & \nearrow r_2 \\
& H^0(\tilde{Y}, f^* \mathcal{L}) &
\end{array}$$

In the above diagram,  $\beta$  is an isomorphism since  $f(Y \setminus \tilde{Y})$  is of codimension  $\geq 2$  and  $Z$  is normal. Thus,  $r_1$  is an isomorphism. Further, since  $r_1$  is an isomorphism and  $r$  and  $r_2$  are injective,  $r$  is an isomorphism as well.  $\square$

### 3. The span of the cone

Before being interested in the faces of  $\Gamma(\mathfrak{g})$ , we describe the span of it.

**Proposition 3.1.** — *The tensor cone  $\Gamma(\mathfrak{g})$  (which is, by definition, a rational cone) has nonempty interior in the following rational vector space*

$$E = E_{\mathfrak{g}} := \{(\lambda_1, \lambda_2, \mu) \in (\mathfrak{h}_{\mathbb{Q}}^*)^3 : \lambda_1 + \lambda_2 - \mu \in \text{Span}_{\mathbb{Q}}(\Delta)\}.$$

Observe that  $E$  has dimension  $2 \dim \mathfrak{h} + \#\Delta$ .

*Proof.* — If  $(\lambda_1, \lambda_2, \mu) \in \Gamma(\mathfrak{g})$  then some integral multiple  $N(\lambda_1 + \lambda_2 - \mu)$  belongs to the root lattice. Hence,

$$(1) \quad \Gamma(\mathfrak{g}) \subset E.$$

Note that, for  $\lambda, \mu$  in  $P^+$ , the point

$$(2) \quad (\lambda, \mu, \lambda + \mu) \in \Gamma(\mathfrak{g}).$$

We claim that for any simple root  $\alpha_i \in \Delta$ ,

$$(3) \quad (\rho, \rho, 2\rho - \alpha_i) \in \Gamma(\mathfrak{g}),$$

where  $\rho \in \mathfrak{h}_{\mathbb{Z}}^*$  is any element satisfying  $\rho(\alpha_i^\vee) = 1$  for all the simple coroots  $\alpha_i^\vee$ . Indeed, fix a highest weight vector  $v_+$  in  $L(\rho)$  and a nonzero  $e_j$  (resp.  $f_j$ ) in  $\mathfrak{g}_{\alpha_j}$  (resp.  $\mathfrak{g}_{-\alpha_j}$ ) for any simple root  $\alpha_j$  with  $[e_j, f_j] = \alpha_j^\vee$ , where  $\mathfrak{g}_\alpha$  denotes the corresponding root space. Consider the element in  $L(\rho) \otimes L(\rho)$ :

$$v = f_i v_+ \otimes v_+ - v_+ \otimes f_i v_+.$$

Clearly,  $e_j v = 0$  for any  $j \neq i$ . Also,

$$\begin{aligned}
e_i v &= (e_i f_i v_+) \otimes v_+ - v_+ \otimes (e_i f_i v_+) \\
&= \alpha_i^\vee v_+ \otimes v_+ - v_+ \otimes \alpha_i^\vee v_+ \\
&= 0.
\end{aligned}$$

It follows that  $v$  is a highest weight vector. But its weight is  $2\rho - \alpha_i$ , proving (3). Combined with (2), we get

$$(4) \quad (0, 0, \alpha_i) \in \langle \Gamma(\mathfrak{g}) \rangle, \quad \forall \alpha_i \in \Delta,$$

where  $\langle \Gamma(\mathfrak{g}) \rangle$  is the  $\mathbb{Q}$ -span of  $\Gamma(\mathfrak{g})$  in  $(\mathfrak{h}_{\mathbb{Q}}^*)^3$ . Now, by (2) and (4),  $\Gamma(\mathfrak{g})$  spans  $E$ .  $\square$

#### 4. On some translated Richardson varieties

Fix a standard parabolic subgroup  $P$  of  $G$  with Levi subgroup  $L \supset T$ , where  $T$  is the (standard) maximal torus of  $G$  with Lie algebra  $\mathfrak{h}$ . For  $w \in W^P$ , let

$$X_P^w := \overline{B^- w P / P} \subset X_P \quad \text{and} \quad X_w^P := \overline{B w P / P} \subset X_P$$

be respectively the *opposite Schubert variety* and the *Schubert variety* associated to  $w$ .

**Proposition 4.1.** — *Fix  $(w, v) \in (W^P)^2$  and  $\alpha \in \Delta$  such that*

- i.  $v \geq w$ ;
- ii.  $\bar{v} := s_\alpha v \leq v$ ;
- iii.  $\bar{v} \not\leq w$ .

*Then,  $X_v^w(P) := X_v^P \cap X_P^w = s_\alpha X_{\bar{v}}^P \cap X_P^w$ .*

*In particular,  $\dot{X}_{\bar{v}}^P \cap s_\alpha \dot{X}_P^w$  is nonempty.*

*Proof.* — The inclusion  $s_\alpha X_{\bar{v}}^P \cap X_P^w \subset X_v^w(P)$  is clear. Moreover,  $X_v^w(P)$  is an irreducible closed subvariety of  $X_v^P$  of dimension  $\ell(v) - \ell(w)$  (cf. [Kum17, Proposition 6.6] and use the surjectivity of  $X_v^w(B)$  onto  $X_v^w(P)$ ). Since  $s_\alpha X_{\bar{v}}^P \cap X_P^w$  is closed in  $X_v^P$ , it is sufficient to prove that

$$(5) \quad \dim(X_{\bar{v}}^P \cap s_\alpha X_P^w) = \ell(v) - \ell(w).$$

Consider the incidence variety  $\mathcal{Y}$  with projection on the second factor:

$$\pi : \mathcal{Y} := \{(x, g\bar{\varrho}^-) \in X_v^P \times G/B^- \mid x \in gX_P^w\} \longrightarrow G/B^-.$$

Set  $\bar{\mathcal{Y}} := \mathcal{Y} \cap (X_{\bar{v}}^P \times G/B^-)$  and  $\bar{\pi}$  the restriction of  $\pi$  to  $\bar{\mathcal{Y}}$ .

Observe first that  $\mathcal{Y}$  and  $\bar{\mathcal{Y}}$  are respectively  $P_\alpha$ -stable and  $B$ -stable closed subsets of  $X_v^P \times G/B^-$  and  $X_{\bar{v}}^P \times G/B^-$  respectively and that  $\pi$  and  $\bar{\pi}$  are equivariant, where  $P_\alpha$  is the minimal parabolic subgroup of  $G$  containing  $B$  and  $s_\alpha$ . Moreover,  $G \times_B \mathcal{Y} \simeq G \times_P (\overline{Pv^{-1}\bar{\varrho}^-} \times \overline{Pw^{-1}\bar{\varrho}^-})$  is ind-irreducible (i.e., admits a filtration by finite-dimensional irreducible closed subsets). Hence  $\mathcal{Y}$  is ind-irreducible. Similarly,  $\bar{\mathcal{Y}}$  is ind-irreducible. Since  $X_v^P$  is projective,  $\pi$  is proper. Similarly,  $\bar{\pi}$  is proper. Hence their images are closed.

Since  $v \geq w$ ,  $\bar{\varrho}^- \in \text{Im } \pi$ . Hence,  $\pi$  is surjective.

Since  $\bar{v} \not\leq w$ ,  $\bar{\varrho}^- \notin \text{Im } \bar{\pi}$ . Hence,  $\text{Im}(\bar{\pi}) \subset \bigcup_{\beta \in \Delta} \overline{Bs_\beta \bar{\varrho}^-}$ . But,  $(\bar{v}P/P, s_\alpha \bar{\varrho}^-) \in \bar{\mathcal{Y}}$ . Indeed  $\bar{v}P/P \in X_{\bar{v}}^P$  and  $s_\alpha \bar{v}P/P = \bar{v}P/P \in s_\alpha X_P^w$ . Hence,  $\text{Im}(\bar{\pi})$  contains  $Bs_\alpha \bar{\varrho}^-$  by  $B$ -equivariance. Since  $\text{Im}(\bar{\pi})$  is closed and irreducible, we get

$$(6) \quad \text{Im}(\bar{\pi}) = \overline{Bs_\alpha \bar{\varrho}^-}.$$

We now restrict  $\mathcal{Y}$  over  $P_\alpha \bar{\varrho}^-$ . Consider the action map

$$\rho : P_\alpha \longrightarrow \text{Aut}(X_v^P).$$

The image  $P_v$  of  $\rho$  is a finite dimensional connected algebraic group of semi-simple rank one. Consider

$$\mathcal{Y}_{\text{red}}^\circ := \{(x, p) \in X_v^P \times P_v \mid p^{-1}x \in X_P^w\}$$

with its two projections  $p_1$  and  $p_2$  on  $X_v^P$  and  $P_v$  respectively. Moreover,  $p_2$  is  $P_v$ -equivariant and hence surjective. Further,  $\mathcal{Y}_{\text{red}}^\circ$  is irreducible since so is  $p_2^{-1}(e)$  and  $P_v$ .

Note that  $p_2^{-1}(e) \simeq X_v^w(P)$  is of dimension  $\ell(v) - \ell(w)$ . Hence,

$$(7) \quad \dim(\mathcal{Y}_{\text{red}}^\circ) = \dim(P_v) + \ell(v) - \ell(w).$$

We already observed that  $(vP/P, e) \in \mathcal{Y}_{\text{red}}^\circ$ . Since  $P_v \cdot v$  is dense in  $X_v^P$  and  $p_1$  is  $P_v$ -equivariant, we conclude that  $p_1$  is dominant. Then, still using the  $P_v$ -equivariance, we get

$$(8) \quad \dim(p_1^{-1}(vP/P)) = \dim(\mathcal{Y}_{\text{red}}^\circ) - \ell(v) = \dim(P_v) - \ell(w), \text{ by equation (7)}$$

Set

$$\bar{\mathcal{Y}}_{\text{red}}^\circ = \mathcal{Y}_{\text{red}}^\circ \cap (X_v^P \times P_v).$$

As observed above,  $\bar{\mathcal{Y}} \neq \mathcal{Y}$ . Similarly  $\bar{\mathcal{Y}}_{\text{red}}^\circ \neq \mathcal{Y}_{\text{red}}^\circ$ . Since  $X_v^P \times P_v$  is a hypersurface in  $X_v^P \times P_v$ , we deduce that

$$(9) \quad \dim(\bar{\mathcal{Y}}_{\text{red}}^\circ) = \dim(\mathcal{Y}_{\text{red}}^\circ) - 1 = \dim(P_v) + \ell(v) - \ell(w) - 1, \text{ by equation (7).}$$

Write now  $P_\alpha = BU_{-\alpha}(\mathbb{C}) \sqcup Bs_\alpha$  and  $P_v = \rho(B)U_{-\alpha}(\mathbb{C}) \sqcup \rho(B)s_\alpha$ , where  $U_{-\alpha}(\mathbb{C})$  is the root subgroup of  $G$  corresponding to the negative root  $-\alpha$ . Let  $\bar{p}_2$  denote the restriction of  $p_2$  to  $\bar{\mathcal{Y}}_{\text{red}}^\circ$ , which is  $\rho(B)$ -equivariant. The discussion about the image of  $\bar{\pi}$  at the beginning of the proof implies that  $\text{Im}(\bar{p}_2) = \rho(B)s_\alpha$ . Hence,

$$(10) \quad \dim(\bar{p}_2)^{-1}(s_\alpha) = \dim(X_v^P \cap s_\alpha X_P^w) = \dim(\bar{\mathcal{Y}}_{\text{red}}^\circ) - \dim(\rho(B)s_\alpha).$$

The expected equality (5) follows from (10), (9) and the identity  $\dim(\rho(B)s_\alpha) = \dim(P_v) - 1$ .

To prove the ‘In particular’ statement of the proposition, observe that

$$\mathring{X}_v^P \cap s_\alpha \mathring{X}_P^w = (X_v^P \cap s_\alpha \mathring{X}_P^w) \cap (\mathring{X}_v^P \cap s_\alpha X_P^w)$$

and the last two open subsets are nonempty in the irreducible variety  $X_v^P \cap s_\alpha X_P^w$  (by the first part of the lemma).  $\square$

**Corollary 4.2.** — *Fix  $(w, v) \in (W^P)^2$  and  $\alpha \in \Delta$  such that*

- i.  $v \geq w$ ;
- ii.  $\bar{w} := s_\alpha w \geq w \in W^P$ ;
- iii.  $v \not\geq \bar{w}$ .

*Then,  $X_v^w(P) = X_v^P \cap s_\alpha X_P^{\bar{w}}$ .*

*Proof.* — Set  $\bar{v} = s_\alpha v$ . Then, by [12, Lemma 1.3.18 and Corollary 1.3.19],  $\bar{v} \in W^P$ ;  $\bar{v} \geq \bar{w}$ ;  $v < \bar{v}$ ; and  $v \not\geq \bar{w}$ . Hence, we can apply Proposition 4.1 to the pair  $(\bar{w}, \bar{v})$  to get

$$(11) \quad X_v^P \cap X_P^{\bar{w}} = s_\alpha X_v^P \cap X_P^{\bar{w}}.$$

In particular, we have

$$\begin{aligned}
 \dim(X_v^w(P)) &= \ell(v) - \ell(w) \\
 &= \ell(\bar{v}) - \ell(\bar{w}) \\
 &= \dim(X_{\bar{v}}^{\bar{w}}(P)) \\
 &= \dim(s_\alpha X_v^P \cap X_P^{\bar{w}}) \\
 &= \dim(X_v^P \cap s_\alpha X_P^{\bar{w}}).
 \end{aligned}$$

Moreover, by [12, Theorem 5.1.3(d)],  $s_\alpha X_P^{\bar{w}} \subset X_P^w$ . Hence,  $X_v^P \cap s_\alpha X_P^{\bar{w}} \subset X_v^w(P)$ . This proves the corollary.  $\square$

## 5. Construction of line bundles

Consider a subvariety  $Z \subset \mathcal{X}$ . If  $G$  and so  $\mathcal{X}$  is finite-dimensional,  $Z$  can be realized as the zero set of a section of some line bundle on  $\mathcal{X}$  if and only if  $Z$  has codimension one. If  $G$  is not finite-dimensional, then  $\mathcal{X}$  is only an ind-variety and the codimension is not so easy to define. Moreover, even if there exists a filtration  $\mathcal{X} = \bigcup_n \mathcal{X}_n$  by finite-dimensional closed subvarieties such that  $Z \cap \mathcal{X}_n$  has codimension one in  $\mathcal{X}_n$ ,  $Z$  is not necessarily the zero locus of a section of some line bundle on  $\mathcal{X}$ .

Nevertheless, if  $Z = F_{\alpha,i}$  (resp.  $Z = E_{\bar{w}_1, \bar{w}_2, \bar{v}}$ ) as defined by Formula (12) (resp. (25)) below, we prove in this section that  $Z$  is the zero locus of a section of some line bundle.

**5.1. First divisors.** — Fix once and for all fundamental weights  $\varpi_{\alpha_0}, \dots, \varpi_{\alpha_l}$  in  $\mathfrak{h}_{\mathbb{Z}}^*$  such that  $\langle \varpi_{\alpha_i}, \alpha_j^\vee \rangle = \delta_i^j$ .

Let  $M$  be a  $\mathfrak{g}$ -module such that, under the action of  $\mathfrak{h}$ ,  $M$  decomposes as  $\bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$  with finite-dimensional weight spaces  $M_\mu$ . Set  $M^\vee = \bigoplus_{\mu} M_\mu^*$ : it is a  $\mathfrak{g}$ -submodule of the full dual space  $M^*$ .

Recall that  $\mathcal{X} = (G/B^-)^2 \times G/B$  and  $\underline{g}^\pm = B^\pm/B^\pm$ . Consider, for  $\alpha \in \Delta$  and  $i = 1, 2$ ,

$$(12) \quad F_{\alpha,i} = \{(x_1, x_2, g\underline{g}) \in \mathcal{X} : g^{-1}x_i \in \overline{Bs_\alpha \underline{g}^-}\}$$

with the reduced ind-scheme structure. It is easy to see that  $F_{\alpha,i}$  is ind-irreducible. Let  $p_1, p_2$  and  $p_3$  denote the projections from  $\mathcal{X}$  to the corresponding factor. Set, for  $i = 1, 2$  and  $\alpha \in \Delta$ ,

$$\mathcal{M}_{\alpha,i} = p_i^*(\mathcal{L}_{\varpi_\alpha}^-) \otimes p_3^*(\mathcal{L}_{\varpi_\alpha}).$$

**Lemma 5.1.** — *The space  $H^0(\mathcal{X}, \mathcal{M}_{\alpha,i})$  contains a unique (up to scalar multiples) nonzero  $G$ -invariant section  $\sigma = \sigma_{\alpha,i}$ . Moreover, scheme-theoretically,*

$$F_{\alpha,i} = \{x \in \mathcal{X} : \sigma(x) = 0\}.$$

*Proof.* — Our construction of  $\mathcal{M}_{\alpha,i}$  and  $\sigma_{\alpha,i}$  is completely explicit.

By the analogue of the Borel-Weil theorem for Kac-Moody groups (cf. [Kum02, Corollary 8.3.12]), we have (cf. [BK14, Proof of Theorem 3.2]):

$$(13) \quad H^0(\mathcal{X}, \mathcal{M}_{\alpha,i}) \simeq \text{Hom}_{\mathbb{C}}(L(\varpi_\alpha)^\vee \otimes L(\varpi_\alpha), \mathbb{C}).$$

Observe that

$$(14) \quad \text{Hom}_{\mathbb{C}}(L(\varpi_{\alpha})^{\vee} \otimes L(\varpi_{\alpha}), \mathbb{C}) \simeq \text{Hom}_{\mathbb{C}}(L(\varpi_{\alpha})^{\vee}, L(\varpi_{\alpha})^*),$$

since  $\text{Hom}_{\mathbb{C}}(V \otimes W, \mathbb{C}) \simeq \text{Hom}_{\mathbb{C}}(V, W^*)$  for any  $\mathbb{C}$ -vector spaces  $V$  and  $W$ . From the Equations (13) and (14) it is easy to see that  $H^0(\mathcal{X}, \mathcal{M}_{\alpha,i})^G$  is one-dimensional spanned by the inclusion of  $L(\varpi_{\alpha})^{\vee}$  in  $L(\varpi_{\alpha})^*$  under the identifications (13) and (14). We now identify the zero locus of nonzero  $\sigma \in H^0(\mathcal{X}, \mathcal{M}_{\alpha,i})^G$ :

Consider the isomorphism

$$\psi : G \times^{B^-} G/B \simeq G/B^- \times G/B, \quad [g, h\underline{\varrho}] \mapsto (g\underline{\varrho}^-, gh\underline{\varrho}), \quad \text{for } g, h \in G,$$

where  $[g, h\underline{\varrho}]$  denotes the  $B^-$ -orbit of  $(g, h\underline{\varrho})$ . Consider the  $B^-$ -equivariant line bundle  $\mathbb{C}_{\varpi_{\alpha}} \otimes \mathcal{L}_{\varpi_{\alpha}}$  over  $G/B$ , where  $\mathbb{C}_{\varpi_{\alpha}}$  denotes the trivial line bundle over  $G/B$  with the  $B^-$ -action given by the character  $e^{\varpi_{\alpha}}$ . It is easy to see that

$$(15) \quad \psi^*(\mathcal{L}_{\varpi_{\alpha}}^- \boxtimes \mathcal{L}_{\varpi_{\alpha}}) = G \times^{B^-} (\mathbb{C}_{\varpi_{\alpha}} \otimes \mathcal{L}_{\varpi_{\alpha}}).$$

Let  $v_-$  be a fixed nonzero vector of  $\mathbb{C}_{-\varpi_{\alpha}}$ . Consider the section  $\sigma_o$  of  $\mathcal{L}_{\varpi_{\alpha}}$  over  $G/B$  given by

$$(16) \quad \sigma_o(g\underline{\varrho}) = [g, v_+^*(gv_+)v_-], \quad \text{for } g \in G,$$

where  $v_+$  is a nonzero highest weight vector of  $L(\varpi_{\alpha})$  and  $v_+^* \in L(\varpi_{\alpha})^*$  is given by

$$v_+^*(v_+) = 1 \text{ and } v_+^*(v) = 0, \text{ for any weight vector } v \text{ of } L(\varpi_{\alpha}) \text{ of weight } \neq \varpi_{\alpha}.$$

By the definition of  $\sigma_o$ , it is a character of  $B^-$  of weight  $-\varpi_{\alpha}$  and hence  $1 \otimes \sigma_o$  thought of as a section of  $\mathbb{C}_{\varpi_{\alpha}} \otimes \mathcal{L}_{\varpi_{\alpha}}$  is  $B^-$ -invariant. Thus, it canonically gives rise to a  $G$ -invariant section  $\hat{\sigma}_o$  of  $G \times^{B^-} (\mathbb{C}_{\varpi_{\alpha}} \otimes \mathcal{L}_{\varpi_{\alpha}})$ .

We next claim that the zero set  $Z(\sigma_o)$  of  $\sigma_o$  is given by

$$(17) \quad Z(\sigma_o) = \overline{B^- s_{\alpha} \underline{\varrho}} \subset G/B.$$

By the definition of  $\sigma_o$ ,  $Z(\sigma_o)$  is left  $B^-$ -stable (since  $v_+^* \in L(\varpi_{\alpha})^*$  is an eigenvector for the action of  $B^-$ ). Take  $w \in W$ . Then,

$$\begin{aligned} w\underline{\varrho} \in Z(\sigma_o) &\Leftrightarrow v_+^*(wv_+) = 0 \\ &\Leftrightarrow w\varpi_{\alpha} \neq \varpi_{\alpha} \\ &\Leftrightarrow w \notin \langle s_{\beta} \rangle_{\beta \in \Delta \setminus \{\alpha\}}, \text{ by [Kum02, Proposition 1.4.2 (a)]} \\ &\Leftrightarrow w \geq s_{\alpha}, \end{aligned}$$

where  $\langle s_{\beta} \rangle \subset W$  denotes the subgroup generated by the elements  $s_{\beta}$ . This proves the Equation (17) by the Birkhoff decomposition [Kum02, Theorem 6.2.8]. Thus, the zero set  $Z(\hat{\sigma}_o)$  of  $\hat{\sigma}_o$  is given by:

$$Z(\hat{\sigma}_o) = G \times^{B^-} \left( \overline{B^- s_{\alpha} \underline{\varrho}} \right).$$

Moreover,

$$\psi \left( G \times^{B^-} \left( \overline{B^- s_{\alpha} \underline{\varrho}} \right) \right) = \{(x, g\underline{\varrho}) \in G/B^- \times G/B : g^{-1}x \in \overline{Bs_{\alpha}\underline{\varrho}^-}\}.$$

From this we obtain that  $Z(\sigma) = F_{\alpha,i}$  set theoretically.

To prove that  $Z(\sigma) = F_{\alpha,i}$  scheme-theoretically, it suffices to show that  $Z(\sigma_o)$  (which is set theoretically  $X^{s_\alpha} = \overline{B^- s_\alpha \underline{o}} \subset G/B$ ) is reduced.

