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**BASIC ANALYSIS OF SOME SECOND MOMENT CLOSURES**  
**PART II : INCOMPRESSIBLE TURBULENT FLOWS INCLUDING BUOYANT EFFECTS**

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**Abstract :**

The suitability of some second moment closures is discussed herein. A general frame of strongly realisable models is exhibited, and the ability of so-called "slow" terms to provide return-to-isotropy is examined. Hyperbolicity of first order differential systems is then investigated. When focusing on a simple so-called Gaussian closure, the solution of the Riemann problem associated to the generalised convection system is detailed, which confirms that realisability still holds when non regular solutions are involved. A priori suitable numerical consequences are then discussed. These results should be related to recent work pertaining to the numerical modelling of compressible flows using second order closures.

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**Nomenclature :**

$f_{,i}$	Partial derivative of $f$ with respect to $x_i$
$f_{,t}$	Partial derivative of $f$ with respect to $t$
$f = \langle f \rangle + f'$	Reynolds decomposition
$D_t f = f_{,t} + \langle U_i \rangle f_{,i}$	Convective derivative of $f$ function
$\langle p \rangle$	Mean pressure
$\langle T \rangle$	Mean temperature
$\langle U_i \rangle$	Mean velocity in $x_i$ direction
$R_{ij} = \langle u_i u_j \rangle \quad (\in R_{3 \times 3})$	Reynolds stress tensor
$X_j = \langle \theta u_j \rangle$	Turbulent heat flux
$\langle \theta^2 \rangle$	Scalar (temperature) variance
$Z_{ij} = \langle \theta^2 \rangle R_{ij} - X_i X_j \quad (\in R_{3 \times 3})$	
$I_R = 2K = \text{trace}(R)$	Turbulent kinetic energy
$II_R = \text{trace}(R^2)$	Second invariant of $R$
$III_R = \text{trace}(R^3)$	Third invariant of $R$
$B_{ij} = U_{i,j}$	
$S_{ij} = U_{i,j} + U_{j,i}$	
$G_{ij} = \langle u_i u_k \rangle U_{k,j} + \langle u_j u_k \rangle U_{k,i} = (R B + B^t R)_{ij}$	
$P_{ij} = \langle u_i u_k \rangle U_{j,k} + \langle u_j u_k \rangle U_{i,k} = (R B^t + B R)_{ij}$	
$P = \frac{1}{2} P_{ii} = \frac{1}{2} G_{ii}$	
$\tilde{\varepsilon} = \nu \langle u_{i,k} u_{i,k} \rangle$	Dissipation rate of $K$
$\varepsilon_\theta = \lambda \langle \theta_{,k} \theta_{,k} \rangle$	Dissipation rate of $\langle \theta^2 \rangle$
$\tau_K = (\tilde{\varepsilon} / K)^{-1}$	Turbulent mechanical time scale
$\tau_\theta = (\varepsilon_\theta / \langle \theta^2 \rangle)^{-1}$	Turbulent scalar time scale
<u>For <math>i=1 \rightarrow 3</math> :</u> $\delta_1^i = R_{ii}$	First fundamental minor of $R$ (no summation)
<u>For <math>i=1 \rightarrow 3</math> :</u> $\delta_2^i = R_{\alpha\alpha} R_{\beta\beta} - (R_{\alpha\beta})^2$	Second fundamental minor of $R$ (no summation), $\alpha$ and $\beta$ non equal, non equal to $i$
$\delta_3 = \det(R_{ij})$	Third fundamental minor of $R$

For  $i=1 \rightarrow 3$  :  $\Delta_1^i = Z_{ii}$

First fundamental minor of  $Z$  (no summation)

For  $i=1 \rightarrow 3$  :

$$\Delta_2^i = Z_{\alpha\alpha} Z_{\beta\beta} - (Z_{\alpha\beta})^2$$

Second fundamental minor of  $Z$  (no summation),  $\alpha$  and  $\beta$  non equal, non equal to  $i$