For any  $v \in W$ , consider  $Z(\sigma_o) \cap X_v = X^{s_\alpha} \cap X_v$ , which is an irreducible subset of codimension one in  $X_v$ . The Chern class of the line bundle  $\mathcal{L}_{\varpi_\alpha|_{X_v}}$  is the Schubert class  $\varepsilon^{s_\alpha} \in H^2(X_v, \mathbb{Z})$ . If  $Z(\sigma_o) \cap X_v$  were not reduced, say

$$Z(\sigma_o) \cap X_v = d(X^{s_\alpha} \cap X_v) \quad (\text{scheme-theoretically}) \text{ for some } d > 1,$$

then  $\frac{1}{d}\varepsilon^{s_\alpha} \in H^2(X_v, \mathbb{Z})$ , which is a contradiction. Hence  $d = 1$ , proving that  $Z(\sigma_o) \cap X_v$  is reduced for any  $v \in W$ . Thus,  $Z(\sigma_o)$  is reduced, proving the lemma.  $\square$

**5.2. Subvarieties of  $\mathcal{X}$  from Schubert varieties.** — Fix a standard parabolic subgroup  $P$  of  $G$  with Levi subgroup  $L \supset T$ , where  $T$  is the (standard) maximal torus of  $G$  with Lie algebra  $\mathfrak{h}$ . For any triple  $(w_1, w_2, v) \in (W^P)^3$ , set

$$\bar{C}_{w_1, w_2, v}^+ = \overline{Pw_1^{-1}\underline{o}^-} \times \overline{Pw_2^{-1}\underline{o}^-} \times \overline{Pv^{-1}\underline{o}} \subset \mathcal{X},$$

and

$$(18) \quad E_{w_1, w_2, v} = G \cdot \bar{C}_{w_1, w_2, v}^+ \subset \mathcal{X} \text{ under the diagonal action of } G.$$

**Lemma 5.2.** — *For any triple  $(w_1, w_2, v) \in (W^P)^3$ , the set  $E_{w_1, w_2, v}$  is closed and ind-irreducible in  $\mathcal{X}$ .*

*Proof.* — Since  $G$  and  $\bar{C}_{w_1, w_2, v}^+$  are ind-irreducible (see [Res21, before Lemma 3] and the argument in the proof of Lemma 5.4), so is  $E_{w_1, w_2, v}$ . Note that

$$(19) \quad E_{w_1, w_2, v} = \{(g_1 \underline{o}^-, g_2 \underline{o}^-, g_3 \underline{o}) \in \mathcal{X} : g_1 X_P^{w_1} \cap g_2 X_P^{w_2} \cap g_3 X_v^P \neq \emptyset\}.$$

Observe that  $E_{w_1, w_2, v} = \mathcal{X}$  if  $\varepsilon_P^v$  occurs in  $\varepsilon_P^{w_1} \cdot \varepsilon_P^{w_2}$  with nonzero coefficient (cf. [BK14, Proposition 3.5]).

By the following isomorphism

$$G \times_B (G/B^-)^2 \longrightarrow \mathcal{X}, [g, x] \mapsto (gx, gB/B),$$

it is sufficient to prove that

$$\tilde{E} = \{(g_1 \underline{o}^-, g_2 \underline{o}^-) : g_1 X_P^{w_1} \cap g_2 X_P^{w_2} \cap X_v^P \neq \emptyset\}$$

is closed in  $\mathcal{X}_\delta := (G/B^-)^2 \simeq (G/B^-)^2 \times \underline{o}$ . Consider

$$\pi_\delta : \mathfrak{X}_\delta \longrightarrow \mathcal{X}_\delta,$$

where

$$\mathfrak{X}_\delta := \{(y, g_1 \underline{o}^-, g_2 \underline{o}^-, \underline{o}) \in G/P \times \mathcal{X} : y \in g_1 X_P^{w_1} \cap g_2 X_P^{w_2} \cap X_v^P\}$$

and  $\pi_\delta$  is the projection to the last three factors. Note that  $\tilde{E}$  is the image of  $\mathfrak{X}_\delta$  and  $\mathfrak{X}_\delta$  is closed in  $X_v^P \times \mathcal{X}_s$ . Consider a filtration  $\mathcal{X}_\delta = \bigcup_n \mathcal{X}_\delta^n$  by closed finite-dimensional subvarieties. Then,  $\pi_\delta^{-1}(\mathcal{X}_\delta^n)$  is closed in  $X_v^P \times \mathcal{X}_s^n$ . Since  $X_v^P$  is projective, it follows that  $\pi_\delta(\pi_\delta^{-1}(\mathcal{X}_\delta^n))$  is closed in  $\mathcal{X}_\delta^n$ . This concludes the proof since  $\pi_\delta(\pi_\delta^{-1}(\mathcal{X}_\delta^n)) = \tilde{E} \cap \mathcal{X}_\delta^n$ .  $\square$

For  $w \in W^P$ , we set  $\mathring{X}_P^w = B^-wP/P$  and  $\mathring{X}_w^P = BwP/P$ . Consider, for any triple  $(w_1, w_2, v) \in (W^P)^3$ ,

$$(20) \quad \begin{aligned} \mathfrak{X} &:= \{(gP/P, x) \in G/P \times \mathcal{X} : g^{-1}x \in \bar{C}^+\} \\ &= \{(y, g_1\mathfrak{d}^-, g_2\mathfrak{d}^-, g_3\mathfrak{d}) \in G/P \times \mathcal{X} : y \in g_1X_P^{w_1} \cap g_2X_P^{w_2} \cap g_3X_v^P\} \end{aligned}$$

and

$$\mathring{\mathfrak{X}} := \{(y, g_1\mathfrak{d}^-, g_2\mathfrak{d}^-, g_3\mathfrak{d}) \in G/P \times \mathcal{X} : y \in g_1\mathring{X}_P^{w_1} \cap g_2\mathring{X}_P^{w_2} \cap g_3\mathring{X}_v^P\},$$

where  $\bar{C}^+ = \bar{C}_{w_1, w_2, v}^+$ . Observe that  $\mathfrak{X}$  is closed in  $G/P \times \mathcal{X}$  and it is ind-irreducible since  $\mathfrak{X} = G \cdot (P/P, \bar{C}^+)$ .

Consider also the set  $\mathring{\mathfrak{X}}^+$  of points  $(y, g_1\mathfrak{d}^-, g_2\mathfrak{d}^-, g_3\mathfrak{d}) \in \mathring{\mathfrak{X}}$  such that the linear map

$$\mathcal{T}_y(g_3\mathring{X}_v^P) \longrightarrow \frac{\mathcal{T}_y(G/P)}{\mathcal{T}_y(g_1\mathring{X}_P^{w_1})} \oplus \frac{\mathcal{T}_y(G/P)}{\mathcal{T}_y(g_2\mathring{X}_P^{w_2})}$$

is injective, i.e.,

$$\mathcal{T}_y(g_1\mathring{X}_P^{w_1}) \cap \mathcal{T}_y(g_2\mathring{X}_P^{w_2}) \cap \mathcal{T}_y(g_3\mathring{X}_v^P) = (0),$$

where  $\mathcal{T}$  denotes the Zariski tangent space.

For  $v \in W^P$ , we denote  $v' \rightarrow v$  if  $v' \in W^P$ ,  $\ell(v') = \ell(v) - 1$  and  $v' \leq v$ .

**Lemma 5.3.** — *The subsets  $\mathring{\mathfrak{X}}$  and  $\mathring{\mathfrak{X}}^+$  are open in  $\mathfrak{X}$  for any triple  $(w_1, w_2, v) \in (W^P)^3$ .*

*In the definition of  $\mathring{\mathfrak{X}}$  and  $\mathring{\mathfrak{X}}^+$  if we replace  $\mathring{X}_P^{w_i}$  (for any  $i = 1, 2$ ) by any  $B^-$ -stable open subset of  $\mathring{X}_P^{w_i} \cup (\bigcup_{w_i \rightarrow w'_i \in W^P} \mathring{X}_P^{w'_i})$  and  $\mathring{X}_v^P$  by any  $B$ -stable open subset of  $\mathring{X}_v^P \cup (\bigcup_{v' \rightarrow v, v' \in W^P} \mathring{X}_{v'}^P)$ , then the lemma still remains true.*

*Proof.* — Consider the projection

$$\pi : G^{\times 4} \rightarrow G/P \times \mathcal{X}, \quad (g, g_1, g_2, g_3) \mapsto (gP/P, g_1\mathfrak{d}^-, g_2\mathfrak{d}^-, g_3\mathfrak{d}),$$

and define  $\tilde{\mathfrak{X}} := \pi^{-1}(\mathfrak{X})$  and  $\tilde{\mathring{\mathfrak{X}}} := \pi^{-1}(\mathring{\mathfrak{X}})$ . Then,

$$(21) \quad \tilde{\mathfrak{X}} = \{(g, g_1, g_2, g_3) \in G^{\times 4} : gP/P \in g_1X_P^{w_1} \cap g_2X_P^{w_2} \cap g_3X_v^P\},$$

and

$$(22) \quad \tilde{\mathring{\mathfrak{X}}} = \{(g, g_1, g_2, g_3) \in G^{\times 4} : gP/P \in g_1\mathring{X}_P^{w_1} \cap g_2\mathring{X}_P^{w_2} \cap g_3\mathring{X}_v^P\}.$$

Define the morphism

$$\beta : \tilde{\mathfrak{X}} \rightarrow X_P^{w_1} \times X_P^{w_2} \times X_v^P, \quad (g, g_1, g_2, g_3) \mapsto (g_1^{-1}gP/P, g_2^{-1}gP/P, g_3^{-1}gP/P).$$

Then,

$$\tilde{\mathring{\mathfrak{X}}} = \beta^{-1} \left( \mathring{X}_P^{w_1} \times \mathring{X}_P^{w_2} \times \mathring{X}_v^P \right)$$

and hence  $\tilde{\mathring{\mathfrak{X}}}$  is open in  $\tilde{\mathfrak{X}}$ . Thus,  $\pi$  being an open map,  $\mathring{\mathfrak{X}}$  is open in  $\mathfrak{X}$ .

We now prove that

$$\mathring{\mathfrak{X}}^+ \text{ is open in } \mathring{\mathfrak{X}} \text{ (and hence in } \mathfrak{X}\text{)}.$$

By the Equation (22)

$$(23) \quad \pi^{-1}(\mathring{\mathfrak{X}}) = \mathring{\mathfrak{X}} = \{(g, g_1, g_2, g_3) \in G^{\times 4} : \\ g^{-1}g_1 \in Pw_1^{-1}U^-, g^{-1}g_2 \in Pw_2^{-1}U^-, g^{-1}g_3 \in Pv^{-1}U\},$$

and

$$(24) \quad \pi^{-1}(\mathring{\mathfrak{X}}^+) = \{(g, g_1, g_2, g_3) \in \pi^{-1}(\mathring{\mathfrak{X}}) : \\ \mathcal{T}_{\dot{e}}(g^{-1}g_1\mathring{X}_P^{w_1}) \cap \mathcal{T}_{\dot{e}}(g^{-1}g_2\mathring{X}_P^{w_2}) \cap \mathcal{T}_{\dot{e}}(g^{-1}g_3\mathring{X}_v^P) = (0)\},$$

where  $\dot{e} := P/P \in G/P$ . Consider the morphism

$$\tilde{\beta} : \mathring{\mathfrak{X}} \rightarrow \mathring{X}_{w_1, w_2, v} := \mathring{X}_P^{w_1} \times \mathring{X}_P^{w_2} \times \mathring{X}_v^P, \quad (g, g_1, g_2, g_3) \mapsto (g_1^{-1}g, g_2^{-1}g, g_3^{-1}g),$$

where  $\mathring{X}_P^{w_i} := B^-w_iP \subset G$  and similarly  $\mathring{X}_v^P := BvP \subset G$ . Define the finite rank vector bundle  $\mathcal{E}_i$  over  $\mathring{X}_P^{w_i}$  ( $i = 1, 2$ ) by

$$\bigcup_{h_i \in \mathring{X}_P^{w_i}} \mathcal{T}_{\dot{e}}(G/P)/\mathcal{T}_{\dot{e}}(h_i^{-1}\mathring{X}_P^{w_i}) \rightarrow \mathring{X}_P^{w_i},$$

and similarly the finite rank vector bundle  $\mathcal{E}_3$  over  $\mathring{X}_v^P$  by

$$\bigcup_{h \in \mathring{X}_v^P} \mathcal{T}_{\dot{e}}(h^{-1}\mathring{X}_v^P) \rightarrow \mathring{X}_v^P,$$

and a morphism over  $\mathring{X}_{w_1, w_2, v}$ :

$$\varphi : \pi_3^*(\mathcal{E}_3) \rightarrow \pi_1^*(\mathcal{E}_1) \oplus \pi_2^*(\mathcal{E}_2)$$

induced by the canonical inclusion of  $\mathcal{T}_{\dot{e}}(h^{-1}\mathring{X}_v^P) \hookrightarrow \mathcal{T}_{\dot{e}}(G/P)$ , where  $\pi_i$  is the projection from  $\mathring{X}_{w_1, w_2, v}$  to the  $i$ -th factor.. The set of points  $Z \subset \mathring{X}_{w_1, w_2, v}$  where  $\varphi$  is injective is clearly open. But, it is easy to see that  $(\tilde{\beta})^{-1}(Z) = \pi^{-1}(\mathring{\mathfrak{X}}^+)$ , and hence  $\pi^{-1}(\mathring{\mathfrak{X}}^+)$  is open in  $\mathring{\mathfrak{X}}$  and thus  $\mathring{\mathfrak{X}}^+$  is open in  $\mathring{\mathfrak{X}}$ . This proves the first part of the lemma.

The proof for the second (stronger) part of the lemma is identical.  $\square$

**5.3. Divisors from Schubert varieties.** — In the remaining part of this Section 5,  $P$  is still a standard parabolic subgroup (and not necessarily maximal). We fix  $(w_1, w_2, v) \in (W^P)^3$  such that  $\varepsilon_P^v$  occurs with coefficient 1 in the deformed product

$$\varepsilon_P^{w_1} \odot_0 \varepsilon_P^{w_2} \in (H^*(X_P, \mathbb{Z}), \odot_0).$$

In particular,  $w_1, w_2 \leq v$ .

**5.4. The setting.** — By a *descent* we mean a pair  $(\alpha, i) \in \Delta \times \{1, 2, 3\}$  such that

–  $i = 1, 2$  and  $\alpha \in \Delta^+(w_i)$ , i.e.,  $s_\alpha w_i \in W^P$  and  $\ell(s_\alpha w_i) = \ell(w_i) + 1$ .

In this case we set  $\bar{w}_i = s_\alpha w_i$ ,  $\bar{w}_{3-i} = w_{3-i}$  and  $\bar{v} = v$ . Or

–  $i = 3$  and  $\alpha \in \Delta^-(v)$ , i.e.,  $\ell(s_\alpha v) = \ell(v) - 1$ .

In this case we set  $\bar{w}_i = w_i$  (for  $i = 1, 2$ ) and  $\bar{v} = s_\alpha v$ .

The set of all descents is denoted by  $\mathcal{D}$ .

We now aim to prove that

$$(25) \quad E_{\alpha, i} := E_{\bar{w}_1, \bar{w}_2, \bar{v}} \text{ defined by equation (18)}$$

is the zero set of a section of some  $G$ -linearized line bundle on  $\mathcal{X}$ . To do this, we consider three different situations of descents:

**Type A:** Descent  $(\alpha, i)$  with  $i = 1$  or  $2$  such that  $v = \bar{v} \not\geq \bar{w}_i$ .

**Type B:** Descent  $(\alpha, i)$  with  $i = 3$  such that  $\bar{v} \not\geq w_j = \bar{w}_j$  for at least one  $j = 1, 2$ .

**Type C:** Descent  $(\alpha, i)$  with  $1 \leq i \leq 3$  such that the relation  $\bar{v} \geq \bar{w}_j$  holds for both  $j = 1, 2$ .

**5.5. Type A descents.** —

**Lemma 5.4.** — For a descent  $(\alpha, i)$  of type A,

$$E_{\alpha, i} := E_{\bar{w}_1, \bar{w}_2, \bar{v}} = F_{\alpha, i},$$

where  $F_{\alpha, i}$  is defined by equation (12).

*Proof.* — Assume that  $i = 2$ . (The proof for  $i = 1$  is identical.) Recall from the Equation (20):

$$\mathfrak{X} := \{(y, g_1 \varrho^-, g_2 \varrho^-, g \varrho) \in G/P \times \mathcal{X} : y \in g_1 X_P^{w_1} \cap g_2 X_P^{w_2} \cap g X_v^P\},$$

for the triple  $(w_1, w_2, v)$ . Consider its analogue for  $w_2$  replaced by  $\bar{w}_2 := s_\alpha w_2$ :

$$\mathfrak{X}' := \mathfrak{X}'_{\alpha, 2} := \{(y, g_1 \varrho^-, g_2 \varrho^-, g \varrho) \in G/P \times \mathcal{X} : y \in g_1 X_P^{w_1} \cap g_2 X_P^{\bar{w}_2} \cap g X_v^P\},$$

and  $\mathfrak{X}'_{\alpha, i}$  has a similar meaning, where we place  $s_\alpha$  in the  $i$ -th factor.

Let  $\eta : G/P \times \mathcal{X} \rightarrow \mathcal{X}$  be the projection. By Lemma 5.2,  $\eta(\mathfrak{X}') = E_{\alpha, 2}$  (cf. the identity (19)) is closed in  $\mathcal{X}$  and ind-irreducible. Define the open subset of  $\mathcal{X}$ :

$$\mathring{\mathcal{X}} := \{(x_1, x_2, x) \in \mathcal{X} : (x_1, x) \in G \cdot (\varrho^-, \varrho)\}.$$

To prove that  $\mathring{\mathcal{X}}$  is open in  $\mathcal{X}$ , use the standard isomorphism  $G \times_{B^-} G/B \simeq G/B^- \times G/B$ . Since  $(\varrho^-, s_\alpha \varrho^-, \varrho) \in \mathring{\mathcal{X}} \cap F_{\alpha, 2}$  and  $F_{\alpha, 2}$  is ind-irreducible (cf. §5.1), we have

$$(26) \quad \overline{\mathring{\mathcal{X}} \cap F_{\alpha, 2}} = F_{\alpha, 2}.$$

Since  $w_1 \leq v$ , the Richardson variety  $X_v^{w_1}(P) := X_v^P \cap X_P^{w_1}$  is nonempty. Take  $x \in X_v^{w_1}(P)$ . There exists  $g \in G$  such that  $g^{-1}x \in X_P^{s_\alpha w_2}$ . Then,  $(\varrho^-, g \varrho^-, \varrho)$  belongs to  $\mathring{\mathcal{X}} \cap \eta(\mathfrak{X}')$ . Since  $\eta(\mathfrak{X}')$  is ind-irreducible, we deduce that

$$(27) \quad \overline{\mathring{\mathcal{X}} \cap \eta(\mathfrak{X}')} = \eta(\mathfrak{X}').$$

By (26) and (27), it is sufficient to prove that

$$(28) \quad \mathring{\mathcal{X}} \cap \eta(\mathfrak{X}') = \mathring{\mathcal{X}} \cap F_{\alpha,2}.$$

But  $G \times_T G/B^- \rightarrow \mathring{\mathcal{X}}, [g : x] \mapsto (g\underline{\varrho}^-, gx, g\underline{\varrho})$  is an isomorphism. Consider the intersection of  $\mathfrak{X}$  with  $G/P \times \underline{\varrho}^- \times G/B^- \times \underline{\varrho}$ :

$$\mathfrak{X}_{\Delta\Delta} = \{(x, g\underline{\varrho}^-) \in X_v^{w_1}(P) \times G/B^- : x \in gX_P^{w_2}\}$$

and

$$\mathfrak{X}'_{\Delta\Delta} = \{(x, g\underline{\varrho}^-) \in X_v^{w_1}(P) \times G/B^- : x \in gX_P^{\bar{w}_2}\}.$$

Since  $\mathfrak{X}$  is closed in  $G/P \times \mathcal{X}$  (see above Lemma 5.3),  $\mathfrak{X}_{\Delta\Delta}$  and  $\mathfrak{X}'_{\Delta\Delta}$  are closed in  $X_v^{w_1}(P) \times G/B^-$ . Note that

$$(29) \quad \mathfrak{X} \cap (G/P \times \mathring{\mathcal{X}}) \simeq G \times_T \mathfrak{X}_{\Delta\Delta}, \quad \mathfrak{X}' \cap (G/P \times \mathring{\mathcal{X}}) \simeq G \times_T \mathfrak{X}'_{\Delta\Delta}$$

under the maps

$$\delta : [g : (x, h\underline{\varrho}^-)] \mapsto (gx, g\underline{\varrho}^-, gh\underline{\varrho}^-, g\underline{\varrho})$$

and  $\mathring{\mathcal{X}} \cap F_{\alpha,2} \simeq G \times_T \overline{Bs_{\alpha}\underline{\varrho}^-}$ . Thus, to prove (28), it is sufficient to prove that

$$(30) \quad \hat{\mathcal{X}}_{\Delta\Delta} = \overline{Bs_{\alpha}\underline{\varrho}^-},$$

where  $\hat{\mathcal{X}}_{\Delta\Delta} := \{g\underline{\varrho}^- \in G/B^- : X_v^{w_1}(P) \cap gX_P^{\alpha w_2} \neq \emptyset\}$ . By Lemma 5.2,  $\hat{\mathcal{X}}_{\Delta\Delta}$  is closed in  $G/B^-$ .

By the identity (30), it suffices to prove that  $\hat{\mathcal{X}}_{ss} = \overline{Bs_{\alpha}\underline{\varrho}^-}$ . By equation (6) applied in the setting of Corollary 4.2 for  $w = w_2$ , we get  $\hat{\mathcal{X}}_{ss} \subset \overline{Bs_{\alpha}\underline{\varrho}^-}$ . Moreover, by Corollary 4.2,  $Bs_{\alpha}\underline{\varrho}^- \subset \hat{\mathcal{X}}_{ss}$  (since  $X_v^P \cap X_P^{w_2} \cap gX_P^{w_1} \neq \emptyset$  for any  $g \in G$  due to the fact that  $\varepsilon_P^v$  occurs in  $\varepsilon_P^{w_1} \cdot \varepsilon_P^{w_1}$  with nonzero coefficient [BK14, Proposition 3.5]). But since  $\hat{\mathcal{X}}_{ss}$  is closed in  $G/B^-$ , we get  $\hat{\mathcal{X}}_{ss} = \overline{Bs_{\alpha}\underline{\varrho}^-}$ .  $\square$

### 5.6. Type B descents. —

**Lemma 5.5.** — *For a descent  $(\alpha, 3)$  of type B such that  $w_j = \bar{w}_j \not\leq \bar{v}$  (for some  $1 \leq j \leq 2$ ),*

$$E_{\alpha,3} := E_{\bar{w}_1, \bar{w}_2, \bar{v}} = F_{\alpha,j},$$

where  $F_{\alpha,j}$  is defined by equation (12).

*Proof.* — Without loss of generality take  $j = 2$ . By Lemma 5.2,  $E_{\alpha,3}$  is closed and ind-irreducible. Define the open subset of  $\mathcal{X}$ :

$$\mathring{\mathcal{X}} := \{(x_1, x_2, x) \in \mathcal{X} : (x_1, x) \in G.(\underline{\varrho}^-, \underline{\varrho})\}.$$

Since  $(\underline{\varrho}^-, s_{\alpha}\underline{\varrho}^-, \underline{\varrho}) \in \mathring{\mathcal{X}} \cap F_{\alpha,2}$  and  $F_{\alpha,2}$  is ind-irreducible (cf. §5.1), we have

$$(31) \quad \overline{\mathring{\mathcal{X}} \cap F_{\alpha,2}} = F_{\alpha,2}.$$

Since  $w_1 \leq \bar{v} := s_{\alpha}v$ , the Richardson variety  $X_{\bar{v}}^{w_1}(P) := X_{\bar{v}}^P \cap X_P^{w_1}$  is nonempty. Take  $x \in X_{\bar{v}}^{w_1}(P)$ . There exists  $g \in G$  such that  $g^{-1}x \in X_P^{w_2}$ . Then,  $(\underline{\varrho}^-, g\underline{\varrho}^-, \underline{\varrho})$  belongs to  $\mathring{\mathcal{X}} \cap \eta(\mathfrak{X}')$ , where  $\mathfrak{X}' := \mathfrak{X}'_{\alpha,3}$ . Since  $\eta(\mathfrak{X}')$  is ind-irreducible, we deduce that

$$(32) \quad \overline{\mathring{\mathcal{X}} \cap \eta(\mathfrak{X}')} = \eta(\mathfrak{X}').$$

By (31) and (32), it is sufficient to prove that

$$(33) \quad \mathring{\mathcal{X}} \cap \eta(\mathfrak{X}') = \mathring{\mathcal{X}} \cap F_{\alpha,2}.$$

But  $G \times_T G/B^- \rightarrow \mathring{\mathcal{X}}, [g : x] \mapsto (g\mathring{\mathcal{Q}}^-, gx, g\mathring{\mathcal{Q}}^-)$  is an isomorphism. Consider the intersection of  $\mathfrak{X}$  with  $G/P \times \mathring{\mathcal{Q}}^- \times G/B^- \times \mathring{\mathcal{Q}}^-$ :

$$\mathfrak{X}_{\Delta\Delta} = \{(x, g\mathring{\mathcal{Q}}^-) \in X_v^{w_1}(P) \times G/B^- : x \in gX_P^{w_2}\}$$

and its closed subset

$$\mathfrak{X}'_{\Delta\Delta} = \{(x, g\mathring{\mathcal{Q}}^-) \in X_v^{w_1}(P) \times G/B^- : x \in gX_P^{w_2}\}.$$

Since  $\mathfrak{X}$  is closed in  $G/P \times \mathcal{X}$  (see above Lemma 5.3),  $\mathfrak{X}_{\Delta\Delta}$  and  $\mathfrak{X}'_{\Delta\Delta}$  are closed in  $X_v^{w_1}(P) \times G/B^-$ . Note that

$$(34) \quad \mathfrak{X} \cap (G/P \times \mathring{\mathcal{X}}) \simeq G \times_T \mathfrak{X}_{\Delta\Delta}, \quad \mathfrak{X}' \cap (G/P \times \mathring{\mathcal{X}}) \simeq G \times_T \mathfrak{X}'_{\Delta\Delta}$$

under the maps

$$\delta : [g : (x, h\mathring{\mathcal{Q}}^-)] \mapsto (gx, g\mathring{\mathcal{Q}}^-, gh\mathring{\mathcal{Q}}^-, g\mathring{\mathcal{Q}}^-)$$

and  $\mathring{\mathcal{X}} \cap F_{\alpha,2} \simeq G \times_T \overline{Bs_{\alpha}\mathring{\mathcal{Q}}^-}$ . Thus, to prove (33), it is sufficient to prove that

$$(35) \quad \hat{\mathcal{X}}_{\Delta\Delta} = \overline{Bs_{\alpha}\mathring{\mathcal{Q}}^-},$$

where  $\hat{\mathcal{X}}_{\Delta\Delta} := \{g\mathring{\mathcal{Q}}^- \in G/B^- : X_{s_{\alpha}v}^{w_1}(P) \cap gX_P^{w_2} \neq \emptyset\}$ . By Lemma 5.2,  $\hat{\mathcal{X}}_{\Delta\Delta}$  is closed in  $G/B^-$ .

By the identity (35), it suffices to prove that  $\hat{\mathcal{X}}_{ss} = \overline{Bs_{\alpha}\mathring{\mathcal{Q}}^-}$ . By equation (6) applied in the setting of Proposition 4.1 for  $w = w_2$ , we get  $\hat{\mathcal{X}}_{ss} \subset \overline{Bs_{\alpha}\mathring{\mathcal{Q}}^-}$ . Moreover, by Proposition 4.1,  $Bs_{\alpha}\mathring{\mathcal{Q}}^- \subset \hat{\mathcal{X}}_{ss}$ . But since  $\hat{\mathcal{X}}_{ss}$  is closed in  $G/B^-$ , we get  $\hat{\mathcal{X}}_{ss} = \overline{Bs_{\alpha}\mathring{\mathcal{Q}}^-}$ .  $\square$

**5.7. Type  $C$  descents.** — Take a descent  $(\alpha, i), 1 \leq i \leq 3$ , of type  $C$ . Thus,

- (i)  $\bar{w}_1 \leq \bar{v}$  and  $\bar{w}_2 \leq \bar{v}$ ;
- (ii)  $\ell(\bar{v}) = \ell(\bar{w}_1) + \ell(\bar{w}_2) - 1$ ;
- (iii) there exist  $l_1, l_2$  and  $l_3$  in  $L$  such that the linear map

$$l_3\mathcal{T}_{\bar{v}} \rightarrow \frac{\mathcal{T}}{l_1\mathcal{T}^{w_1}} \oplus \frac{\mathcal{T}}{l_2\mathcal{T}^{w_2}}$$

is injective, where the Zariski tangent spaces

$$\mathcal{T} = T_{\bar{e}}(G/P), \quad \mathcal{T}^{\bar{w}_i} = T_{\bar{e}}(\bar{w}_i^{-1}X_P^{\bar{w}_i}), \quad \text{and } \mathcal{T}_{\bar{v}} = T_{\bar{e}}(\bar{v}^{-1}X_P^{\bar{v}}).$$

The above condition (iii) follows from the following lemma.

**Lemma 5.6.** — *For any descent  $(\alpha, i)$  of type  $C$ , the triple  $(\bar{w}_1, \bar{w}_2, \bar{v})$  satisfies the above condition (iii).*

*Proof.* — We first prove the lemma for a descent  $(\alpha, 1)$  of type  $C$ . By the proof of [Res21, Lemma 19], there exists  $l_1, l_2, l_3 \in L$  such that

$$l_3\mathcal{T}_{\bar{v}} \cap l_1\mathcal{T}^{w_1} \cap l_2\mathcal{T}^{w_2} = (0).$$

Now,  $\mathcal{T}^{w_1} \supset \mathcal{T}^{\bar{w}_1}$ , since

$$\mathcal{T}^{w_1} = \bigoplus_{\beta \in \Phi_P^+ \cap w_1^{-1}\Phi^+} \mathfrak{g}_{-\beta} \quad \text{and} \quad \mathcal{T}^{\bar{w}_1} = \bigoplus_{\beta \in \Phi_P^+ \cap \bar{w}_1^{-1}\Phi^+} \mathfrak{g}_{-\beta},$$

where  $\Phi^+$  is the set of positive roots of the Kac-Moody Lie algebra  $\mathfrak{g}$  and  $\Phi_P^+ := \Phi^+ \setminus \Phi^+(L)$  ( $\Phi^+(L)$  being the set of positive roots of  $L$ ). Thus,

$$l_3 \mathcal{T}_v \cap l_1 \mathcal{T}^{\bar{w}_1} \cap l_2 \mathcal{T}^{w_2} = (0).$$

The proof of the lemma for any descent  $(\alpha, i)$  of type  $C$  for  $i = 2$  or  $3$  is identical.  $\square$

**Proposition 5.7.** — *Let  $(\alpha, i)$  be any descent of type  $C$ . Then, there exist a  $G$ -linearized line bundle  $\mathcal{L}_{\bar{w}_1, \bar{w}_2, \bar{v}}$  over  $\mathcal{X}$  of the form  $\mathcal{L}_{\bar{w}_1, \bar{w}_2, \bar{v}} = \mathcal{L}^-(\lambda_1) \boxtimes \mathcal{L}^-(\lambda_2) \boxtimes \mathcal{L}(\mu)$  for some  $(\lambda_1, \lambda_2, \mu) \in P_+^3$  and a nonzero  $G$ -invariant section  $\sigma_{\bar{w}_1, \bar{w}_2, \bar{v}}$  of  $\mathcal{L}_{\bar{w}_1, \bar{w}_2, \bar{v}}$  such that*

$$E_{\bar{w}_1, \bar{w}_2, \bar{v}} = \{x \in \mathcal{X} : \sigma_{\bar{w}_1, \bar{w}_2, \bar{v}}(x) = 0\}.$$

Before we come to the proof of the proposition, we need to prove some preparatory results.

Let  $U$  be the commutator subgroup  $[B, B]$  of  $B$  and  $U_{\underline{\varrho}^-}$  the open cell in  $G/B^-$ . Set

$$\Omega = \{(x_1, x_2, g_3 \underline{\varrho}) \in \mathcal{X} : g_3^{-1} x_i \in U_{\underline{\varrho}^-} \text{ for } i = 1, 2\}.$$

It is easy to see that  $\Omega$  is open in  $\mathcal{X}$ .

The construction of  $\mathcal{L}_{\bar{w}_1, \bar{w}_2, \bar{v}}$  and  $\sigma_{\bar{w}_1, \bar{w}_2, \bar{v}}$  is made in two steps:

(1) construct their restrictions to  $\Omega$  by using a slice technique to reduce to the case of finite-dimensional varieties (see Lemma 5.9 below). Now,  $E_{\bar{w}_1, \bar{w}_2, \bar{v}}$  corresponds to the subvariety  $\hat{E}$  (see (37) below) of an affine space. Lemma 5.8 proves that  $\hat{E}$  is a closed divisor using Lemma 5.3.

(2) Twist the restriction  $(\mathcal{L}_{\bar{w}_1, \bar{w}_2, \bar{v}})|_{\Omega}$  to avoid components of the zero locus of  $\sigma_{\bar{w}_1, \bar{w}_2, \bar{v}}$  in the boundary  $\mathcal{X} - \Omega$ . This step uses Lemmas 5.9 and 5.10 below.

Observe that, by the Birkhoff decomposition [Kum02, Theorem 6.2.8],

$$(36) \quad \mathcal{X} = \Omega \sqcup \left( \bigcup_{\alpha \in \Delta, i=1,2} F_{\alpha, i} \right).$$

Consider the group homomorphism  $\theta : U \longrightarrow \text{Aut}(X_v^P)$  given by the action and let  $U_{\bar{v}}$  be its image. Note that  $U_{\bar{v}}$  is a finite-dimensional unipotent group. Set

$$(37) \quad \hat{E} := \{u \in U_{\bar{v}} : (uX_{\bar{v}}^{\bar{w}_1}(P)) \cap X_{\bar{v}}^{\bar{w}_2}(P) \neq \emptyset\}.$$

**Lemma 5.8.** — *The subset  $\hat{E}$  of  $U_{\bar{v}}$  is a closed irreducible divisor of  $U_{\bar{v}}$ .*

*Proof.* — Consider the closed subset of  $U_{\bar{v}} \times X_{\bar{v}}^{\bar{w}_2}(P)$ :

$$\hat{\mathfrak{X}} := \{(u, x) \in U_{\bar{v}} \times X_{\bar{v}}^{\bar{w}_2}(P) : u^{-1}x \in X_{\bar{v}}^{\bar{w}_1}(P)\},$$

with its two projections  $p_1$  and  $p_2$  on  $U_{\bar{v}}$  and  $X_{\bar{v}}^{\bar{w}_2}(P)$  respectively. Since  $X_{\bar{v}}^{\bar{w}_2}(P)$  is projective,  $p_1$  is proper. In particular,  $\hat{E} = p_1(\hat{\mathfrak{X}})$  is closed in  $U_{\bar{v}}$ .

Recall the definition of  $\mathfrak{X}$  from the Equation (20) and as defined earlier in the proof of Lemma 5.2,

$$\begin{aligned} \mathfrak{X}_{\Delta} &:= \mathfrak{X} \cap (G/P \times G/B^- \times G/B^- \times \{\underline{\varrho}\}) \\ &= \{(y, g_1 \underline{\varrho}^-, g_2 \underline{\varrho}^-, \underline{\varrho}) \in G/P \times \mathcal{X} : y \in (g_1 X_P^{\bar{w}_1}) \cap (g_2 X_P^{\bar{w}_2}) \cap X_{\bar{v}}^P\}, \end{aligned}$$

its open subset

$$\mathring{\mathfrak{X}}_1 := \mathfrak{X}_\delta \cap (G/P \times (U \cdot \underline{o}^-) \times (U \cdot \underline{o}^-) \times \{\underline{o}\}),$$

and

$$\hat{\mathfrak{X}}_\delta := \pi_1^{-1}(\mathfrak{X}_\delta), \text{ where } \pi_1 : G \times \mathcal{X} \rightarrow G/P \times \mathcal{X} \text{ is the projection.}$$

Then,

$$(\overline{B\bar{v}P}) \times (\overline{P\bar{w}_1^{-1}B^-/B^-}) \times (\overline{P\bar{w}_2^{-1}B^-/B^-}) \simeq \hat{\mathfrak{X}}_\delta, (g, x_1, x_2) \mapsto (g, gx_1, gx_2, \underline{o}).$$

Hence,  $\hat{\mathfrak{X}}_\delta$  is irreducible and thus so is its quotient  $\mathfrak{X}_\delta$ . By the condition (i) at the beginning of subsection 5.7,  $\mathring{\mathfrak{X}}_1$  is nonempty. By the condition (iii) at the beginning of subsection 5.7 and Lemma 5.3,  $\mathfrak{X}_\delta \cap \mathring{\mathfrak{X}}^+$  is a nonempty open subset of  $\mathfrak{X}_\delta$ . Since  $\mathfrak{X}_\delta$  is irreducible and  $\mathfrak{X}_\delta \cap \mathring{\mathfrak{X}}^+$  and  $\mathring{\mathfrak{X}}_1$  are nonempty open subsets of irreducible  $\mathfrak{X}_\delta$ , their intersection

$$\mathring{\mathfrak{X}}_1^+ := \mathring{\mathfrak{X}}_1 \cap \mathring{\mathfrak{X}}^+ \text{ is nonempty.}$$

Consider the ind-variety  $Y = G/P \times U \times U$  and the morphism

$$\alpha : Y \rightarrow G/P^{\times 3}, (y, u_1, u_2) \mapsto (u_1^{-1}y, u_2^{-1}y, y).$$

Let  $Y' = Y_{(\bar{w}_1, \bar{w}_2, \bar{v})} \subset Y$  be the closed ind-subvariety

$$Y' := \alpha^{-1}(X_P^{\bar{w}_1} \times X_P^{\bar{w}_2} \times X_P^{\bar{v}}).$$

Then, there is an isomorphism

$$\hat{\beta} : \mathring{\mathfrak{X}}_1 \simeq Y', (y, u_1 \underline{o}^-, u_2 \underline{o}^-, \underline{o}) \mapsto (y, u_1, u_2).$$

In particular,  $Y'$  is also irreducible. Let

$$Y'_+ := \hat{\beta}(\mathring{\mathfrak{X}}_1^+) \subset Y' \text{ be the nonempty open subset.}$$

Consider the morphism

$$q : Y' \rightarrow \hat{\mathfrak{X}}, (y, u_1, u_2) \mapsto (\theta(u_2^{-1}u_1), u_2^{-1}y).$$

Clearly,  $q$  is surjective. In particular, we obtain that  $\hat{\mathfrak{X}}$  is irreducible and hence so is  $\hat{E} = p_1(\hat{\mathfrak{X}})$ .

We now determine the image of  $p_2$ : Let  $x \in X_P^{\bar{w}_2}(P)$  and let  $v' \leq \bar{v}$  be such that  $v' \in W^P$  and  $x \in \mathring{X}_{v'}^P$ . Then,  $x \in \text{Im}(p_2)$  if and only if  $Ux \cap X_P^{\bar{w}_1} \neq \emptyset$  if and only if  $\bar{w}_1 \leq v'$  (cf. [Kum02, Lemma 7.1.22]). We deduce that

$$(38) \quad \text{Im}(p_2) = X_P^{\bar{w}_2} \cap \left( \bigcup_{\bar{w}_1 \leq v' \leq \bar{v}; v' \in W^P} \mathring{X}_{v'}^P \right).$$

In particular, it is open in  $X_P^{\bar{w}_2}(P)$ .

We now analyze the fibers of  $p_2$ : Let  $x \in \text{Im}(p_2)$  and  $v'$  be as above. Then,  $p_2^{-1}(x)$  is the set of points  $u \in U_{\bar{v}}$  such that  $u^{-1}x \in X_P^{\bar{w}_1}$ . It is the pullback of  $\mathring{X}_{v'}^P \cap X_P^{\bar{w}_1}$  by the orbit map  $u \mapsto u^{-1}x$ . Since  $\mathring{X}_{v'}^P \cap X_P^{\bar{w}_1}$  is irreducible (cf. [Kum17, Proposition 6.6])

and the stabilizer of  $x$  in  $U_{\bar{v}}$  is, of course, irreducible (being a closed subgroup of a finite-dimensional unipotent group), so is  $p_2^{-1}(x)$ . Moreover,

$$(39) \quad \begin{aligned} \dim(p_2^{-1}(x)) &= \ell(v') + \dim(\text{Stab}_{U_{\bar{v}}}(v'P/P)) - \ell(\bar{w}_1) \\ &= \ell(\bar{v}) + \dim(\text{Stab}_{U_{\bar{v}}}(\bar{v}P/P)) - \ell(\bar{w}_1), \end{aligned}$$

where  $\text{Stab}_{U_{\bar{v}}}(v'P/P)$  denotes the stabilizer of  $v'P/P$  in  $U_{\bar{v}}$ .

Further, by Equations (38) and (39),

$$(40) \quad \begin{aligned} \dim \hat{\mathfrak{X}} &= \ell(\bar{v}) + \dim(\text{Stab}_{U_{\bar{v}}}(\bar{v}P/P)) - \ell(\bar{w}_1) + \ell(\bar{v}) - \ell(\bar{w}_2) \\ &= \dim U_{\bar{v}} - 1, \text{ by the condition (ii) at the beginning of subsection 5.7.} \end{aligned}$$

We return to the surjective map  $q : Y' \rightarrow \hat{\mathfrak{X}}$  defined above. By Chevalley's theorem (cf. [Har77, Chap. II, Exercise 3.19(b)]),  $q(Y'_+)$  contains a nonempty open subset (denoted by  $\hat{\mathfrak{X}}^+$ ) of  $\hat{\mathfrak{X}}$ . By the definition of  $\hat{\mathfrak{X}}_1^+$ , we get the following:

$$(41) \quad \mathcal{T}_x(u\hat{X}_{\bar{v}}^{\bar{w}_1}(P)) \cap \mathcal{T}_x(\hat{X}_{\bar{v}}^{\bar{w}_2}(P)) = (0), \text{ for any } (u, x) \in \hat{\mathfrak{X}}^+ \subset U_{\bar{v}} \times \hat{X}_{\bar{v}}^{\bar{w}_2}(P),$$

where

$$\hat{X}_{\bar{v}}^{\bar{w}}(P) := \hat{X}_P^{\bar{w}} \cap \hat{X}_{\bar{v}}^P.$$

Observe that  $\hat{X}_{\bar{v}}^{\bar{w}_i}(P)$  is smooth (which follows from [Kum02, Lemma 7.3.10]). Consider the projection map

$$p_1^+ : \hat{\mathfrak{X}}^+ \rightarrow U_{\bar{v}}, \text{ where } p_1^+ := p_1|_{\hat{\mathfrak{X}}^+}.$$

From the above Equation (41), we conclude that

$$(p_1^+)^{-1}(p_1^+(u, x)) \subset \{u\} \times \left( (u\hat{X}_{\bar{v}}^{\bar{w}_1}(P)) \cap \hat{X}_{\bar{v}}^{\bar{w}_2}(P) \right)$$

is a finite set for any  $(u, x) \in \hat{\mathfrak{X}}^+$ . In particular,  $\hat{E}$  being irreducible,

$$\dim(\hat{E}) = \dim(\overline{\text{Im } p_1^+}) = \dim(\hat{\mathfrak{X}}^+) = \dim(\hat{\mathfrak{X}}) = \dim(U_{\bar{v}}) - 1,$$

where the last equality follows from the Equation (40). This proves that  $\hat{E}$  is a divisor, proving the lemma.  $\square$

**Lemma 5.9.** — *There exist a  $G$ -equivariant line bundle  $\mathcal{M} \in \text{Pic}(\Omega)$  and nonzero  $\tau \in H^0(\Omega, \mathcal{M})^G$  such that*

$$\Omega \cap E = \{x \in \Omega : \tau(x) = 0\},$$

where  $E = E_{\bar{w}_1, \bar{w}_2, \bar{v}}$  (defined by the Identity (11)–(12)). In fact, we can take  $\mathcal{M} = (p_{3|\Omega})^* \mathcal{L}_\chi$  for a character  $\chi$  of  $B$ .

In particular,  $E \cap \Omega$  is closed in  $\Omega$ .

*Proof.* — By definition,

$$\begin{aligned} E &= \{(g_1 \underline{\varrho}^-, g_2 \underline{\varrho}^-, g_3 \underline{\varrho}) \in \mathcal{X} : g_1 X_P^{\bar{w}_1} \cap g_2 X_P^{\bar{w}_2} \cap g_3 X_{\bar{v}}^P \neq \emptyset\}, \text{ by (12)} \\ &= \{(g_1 \underline{\varrho}^-, g_2 \underline{\varrho}^-, g_3 \underline{\varrho}) : (g_3^{-1} g_1 X_P^{\bar{w}_1}) \cap (g_3^{-1} g_2 X_P^{\bar{w}_2}) \cap X_{\bar{v}}^P \neq \emptyset\}. \end{aligned}$$

Consider the isomorphism  $\iota : U \underline{\varrho}^- \rightarrow U, u \underline{\varrho}^- \mapsto u$ . Then,

$$\begin{aligned} E \cap \Omega &= \{(x_1, x_2, g_3 \underline{\varrho}) \in \Omega : \iota(g_3^{-1} x_1) X_P^{\bar{w}_1} \cap \iota(g_3^{-1} x_2) X_P^{\bar{w}_2} \cap X_{\bar{v}}^P \neq \emptyset\} \\ &= \{(x_1, x_2, g_3 \underline{\varrho}) \in \Omega : (\iota(g_3^{-1} x_1) X_{\bar{v}}^{\bar{w}_1}(P)) \cap (\iota(g_3^{-1} x_2) X_{\bar{v}}^{\bar{w}_2}(P)) \neq \emptyset\}, \end{aligned}$$

since  $X_{\bar{v}}^P$  is  $U$ -stable. Here (as earlier)  $X_{\bar{v}}^{\bar{w}_1}(P) := X_P^{\bar{w}_1} \cap X_{\bar{v}}^P$ . Thus,

$$(42) \quad E \cap \Omega = \{(x_1, x_2, g_3 \underline{\varrho}) \in \Omega : ([\iota(g_3^{-1}x_2)^{-1}\iota(g_3^{-1}x_1)]X_{\bar{v}}^{\bar{w}_1}(P)) \cap X_{\bar{v}}^{\bar{w}_2}(P) \neq \emptyset\}.$$

As earlier, consider the group homomorphism  $\theta : U \rightarrow \text{Aut}(X_{\bar{v}}^P)$  given by the action, and denote by  $U_{\bar{v}}$  its image (which is a finite-dimensional unipotent group). Recall (cf. (19)) that

$$\hat{E} := \{u \in U_{\bar{v}} : (uX_{\bar{v}}^{\bar{w}_1}(P)) \cap X_{\bar{v}}^{\bar{w}_2}(P) \neq \emptyset\}.$$

Note that the torus  $T$  acts by conjugation on  $U_{\bar{v}}$  and that  $\hat{E}$  is  $T$ -stable. Being a finite-dimensional unipotent group,  $U_{\bar{v}}$  is isomorphic as a variety to an affine space. In particular, there exists  $\hat{f} \in \mathbb{C}[U_{\bar{v}}]$ , unique up to scalar multiplication, such that  $\text{div}(\hat{f}) = \hat{E}$  (since  $\hat{E}$  is an irreducible divisor by Lemma 5.8). Moreover, since  $\hat{E}$  is  $T$ -stable,  $\hat{f}$  is an eigenvector of  $T$ ; denote by  $\chi$  the corresponding character. We extend  $\chi$  uniquely to a character of  $B$ .