$$\Delta_3 = \det (Z_{ij} )$$

Third fundamental minor of  $Z$

## Introduction

Statistical modelling of turbulence has generated the need for computation of complex sets of partial differential equations. This is mainly due to the highly non linear terms present in instantaneous Navier Stokes equations on which is based such a modelling process. This is true of course for both compressible and incompressible flow patterns. In both cases, numerical simulation requires applying for stable and accurate enough schemes in order to ensure convergence towards the right solution. When focusing on compressible flows, it is now well admitted that upwinding techniques have enabled gaining sufficient stabilizing effects to allow computation of flows including strong shocks, even when viscous effects are of small amplitude or even vanishing. It does not necessarily mean that these provide the ultimate numerical approach for such a purpose, since the balance between stability and accuracy is clearly in favour of the former, and consequently may penalize improvement of accuracy. Nonetheless, use of standard reconstruction techniques (such as MUSCL approach, or ENO schemes,...) has proved to be an efficient way to handle the whole. From a technical point of view, these upwinding techniques rely on the theory of hyperbolic systems of conservation laws, and have been recently extended to the framework of non linear hyperbolic systems which do not have any conservation form. Clearly, approximate Riemann solvers are almost overwhelmingly present in the frame of compressible flows, but are seldomly used in the frame of incompressible flows. They have been mainly used in the framework of conservative Euler -or Navier Stokes- equations for gas dynamics. More recently, they also have been applied for investigation of magneto hydrodynamics, and also to deal with averaged Navier Stokes equations with statistical Favre averaging. A first step has been devoted to the K or K-e type closures ( $\overline{u'u'}$ ), and in a second step to realisable second-moment closures ( $\overline{u'u'u'}$ ). A unifying approach based on the entropy concept was also proposed (see [1]) in order to cope with first or second order closures. These methods have been successfully implemented in industrial codes using unstructured Finite Volume approach (N3S-NATUR [2], PLEXUS [3], Code\_Saturne [4]). It has also been demonstrated that standard techniques based on Euler type algorithms lead to unstable computations (when focusing on impinging jets on walls for instance, see [5] and [6] among others) which lead to blow up of codes. These techniques may also be used to enforce the coupling between Eulerian and Lagrangian approaches (see [7], [8], [9] and [10] for instance). They are robust enough to be implemented

We examine here the suitability of some second moment (so-called Gaussian) closures involved in systems of partial differential equations which aim at predicting the behaviour of incompressible turbulent flows including buoyant effects. As in [5], special emphasis is given on the realisability conditions and to the link between realisability requirement and hyperbolicity constraints. Present contribution is organised as follows. In section I, governing equations and realisability requirements are briefly recalled. In section II, the class of strongly realisable closures initially defined in [5] is

extended to a more general framework when dealing with isothermal flows ; besides, some features concerning return to isotropy process are also recalled. In section III, a simple closure is proposed, which enables to achieve strong realisability in the non isothermal case. Section IV is devoted to the examination of the general eigenvalue problem, which confirms that realisability is compulsory to gain hyperbolicity. Since the result stated in section III requires the boundedness of two different tensors, which might be violated, especially when dealing with "non-viscous" Gaussian closures, the Riemann problem associated with the initial value problem obtained by getting rid off continuity constraint and eliminating mean pressure variable is examined in section V ; results obtained show that over-realisability still holds (in a standard sense) but that the maximum principle for mean temperature is no longer valid. Some numerical consequences of these results are given in the last section.

## I - Basic set of equations and main requirements :

### I.1 Governing equations

For a wide variety of problems, temperature field is assumed not to have any significant influence on the velocity field, except through external gravity effects, which stands for usual Boussinesq approximation. Hence, instantaneous velocity field remains divergence free and the governing equation for the instantaneous temperature partially decouples with the remaining equations. Thus, no compressibility effect is accounted for in the present study, and the basic set of equations used to construct turbulent model writes :

$$U_{i,i} = 0 \quad (1.1)$$

$$U_{i,t} + U_j U_{i,j} = \left( \frac{-p}{\rho_0} \delta_{ij} + \nu (U_{i,j} + U_{j,i}) \right)_j + \beta_i (T - T_0) \quad (1.2)$$

$$T_{,t} + U_j T_{,j} = (\lambda T_{,j})_{,j} \quad (1.3)$$

Considering standard statistical approach and focusing on one point closures, the latter enable to derive the well-known unclosed set of equations to describe mean velocity, pressure and temperature fields, i.e. :

$$\langle U_i \rangle_{,i} = 0 \quad (2.1)$$

$$\langle U_i \rangle_{,t} + \langle U_j \rangle \langle U_i \rangle_{,j} = (\langle \Sigma_{ij} \rangle)_{,j} - (R_{ij})_{,j} + \beta_i (\langle T \rangle - T_0) \quad (2.2)$$

$$\langle T \rangle_{,t} + \langle U_j \rangle \langle T \rangle_{,j} = (\lambda \langle T \rangle_{,j})_{,j} - X_{j,j} \quad (2.3)$$

where :

$$\langle \Sigma_{ij} \rangle = - \frac{\langle p \rangle}{\rho_0} \delta_{ij} + \nu (\langle U_i \rangle_{,j} + \langle U_j \rangle_{,i})$$

In order to close previous set, governing equations for second moment unknowns  $R_{ij}$ ,  $X_j$  and  $\langle \theta^2 \rangle$  are needed ; these are listed below :

$$(R_{ij})_{,t} + \langle U_k \rangle (R_{ij})_{,k} + P_{ij} - (\beta_i X_j + \beta_j X_i) = \Phi_{ij} - (d_{ijk})_{,k} \quad (2.4)$$

$$(X_i)_{,t} + \langle U_k \rangle (X_i)_{,k} + X_k \langle U_i \rangle_{,k} + R_{ik} \langle T \rangle_{,k} - \beta_i \langle \theta^2 \rangle = \Phi_i - (d_{ik})_{,k} \quad (2.5)$$

$$\langle \theta^2 \rangle_{,t} + \langle U_k \rangle \langle \theta^2 \rangle_{,k} + 2 X_k \langle T \rangle_{,k} = \Phi - (d_k)_{,k} \quad (2.6)$$

while noting :

$$\Phi_{ij} = \langle u_j \Sigma'_{ik,k} + u_i \Sigma'_{jk,k} \rangle \quad d_{ijk} = \langle u_i u_j u_k \rangle \quad (3)$$

$$\Phi_i = \langle \theta \Sigma'_{ik,k} \rangle + \lambda \langle u_i \theta_{,kk} \rangle \quad d_{ik} = \langle u_i \theta u_k \rangle$$

$$\Phi = 2 \lambda \langle \theta \theta_{,kk} \rangle \quad d_k = \langle \theta \theta u_k \rangle$$

Hence, set (2) requires the modelling of the  $\Phi_{ij}$ ,  $\Phi_i$ ,  $\Phi$ ,  $d_{ijk}$ ,  $d_{ik}$ ,  $d_k$  contributions.

### I.2 Realisability requirements



When dealing with non isothermal flows, the following realisability condition should hold (see /8/), which is:

$$A = \begin{pmatrix} R_{11} & R_{12} & R_{13} & X_1 \\ R_{21} & R_{22} & R_{23} & X_2 \\ R_{31} & R_{32} & R_{33} & X_3 \\ X_1 & X_2 & X_3 & \langle \theta^2 \rangle \end{pmatrix} = \begin{pmatrix} \mathbf{R} & \mathbf{X} \\ \mathbf{X}^t & \langle \theta^2 \rangle \end{pmatrix}$$

must be positive half definite. Provided that  $\langle \theta^2 \rangle$  is non zero, the latter may be premultiplied on the left by the non singular matrix :

$$B = \begin{pmatrix} \mathbf{I} & -\langle \theta^2 \rangle^{-1} \mathbf{X} \\ 0 & 1 \end{pmatrix}$$

and on the right by  $B^t$ ; thus :

$$\forall Z \in R_4, Z^t A Z \geq 0$$

is achieved as soon as (see /8/ for similar statement) :

$$\forall Y \in R_4, Y^t \begin{pmatrix} \mathbf{R} - \langle \theta^2 \rangle^{-1} \mathbf{X} \mathbf{X}^t & 0 \\ 0 & \langle \theta^2 \rangle \end{pmatrix} Y \geq 0$$

Hence, provided that  $\langle \theta^2 \rangle$  is non zero :

$$(i) \mathbf{Z} = (\langle \theta^2 \rangle \mathbf{R} - \mathbf{X} \mathbf{X}^t) \text{ must be half definite positive} \quad (4.1)$$

$$(ii) \langle \theta^2 \rangle \text{ must be positive} \quad (4.2)$$

Moreover, if  $\langle \theta^2 \rangle$  is zero :

$$(i)' \mathbf{R} \text{ must be half definite positive}$$

$$(ii)' \mathbf{X}^t \mathbf{X} \text{ must be zero}$$

Note that (i) implies :

$$\text{trace}(\mathbf{Z}) = \langle \theta^2 \rangle \text{trace}(\mathbf{R}) - \mathbf{X}^t \mathbf{X} \geq 0$$

Thus, (ii)' is contained in (i). Note also that for non vanishing value of  $\langle \theta^2 \rangle$  which fulfills (ii), (i)' holds since :

$$\mathbf{R} = \langle \theta^2 \rangle^{-1} (\mathbf{Z} + \mathbf{X} \mathbf{X}^t)$$

From now on, the limit case of vanishing value of scalar variance will be omitted, and (4.1) and (4.2) will be accounted for while ensuring that  $\langle \theta^2 \rangle$  and the three fundamental minors of  $Z$  (namely  $\Delta_1^1, \Delta_2^3, \Delta_3$ ) remain positive (which stands for over-realisability concept as defined in /5/). Moreover, closures will be chosen in such a way that :

$$f = 0 \Rightarrow (D_t f = 0 \text{ and } :D_t(D_t f) \geq 0) \text{ for } :f = \Delta_1^1, \Delta_2^3, \Delta_3 \quad (5)$$

Obviously, closures involving time scales  $\tau_K$  and  $\tau_\theta$  should be such that :

$$\tau_K > 0 \text{ and } : \tau_\theta > 0$$

### I.3 Maximum principle for mean temperature and scalar variance

Going back to the original system (1.1, 1.2, 1.3), and considering micro-gravity effects for sake of simplicity, we may expect that slightly different initial (or boundary) conditions will provide different solutions for (k) or (k+1) experiments ; denoting by  $U^{(k)}(\mathbf{x},t)$ ,  $T^{(k)}(\mathbf{x},t)$ ,  $P^{(k)}(\mathbf{x},t)$  solutions associated to (k) experiment, it occurs that  $T^{(k)}(\mathbf{x},t)$  will be solution of :

$$T_{,t}^{(k)} + U_j^{(k)} T_{,j}^{(k)} = (\lambda T_{,j}^{(k)})_{,j} \text{ where : } (\mathbf{x}, t) \in \Omega \times (0, T)$$

with given initial and boundary conditions, which may be of Dirichlet type i.e. :

$$\begin{aligned} T^{(k)}(\mathbf{x}, t = 0) &= T_0^{(k)}(\mathbf{x}) & \text{for : } \mathbf{x} \in \Omega \\ T^{(k)}(\mathbf{x}, t > 0) &= T_{\delta\Omega}^{(k)}(\mathbf{x},t) & \text{for : } (\mathbf{x},t) \in \delta\Omega \times (0, T) \end{aligned}$$