Set  $\tilde{E} = \tilde{\pi}^{-1}(E)$  and  $\tilde{\Omega} := \tilde{\pi}^{-1}(\Omega)$ , where  $\tilde{\pi} : \tilde{\mathcal{X}} := G/B^- \times G/B^- \times G \rightarrow \mathcal{X}$  is the projection. Then,  $\tilde{\Omega}$  and  $\tilde{E}$  are stable by the following action of  $G \times B$ :

$$(g, b) \cdot (x_1, x_2, g') := (gx_1, gx_2, gg'b^{-1}).$$

Consider  $\tilde{f} : \tilde{\Omega} \rightarrow \mathbb{C}$  defined by

$$\tilde{f}(x_1, x_2, g) = \hat{f} \circ \theta(\iota(g^{-1}x_2)^{-1}\iota(g^{-1}x_1)).$$

Then, by the Equation (42),  $\tilde{E} \cap \tilde{\Omega}$  is the zero locus  $Z(\tilde{f})$  of  $\tilde{f}$  and for  $b = ut \in B$  (where  $u \in U, t \in T$ ):

$$(43) \quad \begin{aligned} \tilde{f}(x_1, x_2, gb) &:= \hat{f} \circ \theta(t^{-1}[\iota(g^{-1}x_2)^{-1}\iota(g^{-1}x_1)]t) \\ &= \chi(t)\tilde{f}(x_1, x_2, g) = \chi(b)\tilde{f}(x_1, x_2, g). \end{aligned}$$

We claim that  $\tilde{f}$  induces a section  $\tau_{\tilde{f}}$  of  $(p_{3|\Omega})^*(\mathcal{L}_{\mathcal{X}})$ , where  $p_3 : \mathcal{X} \rightarrow G/B$  is the projection onto the third factor.

By the Equation (43),  $\tilde{f}$  gives rise to a section  $\tau_{\tilde{f}}$  of the line bundle  $\mathcal{L}_{\Omega}(\chi)$  associated to the principal  $B$ -bundle  $\tilde{\Omega} \rightarrow \Omega$  (induced from the right  $\cdot$  action of  $B$  on  $\tilde{\Omega}$ ) via the character  $\chi^{-1}$  of  $B$ . Clearly,

$$\mathcal{L}_{\Omega}(\chi) = (p_{3|\Omega})^*(\mathcal{L}_{\mathcal{X}}).$$

By construction, the zero set  $Z(\tau_{\tilde{f}}) = E \cap \Omega$ . By the definition of  $\tau_{\tilde{f}}$ , it is easy to see that it is a  $G$ -invariant section. Taking  $\tau = \tau_{\tilde{f}}$ , we get the lemma.  $\square$

We now have a line bundle and a section  $\tau$  on  $\Omega$  with the expected zero locus. To avoid extra zero locus in the boundary  $\mathcal{X} \setminus \Omega$  we need to twist by some line bundles given by Lemma 5.1. The key point to do this is the following finiteness result:

**Lemma 5.10.** — *The valuation  $v_{F_{\beta,i}}(\tau)$  is finite for any  $\beta \in \Delta$  and  $i = 1, 2$ , where  $\tau$  is the section taken from Lemma 5.9. (In the proof below we see that  $F_{\beta,i}$  is ind-irreducible.)*

*Proof.* — We are going to prove that  $v_{F_{\beta,i}}(\tau)$  can be computed in some finite-dimensional variety after taking a quotient by a unipotent group.

Fix a simple root  $\beta \in \Delta$  and  $i = 1$  and consider

$$F = F_{\beta,1} = \{(x_1, x_2, g_3 \underline{\varrho}) \in \mathcal{X} : g_3^{-1}x_1 \in \overline{Bs_{\beta}\underline{\varrho}}\}.$$

Consider the isomorphism

$$\varphi : \tilde{\mathcal{X}} \rightarrow \mathcal{X}, \quad (x_1, x_2, g) \mapsto (gx_1, gx_2, g).$$

Endow  $\tilde{\mathcal{X}}$  with the following two right actions of  $B$ :

$$(x_1, x_2, g) \odot b = (b^{-1}x_1, b^{-1}x_2, gb)$$

and

$$(x_1, x_2, g) \cdot b = (x_1, x_2, gb).$$

Then, the morphism  $\varphi$  is  $B$ -equivariant with respect to the action  $\odot$  on the domain and the action  $\cdot$  on the range.

Clearly,  $\tilde{\pi} : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  is a principal  $B$ -bundle with respect to the action  $\cdot$ . Define

$$\tilde{\Omega}' := \varphi^{-1}(\tilde{\Omega}).$$

By the definition of  $\Omega$ ,

$$(44) \quad \tilde{\Omega}' = U_{\underline{\varrho}}^- \times U_{\underline{\varrho}}^- \times G.$$

Let  $\hat{f}$  and  $\tilde{f}$  be as in the proof of Lemma 5.9. Set  $\tilde{f}' = \tilde{f} \circ \varphi : \tilde{\Omega}' \rightarrow \mathbb{C}$ . Thus,

$$(45) \quad \tilde{f}'(u_1 \underline{\varrho}^-, u_2 \underline{\varrho}^-, g) = \hat{f} \circ \theta(u_2^{-1} u_1), \quad \text{for } u_1, u_2 \in U \text{ and } g \in G.$$

Set  $F' := (\tilde{\pi} \circ \varphi)^{-1}(F) = \overline{Us_{\beta}\underline{\varrho}^-} \times G/B^- \times G$ . Consider  $V^{\beta} := U_{\underline{\varrho}}^- \cup Us_{\beta}\underline{\varrho}^-$ . It is an open subset of  $G/B^-$  (containing  $\overline{Us_{\beta}\underline{\varrho}^-}$ ). By [Kum17, Lemma 6.1] and [KS09, Proposition 3.2], there exists a closed normal subgroup  $\mathcal{U}$  of  $U$  such that  $V^{\beta} \rightarrow \mathcal{U} \backslash V^{\beta} =: Y^{\beta}$  is a principal  $\mathcal{U}$ -bundle and  $Y^{\beta}$  is a smooth finite-dimensional variety. Moreover, by intersecting with  $\text{Ker } \theta$ , one can assume that  $\mathcal{U}$  acts trivially on  $X_{\bar{v}}$ .

Let  $h_1, h_2 \in \mathcal{U}$ . We have, for any  $u_1, u_2 \in U$  and  $g \in G$ ,

$$\begin{aligned} \tilde{f}'(h_1 u_1 \underline{\varrho}^-, h_2 u_2 \underline{\varrho}^-, g) &= \hat{f} \circ \theta(u_2^{-1} h_2^{-1} h_1 u_1), \text{ by Equation (45)} \\ &= \hat{f} \circ \theta(u_2^{-1} u_1), \text{ since } \theta \text{ is a group homomorphism} \\ &\quad \text{and } h_1, h_2 \in \mathcal{U} \subset \text{Ker } \theta \\ (46) \quad &= \tilde{f}'(u_1 \underline{\varrho}^-, u_2 \underline{\varrho}^-, g). \end{aligned}$$

Since the line bundle  $p_3^*(\mathcal{L}_{\chi})$  over  $\mathcal{X}$  pulled to the principal  $B$ -bundle  $\pi : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  is trivialized, to prove the finiteness of  $v_F(\tau)$ , it suffices to show that the function  $\tilde{f} : \tilde{\Omega} \rightarrow \mathbb{C}$  has a pole of finite order along  $\pi^{-1}(F)$ . Equivalently, considering the isomorphism  $\varphi : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ , it suffices to show that the function

$$\tilde{f}' : \tilde{\Omega}' = U_{\underline{\varrho}}^- \times U_{\underline{\varrho}}^- \times G \rightarrow \mathbb{C}$$

has a pole of finite order along  $F' = \overline{Us_{\beta}\underline{\varrho}^-} \times G/B^- \times G$ , since  $F' = \varphi^{-1}(F)$ ; in particular,  $F$  is ind-irreducible.

The diagonal action of  $G$  on  $\tilde{\mathcal{X}}$  pulled back via  $\varphi$  to the action  $\odot$  of  $G$  on  $\tilde{\mathcal{X}}$  is given by:

$$g \odot (x_1, x_2, h) = (x_1, x_2, gh), \text{ for } x_1, x_2 \in G/B^- \text{ and } g, h \in G.$$

The function  $\tilde{f}' : U_{\underline{\varrho}}^- \times U_{\underline{\varrho}}^- \times G \rightarrow \mathbb{C}$  descends to a function  $\hat{f}'$  on  $U_{\underline{\varrho}}^- \times U_{\underline{\varrho}}^-$  by Equation (45). So, to prove that the function  $\tilde{f}'$  has a pole of finite order along  $F'$ , it suffices to show that the function  $\hat{f}' : U_{\underline{\varrho}}^- \times U_{\underline{\varrho}}^- \rightarrow \mathbb{C}$  has a pole of finite order along  $(\overline{Us_{\beta}\underline{\varrho}}^-) \times G/B^-$ . Consider the open embedding

$$(\mathcal{U} \setminus U_{\underline{\varrho}}^-) \times (\mathcal{U} \setminus U_{\underline{\varrho}}^-) \hookrightarrow (\mathcal{U} \setminus V^{\beta}) \times (\mathcal{U} \setminus U_{\underline{\varrho}}^-).$$

By the Equation (46), the function  $\hat{f}'$  descends to a function  $\hat{\phi}'$  on  $(\mathcal{U} \setminus U_{\underline{\varrho}}^-) \times (\mathcal{U} \setminus U_{\underline{\varrho}}^-)$ . Since  $(\mathcal{U} \setminus V^{\beta}) \times (\mathcal{U} \setminus U_{\underline{\varrho}}^-)$  is a (smooth) scheme of finite type over  $\mathbb{C}$ , the function  $\hat{\phi}'$  has a pole of finite order along the divisor  $(\mathcal{U} \setminus (Us_{\beta}\underline{\varrho}^-)) \times (\mathcal{U} \setminus U_{\underline{\varrho}}^-)$  and hence  $\hat{f}'$  has a pole of finite order along the divisor  $(Us_{\beta}\underline{\varrho}^-) \times U_{\underline{\varrho}}^-$ . Since  $Us_{\beta}\underline{\varrho}^-$  is an open subset of  $\overline{Us_{\beta}\underline{\varrho}}^-$ , we get that  $\hat{f}'$  has a pole of finite order along  $(\overline{Us_{\beta}\underline{\varrho}}^-) \times U_{\underline{\varrho}}^-$ . This proves the finiteness of  $v_{F_{\beta,1}}(\tau)$  for any  $\beta \in \Delta$ . The proof of the finiteness of  $v_{F_{\beta,2}}(\tau)$  is identical.  $\square$

*Proof of Proposition 5.7.* — Observe that  $E \neq \mathcal{X}$  by Lemma 4.5. By Lemma 5.9, there exist a  $G$ -equivariant line bundle  $\mathcal{M}$  over  $\Omega$  and a nonzero  $G$ -invariant section  $\tau$  over  $\Omega$  with its zero set  $Z(\tau) = E \cap \Omega$ . Moreover, the line bundle  $\mathcal{M}$  is the restriction of the line bundle  $p_3^*(\mathcal{L}_{\chi})$  over  $\mathcal{X}$ . Then,  $\tau$  is a (rational) section of  $\mathcal{M}' := p_3^*(\mathcal{L}_{\chi})$  regular over  $\Omega$ .

Lemma 5.10 allows to consider the  $G$ -linearized line bundle

$$\mathcal{L}_{\bar{w}_1, \bar{w}_2, \bar{v}} := \mathcal{M}' \otimes \left( \bigotimes_{\beta \in \Delta, i=1,2} \mathcal{M}_{\beta,i}^{v_{F_{\beta,i}}(\tau)} \right) \text{ over } \mathcal{X},$$

where the line bundles  $\mathcal{M}_{\beta,i}$  are as in Lemma 5.1. In particular,  $\mathcal{L}_{\bar{w}_1, \bar{w}_2, \bar{v}}$  is of the form  $\mathcal{L}^-(\lambda_1) \boxtimes \mathcal{L}^-(\lambda_2) \boxtimes \mathcal{L}(\mu)$  for some  $\lambda_1, \lambda_2, \mu \in \mathfrak{h}_{\mathbb{Z}}^*$ .

By Lemmas 5.1, 4.6, 4.7 and the decomposition (36), it has a nonzero  $G$ -invariant section

$$(47) \quad \sigma_{\bar{w}_1, \bar{w}_2, \bar{v}} = \tau \otimes \left( \bigotimes_{\beta \in \Delta, i=1,2} \sigma_{\beta,i}^{v_{F_{\beta,i}}(\tau)} \right).$$

Thus, by [Kum02, Corollary 8.3.12],  $(\lambda_1, \lambda_2, \mu) \in P_+^3$ . This proves the proposition by using the following Lemma 5.11.  $\square$

Observe that  $\overline{E \cap \Omega} \subset E$  (since  $E$  is closed by Lemma 5.2). Moreover, since  $E$  is irreducible and  $E \cap \Omega \neq \emptyset$  (as  $(\underline{\varrho}^-, \underline{\varrho}^-, \underline{\varrho}) \in E \cap \Omega$ ),

$$(48) \quad \overline{E \cap \Omega} = E.$$

**Lemma 5.11.** — *The zero set  $Z(\sigma_{\bar{w}_1, \bar{w}_2, \bar{v}}) := \{x \in \mathcal{X} : \sigma_{\bar{w}_1, \bar{w}_2, \bar{v}}(x) = 0\}$  is equal to  $E$ .*

*Proof.* — Consider the map

$$\psi : \tilde{\mathcal{X}} := (G/B^-)^2 \times G \rightarrow \mathcal{X} := (G/B^-)^2 \times G/B, \quad (x_1, x_2, g) \mapsto (gx_1, gx_2, g\varrho).$$

For any subset  $Y \subset \mathcal{X}$ , we set  $\hat{Y}' := \psi^{-1}(Y)$ . Then,

$$\hat{F}'_{\beta,1} = \overline{Bs_\beta \varrho^-} \times G/B^- \times G.$$

Take an increasing cofinal sequence  $w_n \in W$  (i.e.,  $\bar{w}_1 < \bar{w}_2 < w_3 < \dots$  and for any  $w \in W$  there exists a  $w_n$  such that  $w \leq w_n$ ). Take a filtration  $(G_n)_{n \geq 0}$  of  $G$  by finite-dimensional irreducible subvarieties compatible with its ind-variety structure (cf. [Res21, above Lemma 2.3]). Now, define the increasing filtration

$$\tilde{\mathcal{X}}_n := X_{w_n}^- \times X_{w_n}^- \times G_n \text{ of } \tilde{\mathcal{X}}, \text{ where } X_w^- := \overline{B^- w \varrho^-}.$$

Then,

$$(49) \quad \tilde{\mathcal{X}}_n \cap \hat{F}'_{\beta,1} = (X_{w_n}^- \cap \overline{Bs_\beta \varrho^-}) \times X_{w_n}^- \times G_n,$$

and a similar expression for  $\tilde{\mathcal{X}}_n \cap \hat{F}'_{\beta,2}$ . Thus,  $\tilde{\mathcal{X}}_n \cap \hat{F}'_{\beta,i}$  is irreducible. Abbreviate  $Z = Z(\sigma_{\bar{w}_1, \bar{w}_2, \bar{v}})$ . Then, by Lemmas 5.1 and 5.9 and the identity (36),  $Z \cap \Omega = E \cap \Omega$  and hence  $Z \supset E$  by the identity (48). Write

$$\hat{Z}' = \hat{E}' \cup \left( \bigcup_{(\beta,i) \in \Delta \times \{1,2\}} (\hat{Z}' \cap \hat{F}'_{\beta,i}) \right), \text{ by the identity (36).}$$

Thus, for any  $n \geq 0$ ,

$$(50) \quad \hat{Z}' \cap \tilde{\mathcal{X}}_n = (\hat{E}' \cap \tilde{\mathcal{X}}_n) \cup \left( \bigcup_{(\beta,i) \in \Delta \times \{1,2\}} (\hat{Z}' \cap \hat{F}'_{\beta,i} \cap \tilde{\mathcal{X}}_n) \right).$$

But, being the zero set of a section of a line bundle,  $\hat{Z}' \cap \tilde{\mathcal{X}}_n$  is a divisor in  $\tilde{\mathcal{X}}_n$  for large  $n$  and so is  $\hat{F}'_{\beta,i} \cap \tilde{\mathcal{X}}_n$  and the latter is irreducible (divisor of  $\tilde{\mathcal{X}}_n$ ) by the Equation (49). From the definition of  $\sigma$  given by the Equation (47), we get (for any  $(\beta, i) \in \Delta \times \{1, 2\}$ )

$$(51) \quad \hat{Z}' \cap \hat{F}'_{\beta,i} \cap \tilde{\mathcal{X}}_n \subsetneq \hat{F}'_{\beta,i} \cap \tilde{\mathcal{X}}_n, \text{ for large enough } n.$$

Thus,  $\hat{Z}' \cap \hat{F}'_{\beta,i} \cap \tilde{\mathcal{X}}_n$  is of codimension  $\geq 2$  in  $\tilde{\mathcal{X}}_n$  for large enough  $n$ . But, since  $\hat{Z}' \cap \tilde{\mathcal{X}}_n$  is a divisor in  $\tilde{\mathcal{X}}_n$ , we get from the Equation (50) that

$$\hat{Z}' \cap \hat{F}'_{\beta,i} \cap \tilde{\mathcal{X}}_n \subset \hat{E}' \cap \tilde{\mathcal{X}}_n, \text{ for large enough } n.$$

Thus,

$$\hat{Z}' \cap \tilde{\mathcal{X}}_n = \hat{E}' \cap \tilde{\mathcal{X}}_n, \text{ for large enough } n \text{ which gives } \hat{Z}' = \hat{E}'.$$

Hence,  $Z = E$  proving the lemma.  $\square$

Combining Lemmas 5.1, 5.4 and 5.5 and Proposition 5.7, we get the following.

**Corollary 5.12.** — *For any  $(\alpha, i) \in \mathcal{D}$ , there exists a  $G$ -linearized line bundle  $\mathcal{N}_{\alpha,i}$  of the form  $\mathcal{L}^-(\lambda_1) \otimes \mathcal{L}^-(\lambda_2) \otimes \mathcal{L}(\mu)$ , for some  $\lambda_1, \lambda_2, \mu \in P_+^3$ , together with a nonzero  $G$ -invariant section  $\mu_{\alpha,i}$  of  $\mathcal{N}_{\alpha,i}$  such that the zero set*

$$Z(\mu_{\alpha,i}) = E_{\alpha,i}, \text{ where } E_{\alpha,i} := E_{\bar{w}_1, \bar{w}_2, \bar{v}}.$$

**Remark 5.13.** — Observe that for a descent  $(\alpha, 3)$  of type  $B$ , there exists exactly one  $1 \leq j \leq 2$  such that  $w_j = \bar{w}_j \not\leq \bar{v}$ . To show this, assume for contradiction, that  $w_i \not\leq \bar{v}$  for both  $i = 1, 2$ . Then, by Lemma 5.5,  $E_{\alpha,3} = F_{\alpha,1} = F_{\alpha,2}$ . This is a contradiction since  $F_{\alpha,1} \neq F_{\alpha,2}$ . This proves the claim.

## 6. Proof of Theorem 1.5

In this section, we fix  $P$ ,  $(w_1, w_2, v)$  and  $\mathcal{L}$  as in the theorem.

As earlier, let  $\mathcal{D}$  denote the set of descents  $(\alpha, i) \in \Delta \times \{1, 2, 3\}$  coming from  $\Delta^+(w_1)$ ,  $\Delta^+(w_2)$  and  $\Delta^-(v)$ , i.e.,

$$\mathcal{D} \cap (\Delta \times \{i\}) = \Delta^+(w_i) \text{ for } i = 1, 2 \text{ and } \mathcal{D} \cap (\Delta \times \{3\}) = \Delta^-(v),$$

where  $\Delta^+(w_i)$  and  $\Delta^-(v)$  are defined in the Introduction.