Provided that there exists two real constant values (independent of statistics) such that :

$$T_{low} \leq T_{\delta\Omega}^{(k)}(\mathbf{x},t) \leq T_{up} \quad \text{and : } T_{low} \leq T_0^{(k)}(\mathbf{x}) \leq T_{up}$$

then  $T^{(k)}(\mathbf{x},t)$  will be such that :

$$T_{low} \leq T^{(k)}(\mathbf{x},t) \leq T_{up}$$

Consequently, the following constraint for the average value  $\langle T \rangle(\mathbf{x},t)$  should hold :

$$T_{low} \leq \langle T \rangle(\mathbf{x},t) \leq T_{up} \tag{6}$$

Moreover,  $(\langle T \rangle - T^{(k)})(\mathbf{x},t)$  will be such that :

$$0 \leq (T^{(k)} - \langle T \rangle)^2(\mathbf{x},t) \leq (T_{low} - T_{up})^2$$

Hence, scalar variance  $\langle \theta^2 \rangle$  should fulfill :

$$0 \leq \langle \theta^2 \rangle(\mathbf{x},t) \leq (T_{low} - T_{up})^2 \tag{7}$$

It must be emphasized that the latter still holds when external (buoyancy) forces are accounted for. Note anyway that constraints (6) and (7) are not intrinsic to mean temperature and scalar variance entities, unlike (4.1) and (4.2) ; they occur in fact to stand for some counterpart of (5), in the sense that they arise while focusing on the instantaneous governing equations (1.1) to (1.3).

## II Some more about isothermal turbulent flows

Before going ahead, let us go back to the isothermal case. In a previous work, it was suggested that the following closure of Reynolds stress governing equations enable to achieve strong realisability :

$$\begin{aligned} \Phi_{ij} = & \alpha_2 R_{ij} + \alpha_3 R_{ij}^2 + \alpha_5 ( R_{ik} \langle U_j \rangle_{,k} + R_{jk} \langle U_i \rangle_{,k} ) \\ & + \alpha_6 ( R_{ik} \langle U_k \rangle_{,j} + R_{jk} \langle U_k \rangle_{,i} ) \end{aligned} \quad (8.1)$$

However, a few questions naturally arise. First of all, do "slow terms" enable to retrieve "return to isotropy" process which appears to be an important feature actually occurring in basic experimental apparatus ? Second, this closure obviously no longer contains tensorial forms proportionnal to either  $(\delta_{ij})$  or  $(\langle U_i \rangle_{,j} + \langle U_j \rangle_{,i})$  ; does that mean that the latter may not appear in any formal development ? Third, and though it also does not contain any  $(\delta_{ij})$  or  $(\langle U_i \rangle_{,j} + \langle U_j \rangle_{,i})$  terms, another proposal described in /11/ (see /12/ also), which fulfils realisability requirement writes :

$$\begin{aligned} \Phi_{ij}^{SL} = & \frac{-6}{5 q^2} ( R_{lk} \langle U_l \rangle_{,k} ) R_{ij} + \frac{3}{5} ( R_{ik} \langle U_j \rangle_{,k} + R_{jk} \langle U_i \rangle_{,k} ) \\ & + \frac{2}{5 q^2} ( R_{ik}^2 \langle U_j \rangle_{,k} + R_{jk}^2 \langle U_i \rangle_{,k} - (R_{jk} R_{li} + R_{ik} R_{lj}) \langle U_l \rangle_{,k} ) \end{aligned} \quad (8.2)$$

which actually differs from the one above ; does the whole make emerge any contradiction ?

As far as second and third items are concerned, it seems that the answer dwells in a broadened frame of strongly realisable closures :

$$\begin{aligned} \phi_{ij} = & 2\beta_0 R_{ij} + 2\beta_1 R_{ij}^2 + 2\beta_2 R_{ij}^3 + \beta_3 (RB + B^t R)_{ij} \\ & + \beta_4 (B R + R B^t)_{ij} + \beta_5 (R^2 B + B^t R^2)_{ij} \\ & + (\beta_6 + \beta_8) (R(B + B^t)R)_{ij} + \beta_7 (R^2 B^t + B R^2)_{ij} + \beta_9 (R^3 B + B^t R^3)_{ij} \\ & + \beta_{10} (R^2 B^t R + R B R^2)_{ij} + \beta_{11} (R^3 B^t + B R^3)_{ij} + \beta_{12} (R^2 B R + R B^t R^2)_{ij} \\ & + \beta_{13} (R^2 B R + R B^t R^2)_{ij} + \beta_{14} (R^2 B^t R + R B R^2)_{ij} \end{aligned} \quad (8.3)$$

where  $\beta_i$  's functions should obey some constraints and  $\mathbf{B}$  should remain bounded (see appendix 1 for proof). Using Cayley-Hamiltonian identity provides a counterpart of the latter which makes "explicitely" arise  $(\delta_{ij})$  or  $(\langle U_i \rangle_{,j} + \langle U_j \rangle_{,i})$  contributions. Even more, it must be noted that both (8.1) and (8.2) are included in the latter.