**6.1. Strategy.** — We set

$$\begin{aligned} C &= Lw_1^{-1}\underline{\varrho}^- \times Lw_2^{-1}\underline{\varrho}^- \times Lv^{-1}\underline{\varrho}, \\ C^+ &= Pw_1^{-1}\underline{\varrho}^- \times Pw_2^{-1}\underline{\varrho}^- \times Pv^{-1}\underline{\varrho}, \end{aligned}$$

and (as earlier)

$$\bar{C}^+ = \bar{C}_{w_1, w_2, v}^+ := \overline{Pw_1^{-1}\underline{\varrho}^-} \times \overline{Pw_2^{-1}\underline{\varrho}^-} \times \overline{Pv^{-1}\underline{\varrho}}.$$

Recall from Equation (20):

$$\begin{aligned} \mathfrak{X} &:= \{(gP/P, x) \in G/P \times \mathcal{X} : g^{-1}x \in \bar{C}^+\} \\ &= \{(y, g_1\underline{\varrho}^-, g_2\underline{\varrho}^-, g_3\underline{\varrho}) \in G/P \times \mathcal{X} : y \in g_1X_P^{w_1} \cap g_2X_P^{w_2} \cap g_3X_v^P\}. \end{aligned}$$

As a closed subset of  $G/P \times \mathcal{X}$ , it is a  $G$ -ind-variety with the diagonal action of  $G$ . Consider the projection

$$\eta : \mathfrak{X} \rightarrow \mathcal{X}, \quad (y, x) \mapsto x.$$

For each  $(\alpha, i) \in \mathcal{D}$ , consider the associated  $P^3$ -orbit  $\partial C_{\alpha, i}^+$  in  $\mathcal{X}$ , where

$$\partial C_{\alpha, 1}^+ := Pw_1^{-1}s_\alpha\underline{\varrho}^- \times Pw_2^{-1}\underline{\varrho}^- \times Pv^{-1}\underline{\varrho}$$

and  $\partial C_{\alpha, i}^+$  ( $i = 2, 3$ ) are defined similarly. Then,  $\partial C_{\alpha, i}^+$  is open in an irreducible component of  $\bar{C}^+ \setminus C^+$ . Set

$$\tilde{C}^+ = \tilde{Y}^{w_1} \times \tilde{Y}^{w_2} \times \tilde{Y}_v,$$

where

$$\tilde{Y}^{w_i} := (Pw_i^{-1}\underline{\varrho}^-) \cup \left( \bigcup_{(\alpha, i) \in \mathcal{D}} Pw_i^{-1}s_\alpha\underline{\varrho}^- \right)$$

and

$$\tilde{Y}_v := (Pv^{-1}\underline{\varrho}) \cup \left( \bigcup_{(\alpha, 3) \in \mathcal{D}} Pv^{-1}s_\alpha\underline{\varrho} \right).$$

It is open in  $\bar{C}^+$ . Similarly, we define the open subset of  $X_P^{w_i}$ :

$$\tilde{X}_P^{w_i} := (B^- w_i P/P) \cup \left( \bigcup_{\alpha \in \Delta^+(w_i)} B^- s_\alpha w_i P/P \right) \quad (\text{for } i = 1, 2)$$

and the open subset of  $X_v^P$ :

$$\tilde{X}_v^P := (BvP/P) \cup \left( \bigcup_{\alpha \in \Delta^-(v)} Bs_\alpha vP/P \right).$$

We also set

$$\begin{aligned} \tilde{\mathfrak{X}}' &:= \{(gP/P, x) \in G/P \times \mathcal{X} : g^{-1}x \in \tilde{C}^+\} \\ &= \{(y, g_1 \varrho^-, g_2 \varrho^-, g_3 \varrho) \in G/P \times \mathcal{X} : y \in g_1 \tilde{X}_P^{w_1} \cap g_2 \tilde{X}_P^{w_2} \cap g_3 \tilde{X}_v^P\}, \end{aligned}$$

which is an open subset of  $\mathfrak{X}$  and hence ind-irreducible (since so is  $\mathfrak{X}$  as observed earlier below the Equation (20)). We make use of a slice by setting

$$\mathcal{X}_\delta := (G/B^-)^2 \times \{\varrho\} \subset \mathcal{X},$$

and its  $B$ -stable open subset

$$\mathring{\mathcal{X}}_\delta := (B\varrho^- \cup \bigcup_{\alpha \in \Delta} s_\alpha B\varrho^-)^2 \times \{\varrho\} = \left( \bigcup_{\ell(w) \leq 1} Bw\varrho^- \right)^2 \times \{\varrho\}.$$

Then, we have a  $G$ -equivariant isomorphism:

$$(52) \quad G \times^B \mathcal{X}_\delta \simeq \mathcal{X}, \quad [g, x] \mapsto gx.$$

As defined in the proof of Lemma 5.2,

$$\mathfrak{X}_\delta := \{(y, g_1 \varrho^-, g_2 \varrho^-, \varrho) \in G/P \times \mathcal{X}_\delta : y \in g_1 X_P^{w_1} \cap g_2 X_P^{w_2} \cap X_v^P\} \subset \mathfrak{X}.$$

We also set

$$\mathring{\mathfrak{X}}_\delta := \mathfrak{X}_\delta \cap (G/P \times \mathring{\mathcal{X}}_\delta)$$

and

$$\tilde{\mathfrak{X}}_\delta := \{(y, g_1 \varrho^-, g_2 \varrho^-, \varrho) \in G/P \times \mathcal{X}_\delta : y \in g_1 \tilde{X}_P^{w_1} \cap g_2 \tilde{X}_P^{w_2} \cap \tilde{X}_v^P\}.$$

Then,

$$(53) \quad G \times^B \mathfrak{X}_\delta \simeq \mathfrak{X}, \quad [g, x] \mapsto gx.$$

In particular,  $\mathfrak{X}_\delta$  is irreducible since so is  $\mathfrak{X}$ . Hence,  $\tilde{\mathfrak{X}}_\delta$  and  $\mathring{\mathfrak{X}}_\delta$  (being open subsets of  $\mathfrak{X}_\delta$ ) are irreducible.

We now consider the following commutative diagram ( $\diamond$ ) for any  $G$ -equivariant line bundle  $\mathcal{L}$  over  $\mathcal{X}$  as in Theorem 1.5:

$$\begin{array}{ccccccc}
H^0(\mathcal{X}, \mathcal{L})^G & \xrightarrow{\eta^*} & H^0(\mathfrak{X}, \mathcal{L})^G & \xrightarrow{\alpha^*} & H^0(G \times^P \tilde{C}^+, \mathcal{L})^G & \xrightarrow{\beta^*} & H^0(C, \mathcal{L})^L \\
\downarrow i_1^* & & \downarrow i_2^* & & \downarrow \gamma^* & & \\
H^0(\mathcal{X}_\flat, \mathcal{L})^B & \xrightarrow{\eta_1^*} & H^0(\mathfrak{X}_\flat, \mathcal{L})^B & \xrightarrow{i_3^*} & H^0(\tilde{\mathfrak{X}}_\flat, \mathcal{L})^B & & \\
\downarrow i_4^* & & \downarrow i_5^* & & \downarrow i_6^* & & \\
H^0(\tilde{\mathcal{X}}_\flat, \mathcal{L})^B & \xrightarrow{\eta_2^*} & H^0(\tilde{\mathfrak{X}}_\flat, \mathcal{L})^B & \xrightarrow{i_7^*} & H^0(\tilde{\mathfrak{X}}_\flat \cap \tilde{\mathfrak{X}}_\flat^\infty, \mathcal{L})^B, & & 
\end{array}$$

where

$$\alpha : G \times^P \tilde{C}^+ \rightarrow \mathfrak{X}, \quad [g, (x_1, x_2, x_3)] \mapsto (gP, gx_1, gx_2, gx_3)$$

is a  $G$ -equivariant open embedding with image  $\tilde{\mathfrak{X}}'$ ,

$$\begin{aligned}
\beta : C &\hookrightarrow G \times^P \tilde{C}^+ \text{ is the } L\text{-equivariant morphism } x \mapsto [1, x], \\
\gamma : \tilde{\mathfrak{X}}_\flat &\longrightarrow G \times^P \tilde{C}^+, \quad (gP, g_1 \underline{\varrho}^-, g_2 \underline{\varrho}^-, \underline{\varrho}) \mapsto [g, (g^{-1}g_1 \underline{\varrho}^-, g^{-1}g_2 \underline{\varrho}^-, g^{-1}\underline{\varrho})],
\end{aligned}$$

is the morphism (which is  $\alpha|_{\tilde{\mathfrak{X}}_\flat}^{-1}$ ),  $\eta_1, \eta_2$  are restrictions of  $\eta$  to  $\mathfrak{X}_\flat$  and  $\tilde{\mathfrak{X}}_\flat^\infty$  respectively. All the maps  $i_j$  are appropriate inclusion maps. In the above diagram  $\mathcal{L}$  also denotes the induced line bundle on each of the above ind-varieties by pullback. Note that the ind-varieties with  $\flat$  as subscript are  $B$ -ind-varieties with the  $B$ -action induced from the  $G$ -action of the ambient  $G$ -ind-varieties; in particular, the line bundle  $\mathcal{L}$  over them is endowed with a natural  $B$ -action.

We now prove that all the maps in the above commutative diagram are isomorphisms.

**6.2. Various isomorphisms.** — We first prove the following lemma for its use in the proof of Lemma 6.2.

**Lemma 6.1.** — *Let  $U_P$  be the unipotent radical of  $P$ . Then,*

- (a) *Any regular map  $U_P \rightarrow \mathbb{C}^*$  is constant.*
- (b)  $\text{Pic}(U_P) = (0)$ .

*Proof.* — (a) Consider the parabolic subgroup  $P^-$  opposite to  $P$  and the homogeneous space  $G/P^-$ . Then  $U_P$  can be seen as an open subset of  $G/P^-$ . For any Schubert variety  $X_w^- = X_w^-(P) := \overline{B^- w P^- / P^-} \subset G/P^-$  (with  $w \in W^P$ ),  $X_w^- \cap U_P$  is contractible in the analytic topology (cf. [Kum02, Proposition 7.4.17 and its proof]). Now, by [KNR94, Lemma 2.5], we get that any regular map  $X_w^- \cap U_P \rightarrow \mathbb{C}^*$  is a constant. From this (a) follows.

(b) By induction on  $\ell(w)$ , we show that the group of  $k$ -cycles modulo rational equivalence  $A_k(X_w^- \cap U_P)$  is a finitely generated group. By [Ful98, Proposition 1.8], we have an exact sequence:

$$A_k((\partial X_w^-) \cap U_P) \rightarrow A_k(X_w^- \cap U_P) \rightarrow A_k((B^- w P^- / P^-) \cap U_P) \rightarrow 0.$$

Writing  $\partial X_w^-$  as a union  $\bigcup_{\ell(v)=\ell(w)-1} X_v^-$  and applying [Ful98, Example 1.3.1(c)] and the induction hypothesis, we get that  $A_k(\partial X_w^- \cap U_P)$  is finitely generated. Also, applying [Ful98, Proposition 1.8] again to the open subset  $(B^-wP^-/P^-) \cap U_P$  of the affine space  $B^-wP^-/P^-$ , we get that  $A_k((B^-wP^-/P^-) \cap U_P)$  is finitely generated since so is  $A_k(B^-wP^-/P^-)$  (cf. [Ful98, Proposition 1.9]). Thus, from the above exact sequence, we get that  $A_k(X_w^- \cap U_P)$  is finitely generated, completing the induction.

Consider the cohomology exact sequence (since  $X_w^- \cap U_P$  is contractible in the analytic topology)

$$\begin{aligned} H^1(X_w^- \cap U_P, \mathbb{Z}_m) = 0 \rightarrow H^1(X_w^- \cap U_P, \mathcal{O}^*) &= \text{Pic}(X_w^- \cap U_P) \\ &\rightarrow H^1(X_w^- \cap U_P, \mathcal{O}^*) = \text{Pic}(X_w^- \cap U_P) \rightarrow H^2(X_w^- \cap U_P, \mathbb{Z}_m) = 0, \end{aligned}$$

induced from the sheaf exact sequence:

$$\mathbb{Z}_m \rightarrow \mathcal{O}^* \rightarrow \mathcal{O}^* \rightarrow 0,$$

where the map  $\mathcal{O}^* \rightarrow \mathcal{O}^*$  takes  $f \mapsto f^m$ . From the above cohomology exact sequence we see that  $\text{Pic}(X_w^- \cap U_P)$  is a divisible group. But, since it is also a finitely generated abelian group (by [Ful98, Example 2.1.1]), it must be trivial. From this, taking limit, we obtain (b).  $\square$

Since  $\mathfrak{X}$  is irreducible and  $\text{Im } \alpha = \tilde{\mathfrak{X}}'$  is open in  $\mathfrak{X}$ , the restriction map

$$H^0(\mathfrak{X}, \mathcal{L}) \longrightarrow H^0(G \times^P \tilde{C}^+, \mathcal{L}) \text{ is injective and hence so is } \alpha^*.$$

**Lemma 6.2.** — (a) *The pullback induces an isomorphism:*

$$\eta^* : H^0(\mathcal{X}, \mathcal{L})^G \simeq H^0(\mathfrak{X}, \mathcal{L})^G.$$

(b) *The restriction map*

$$H^0(\tilde{C}^+, \mathcal{L})^P \longrightarrow H^0(C^+, \mathcal{L})^P$$

*is an isomorphism.*

(c) *The restriction map*

$$H^0(C^+, \mathcal{L})^P \rightarrow H^0(C, \mathcal{L})^L$$

*is an isomorphism.*

*Proof.* — (a) follows by [Res21, Lemma 11].

The proof of (b) is analogous to the proof of [Res21, Lemma 13]. We sketch the proof: the map  $H^0(\tilde{C}^+, \mathcal{L})^P \longrightarrow H^0(C^+, \mathcal{L})^P$  is obviously injective. Hence, it remains to prove that any  $P$ -invariant section  $\sigma$  of  $\mathcal{L}$  on  $C^+$  extends to  $\tilde{C}^+$ .

For  $x \in W^P$ ,  $Px^{-1}\underline{\varrho}^-$  is contained in  $\overline{Pw_i^{-1}\underline{\varrho}^-}$  if and only if  $x \geq w_i$ . Moreover,  $\{z \in W : z\underline{\varrho}^- \in Px^{-1}\underline{\varrho}^-\}$  is the set of  $z \in W$  that can be written as  $z = yx^{-1}$  for some  $y \in W_P$ . Since  $xy^{-1} \geq x$ , such a point  $z\underline{\varrho}^-$  belongs to  $\overline{Bw_i^{-1}\underline{\varrho}^-}$ . Then,  $\overline{Pw_i^{-1}\underline{\varrho}^-}$  and  $\overline{Bw_i^{-1}\underline{\varrho}^-}$  are  $B$ -stable and contain the same  $T$ -fixed points. We deduce that

$$(54) \quad \overline{Pw_i^{-1}\underline{\varrho}^-} = \overline{Bw_i^{-1}\underline{\varrho}^-}.$$

Yet,  $\overline{Pv^{-1}\underline{\varrho}} = \bigcup_{v_n \in W_P} X_{v_n v^{-1}}$ , where  $v_n$  is an increasing cofinal sequence in  $W_P$ . We now construct an increasing filtration of  $\bar{C}^+$  by products of finite-dimensional Richardson varieties:

$$\bar{C}^+ = \bigcup_{n \in \mathbb{N}} \bar{C}_n^+.$$

Explicitly

$$\bar{C}_n^+ := (X_{-}^{w_1} \cap X_{w_n}^-) \times (X_{-}^{w_2} \cap X_{w_n}^-) \times X_{v_n v^{-1}},$$

where  $\{w_n\}$  is a cofinal increasing sequence in  $W$  and  $\overline{Pw_i^{-1}\underline{\varrho}^-} = X_{-}^{w_i}$  by the Equation (54), where  $X_{-}^w := \overline{Bw^{-1}\underline{\varrho}^-}$  and  $X_w^- := \overline{B^-w\underline{\varrho}^-}$ . In particular,  $\bar{C}_n^+$  are irreducible and normal (cf. [Kum17, Proposition 6.6]). Of course,  $\bar{C}_n^+ \cap C^+$  is open in  $\bar{C}_n^+$  and nonempty for large enough  $n$ . It remains to prove that  $\sigma|_{\bar{C}_n^+ \cap C^+}$  extends to a regular section on  $\bar{C}_n^+ \cap \tilde{C}^+$ , for any  $n$ .

Fix  $(\alpha, i) \in \mathcal{D}$ . The irreducibility of the Richardson varieties implies that the intersection  $\bar{C}_n^+ \cap \overline{\partial C_{\alpha, i}^+}$  is either empty or irreducible. Since  $\bar{C}_n^+$  is normal, to prove that  $\sigma|_{\bar{C}_n^+ \cap C^+}$  extends to  $\bar{C}_n^+ \cap \tilde{C}^+$ , it is sufficient to prove that  $\sigma|_{\bar{C}_n^+ \cap C^+}$  has no pole along  $\bar{C}_n^+ \cap \overline{\partial C_{\alpha, i}^+}$  if  $\bar{C}_n^+ \cap \overline{\partial C_{\alpha, i}^+}$  has codimension 1 in  $\bar{C}_n^+$ .

Assume that  $D_n := \bar{C}_n^+ \cap \overline{\partial C_{\alpha, i}^+}$  has codimension 1 in  $\bar{C}_n^+$ . Then,  $D_n$  is equal to either

- ( $\alpha$ )  $(X_{-}^{\bar{u}_1} \cap X_{w_n}^-) \times (X_{-}^{w_2} \cap X_{w_n}^-) \times X_{v_n v^{-1}}$ , for some  $\bar{u}_1 \geq w_1 \in W^P$  and  $\ell(\bar{u}_1) = \ell(w_1) + 1$ ; or
- ( $\alpha'$ )  $(X_{-}^{w_1} \cap X_{w_n}^-) \times (X_{-}^{\bar{u}_2} \cap X_{w_n}^-) \times X_{v_n v^{-1}}$ , for some  $\bar{u}_2 \geq w_2 \in W^P$  and  $\ell(\bar{u}_2) = \ell(w_2) + 1$ ; or
- ( $\beta$ )  $(X_{-}^{w_1} \cap X_{w_n}^-) \times (X_{-}^{w_2} \cap X_{w_n}^-) \times X_{v_n v^{-1} s_{\alpha}}$ .

Now, we construct an explicit affine open subset  $\Omega_n$  in  $\bar{C}_n^+$  that intersects  $D_n$ .

In case ( $\alpha$ ), set

$$\Omega_n = (X_{-}^{w_1} \cap X_{w_n}^- \cap ((\bar{u}_1)^{-1} B \underline{\varrho}^-)) \times (X_{-}^{w_2} \cap \dot{X}_{w_n}^-) \times \dot{X}_{v_n v^{-1}},$$

where  $\dot{X}_w^- := B^-w\underline{\varrho}^-$  and  $\dot{X}_w := Bw\underline{\varrho}$  and similarly for the case ( $\alpha'$ ). In case ( $\beta$ ),

$$\Omega_n = (X_{-}^{w_1} \cap \dot{X}_{w_n}^-) \times (X_{-}^{w_2} \cap \dot{X}_{w_n}^-) \times (X_{v_n v^{-1}} \cap (v_n v^{-1} s_{\alpha} B^- \underline{\varrho})).$$

Fix  $\tau = z^{\sum_{\alpha_i \notin \Delta(P)} d_i x_i} : \mathbb{C}^* \longrightarrow T$ , where  $d_i > 0$  is an integer such that  $d_i x_i \in \mathfrak{h}_{\mathbb{Z}}$ . We now apply [Res21, Lemma 33] to  $\Omega_n$  endowed with the action of  $\mathbb{C}^*$  induced by  $\tau$ . The checking of the assumptions (i)–(iv) of [Res21, Lemma 33] are done in the proof of [Res21, Lemma 13]. The only remaining point, with the notation of [Res21, Lemma 33], is to prove that  $k \geq 0$ . This is done as in [Res21, Proof of Lemma 13, specifically the part ‘The line bundle on the affine subvarieties’]. Here, the non-negativity of  $k$  is due to the fact that  $\mathcal{L}$  is nonnegative restricted to the projective lines  $\ell_{\alpha, i}$  for any  $(\alpha, i) \in \mathcal{D}$ , which is our assumption (cf. Theorem 1.5). This proves (b).

We now come to the proof of (c). Since  $H^0(C^+, \mathcal{L})^P$  is contained in  $H^0(C^+, \mathcal{L})^\tau$ , [Res21, Lemma 14] implies that the map (c) of the lemma is injective. We now prove its surjectivity:

Consider the map  $\theta : P \rightarrow L$ ,  $p \mapsto \lim_{t \rightarrow 0} \tau(t)p\tau(t^{-1})$ , which is a surjective group homomorphism. This provides an action of  $P^3$  on  $C$  through the homomorphism  $\theta$ . Then, the regular map  $\gamma : C^+ \rightarrow C$ ,  $x \mapsto \lim_{t \rightarrow 0} \tau(t)x$  is  $P^3$ -equivariant.

Take the canonical  $G^3$ -equivariant structure on  $\mathcal{L}$  over  $\mathcal{X}$  under the componentwise action of  $G^3$  on  $\mathcal{X}$ . Thus, we will think of  $\mathcal{L}$  as a  $G^3$ -equivariant line bundle over  $\mathcal{X}$ . Denote

$$x = (w_1^{-1}\underline{e}^-, w_2^{-1}\underline{e}^-, v^{-1}\underline{e}) \in C.$$

Then,  $C = L^3 \cdot x$  and  $C^+ = P^3 \cdot x$ . Thus,

$$(55) \quad \text{Pic}^{P^3}(C^+) \simeq X(P_x^3) \text{ and } \text{Pic}^{L^3}(C) \simeq X(L_x^3),$$

where  $X(\cdot)$  denotes the character group and  $P_x^3$  (resp.  $L_x^3$ ) denotes the isotropy subgroup of  $P^3$  (resp.  $L^3$ ) at  $x$ . Now, it is easy to see that (by considering  $P \cap w_i^{-1}B^-w_i$  and  $P \cap v^{-1}Bv$ )

$$(56) \quad P_x^3 = L_x^3 \cdot (U_{w_1} \times U_{w_2} \times U'_v),$$

where  $U_w$  (resp.  $U'_v$ ) is the finite-dimensional (resp. finite-codimensional) subgroup of the *unipotent radical*  $U_P$  of  $P$  with Lie algebra  $\bigoplus_{\beta \in \Phi^+ \cap w^{-1}\Phi^-} \mathfrak{g}_\beta$  (resp.  $\bigoplus_{\beta \in (\Phi^+ \setminus \Phi_L^+) \cap v^{-1}\Phi^+} \mathfrak{g}_\beta$ ), where  $\Phi^+$  (resp.  $\Phi_L^+$ ) is the set of positive roots of  $G$  (resp.  $L$ ). Moreover, since  $L^3$  normalizes  $U_P^3$ ,  $L_x^3$  normalizes  $U_{w_1} \times U_{w_2} \times U'_v$ . Now, for a finite-dimensional unipotent group, any character is trivial and similarly  $U'_v$  has no nontrivial characters by the same proof as that of Lemma 6.1(a). Thus,

$$X(P_x^3) = X(L_x^3).$$

Hence, by combining the Equations (55) and (56), we get

$$(57) \quad \text{Pic}^{P^3}(C^+) \simeq \text{Pic}^{L^3}(C).$$

We define the  $P^3$ -action on  $\mathcal{L}|_C$  compatible with the action of  $P^3$  on  $C$  by demanding that  $U_P^3$  acts trivially on  $\mathcal{L}|_C$ . Thus, we get a  $P^3$ -equivariant line bundle  $\gamma^*(\mathcal{L}|_C)$  over  $C^+$ . We also have a  $P^3$ -equivariant line bundle  $\mathcal{L}|_{C^+}$ . By the Equation (57), we readily see that

$$\mathcal{L}|_{C^+} \simeq \gamma^*(\mathcal{L}|_C), \text{ as } P^3\text{-equivariant line bundles;}$$

in particular, as diagonal  $P$ -equivariant line bundles.

Thus, for  $\sigma \in H^0(C, \mathcal{L})^L$ ,  $\gamma^*(\sigma) \in H^0(C^+, \mathcal{L})^P$  and  $\gamma^*(\sigma)|_C = \sigma$ . We deduce thus that the restriction map  $H^0(C^+, \mathcal{L})^P \rightarrow H^0(C, \mathcal{L})^L$  is surjective. This proves (c).  $\square$

We thus conclude that the first horizontal line in the above diagram  $(\diamond)$  satisfies:

$$\begin{array}{ccccccc}
H^0(\mathcal{X}, \mathcal{L})^G & \xrightarrow[\eta^*]{\sim} & H^0(\mathfrak{X}, \mathcal{L})^G & \xrightarrow{\alpha^*} & H^0(G \times^P \tilde{C}^+, \mathcal{L})^G & \xrightarrow{\sim} & H^0(\tilde{C}^+, \mathcal{L})^P \\
& & & & \searrow \beta^* & & \downarrow \wr \\
& & & & & & H^0(C, \mathcal{L})^L,
\end{array}$$

where  $\eta^*$  is an isomorphism and the last vertical map is an isomorphism (which follows from Lemma 6.2).