If we turn now to return-to-isotropy topic, it may be checked that the behaviour of (realisable) solutions of :

$$(R_{ij})_{,t} = \alpha_2 R_{ij} + \alpha_3 R_{ij}^2 \quad (9)$$

is such that return to isotropy is achieved in the purely two-dimensional case, and in a weaker sense in the three-dimensional case, provided that first and second (negative) co-factor

functions are chosen in a suitable way. Previous result still holds in the three dimensional case, when dealing with the most general expansion of "slow" part :

$$(R_{ij})_{,t} = \alpha_2 R_{ij} + \alpha_3 R_{ij}^2 + \alpha_4 R_{ij}^3$$

provided that third co-factor function remains negative. Proof is given in appendix 3. Note that the present analysis also confirms the need for boundedness of the inverse of turbulent (mechanical) time scale  $\tau_k$ . The anisotropy of the Reynolds stress tensor is indeed a very important feature since traceless part of  $R$  which writes :

$$a_{ij} = R_{ij} - \frac{I}{3} \delta_{ij}$$

explicitly contributes to the budget of rotational of mean velocity field :

$$\Omega_{i,t} + \langle U_j \rangle \Omega_{i,j} + \Omega_l \langle U_i \rangle_{,l} - \nu \Omega_{i,ll} = - \varepsilon_{ijk} (a_{lj})_{,lk}$$

(noting :  $\Omega_i = \varepsilon_{ijk} (\langle U_j \rangle)_{,k}$ )

### III A simple Gaussian closure for non isothermal turbulent flows

We examine here a very simple so-called Gaussian closure, setting :

$$d_{ijk} = \langle u_i u_j u_k \rangle = 0 ; d_{ik} = \langle u_i \theta u_k \rangle = 0 ; d_k = \langle \theta \theta u_k \rangle = 0$$

and :

$$\Phi_{ij} = \alpha_2 R_{ij} + \alpha_3 R_{ij}^2 + \alpha_5 (R_{ik} \langle U_j \rangle_{,k} + R_{jk} \langle U_i \rangle_{,k}) + \alpha_6 (R_{ik} \langle U_k \rangle_{,j} + R_{jk} \langle U_k \rangle_{,i})$$

$$\Phi_i = \frac{(\alpha_2 + \beta_1)}{2} X_i + \alpha_3 (R_{ik} X_k - \frac{(X_k X_k)}{2 \langle \theta^2 \rangle} X_i) + \alpha_5 X_k \langle U_i \rangle_{,k} + \alpha_6 X_k \langle U_k \rangle_{,i}$$

$$\Phi = \beta_1 \langle \theta^2 \rangle$$

with :

$$\alpha_2 = \alpha_2' (I_R, II_R, III_R) \frac{\tilde{\varepsilon}}{I_R} ; \alpha_3 = \alpha_3' (I_R, II_R, III_R) \frac{\tilde{\varepsilon}}{II_R}$$

$$\alpha_i = \alpha_i' (I_R, II_R, III_R) \text{ for } i = 5, 6$$

$$\beta_1 = \beta_1' (I_R, II_R, III_R) \frac{\varepsilon \theta}{\langle \theta^2 \rangle}$$

where all primed functions stand for non dimensional bounded functions. We have the following result, defining :

$$C_{ij} = X_i \langle T \rangle_{,j} / \langle \theta^2 \rangle \quad (10)$$

#### **Prop. III :**

Provided that the inverses of turbulent mechanical and scalar time scales, and that  $B_{ij}$ 's and  $C_{ij}$ 's remain bounded, set (2) associated to above Gaussian closure is strongly realisable.