**6.3. Isomorphisms induced from slice.** — Since  $G \times^B \mathcal{X}_\delta \simeq \mathcal{X}$  (cf. Equation (52)), we get that  $i_1^* : H^0(\mathcal{X}, \mathcal{L})^G \rightarrow H^0(\mathcal{X}_\delta, \mathcal{L})^B$  is an isomorphism. Similarly,  $i_2^*$  is an isomorphism by using Equation (53). Further,  $\gamma^*$  is an isomorphism since  $\alpha : G \times^P \tilde{C}^+ \rightarrow \tilde{\mathfrak{X}}'$  is a  $G$ -equivariant isomorphism and so is

$$(58) \quad G \times^B \tilde{\mathfrak{X}}_\delta \simeq \tilde{\mathfrak{X}}', \quad [g, x] \mapsto gx.$$

**6.4. Isomorphisms obtained from restriction to some open subsets.** —

**Lemma 6.3.** — *The restriction map  $H^0(\mathcal{X}_\delta, \mathcal{L}) \rightarrow H^0(\tilde{\mathcal{X}}_\delta, \mathcal{L})$  is an isomorphism and hence  $i_4^*$  is an isomorphism.*

*Proof.* — For any  $w \in W$ , consider the Schubert variety

$$X_w^- := \overline{B^- w B^- / B^-} \subset G / B^-.$$

For any  $w_1, w_2 \in W$ , consider the open embedding

$$i_{w_1, w_2} : \tilde{\mathcal{X}}_\delta \cap (X_{w_1}^- \times X_{w_2}^- \times \{\varrho\}) \hookrightarrow X_{w_1}^- \times X_{w_2}^- \times \{\varrho\}.$$

The complement

$$Y_{w_1, w_2} := (X_{w_1}^- \times X_{w_2}^- \times \{\varrho\}) \setminus \text{Im}(i_{w_1, w_2})$$

has its irreducible components of the form

$$(X_{w_1}^- \cap \overline{BuB^- / B^-}) \times X_{w_2}^- \times \{\varrho\} \text{ or } X_{w_1}^- \times (X_{w_2}^- \cap \overline{BuB^- / B^-}) \times \{\varrho\}$$

for some  $\ell(u) = 2$ . But, by [Kum02, Lemma 7.3.10], each of these irreducible components have codimension 2 in (the finite-dimensional)  $X_{w_1}^- \times X_{w_2}^- \times \{\varrho\}$ . Thus, by the normality of  $X_w^-$  (cf. [Kum02, Theorem 8.2.2(b)]), we see that the restriction map

$$H^0(X_{w_1}^- \times X_{w_2}^- \times \{\varrho\}, \mathcal{L}) \rightarrow H^0(\tilde{\mathcal{X}}_\delta \cap (X_{w_1}^- \times X_{w_2}^- \times \{\varrho\}), \mathcal{L})$$

is an isomorphism. Taking limits over  $w_1, w_2$ , we get the lemma.  $\square$

As observed earlier,  $\tilde{\mathfrak{X}}'$  is irreducible and hence so is  $\tilde{\mathfrak{X}}_\delta$  by the isomorphism (58) and  $\tilde{\mathfrak{X}}_\delta \cap \tilde{\mathfrak{X}}_\delta^{\circ\circ}$  is open in  $\tilde{\mathfrak{X}}_\delta$ . It follows thus that the map

$$i_6^* : H^0(\tilde{\mathfrak{X}}_\delta, \mathcal{L})^B \rightarrow H^0(\tilde{\mathfrak{X}}_\delta \cap \tilde{\mathfrak{X}}_\delta^{\circ\circ}, \mathcal{L})^B$$

is injective.

We now prove that the maps  $\eta_2^*$  and  $i_7^*$  are isomorphisms.

**Lemma 6.4.** — *The map  $H^0(\tilde{\mathcal{X}}_\delta, \mathcal{L}) \rightarrow H^0(\tilde{\mathfrak{X}}_\delta, \mathcal{L})$  induced from  $\eta_2$  is an isomorphism and hence so is  $\eta_2^*$ .*

*Proof.* — It is easy to see that the map  $\eta_2$  is proper. Moreover, it is birational by [Res21, Lemma 10]. In particular, it is surjective. If  $\tilde{\mathcal{X}}_\delta$  and  $\tilde{\mathfrak{X}}_\delta$  are finite-dimensional, the lemma follows from Zariski's main theorem (see, e.g., [Har77, Chap. III, Corollary 11.4]). The argument used to prove [Res21, Lemma 11] allows us to prove that the above map  $H^0(\tilde{\mathcal{X}}_\delta, \mathcal{L}) \rightarrow H^0(\tilde{\mathfrak{X}}_\delta, \mathcal{L})$  is an isomorphism. To prove this we first show that  $\tilde{\mathfrak{X}}_\delta$  is ind-irreducible. To prove this, since  $\tilde{\mathfrak{X}}_\delta$  is an open subset of  $\mathfrak{X}_\delta$ , by the isomorphism (53), it suffices to show that  $\mathfrak{X}$  is ind-irreducible. Further, since  $\mathfrak{X} = G \cdot (P/P, \bar{C}^+)$  (see above Lemma 5.3), it suffices to show that  $\overline{Pw_i^{-1}\underline{\varrho}^-}$  and  $\overline{Pv^{-1}\underline{\varrho}^-}$  are ind-irreducible. But, as observed earlier in the proof of Lemma 6.2 equality (54),  $\overline{Pw_i^{-1}\underline{\varrho}^-} = \overline{Bw_i^{-1}\underline{\varrho}^-}$ . So, it is ind-irreducible. Similarly,  $\overline{Pv^{-1}\underline{\varrho}^-} = \bigcup_{v_n \in W_P} X_{v_n v^{-1}}$ , where  $v_n$  is an increasing cofinal sequence in  $W_P$ . This shows that  $\overline{Pv^{-1}\underline{\varrho}^-}$  is also ind-irreducible. Thus,  $\mathfrak{X}$  is ind-irreducible. Take an open set  $\Omega'_\delta \subset \tilde{\mathfrak{X}}_\delta$  and open set  $\Omega_\delta \subset \tilde{\mathcal{X}}_\delta$  such that  $\eta_2 : \Omega'_\delta \rightarrow \Omega_\delta$  is an isomorphism. Let  $\xi : \Omega_\delta \rightarrow \Omega'_\delta$  be its inverse. Now, take an increasing cofinal sequence  $\{u_n\}_{n \geq 0}$  in  $W$  and take the filtration  $(Y_n)_{n \geq 0}$  of  $\tilde{\mathcal{X}}_\delta$  by

$$Y_n := \tilde{\mathcal{X}}_\delta \cap (X_{u_n}^- \times X_{u_n}^-),$$

where  $X_{u_n}^- := \overline{B^- u_n \underline{\varrho}^-} \subset G/B^-$ . Set  $Z_n := \overline{\xi(\Omega_\delta \cap Y_n)} \subset \tilde{\mathfrak{X}}_\delta$ . Then,  $Y_n$  is irreducible and normal (since so is  $X_{u_n}^-$ ) and  $Z_n$  is closed and irreducible in  $\tilde{\mathfrak{X}}_\delta$  (by definition). Moreover,  $\bigcup_n Z_n \supset \Omega'_\delta$ . Thus, by [Res21, Lemma 3],  $(Z_n)_{n \geq 0}$  provides an irreducible filtration for  $\tilde{\mathfrak{X}}_\delta$ . Apply now Zariski's main theorem to the morphism  $Z_n \rightarrow Y_n$  to conclude that  $H^0(Y_n, \mathcal{L}) \rightarrow H^0(Z_n, \mathcal{L})$  is an isomorphism for all  $n \geq 0$ . Taking the inverse limit, we get the lemma.  $\square$

**Lemma 6.5.** — *The restriction map  $H^0(\tilde{\mathfrak{X}}_\delta, \mathcal{L}) \rightarrow H^0(\tilde{\mathfrak{X}}_\delta \cap \tilde{\mathfrak{X}}_\delta, \mathcal{L})$  is an isomorphism and hence so is  $i_7^*$ .*

*Proof.* — As earlier, consider the action of  $U$  on  $X_v^P$ :

$$\theta : U \rightarrow \text{Aut}(X_v^P).$$

Then,  $\text{Im } \theta$  is a finite-dimensional unipotent group  $U_v$ . As a consequence,  $\text{Ker } \theta$  is a normal subgroup of  $U$  of finite-codimension.

Consider now the group

$$U_1 = \text{Ker } \theta \cap \left( \bigcap_{\alpha \in \Delta} s_\alpha U s_\alpha \right).$$

Then,  $U_1$  is again a normal subgroup of  $U$  of finite-codimension (i.e.,  $U/U_1$  is a finite-dimensional group). There exists a closed subgroup  $\mathcal{U}$  of  $U_1$  of finite-codimension such that  $\mathcal{U}$  is normal in  $U$ ,  $\mathcal{U}^2 := \mathcal{U} \times \mathcal{U}$  acts freely and properly on  $\tilde{\mathcal{X}}_\delta$  (under the action  $(u_1, u_2) \cdot (x_1, x_2, \underline{\varrho}) = (u_1 x_1, u_2 x_2, \underline{\varrho})$ ) and the quotient map  $\pi_{\mathcal{X}} : \tilde{\mathcal{X}}_\delta \rightarrow \mathcal{U}^2 \backslash \tilde{\mathcal{X}}_\delta$  is a principal  $\mathcal{U}^2$ -bundle (cf. [Kum17, Lemma 6.1]). Moreover, since  $\eta_2$  is proper (cf. Proof of Lemma 6.4),  $\mathcal{U}^2$  acts freely and properly on  $\tilde{\mathfrak{X}}_\delta$ .

Consider the action of  $\mathcal{U}^2$  on  $X_v^P \times \mathcal{X}_\delta$  given by

$$(59) \quad (u_1, u_2) \cdot (y, g_1 \underline{\varrho}^-, g_2 \underline{\varrho}^-, \underline{\varrho}) = (y, u_1 g_1 \underline{\varrho}^-, u_2 g_2 \underline{\varrho}^-, \underline{\varrho}).$$

Since  $\mathcal{U}$  acts trivially on  $X_v^P$  and  $y \in X_v^P$ , the condition  $y \in u_i g_i X_P^{w_i}$  is equivalent to  $y \in g_i X_P^{w_i}$ . In particular,  $\mathfrak{X}_\delta$ ,  $\mathfrak{X}_\delta^\circ$  and  $\tilde{\mathfrak{X}}_\delta$  are all stable by the action of  $\mathcal{U}^2$ . Moreover,  $\eta_2 : \mathfrak{X}_\delta^\circ \rightarrow \mathfrak{X}_\delta^\circ$  is  $\mathcal{U}^2$ -equivariant.

We consider the associated quotients:

$$\begin{array}{ccccc} \mathfrak{X}_\delta^\circ & \xleftarrow{\eta_2} & \mathfrak{X}_\delta^\circ & \xleftarrow{\quad} & \tilde{\mathfrak{X}}_\delta \cap \mathfrak{X}_\delta^\circ \\ \pi_{\mathcal{X}} \downarrow & & \pi_{\mathfrak{X}} \downarrow & & \downarrow \\ \mathcal{U}^2 \backslash \mathfrak{X}_\delta^\circ & \xleftarrow{\bar{\eta}_2} & \mathcal{U}^2 \backslash \mathfrak{X}_\delta^\circ & \xleftarrow{\quad} & \mathcal{U}^2 \backslash (\tilde{\mathfrak{X}}_\delta \cap \mathfrak{X}_\delta^\circ). \end{array}$$

Let  $\Omega_{\mathcal{X}}$  be an open subset of  $\mathcal{U}^2 \backslash \mathfrak{X}_\delta^\circ$  such that the quotient  $\pi_{\mathcal{X}}$  is trivial over  $\Omega_{\mathcal{X}}$ . (It can be seen that  $\pi_{\mathcal{X}}$  is locally trivial.) Set  $\Omega_{\mathfrak{X}} = \bar{\eta}_2^{-1}(\Omega_{\mathcal{X}})$ . Choosing a section of  $\pi_{\mathfrak{X}}$  over  $\Omega_{\mathcal{X}}$  and taking the induced section of  $\pi_{\mathfrak{X}}$  over  $\Omega_{\mathfrak{X}}$ , we get

$$(60) \quad \pi_{\mathcal{X}}^{-1}(\Omega_{\mathcal{X}}) \simeq \mathcal{U}^2 \times \Omega_{\mathcal{X}} \text{ and } \pi_{\mathfrak{X}}^{-1}(\Omega_{\mathfrak{X}}) \simeq \mathcal{U}^2 \times \Omega_{\mathfrak{X}}$$

such that  $\eta_2|_{\pi_{\mathfrak{X}}^{-1}(\Omega_{\mathfrak{X}})}$  under the above isomorphism is given by

$$\eta_2(\tilde{u}, x) = (\tilde{u}, \bar{\eta}_2(x)), \text{ for } \tilde{u} \in \mathcal{U}^2 \text{ and } x \in \Omega_{\mathfrak{X}}.$$

Since  $\mathcal{L}$  is  $G^3$ -equivariant with  $G^3$  acting on  $\mathcal{X}$  componentwise, we get that  $\mathcal{L}|_{\mathfrak{X}_\delta^\circ}$  and  $\mathcal{L}|_{\tilde{\mathfrak{X}}_\delta}$  are  $\mathcal{U}^2$ -equivariant. Since  $\mathcal{U}^2$  acts freely on  $\mathfrak{X}_\delta^\circ$  (resp.  $\tilde{\mathfrak{X}}_\delta$ ),  $\mathcal{L}|_{\mathfrak{X}_\delta^\circ}$  (resp.  $\mathcal{L}|_{\tilde{\mathfrak{X}}_\delta}$ ) descends to a unique line bundle  $\bar{\mathcal{L}}$  over  $\mathcal{U}^2 \backslash \mathfrak{X}_\delta^\circ$  (resp.  $\mathcal{U}^2 \backslash \tilde{\mathfrak{X}}_\delta$ ). Hence, under the decompositions (60),

$$(61) \quad \mathcal{L}|_{\mathcal{U}^2 \times \Omega_{\mathcal{X}}} = \mathcal{O}_{\mathcal{U}^2} \boxtimes \bar{\mathcal{L}}|_{\Omega_{\mathcal{X}}}, \text{ and } \mathcal{L}|_{\mathcal{U}^2 \times \Omega_{\mathfrak{X}}} = \mathcal{O}_{\mathcal{U}^2} \boxtimes \bar{\mathcal{L}}|_{\Omega_{\mathfrak{X}}}.$$

Now, the map

$$\bar{\eta}_2 : \mathcal{U}^2 \backslash \tilde{\mathfrak{X}}_\delta \rightarrow \mathcal{U}^2 \backslash \mathfrak{X}_\delta^\circ$$

is proper. To prove this, consider the projection

$$\pi_2 : \mathcal{U}^2 \backslash (X_v^P \times \mathfrak{X}_\delta^\circ) = X_v^P \times (\mathcal{U}^2 \backslash \mathfrak{X}_\delta^\circ) \rightarrow \mathcal{U}^2 \backslash \mathfrak{X}_\delta^\circ$$

with  $\mathcal{U}^2$  acting on  $X_v^P \times \mathfrak{X}_\delta^\circ$  as in (59). This is clearly a projective morphism. Now,

$$\bar{\eta}_2 = (\pi_2)|_{\mathcal{U}^2 \backslash \tilde{\mathfrak{X}}_\delta}.$$

Moreover,  $\mathcal{U}^2 \backslash \tilde{\mathfrak{X}}_\delta$  is a closed subset of  $\mathcal{U}^2 \backslash (X_v^P \times \mathfrak{X}_\delta^\circ)$  (as can easily be seen) and hence  $\bar{\eta}_2$  is a projective morphism.

Further,  $\bar{\eta}_2$  is a birational map since so is  $\eta_2$  (cf. Proof of Lemma 6.4).

By the following lemma,  $\bar{\eta}_2(\mathcal{U}^2 \backslash (\tilde{\mathfrak{X}}_\delta \setminus \mathfrak{X}_\delta^\circ))$  is of codimension  $\geq 2$  in  $\mathcal{U}^2 \backslash \mathfrak{X}_\delta^\circ$ . Moreover,  $\mathcal{U}^2 \backslash \mathfrak{X}_\delta^\circ$  is normal (cf. [KS09, Proposition 3.2]). In fact, it is smooth (cf. [Kum17, §10]). Thus, by Proposition 2.1, the restriction map

$$(62) \quad H^0(\Omega_{\mathfrak{X}}, \bar{\mathcal{L}}) \rightarrow H^0(\Omega'_{\mathfrak{X}}, \bar{\mathcal{L}}) \text{ is an isomorphism,}$$

for any open subset  $\Omega_{\mathcal{X}} \subset \mathcal{U}^2 \setminus \check{\mathcal{X}}_{\delta}^{\circ\circ}$  over which  $\pi_{\mathcal{X}}$  admits a section and  $\Omega_{\mathfrak{X}} := \bar{\eta}_2^{-1}(\Omega_{\mathcal{X}})$ , where  $\Omega'_{\mathfrak{X}} := \Omega_{\mathfrak{X}} \cap (\mathcal{U}^2 \setminus (\check{\mathfrak{X}}_{\delta}^{\circ\circ} \cap \check{\mathfrak{X}}_{\delta}))$ . But, by the decomposition (61)

$$(63) \quad \begin{aligned} H^0(\pi_{\mathfrak{X}}^{-1}(\Omega_{\mathfrak{X}}), \mathcal{L}) &\simeq H^0(\mathcal{U}^2 \times \Omega_{\mathfrak{X}}, \mathcal{O}_{\mathcal{U}^2} \boxtimes \bar{\mathcal{L}}) \\ &= \varprojlim_n \mathbb{C}[\mathcal{U}_n^2] \otimes H^0(\Omega_{\mathfrak{X}}, \bar{\mathcal{L}}), \end{aligned}$$

where  $\{\mathcal{U}_n\}_{n \geq 0}$  is a filtration of  $\mathcal{U}$  giving the ind-variety structure. Similarly,

$$(64) \quad H^0(\pi_{\mathfrak{X}}^{-1}(\Omega'_{\mathfrak{X}}), \mathcal{L}) = \varprojlim_n \mathbb{C}[\mathcal{U}_n^2] \otimes H^0(\Omega'_{\mathfrak{X}}, \bar{\mathcal{L}}).$$

Combining the Equations (62) - (64), we get that the restriction map

$$H^0(\pi_{\mathfrak{X}}^{-1}(\Omega_{\mathfrak{X}}), \mathcal{L}) \rightarrow H^0(\pi_{\mathfrak{X}}^{-1}(\Omega'_{\mathfrak{X}}), \mathcal{L})$$

is an isomorphism (use [Har77, Chap. II, Proposition 9.1]). Since  $\{\pi_{\mathfrak{X}}^{-1}(\Omega_{\mathfrak{X}})\}$  provides an open cover of  $\check{\mathfrak{X}}_{\delta}^{\circ\circ}$ , we get that the restriction map

$$H^0(\check{\mathfrak{X}}_{\delta}^{\circ\circ}, \mathcal{L}) \rightarrow H^0(\check{\mathfrak{X}}_{\delta} \cap \check{\mathfrak{X}}_{\delta}^{\circ\circ}, \mathcal{L})$$

is an isomorphism. This proves the lemma modulo Lemma 6.6 below.  $\square$

**6.5. Smallness of the boundary of  $\check{\mathfrak{X}}_{\delta}$ .** — The goal of this subsection is to prove the following lemma. We refer to [BKR12, §7] for some parallel arguments.

**Lemma 6.6.** — *With the notation as in the proof of Lemma 6.5, the image  $\bar{\eta}_2(\mathcal{U}^2 \setminus (\check{\mathfrak{X}}_{\delta}^{\circ\circ} \setminus \check{\mathfrak{X}}_{\delta}))$  is of codimension  $\geq 2$  in  $\mathcal{U}^2 \setminus \check{\mathcal{X}}_{\delta}^{\circ\circ}$ .*

This lemma will be a consequence of the nontransversality Corollary 6.8, which in turn is a consequence of Proposition 6.7.

Set, for  $i \in \mathbb{N} := \{0, 1, 2, \dots\}$ ,

$$(65) \quad (\mathfrak{g}/\mathfrak{p})_i := \{\xi \in \mathfrak{g}/\mathfrak{p} : \text{ad}(x_P) \cdot \xi = -i\xi\}, \text{ where } x_P := \sum_{\alpha_j \in \Delta \setminus \Delta(P)} x_j$$

and

$$(66) \quad (\mathfrak{g}/\mathfrak{p})_{\leq i} := \bigoplus_{j \leq i} (\mathfrak{g}/\mathfrak{p})_j.$$

Note that the  $(\mathfrak{g}/\mathfrak{p})_{\leq i}$ 's form a  $P$ -stable filtration of  $\mathfrak{g}/\mathfrak{p}$ .

Let  $Z \subset G/P$  be a locally closed finite-dimensional subvariety of  $G/P$  and let  $z$  be a point of  $Z$ . Write  $z = gP/P$ . Set, for  $i \in \mathbb{N}$ ,

$$(67) \quad d_i(z, Z) := \dim(T_{\dot{e}}(g^{-1}Z) \cap (\mathfrak{g}/\mathfrak{p})_{\leq i}), \text{ where } \dot{g} := gP/P \in G/P.$$

This indeed does not depend on the choice of  $g$  such that  $z = gP/P$ . Observe that  $d_0(z, Z) = 0$ ,  $d_n(z, Z) = \dim T_z Z$  for  $n$  large enough, and that  $i \mapsto d_i(z, Z)$  is non-decreasing. Define, for any  $i \in \mathbb{N}$ ,

$$\bar{d}_i(z, Z) = d_i(z, Z) - d_{i-1}(z, Z),$$

where we declare  $d_{-1}(z, Z) = 0$ . Thus,  $\bar{d}_m(z, Z) = 0$ , for  $m > n$ .

Similarly, let  $z \in Z \subset G/P$ , where  $Z$  has finite-codimension. Write  $z = gP/P$ . Set, for  $i \in \mathbb{N}$ ,

$$(68) \quad d^i(z, Z) := \dim \left( \frac{T_{\dot{e}}(g^{-1}Z) + (\mathfrak{g}/\mathfrak{p})_{\leq i}}{T_{\dot{e}}(g^{-1}Z)} \right).$$

Again this does not depend on the choice of  $g$  such that  $z = gP/P$ . Observe that  $d^0(z, Z) = 0$ , that  $d^n(z, Z)$  is the codimension of  $T_z Z$  for  $n$  large enough, and that  $i \mapsto d^i(z, Z)$  is non-decreasing.

**Proposition 6.7.** — *Let  $v \in W^P$  and  $\beta$  be a positive real root such that  $w = s_\beta v \in W^P$ .*

(i) *If  $\ell(w) = \ell(v) - 1$ , then*

$$d_i(\dot{w}, X_v^P) \geq d_i(\dot{v}, X_v^P), \quad \forall i \in \mathbb{N}.$$

*Moreover, if  $\beta$  is **not** a simple root,*

$$d_{i_o}(\dot{w}, X_v^P) > d_{i_o}(\dot{v}, X_v^P), \quad \text{for some } i_o \in \mathbb{N}.$$

(ii) *If  $\ell(w) = \ell(v) + 1$ , then*

$$d^i(\dot{w}, X_v^P) \leq d^i(\dot{v}, X_v^P), \quad \forall i \in \mathbb{N}.$$

*Moreover, if  $\beta$  is **not** a simple root,*

$$d^{i_o}(\dot{w}, X_v^P) < d^{i_o}(\dot{v}, X_v^P), \quad \text{for some } i_o \in \mathbb{N}.$$

*Proof.* — We first translate the first assertion in a combinatorial statement in terms of roots. Given a  $T$ -vector space  $E$ , we denote by  $\Phi(E)$  the set of weights of  $T$  acting on  $E$ .