This represents a straightforward extension of the one stated in /5/. The proof may be obtained as follows. First, (2.4, 2.5, 2.6) may be rewritten in :

$$(Z_{ij})_{,t} + \langle U_k \rangle (Z_{ij})_{,k} + Z_{ik} \langle U_j \rangle_{,k} + Z_{jk} \langle U_i \rangle_{,k} + 2 Z_{ij} \left( \frac{X_k \langle T \rangle_{,k}}{\langle \theta^2 \rangle} \right) - Z_{ik} \left( \frac{X_j \langle T \rangle_{,k}}{\langle \theta^2 \rangle} \right) - Z_{jk} \left( \frac{X_i \langle T \rangle_{,k}}{\langle \theta^2 \rangle} \right) = RHS_{ij}$$

$$\langle \theta^2 \rangle_{,t} + \langle U_k \rangle \langle \theta^2 \rangle_{,k} + 2 X_k \langle T \rangle_{,k} = \Phi$$

$$(X_i)_{,t} + \langle U_k \rangle (X_i)_{,k} + X_k \langle U_i \rangle_{,k} + R_{ik} \langle T \rangle_{,k} - \beta_i \langle \theta^2 \rangle = \Phi_i$$

noting :

$$RHS_{ij} = \alpha_5 (Z_{ik} \langle U_j \rangle_{,k} + Z_{jk} \langle U_i \rangle_{,k}) + \alpha_6 (Z_{ik} \langle U_k \rangle_{,j} + Z_{jk} \langle U_k \rangle_{,i}) + (\alpha_2 + \beta_1) Z_{ij} + \alpha_3 \frac{Z_{ij}^2}{\langle \theta^2 \rangle}$$

Introducing :

$$H_{kj} = (\alpha_5 - 1) \langle U_j \rangle_{,k} + \alpha_6 \langle U_k \rangle_{,j} + C_{jk} + \left( \frac{\alpha_2 + \beta_1}{2} - C_{ll} \right) \delta_{jk} + \alpha_3 \frac{Z_{kj}}{2 \langle \theta^2 \rangle}$$

enables to rewrite :

$$(Z_{ij})_{,t} + \langle U_k \rangle (Z_{ij})_{,k} = Z_{ik} H_{kj} + Z_{jk} H_{ki}$$

and :

$$\langle \theta^2 \rangle_{,t} + \langle U_k \rangle \langle \theta^2 \rangle_{,k} = (\beta_1 - 2 C_{ll}) \langle \theta^2 \rangle$$

The latter enables to ensure that the scalar variance will remain positive, due to the boundedness of both inverse of turbulent scalar time scale and  $C_{ij}$  's components. Moreover, applying for appendix 1 permits to conclude that the model is strongly realisable, provided that, in addition, inverse of turbulent mechanical time scale and the  $B_{ij}$  's remain bounded, since :

$$\alpha_3 \frac{Z_{kj}}{2 \langle \theta^2 \rangle} = \alpha'_3 (I_R, II_R, III_R) \left( \frac{\tilde{\varepsilon}}{I_R} \right) \left( \frac{I_R^2}{2II_R} \right) \left( \frac{R_{kj}}{I_R} - \frac{X_j X_k}{I_R \langle \theta^2 \rangle} \right)$$

with :

$$1 \leq \frac{I_R^2}{II_R} \leq 3 \quad \text{and :} \quad \text{abs} \left( \frac{X_j X_k}{I_R \langle \theta^2 \rangle} \right) \leq \frac{R_{jj}^{1/2} R_{kk}^{1/2}}{I_R} \leq \frac{1}{2}$$

It must be underlined anyway that the boundedness of  $C_{ij}$  components does not arise naturally since Schwarz inequalities only provide :

$$\text{abs} (C_{ij}) \leq R_{ii}^{1/2} \frac{\text{abs} (\langle T \rangle_{,j})}{\langle \theta^2 \rangle^{1/2}}$$

This will be implicitly revisited in section V.

#### IV - The eigenvalue problem

We focus now on the following non viscous realisable system while neglecting zeroth order terms associated to gravity, which writes :

$$\langle U_i \rangle_{,i} = 0 \quad (11)$$

$$\langle U_i \rangle_{,t} + \langle U_j \rangle \langle U_i \rangle_{,j} + (R_{ij})_{,j} + \langle p \rangle_{,i} = 0$$

$$(R_{ij})_{,t} + \langle U_k \rangle (R_{ij})_{,k} + P_{ij} - \Phi_{ij}^r = 0$$

$$(X_i)_{,t} + \langle U_k \rangle (X_i)_{,k} + X_k \langle U_i \rangle_{,k} + R_{ik} \langle T \rangle_{,k} - \Phi_i^r = 0$$

$$\langle \theta^2 \rangle_{,t} + \langle U_k \rangle \langle \theta^2 \rangle_{,k} + 2 X_k \langle T \rangle_{,k} = 0$$

$$\langle T \rangle_{,t} + \langle U_j \rangle \langle T \rangle_{,j} + X_{j,j} = 0$$

noting :

$$\Phi_{ij}^r = \alpha_5 (R_{ik} \langle U_j \rangle_{,k} + R_{jk} \langle U_i \rangle_{,k}) + \alpha_6 (R_{ik} \langle U_k \rangle_{,j} + R_{jk} \langle U_k \rangle_{,i})$$

$$\Phi_i^r = \alpha_5 X_k \langle U_i \rangle_{,k} + \alpha_6 X_k \langle U_k \rangle_{,i}$$