Let  $\Phi^+$  (resp.  $\Phi^-$ ) be the set of positive (resp. negative) roots. Since  $T_{\dot{e}}(v^{-1}X_v^P)$  is multiplicity free as a  $T$ -module, and  $\Phi(T_{\dot{e}}(v^{-1}X_v^P)) = \{\theta \in \Phi^- : v\theta \in \Phi^+\}$ , we have

$$(69) \quad d_i(\dot{v}, X_v^P) = \#\{\theta \in \Phi^- : v\theta \in \Phi^+ \text{ and } -\theta(x_P) \leq i\}, \quad \forall i \geq 1.$$

Consider the unique  $T$ -stable curve  $\ell$  containing both  $\dot{v}$  and  $\dot{w}$ . Observe that  $\ell$  is isomorphic to  $\mathbb{P}^1$ ,  $\Phi(T_{\dot{v}}\ell) = \{\beta\}$ ,  $\Phi(T_{\dot{w}}\ell) = \{-\beta\}$  and  $\ell$  is contained in  $X_v^P$ . Moreover,  $X_w^P$  is contained in  $X_v^P$  and

$$(70) \quad T_{\dot{w}}X_w^P = T_{\dot{w}}X_v^P \oplus T_{\dot{w}}\ell.$$

After translating by  $w^{-1}$ , equality (70) implies that

$$\Phi(T_{\dot{e}}(w^{-1}X_v^P)) = \Phi(T_{\dot{e}}(w^{-1}X_w^P)) \cup \{-w^{-1}\beta\}.$$

It follows that

$$(71) \quad d_i(\dot{w}, X_v^P) = \#\{\theta \in \Phi^- : w\theta \in \Phi^+ \text{ and } -\theta(x_P) \leq i\} + \delta_i^{(w^{-1}\beta)(x_P)}, \quad \forall i \geq 1,$$

where  $\delta_i^m = 1$  if  $m \leq i$  and 0 otherwise.

We deduce that the first assertion of the proposition is equivalent to  $\forall i \geq 1$ :

$$(72) \quad \begin{aligned} & \#\{\theta \in \Phi^- : w\theta \in \Phi^+ \text{ and } -\theta(x_P) \leq i\} + \delta_i^{(w^{-1}\beta)(x_P)} \\ & \geq \#\{\theta \in \Phi^- : v\theta \in \Phi^+ \text{ and } -\theta(x_P) \leq i\}, \end{aligned}$$

and the existence of  $i_o$  with a strict inequality (72) if  $\beta$  is not simple.

We now translate the second assertion of the proposition in a combinatorial statement. First observe that, since  $v \in W^P$ ,

$$\Phi\left(\frac{T_{\dot{e}}(G/P)}{T_{\dot{e}}(v^{-1}X_P^v)}\right) = \{\theta \in \Phi^- : v\theta \in \Phi^+\}.$$

We deduce that

$$(73) \quad d^i(\dot{v}, X_P^v) = \#\{\theta \in \Phi^- : v\theta \in \Phi^+ \text{ and } -\theta(x_P) \leq i\}, \quad \forall i \geq 1.$$

Now, since  $\ell(w) = \ell(v) + 1$ ,  $X_P^v \supset X_P^w$ ,  $\ell$  is contained in  $X_P^v$ ,  $\Phi(T_{\dot{w}}\ell) = \{\beta\}$  and  $\Phi(T_{\dot{v}}\ell) = \{-\beta\}$ . Moreover, we have the following exact sequence

$$0 \longrightarrow T_{\dot{w}}\ell \longrightarrow \frac{T_{\dot{w}}(G/P)}{T_{\dot{w}}X_P^w} \longrightarrow \frac{T_{\dot{w}}(G/P)}{T_{\dot{w}}X_P^v} \longrightarrow 0.$$

After translation by  $w^{-1}$ , we obtain that

$$\Phi\left(\frac{T_{\dot{e}}(G/P)}{T_{\dot{e}}(w^{-1}X_P^v)}\right) = \Phi\left(\frac{T_{\dot{e}}(G/P)}{T_{\dot{e}}(w^{-1}X_P^w)}\right) \sqcup \{w^{-1}\beta\}.$$

This implies that

$$(74) \quad d^i(\dot{w}, X_P^v) = \#\{\theta \in \Phi^- : w\theta \in \Phi^+ \text{ and } -\theta(x_P) \leq i\} - \delta_i^{-(w^{-1}\beta)(x_P)}, \quad \forall i \geq 1.$$

With (73) and (74), the second assertion of the proposition is equivalent to  $\forall i \geq 1$ :

$$(75) \quad \begin{aligned} & \#\{\theta \in \Phi^- : v\theta \in \Phi^+ \text{ and } -\theta(x_P) \leq i\} \\ & \geq \#\{\theta \in \Phi^- : w\theta \in \Phi^+ \text{ and } -\theta(x_P) \leq i\} - \delta_i^{-(w^{-1}\beta)(x_P)}, \end{aligned}$$

with a strict inequality for some  $i_o$ , if  $\beta$  is not simple.

Now, observe that given  $(v, w)$  such that  $w = s_\beta v$  and  $\ell(w) = \ell(v) + 1$ , one gets  $(v', w')$  such that  $w' = s_\beta v'$  and  $\ell(w') = \ell(v') - 1$  by setting  $w' = v$  and  $v' = w$ . By (72) and (75), the first assertion for  $(v', w')$  implies the second one for  $(v, w)$  (note that  $w'^{-1}\beta = -w^{-1}\beta$ ). It is now sufficient to prove the first assertion.

From now on, we assume that  $\ell(w) = \ell(v) - 1$ . Recall that we denote  $v' \rightarrow v$  if  $v' \in W^P$ ,  $\ell(v') = \ell(v) - 1$  and  $v' \leq v$ . Set

$$\hat{X}_v^P = \hat{X}_v^P \cup \left( \bigcup_{v' \rightarrow v} \hat{X}_{v'}^P \right), \quad \text{where } \hat{X}_v^P := BvP/P.$$

Then,  $\hat{X}_v^P$  is a smooth open subset of  $X_v^P$ . Set

$$\hat{Y}_v^P = \pi^{-1}(\hat{X}_v^P),$$

where  $\pi : G \rightarrow G/P$  is the natural projection. Define two vector bundles over  $\hat{Y}_v^P$ :

$$\mathcal{V} := \bigcup_{g \in \hat{Y}_v^P} (\{g\} \times T_{\dot{e}}(g^{-1}X_v^P)) \longrightarrow \hat{Y}_v^P,$$

and the trivial bundle

$$\varepsilon_i := \hat{Y}_v^P \times \frac{T_{\dot{e}}(G/P)}{(\mathfrak{g}/\mathfrak{p})_{\leq i}},$$

for any fixed  $i \in \mathbb{N}$ . The inclusion  $T_{\dot{e}}(g^{-1}X_v^P) \subset T_{\dot{e}}(G/P)$  induces a bundle map

$$\varphi_i : \mathcal{V} \longrightarrow \varepsilon_i.$$

On the open subset  $BvP$ , the rank of  $\varphi_i$  is constant since

$$T_{\dot{e}}((bvp)^{-1}X_v^P) = T_{\dot{e}}(p^{-1}v^{-1}b^{-1}X_v^P) = p^{-1}T_{\dot{e}}(v^{-1}X_v^P),$$

and  $(\mathfrak{g}/\mathfrak{p})_{\leq i}$  is  $P$ -stable.

On the other hand, the subset of points in  $\hat{Y}_v^P$ , where the rank of  $\varphi_i$  is maximum is open. Hence, this rank is maximum at any point of  $BvP \subset \hat{Y}_v^P$ ; in particular, at  $v$ . This shows the inequalities of the first assertion of the proposition.

Note that

$$\rho - v^{-1}\rho = - \sum_{\theta \in \Phi^- \cap v^{-1}\Phi^+} \theta.$$

Hence,

$$(\rho - v^{-1}\rho)(x_P) = \sum_{\theta \in \Phi^- \cap v^{-1}\Phi^+} -\theta(x_P).$$

But by the Equation (69),

$$\#\{\theta \in \Phi^- : v\theta \in \Phi^+ \text{ and } -\theta(x_P) = i\} = d_i(\dot{v}, X_v^P) - d_{i-1}(\dot{v}, X_v^P), \quad \forall i \geq 1.$$

Hence,

$$\begin{aligned} (\rho - v^{-1}\rho)(x_P) &= \sum_{j \geq 1} j \bar{d}_j(\dot{v}, X_v^P) \\ &= \ell(v) + \sum_{j \geq 2} (j-1) \bar{d}_j(\dot{v}, X_v^P), \end{aligned}$$

since, by the Equation (69),  $d_m = \ell(v)$  for large enough  $m$ . Similarly,

$$(\rho - w^{-1}\rho)(x_P) = \ell(w) + \sum_{j \geq 2} (j-1) \bar{d}_j(\dot{w}, X_w^P).$$

Since  $\ell(w) = \ell(v) - 1$ , we get

$$(76) \quad (\rho - w^{-1}\rho - (\rho - v^{-1}\rho))(x_P) = -1 + \sum_{j \geq 2} (j-1) (\bar{d}_j(\dot{w}, X_w^P) - \bar{d}_j(\dot{v}, X_v^P)).$$

On the other hand, since  $w = s_\beta v$ , we get

$$\begin{aligned} (77) \quad \rho - w^{-1}\rho - (\rho - v^{-1}\rho) &= -w^{-1}\rho + w^{-1}s_\beta\rho \\ &= w^{-1}(s_\beta\rho - \rho) \\ &= -\langle \rho, \beta^\vee \rangle w^{-1}\beta. \end{aligned}$$

Combining the Equations (76) and (77), we get

$$1 + \sum_{j \geq 2} (j-1) (\bar{d}_j(\dot{v}, X_v^P) - \bar{d}_j(\dot{w}, X_w^P)) = \langle \rho, \beta^\vee \rangle (w^{-1}\beta)(x_P).$$

But by the Equation (69) (for  $v$  replaced by  $w$ ) and the Equation (71), we have

$$d_i(\dot{w}, X_w^P) = d_i(\dot{w}, X_w^P) + \delta_i^{(w^{-1}\beta)(x_P)}, \quad \forall i \geq 1$$

and hence (for  $k := (w^{-1}\beta)(x_P)$ )

$$(78) \quad 1 + \sum_{j \geq 2, j \neq k} (j-1) (\bar{d}_j(\dot{v}, X_v^P) - \bar{d}_j(\dot{w}, X_v^P)) + (k-1) (\bar{d}_k(\dot{v}, X_v^P) - \bar{d}_k(\dot{w}, X_v^P) + 1) = \langle \rho, \beta^\vee \rangle (w^{-1}\beta)(x_P).$$

If possible, assume that

$$(79) \quad d_j(\dot{v}, X_v^P) = d_j(\dot{w}, X_v^P), \quad \forall j \geq 1.$$

Equivalently,

$$\bar{d}_j(\dot{v}, X_v^P) = \bar{d}_j(\dot{w}, X_v^P), \quad \forall j \geq 1.$$

Then, the Equation (78) implies that

$$k = \langle \rho, \beta^\vee \rangle k.$$

But,  $\langle \rho, \beta^\vee \rangle \geq 1$ . Hence,

$$\langle \rho, \beta^\vee \rangle = 1.$$

Observe that  $k \neq 0$  since  $v = ws_{w^{-1}\beta}$  and  $v, w \in W^P$ . We deduce that  $\beta$  is simple if (79) holds. This ends the proof of the proposition.  $\square$

**Corollary 6.8.** — *Let  $w_1, w_2, v \in W^P$  be as in Theorem 1.5. In particular,  $\ell(v) = \ell(w_1) + \ell(w_2)$ . Let  $x \in G/P$  and  $g, g_1, g_2$  in  $G$  be such that  $x$  belongs to  $g\hat{X}_v^P \cap g_1\hat{X}_P^{w_1} \cap g_2\hat{X}_P^{w_2}$ , where  $\hat{X}_v^P$  is as defined in the proof of Proposition 6.7 and*

$$\hat{X}_P^w := \hat{X}_P^w \cup \left( \bigcup_{w \rightarrow w'} \hat{X}_P^{w'} \right), \quad \text{where } \hat{X}_P^{w'} := B^- w' P / P.$$

*We assume that there exists a non-simple real root  $\beta$  such that one of the following two conditions holds:*

- (i)  $\ell(s_\beta v) = \ell(v) - 1$ ,  $s_\beta v \in W^P$  and  $x \in g\hat{X}_{s_\beta v}^P$ .
- (ii)  $\ell(s_\beta w_1) = \ell(w_1) + 1$ ,  $s_\beta v \in W^P$  and  $x \in g_1\hat{X}_P^{s_\beta w_1}$ .

*Then, the intersection  $g\hat{X}_v^P \cap g_1\hat{X}_P^{w_1} \cap g_2\hat{X}_P^{w_2}$  is not transverse at  $x$ .*

*Proof.* — It suffices to prove that the standard linear map

$$\theta : T_x(gX_v^P) \longrightarrow \frac{T_x(G/P)}{T_x(g_1X_P^{w_1})} \oplus \frac{T_x(G/P)}{T_x(g_2X_P^{w_2})}$$

is not an isomorphism. Write  $x = hP/P$ . Up to changing  $(g, g_1, g_2)$  by  $(h^{-1}g, h^{-1}g_1, h^{-1}g_2)$ , we may assume that  $h = e$ .

Observe that

$$(*) \quad \theta(T_e(gX_v^P) \cap (\mathfrak{g}/\mathfrak{p})_{\leq i}) \subset \frac{(T_e(g_1X_P^{w_1})) + (\mathfrak{g}/\mathfrak{p})_{\leq i}}{T_e(g_1X_P^{w_1})} \oplus \frac{(T_e(g_2X_P^{w_2})) + (\mathfrak{g}/\mathfrak{p})_{\leq i}}{T_e(g_2X_P^{w_2})}.$$

Moreover, since  $\varepsilon_P^v$  occurs with coefficient 1 (in particular, nonzero) in the deformed product  $\varepsilon_P^{w_1} \odot_0 \varepsilon_P^{w_2}$  by assumption,  $\forall i \in \mathbb{N}$ ,

$$\begin{aligned} \dim (T_{\dot{e}}(v^{-1}X_v^P) \cap (\mathfrak{g}/\mathfrak{p})_{\leq i}) &= \dim \left( \frac{T_{\dot{e}}(w_1^{-1}X_P^{w_1}) + (\mathfrak{g}/\mathfrak{p})_{\leq i}}{T_{\dot{e}}(w_1^{-1}X_P^{w_1})} \right) \\ &\quad + \dim \left( \frac{T_{\dot{e}}(w_2^{-1}X_P^{w_2}) + (\mathfrak{g}/\mathfrak{p})_{\leq i}}{T_{\dot{e}}(w_2^{-1}X_P^{w_2})} \right) \end{aligned}$$

(cf. [Res21, Lemma 19]). But, Proposition 6.7 implies that, for some  $i_o$ , the dimension of the first space  $T_{\dot{e}}(gX_v^P) \cap (\mathfrak{g}/\mathfrak{p})_{i_o}$  in  $(*)$  is greater than that of the direct sum. Hence, the restriction of  $\theta$  to  $T_{\dot{e}}(gX_v^P) \cap (\mathfrak{g}/\mathfrak{p})_{\leq i}$  can not be injective. Thus,  $\theta$  can not be an isomorphism.  $\square$

**Lemma 6.9.** — *Let  $f : Y \rightarrow X$  be a dominant morphism between two quasi-projective irreducible varieties of the same dimension. Let  $D \subset Y$  be an irreducible proper closed subset.*

*Then, if  $\overline{f(D)}$  has codimension one in  $X$ , then, for  $x \in D$  general,  $f^{-1}(f(x))$  is finite.*

*Proof.* — Otherwise, the general fibers of the restriction of  $f$  to  $f^{-1}(\overline{f(D)})$  would have positive dimension. Since  $\overline{f(D)}$  has codimension one, this implies that  $\dim(f^{-1}(\overline{f(D)})) = \dim(Y)$  and hence  $f^{-1}(\overline{f(D)}) = Y$ . But,  $f$  is assumed to be dominant. A contradiction.  $\square$

*Proof of Lemma 6.6.* — For  $(w'_1, w'_2, v') \in (W^P)^3$ , we set

$$\overset{\circ}{\mathfrak{X}}_{\delta}(w'_1, w'_2, v') := \{(x, g_1 \underline{q}^-, g_2 \underline{q}_-, \underline{q}) \in X_{v'}^P \times \overset{\circ}{\mathcal{X}}_{\delta} : x \in g_1 X_P^{w'_1} \cap g_2 X_P^{w'_2}\}$$

and

$$\mathfrak{X}_{\delta}(w'_1, w'_2, v') := \{(x, g_1 \underline{q}^-, g_2 \underline{q}_-, \underline{q}) \in X_{v'}^P \times \mathcal{X}_{\delta} : x \in g_1 X_P^{w'_1} \cap g_2 X_P^{w'_2}\}.$$

The set  $\overset{\circ}{\mathfrak{X}}_{\delta} \setminus \mathfrak{X}_{\delta}$  is the union of finitely many subsets of one of the following types:

*Type I.* —  $\overset{\circ}{\mathfrak{X}}_{\delta}(w'_1, w'_2, v')$ , where  $(w'_1, w'_2, v') \in (W^P)^3$ ,  $w'_1 \geq w_1$ ,  $w'_2 \geq w_2$ ,  $v' \leq v$  and  $\ell(w'_1) + \ell(w'_2) - \ell(v') \geq 2$ .

*Type II.* —  $\overset{\circ}{\mathfrak{X}}_{\delta}(w_1, w_2, v')$ , where  $v' \in W^P$ ,  $v' \leq v$ ,  $\ell(v') = \ell(v) - 1$  and  $v'v^{-1}$  is not a simple reflection.

*Type III.* —  $\overset{\circ}{\mathfrak{X}}_{\delta}(w'_1, w_2, v)$ , where  $w'_1 \in W^P$ ,  $w'_1 \geq w_1$ ,  $\ell(w'_1) = \ell(w_1) + 1$  and  $w'_1 w_1^{-1}$  is not a simple reflection.

*Type IV.* — Like type III after exchanging  $w_1$  and  $w_2$ .

It is sufficient to prove that the image by  $\bar{\eta}_2$  of each one of these subsets has codimension at least two in  $\mathcal{U}^2 \setminus \overset{\circ}{\mathcal{X}}_{\delta}$ .

Consider  $(w'_1, w'_2, v')$  as in type I. There exists  $(w''_1, w''_2, v'')$  such that  $w'_1 \geq w''_1 \geq w_1$ ,  $w'_2 \geq w''_2 \geq w_2$ ,  $v' \leq v'' \leq v$  and  $\ell(w''_1) + \ell(w''_2) - \ell(v'') = 1$ .

The point  $(v'', v''(w''_1)^{-1} \underline{q}^-, v''(w''_2)^{-1} \underline{q}_-)$  belongs to  $\mathfrak{X}_{\delta}(w''_1, w''_2, v'')$  and does not belong to  $\mathfrak{X}_{\delta}(w'_1, w'_2, v')$ . Hence,  $\mathfrak{X}_{\delta}(w''_1, w''_2, v'') \setminus \mathfrak{X}_{\delta}(w'_1, w'_2, v')$  is open and nonempty in  $\mathfrak{X}_{\delta}(w''_1, w''_2, v'')$ .

To prove the lemma in this type, we can assume that  $\tilde{\mathfrak{X}}_\delta(w'_1, w'_2, v')$  is nonempty, then so is  $\tilde{\mathfrak{X}}_\delta(w''_1, w''_2, v'')$ . Since  $\mathfrak{X}_\delta(w''_1, w''_2, v'')$  is irreducible (cf. §6.1), we deduce that the intersection  $(\mathfrak{X}_\delta(w''_1, w''_2, v'') \setminus \tilde{\mathfrak{X}}_\delta(w'_1, w'_2, v')) \cap (G/P \times \tilde{\mathcal{X}}_s)$  is nonempty.

Thus, we have a strict inclusion  $\tilde{\mathfrak{X}}_\delta(w'_1, w'_2, v') \subset \tilde{\mathfrak{X}}_\delta(w''_1, w''_2, v'')$ . Similarly, we have the strict inclusion:

$$\tilde{\mathfrak{X}}_\delta(w''_1, w''_2, v'') \subset \tilde{\mathfrak{X}}_\delta(w_1, w_2, v).$$

Combining the above two, we get the strict inclusions:

$$\tilde{\mathfrak{X}}_\delta(w'_1, w'_2, v') \subset \tilde{\mathfrak{X}}_\delta(w''_1, w''_2, v'') \subset \tilde{\mathfrak{X}}_\delta(w_1, w_2, v).$$

Since these varieties are irreducible and  $\mathcal{U}^2$ -stable, we deduce that  $\mathcal{U}^2 \setminus \tilde{\mathfrak{X}}_\delta(w'_1, w'_2, v')$  is of codimension at least two in  $\mathcal{U}^2 \setminus \tilde{\mathfrak{X}}_\delta$ .

The lemma follows in this case since  $\dim(\mathcal{U}^2 \setminus \tilde{\mathfrak{X}}_\delta) = \dim(\mathcal{U}^2 \setminus \tilde{\mathcal{X}}_\delta)$  since  $\bar{\eta}_2$  is a birational map (cf. Proof of Lemma 6.5).

Let now  $(w_1, w_2, v')$  be as in type II.

Assume, for contradiction, that  $\bar{\eta}_2(\mathcal{U}^2 \setminus \tilde{\mathfrak{X}}_\delta(w_1, w_2, v'))$  is a divisor. By Lemma 6.9, there exists  $(g_1, g_2) \in G^2$  such that  $X_v^P \cap g_1 X_P^{w_1} \cap g_2 X_P^{w_2}$  is finite and there exists  $x \in \hat{X}_v^P \cap g_1 \hat{X}_P^{w_1} \cap g_2 \hat{X}_P^{w_2}$ ; in particular,  $(x, g_1 \varrho^-, g_2 \varrho^-, \varrho) \in \tilde{\mathfrak{X}}_\delta(w_1, w_2, v')$ .

By Corollary 6.8, the intersection  $\hat{X}_v^P \cap g_1 \hat{X}_P^{w_1} \cap g_2 \hat{X}_P^{w_2}$  is not transverse at  $x$ . Hence, the multiplicity of  $x$  in  $X_v^P \cap g_1 X_P^{w_1} \cap g_2 X_P^{w_2}$  is at least 2. Since this intersection is finite, this implies that the coefficient of  $\varepsilon_P^v$  in  $\varepsilon_P^{w_1} \cdot \varepsilon_P^{w_2} : n_{w_1, w_2}^v \geq 2$  (cf. [BK14, Proof of Proposition 3.5]). A contradiction!