We assume here that :

$$\alpha_i = \alpha'_i (I_R, II_R, III_R) \text{ for } : i = 5, 6$$

We consider two-dimensional turbulence i.e. :

$$\langle U_3 \rangle (x, y, z, t) = 0 \quad (H1)$$

$$\langle u_1 u_3 \rangle (x, y, z, t) = \langle u_2 u_3 \rangle (x, y, z, t) = \langle \theta u_3 \rangle (x, y, z, t) = 0 \quad (H2)$$

$$\langle \phi \rangle_{,3} (x, y, z, t) = 0, \text{ whatever } \phi \text{ stands for.} \quad (H3)$$

Provided that zero Dirichlet boundary conditions and initial conditions are retained, both (H1)

and (H2) are consistent with (11), which may be rewritten in the form :

$$A W_{,t} + A_x W_{,x} + A_y W_{,y} = 0$$

while defining :

$$W^t = (\langle U_1 \rangle, \langle U_2 \rangle, \langle u_1 u_1 \rangle, \langle u_2 u_2 \rangle, \langle u_1 u_2 \rangle, \langle u_3 u_3 \rangle, \langle p \rangle, \langle \theta u_1 \rangle, \langle \theta u_2 \rangle, \langle \theta^2 \rangle, \langle T \rangle) \in R_{11}$$

$$A \in R_{11 \times 11}, A_x \in R_{11 \times 11}, A_y \in R_{11 \times 11}$$

This enables to state :

#### **Prop. IV :**

System (11) is such that :

$$\forall (n_x, n_y) \in R^2 / n_x^2 + n_y^2 = 1, \det(n_x A_x + n_y A_y - \lambda A) = 0 \Rightarrow \lambda \in R$$

if and only if :

$$\forall (n_x, n_y) \in R_2, (n_x, n_y) \begin{pmatrix} \langle u_1 u_1 \rangle & \langle u_1 u_2 \rangle \\ \langle u_1 u_2 \rangle & \langle u_2 u_2 \rangle \end{pmatrix} \begin{pmatrix} n_x \\ n_y \end{pmatrix} \geq 0$$

and :

$$\forall n^t = (n_x, n_y) \in R^2, n^t C n \geq 0$$

noting :

$$C = (1 - \alpha_5) \begin{pmatrix} \langle u_1 u_1 \rangle & \langle u_1 u_2 \rangle \\ \langle u_1 u_2 \rangle & \langle u_2 u_2 \rangle \end{pmatrix} - \alpha_6 \begin{pmatrix} \langle u_2 u_2 \rangle & -\langle u_1 u_2 \rangle \\ -\langle u_1 u_2 \rangle & \langle u_1 u_1 \rangle \end{pmatrix}$$

This deserves a few remarks. First condition stands for standard over-realisability requirement ; besides, second condition identifies with the one associated to the purely isothermal turbulent case. Moreover, both of these only require positivity of  $R$ -eigenvalues, which means that system (11) will remain hyperbolic even if determinant of submatrix below

$$\begin{pmatrix} \langle u_1 u_1 \rangle & \langle u_1 u_2 \rangle & \langle \theta u_1 \rangle \\ \langle u_1 u_2 \rangle & \langle u_2 u_2 \rangle & \langle \theta u_2 \rangle \\ \langle \theta u_1 \rangle & \langle \theta u_2 \rangle & \langle \theta^2 \rangle \end{pmatrix}$$

takes (strictly) negative values. Note that the independance of the  $\lambda_i$ 's with respect to thermal correlations might be expected since governing equation for instantaneous temperature weakly couples with the dynamical field. It should also be underlined that above proposition still holds when accounting for  $\varepsilon$  variable while adding

$$(\varepsilon)_{,t} + \langle U_k \rangle (\varepsilon)_{,k} + C_{\varepsilon 1} \frac{\varepsilon}{K} P_{kk} = 0 \quad (12)$$

to previous system (11). Before going further on, we now set :

$$(1 - \alpha_5) \geq 0$$

$$- \alpha_6 \geq 0$$



### V Analysis of the generalised convection system

It looks of prime importance to have deeper insight in the behaviour of Reynolds stress tensor (and also thermal correlations). During past years, considerable efforts of workers have been devoted to the analysis of Reynolds stress asymptotical behaviour within the frame of homogeneous turbulence, i.e. discussing basic system :

$$(R_{ij})_{,t} = \phi_{ij}^{slow} (R_{kl}, \varepsilon)$$

However, very little is known about (non linear) generalised convection system issuing from (11). In section III, it was suggested that suitable forms for  $\phi_{ij}^{rapid}$ ,  $\phi_i^{rapid}$  enable to ensure realisability of Reynolds stress tensor and thermal correlations. However, this is true under the assumption of boundedness of turbulent mechanical and thermal time scales  $\tau_K$  and  $\tau_\theta$  (involved in slow part), but also requires that all Bij 's and Cij 's are bounded, even when so-called "rapid terms" are neglected (due to the presence of  $P_{ij}$ ). Hence, what can be said about behaviour, and moreover about realisability of Reynolds stress tensor, viewed as a solution of the first order initial-value problem issuing from (11), which writes :