The last types III and IV work similarly.  $\square$

**6.6. Conclusion of the proof of Theorem 1.5.** — Observe that  $\tilde{\mathfrak{X}}_\delta \cap \tilde{\mathfrak{X}}_\delta^\circ$  being open in the irreducible  $\tilde{\mathfrak{X}}_\delta$ ,  $i_6^*$  is injective. Combining the results from Subsections 6.2 - 6.4, we get that

$$i_6^* \circ \gamma^* \circ \alpha^* \circ \eta^* : H^0(\mathcal{X}, \mathcal{L})^G \rightarrow H^0(\tilde{\mathfrak{X}}_\delta \cap \tilde{\mathfrak{X}}_\delta^\circ, \mathcal{L})^B \text{ is injective}$$

and

$$i_7^* \circ \eta_2^* \circ i_4^* \circ i_1^* : H^0(\mathcal{X}, \mathcal{L})^G \rightarrow H^0(\tilde{\mathfrak{X}}_\delta \cap \tilde{\mathfrak{X}}_\delta^\circ, \mathcal{L})^B \text{ is an isomorphism.}$$

From the commutative diagram ( $\diamond$ ) of Subsection 6.1, these two composite maps are equal forcing  $\alpha^*$  to be an isomorphism. Thus, we get (from the top horizontal line of the commutative diagram ( $\diamond$ )) that the restriction map

$$H^0(\mathcal{X}, \mathcal{L})^G \rightarrow H^0(C, \mathcal{L})^L \text{ is an isomorphism.}$$

This ends the proof of the theorem.  $\square$

## 7. Proof of Theorem 1.3

In this section,  $P$  is still a standard parabolic subgroup (and not necessarily maximal). We fix  $(w_1, w_2, v) \in (W^P)^3$  such that  $\varepsilon_P^v$  occurs with coefficient 1 in the deformed product

$$\varepsilon_P^{w_1} \odot_0 \varepsilon_P^{w_2} \in (H^*(X_P, \mathbb{Z}), \odot_0).$$

In particular,  $w_1, w_2 \leq v$ . Recall the definition of  $\mathcal{D}$  and  $E_{\alpha,i}$  (for  $(\alpha, i) \in \mathcal{D}$ ) from subsection 5.4.

**7.1. On the relative position of  $E_{\alpha,i}$  and  $C$ . —**

**Proposition 7.1.** — *For any  $(\alpha, i) \in \mathcal{D}$ , the ind-variety*

$$C := Lw_1^{-1}\underline{\varrho}^- \times Lw_2^{-1}\underline{\varrho}^- \times Lv^{-1}\underline{\varrho}$$

*is not contained in  $E_{\alpha,i}$ .*

To prove Proposition 7.1 we need the following lemma.

**Lemma 7.2.** — *Let  $x \in G/P$  and  $(g, g_1, g_2) \in G^3$  be such that the intersection*

$$g_1\mathring{X}_P^{w_1} \cap g_2\mathring{X}_P^{w_2} \cap g\mathring{X}_v^P$$

*contains  $x$  and is transverse at this point. Such a choice is possible by [Res21, Lemmas 6 and 7]. Then,*

$$g_1X_P^{w_1} \cap g_2X_P^{w_2} \cap gX_v^P = \{x\}.$$

*In fact, the lemma remains true if we replace  $\mathring{X}_P^{w_i}$  (for any  $i = 1, 2$ ) by any  $B^-$ -stable open subset of  $\mathring{X}_P^{w_i} \cup \left( \bigcup_{w_i \rightarrow w'_i \in W^P} \mathring{X}_P^{w'_i} \right)$  and  $\mathring{X}_v^P$  by any  $B$ -stable open subset of  $\mathring{X}_v^P \cup \left( \bigcup_{v' \rightarrow v, v' \in W^P} \mathring{X}_{v'}^P \right)$ .*

*Proof.* — The strategy of the proof is to reduce the problem to a finite-dimensional situation (by quotient), and then to apply Zariski's main theorem.

Up to a translation, we may assume that  $g$  is trivial. Since  $G/B^- = \bigcup_{w \in W} wU\underline{\varrho}^-$ , there exists, for  $i = 1, 2$ ,  $u_i \in W$  such that  $g_i\underline{\varrho}^- \in u_iU\underline{\varrho}^-$ . Consider now

$$\mathring{\mathfrak{X}}_\delta = \{(y, h_1\underline{\varrho}^-, h_2\underline{\varrho}^-) \in X_v^P \times u_1U\underline{\varrho}^- \times u_2U\underline{\varrho}^- : y \in h_1X_P^{w_1} \cap h_2X_P^{w_2}\}$$

and its projection  $\eta$  to  $u_1U\underline{\varrho}^- \times u_2U\underline{\varrho}^-$ .

Consider  $\theta : U \rightarrow \text{Aut}(X_v^P)$  obtained by the action as before. Fix  $i \in \{1, 2\}$ . Then,  $\text{Ker } \theta$  has finite-codimension in  $U$  and  $U \cap u_iUu_i^{-1}$  has finite-codimension in  $u_iUu_i^{-1}$ . It follows that there exists a closed normal subgroup  $\mathcal{U}_i$  of  $u_iUu_i^{-1}$  of finite-codimension such that

$$\mathcal{U}_i \subset u_iUu_i^{-1} \cap \text{Ker } \theta.$$

Such a  $\mathcal{U}_i$  can be obtained as a closed subgroup of  $U$  with Lie algebra

$$\text{Lie } \mathcal{U}_i = \bigoplus_{\beta \in \Phi^+, |\beta| > N} \mathfrak{g}_\beta,$$

for large enough  $N$  (depending upon  $v$  and  $u_i$ ), where, for  $\beta = \sum_j n_j \alpha_j$ ,  $|\beta| := \sum n_j$ .

The group  $\mathcal{U}_1 \times \mathcal{U}_2$  acts freely and properly on  $u_1U\underline{\varrho}^- \times u_2U\underline{\varrho}^-$  (and hence on  $\mathring{\mathfrak{X}}_\delta$ ). Moreover,  $\eta$  is  $(\mathcal{U}_1 \times \mathcal{U}_2)$ -equivariant. After quotient, one gets

$$\bar{\eta} : (\mathcal{U}_1 \times \mathcal{U}_2) \backslash \mathring{\mathfrak{X}}_\delta \rightarrow (\mathcal{U}_1 \times \mathcal{U}_2) \backslash (u_1U\underline{\varrho}^- \times u_2U\underline{\varrho}^-).$$

Observe that  $\mathcal{U}_i$  being closed subgroups of finite-codimension in  $u_iUu_i^{-1}$  and  $X_v^P$  being finite-dimensional, the domain and the range of  $\bar{\eta}$  are finite-dimensional varieties and the range of  $\bar{\eta}$  is smooth and irreducible.

Since the coefficient of  $\varepsilon_v^P$  in  $\varepsilon_{w_1}^P \cdot \varepsilon_{w_2}^P : n_{w_1, w_2}^v = 1$ , the general fiber of  $\eta$  is one point (see [Res21, §4.2]). Further, as observed below the Equation (53),  $\mathfrak{X}_\delta$  is irreducible and hence so is  $(\mathcal{U}_1 \times \mathcal{U}_2) \setminus \mathfrak{X}_\delta$ . Since the base field is  $\mathbb{C}$ , this implies that  $\bar{\eta}$  is birational. Since  $X_v^P$  is projective and  $X_P^{w_1}$  and  $X_P^{w_2}$  are closed in  $G/P$ , it is easy to see that the map  $\bar{\eta}$  is proper. Now, we can apply Zariski's main theorem [Har77, Chap. III, Corollary 11.4] to conclude that the fibers of  $\bar{\eta}$  are connected. But, by assumption,  $[x, g_1 \underline{\varrho}^-, g_2 \underline{\varrho}^-]$  is isolated in the fiber  $\bar{\eta}^{-1}[g_1 \underline{\varrho}^-, g_2 \underline{\varrho}^-]$ , where  $[g_1 \underline{\varrho}^-, g_2 \underline{\varrho}^-]$  denotes the  $(\mathcal{U}_1 \times \mathcal{U}_2)$ -orbit of  $(g_1 \underline{\varrho}^-, g_2 \underline{\varrho}^-)$ . Then,  $\bar{\eta}^{-1}[g_1 \underline{\varrho}^-, g_2 \underline{\varrho}^-] = \{[x, g_1 \underline{\varrho}^-, g_2 \underline{\varrho}^-]\}$ , that is

$$g_1 X_P^{w_1} \cap g_2 X_P^{w_2} \cap X_v^P = \{x\}.$$

This proves the first part of the lemma.

The proof for the 'In fact' statement in the lemma is identical.  $\square$

*Proof of Proposition 7.1.* — Since  $\varepsilon_P^v$  occurs with coefficient 1 in the deformed product  $\varepsilon_P^{w_1} \odot_0 \varepsilon_P^{w_2}$ , by the proof of [Res21, Lemma 19], there exist  $l_1, l_2, l_3 \in L$  such that the intersection

$$(80) \quad (l_1 w_1^{-1} \hat{X}_P^{w_1}) \cap (l_2 w_2^{-1} \hat{X}_P^{w_2}) \cap (l_3 v^{-1} \hat{X}_v^P)$$

is transverse at  $P/P$ . Then, Lemma 7.2 implies that the intersection

$$(l_1 w_1^{-1} X_P^{w_1}) \cap (l_2 w_2^{-1} X_P^{w_2}) \cap (l_3 v^{-1} X_v^P)$$

is reduced to  $\{P/P\}$ . In particular, if  $w_1 \leq s_\alpha w_1$  and  $s_\alpha w_1 \in W^P$ ,

$$(81) \quad (l_1 w_1^{-1} X_P^{s_\alpha w_1}) \cap (l_2 w_2^{-1} X_P^{w_2}) \cap (l_3 v^{-1} X_v^P) = \emptyset.$$

Then,

$$(82) \quad (l_1 w_1^{-1} \underline{\varrho}^-, l_2 w_2^{-1} \underline{\varrho}^-, l_3 v^{-1} \underline{\varrho}) \notin G \cdot \left( \overline{P w_1^{-1} s_\alpha \underline{\varrho}^-} \times \overline{P w_2^{-1} \underline{\varrho}^-} \times \overline{P v^{-1} \underline{\varrho}} \right).$$

This proves that  $(l_1 w_1^{-1} \underline{\varrho}^-, l_2 w_2^{-1} \underline{\varrho}^-, l_3 v^{-1} \underline{\varrho})$  does not belong to  $E_{\alpha, 1}$ . The proposition follows for  $(\alpha, 1)$ . The proof for  $(\alpha, i) \in \mathcal{D}$  for  $i = 2, 3$  is identical.  $\square$

**7.2. The line bundles  $\mathcal{N}_{\alpha, i}$ .** — The goal of this subsection is to prove that  $\mathcal{N}_{\alpha, i}$  belongs to the face considered in Theorem 1.3:

**Proposition 7.3.** — *For any  $(\alpha, i) \in \mathcal{D}$ , the center  $Z(L)$  of  $L$  acts trivially on the restriction of  $\mathcal{N}_{\alpha, i}$  to  $C$ , where  $C$  is as in Proposition 7.1.*

*In fact, for any  $L$ -equivariant line bundle  $\mathcal{L}$  over  $C$  with  $H^0(C, \mathcal{L})^L \neq 0$ ,  $Z(L)$  acts trivially on  $\mathcal{L}$ . In particular, if we write  $\mathcal{N}_{\alpha, i} = \mathcal{L}^-(\lambda_1) \otimes \mathcal{L}^-(\lambda_2) \otimes \mathcal{L}(\mu)$ , then for all  $\alpha_j \notin \Delta(P)$ ,*

$$(I_{(w_1, w_2, v)}^j) \quad \lambda_1(w_1 x_j) + \lambda_2(w_2 x_j) - \mu(v x_j) = 0.$$

*Proof.* — Consider a  $G$ -invariant section  $\mu_{\alpha, i}$  of  $\mathcal{N}_{\alpha, i}$  as guaranteed by Corollary 5.12. For any  $(\alpha, i) \in \mathcal{D}$ , by Corollary 5.12,  $Z(\mu_{\alpha, i}) = E_{\alpha, i}$ . Then Proposition 7.1 implies that  $\mu_{\alpha, i}$  restricts to a nonzero  $L$ -invariant section on  $C$ .

Since  $Z(L)$  acts trivially on  $C$ , it acts by a character on any line bundle over  $C$ . The existence of a nonzero  $Z(L)$ -invariant section implies that this character is trivial for the restriction of  $\mathcal{N}_{\alpha, i}$ .

Write  $\mathcal{N}_{\alpha,i} = \mathcal{L}^-(\lambda_1) \otimes \mathcal{L}^-(\lambda_2) \otimes \mathcal{L}(\mu)$  and fix  $\alpha_j \notin \Delta(P)$ . There exists  $d > 0$ , such that  $dx_j$  is the differential at 1 of a one parameter subgroup of  $Z(L)$ . This one parameter subgroup acts with weight  $\lambda_1(w_1x_j)$ ,  $\lambda_2(w_2x_j)$  and  $-\mu(vx_j)$  on the fiber over  $w_1^{-1}\underline{\varrho}^-$ ,  $w_2^{-1}\underline{\varrho}^-$  and  $v^{-1}\underline{\varrho}$  in  $\mathcal{L}^-(\lambda_1)$ ,  $\mathcal{L}^-(\lambda_2)$  and  $\mathcal{L}(\mu)$  respectively. Thus, the equality  $I_{(w_1,w_2,v)}^j$  follows proving Proposition 7.3.  $\square$

**7.3. The line bundles  $\mathcal{N}_{\alpha,i}$  and the lines  $\ell_{\beta,j}$ .** — Recall the definition of the line  $\ell_{\beta,j}$  from §1. We now study the restriction of the line bundle  $\mathcal{N}_{\alpha,i}$  to the lines  $\ell_{\beta,j}$ . This will be used to apply Theorem 1.5.

**Lemma 7.4.** — *Let  $(\alpha, i) \in \mathcal{D}$  and  $(\beta, j) \in \mathcal{D}$  be two distinct elements. Then,*

- (i) *the degree of the restriction of  $\mathcal{N}_{\alpha,i}$  to  $\ell_{\alpha,i}$  is positive.*
- (ii) *the degree of the restriction of  $\mathcal{N}_{\alpha,i}$  to  $\ell_{\beta,j}$  is nonnegative.*

*Proof.* — Take  $(\alpha, 1) \in \mathcal{D}$ . Then, as in Section 1,

$$\ell_{\alpha,1} = (w_1^{-1}P_{\alpha}^-\underline{\varrho}^-, w_2^{-1}\underline{\varrho}^-, v^{-1}\underline{\varrho}).$$

Since the line bundle  $\mathcal{N}_{\alpha,i}$  has the form  $\mathcal{L}^-(\lambda_1) \boxtimes \mathcal{L}^-(\lambda_2) \boxtimes \mathcal{L}(\mu)$  for some  $(\lambda_1, \lambda_2, \mu) \in P_+^3$  (cf. Corollary 5.12),

$$\mathcal{N}_{\alpha,1}|_{\ell_{\alpha,1}} \simeq \mathcal{L}^-(\lambda_1)|_{w_1^{-1}P_{\alpha}^-\underline{\varrho}^-},$$

which is of degree

$$(w_1^{-1}\lambda_1)(w_1^{-1}\alpha^{\vee}) = \lambda_1(\alpha^{\vee}) \geq 0.$$

Assume, if possible, that  $\lambda_1(\alpha^{\vee}) = 0$ . Then, the zero set  $Z(\mu_{\alpha,1})$  would be of the form  $\pi_{\alpha}^{-1}(S)$  for some  $S \subset G/P_{\alpha}^- \times G/B^- \times G/B$ , where

$$\pi_{\alpha} : G/B^- \times G/B^- \times G/B \rightarrow G/P_{\alpha}^- \times G/B^- \times G/B$$

is the projection.

Then, by Corollary 5.12 and Equation (18),

$$Z(\mu_{\alpha,1}) = E_{\alpha,1} = G \cdot \bar{C}_{s_{\alpha}w_1, w_2, v}^+$$

and hence we would have

$$Z(\mu_{\alpha,1}) \supset G \cdot \bar{C}_{w_1, w_2, v}^+ = \mathcal{X},$$

where the last equality follows from [BK14, Proposition 3.5] since  $\varepsilon_P^v$  occurs with nonzero coefficient in  $\varepsilon_P^{w_1} \cdot \varepsilon_P^{w_2}$ . This contradicts the nonvanishing of  $\mu_{\alpha,1}$ . Thus,  $\lambda_1(\alpha^{\vee}) > 0$ , proving (i) for  $(\alpha, 1) \in \mathcal{D}$ . The same proof works for any  $(\alpha, i) \in \mathcal{D}$  to prove (i).

To prove (ii), we still take  $(\alpha, 1) \in \mathcal{D}$  and  $(\beta, j) \in \mathcal{D}$  for  $j = 1, 2$ . Then,

$$\mathcal{N}_{\alpha,1}|_{\ell_{\beta,j}} \simeq \mathcal{L}^-(\lambda_j)|_{w_j^{-1}P_{\beta}^-\underline{\varrho}^-},$$

which is of degree

$$(w_j^{-1}\lambda_j)(w_j^{-1}\beta^{\vee}) = \lambda_j(\beta^{\vee}) \geq 0.$$

For  $(\beta, 3) \in \mathcal{D}$ ,

$$\mathcal{N}_{\alpha,1}|_{\ell_{\beta,3}} \simeq \mathcal{L}(\mu)|_{v^{-1}P_{\beta}\underline{\varrho}},$$

which is of degree

$$(v^{-1}\mu)(v^{-1}\beta^{\vee}) = \mu(\beta^{\vee}) \geq 0.$$

This proves (ii) for  $(\alpha, 1) \in \mathcal{D}$ . The same proof gives (ii) for any  $(\alpha, i) \in \mathcal{D}$ .  $\square$

**7.4. Conclusion of the proof of Theorem 1.3.** — Let  $w_1, w_2, v \in W^P$  be as in Theorem 1.3, i.e.,  $\varepsilon_P^v$  occurs with coefficient 1 in the deformed product  $\varepsilon_P^{w_1} \odot_0 \varepsilon_P^{w_2}$ .

Set  $d = 2 \dim \mathfrak{h} + \sharp \Delta(P)$ . Let  $\mathcal{F} = \mathcal{F}_{w_1, w_2, v}^P$  be the convex cone generated by the weights  $(\lambda_1, \lambda_2, \mu) \in \Gamma(\mathfrak{g})$  satisfying Identity  $I_{(w_1, w_2, v)}^j$  for all  $\alpha_j \notin \Delta(P)$  as in Theorem 1.3. Since the linear forms  $\{I_{(w_1, w_2, v)}^j\}_{\alpha_j \in \Delta \setminus \Delta(P)}$  restricted to  $E_{\mathfrak{g}}$  (cf. Proposition 3.1) defining  $\mathcal{F}$  are linearly independent, the dimension of  $\mathcal{F}$  is at most  $d$ .

We now have to produce ‘enough’ points in  $\mathcal{F}$ . To do this we consider the restriction map  $\text{Pic}^{G^3}(\mathcal{X}) \rightarrow \text{Pic}^{L^3}(C)$  and we apply Theorem 1.5 to sufficiently many line bundles  $\mathcal{L}$  such that  $H^0(C, \mathcal{L}|_C)^L \neq \{0\}$ .

Observe that, for any  $w \in W^P$ , the map

$$L/B_L^- \rightarrow Lw^{-1}\underline{\varrho}^- \subset G/B^-, \quad lB_L^- \mapsto lw^{-1}\underline{\varrho}^- \text{ is an } L\text{-equivariant isomorphism}$$

and also the map

$$L/B_L \rightarrow Lw^{-1}\underline{\varrho} \subset G/B, \quad lB_L \mapsto lw^{-1}\underline{\varrho} \text{ is an isomorphism,}$$

where  $B_L := B \cap L$  is the standard Borel subgroup of  $L$  and  $B_L^- := B^- \cap L$  is the standard opposite Borel subgroup of  $L$ . (To prove the above two isomorphisms, use the fact that  $w\Delta_P \subset \Phi^+$ .) Thus, the restriction map  $\text{Pic}^{G^3}(\mathcal{X}) \simeq (\mathfrak{h}_{\mathbb{Z}}^*)^3 \rightarrow \text{Pic}^{L^3}(C) \simeq (\mathfrak{h}_{\mathbb{Z}}^*)^3$  is an isomorphism. Let  $\mathfrak{l}$  denote the Lie algebra of  $L$ .

**Lemma 7.5.** — *There exist  $\mathcal{L}_1, \dots, \mathcal{L}_d \in \text{Pic}^{G^3}(\mathcal{X})$  such that*

- (i)  $\mathcal{L}_1, \dots, \mathcal{L}_d \in \text{Pic}^{G^3}(\mathcal{X}) \otimes \mathbb{Q}$  are linearly independent;
- (ii) The restriction of each  $\mathcal{L}_i$  to  $C$  belongs to  $\Gamma(\mathfrak{l})$ .

*Proof.* — By Proposition 3.1,  $\Gamma(\mathfrak{l})$  has dimension  $d$ . (Observe that Proposition 3.1 remains valid for  $\mathfrak{l}$  by the same proof.) Hence,  $\Gamma(\mathfrak{l}) \subset \text{Pic}^{L^3}(C) \otimes_{\mathbb{Z}} \mathbb{Q}$  contains  $d$  linearly independent elements. Then, the lemma follows from the isomorphism  $\text{Pic}^{G^3}(\mathcal{X}) \simeq \text{Pic}^{L^3}(C)$ .  $\square$

*Proof of Theorem 1.3.* — Up to taking tensor powers, we may assume that the restriction of  $\mathcal{L}_i$  to  $C$  admits a nonzero  $L$ -invariant section  $\sigma_i$  (cf. [BK14, Proof of Theorem 3.2]).

By Lemma 7.4, there exists  $(a_{\alpha, i})_{(\alpha, i) \in \mathcal{D}} \in \mathbb{N}^{\mathcal{D}}$  such that  $\mathcal{N} := \sum_{(\alpha, i) \in \mathcal{D}} a_{\alpha, i} \mathcal{N}_{\alpha, i}$  satisfies:

$$\mathcal{L}_k \otimes \mathcal{N} \text{ is nonnegative for all } k \text{ when restricted to any } \ell_{\beta, j} \text{ for } (\beta, j) \in \mathcal{D}.$$

Moreover, up to changing  $\mathcal{N}$  by  $2\mathcal{N}$  if necessary, we may assume that

$$\mathcal{L}_1 \otimes \mathcal{N}, \dots, \mathcal{L}_d \otimes \mathcal{N} \in \text{Pic}^{G^3}(\mathcal{X}) \otimes \mathbb{Q}$$

are linearly independent.

By Corollary 5.12,  $\mathcal{N}$  has a  $G$ -invariant section  $\sigma_{\mathcal{N}}$  that does not vanish identically on  $C$ . Then,

$$\tilde{\sigma}_i \in H^0(C, \mathcal{L}_i \otimes \mathcal{N})^L \setminus \{0\}, \text{ where } \tilde{\sigma}_i := (\sigma_i \otimes \sigma_{\mathcal{N}})|_C.$$

Moreover, since  $\tilde{\sigma}_i$  is not identically zero on  $C$ , by Proposition 7.3, each  $\mathcal{L}_i \otimes \mathcal{N}$  satisfies the identity  $I_{(w_1, w_2, v)}^j$  of Theorem 1.3 for all  $\alpha_j \in \Delta \setminus \Delta(P)$ .

By Theorem 1.5, each  $\tilde{\sigma}_i$  can be extended to a  $G$ -invariant section  $\tilde{\sigma}_i$  of  $\mathcal{L}_i \otimes \mathcal{N}$ . In particular,  $\mathcal{L}_i \otimes \mathcal{N}$  belongs to  $\Gamma(\mathfrak{g})$ . Thus, the dimension of  $\mathcal{F}$  is at least  $d$  and hence it is exactly  $d$ . This proves the theorem.  $\square$

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