$$\langle U_i \rangle_{,t} + \langle U_j \rangle \langle U_i \rangle_{,j} + (R_{ij})_{,j} = 0 \quad (13)$$

$$(R_{ij})_{,t} + \langle U_k \rangle (R_{ij})_{,k} + P_{ij} - \Phi_{ij}^r = 0$$

$$(X_i)_{,t} + \langle U_k \rangle (X_i)_{,k} + X_k \langle U_i \rangle_{,k} + R_{ik} \langle T \rangle_{,k} - \Phi_i^r = 0$$

$$\langle \theta^2 \rangle_{,t} + \langle U_k \rangle \langle \theta^2 \rangle_{,k} + 2 X_k \langle T \rangle_{,k} = 0$$

$$\langle T \rangle_{,t} + \langle U_j \rangle \langle T \rangle_{,j} + X_{j,j} = 0$$

This will be discussed afterwards, assuming that :

$$\Phi_{ij}^r = \alpha_5 (R_{ik} \langle U_j \rangle_{,k} + R_{jk} \langle U_i \rangle_{,k}) + \alpha_6 (R_{ik} \langle U_k \rangle_{,j} + R_{jk} \langle U_k \rangle_{,i})$$

$$\Phi_i^r = \alpha_5 X_k \langle U_i \rangle_{,k} + \alpha_6 X_k \langle U_k \rangle_{,i}$$

and still setting :

$$\alpha_i = \alpha'_i (I_R, II_R, III_R) \text{ for } : i = 5, 6$$

Moreover, an additional hypothesis will be made (to ensure objectivity requirement) :

$$\alpha'_5 (I_R, II_R, III_R) = \alpha'_6 (I_R, II_R, III_R) \quad (14)$$

We even more restrain our attention to the two-dimensional case, as defined in section IV, and use positivity constraint (see section IV) :

$$- \alpha_6 \geq 0$$

Now, we note :

$$W^t = (\langle U \rangle, \langle V \rangle, \langle u^2 \rangle, \langle v^2 \rangle, \langle uv \rangle, \langle \theta u \rangle, \langle \theta v \rangle, \langle \theta^2 \rangle, \langle T \rangle) \quad (15)$$

$$\text{and } : \beta = 1 - 2 \alpha_5$$

and introduce the non viscous problem :

$$\langle U \rangle_{,t} + \langle U \rangle \langle U \rangle_{,x} + \langle V \rangle \langle U \rangle_{,y} + \langle u^2 \rangle_{,x} + \langle uv \rangle_{,y} = 0 \quad (16)$$

$$\langle V \rangle_{,t} + \langle U \rangle \langle V \rangle_{,x} + \langle V \rangle \langle V \rangle_{,y} + \langle uv \rangle_{,x} + \langle v^2 \rangle_{,y} = 0$$

$$\langle u^2 \rangle_{,t} + \langle U \rangle \langle u^2 \rangle_{,x} + \langle V \rangle \langle u^2 \rangle_{,y} + 2\beta \langle u^2 \rangle \langle U \rangle_{,x} \dots$$

$$+ 2(1 - \alpha_5) \langle uv \rangle \langle U \rangle_{,y} - 2\alpha_5 \langle uv \rangle \langle V \rangle_{,x} = 0$$

$$\langle v^2 \rangle_{,t} + \langle U \rangle \langle v^2 \rangle_{,x} + \langle V \rangle \langle v^2 \rangle_{,y} + 2\beta \langle v^2 \rangle \langle V \rangle_{,y}$$

$$+ 2(1 - \alpha_5) \langle uv \rangle \langle V \rangle_{,x} - 2\alpha_5 \langle uv \rangle \langle U \rangle_{,y} = 0$$

$$\langle uv \rangle_{,t} + \langle U \rangle \langle uv \rangle_{,x} + \langle V \rangle \langle uv \rangle_{,y} + \beta \langle uv \rangle (\langle U \rangle_{,x} + \langle V \rangle_{,y})$$

$$+ (1 - \alpha_5) (\langle v^2 \rangle \langle U \rangle_{,y} + \langle u^2 \rangle \langle V \rangle_{,x}) - \alpha_5 (\langle v^2 \rangle \langle V \rangle_{,x} + \langle u^2 \rangle \langle U \rangle_{,y}) = 0$$

$$\langle \theta u \rangle_{,t} + \langle U \rangle \langle \theta u \rangle_{,x} + \langle V \rangle \langle \theta u \rangle_{,y} + \beta \langle \theta u \rangle \langle U \rangle_{,x} + \langle u^2 \rangle \langle T \rangle_{,x} + \langle uv \rangle \langle T \rangle_{,y}$$

$$- \alpha_5 \langle \theta v \rangle \langle V \rangle_{,x} + (1 - \alpha_5) \langle \theta v \rangle \langle U \rangle_{,y} = 0$$

$$\langle \theta v \rangle_{,t} + \langle U \rangle \langle \theta v \rangle_{,x} + \langle V \rangle \langle \theta v \rangle_{,y} + \beta \langle \theta v \rangle \langle V \rangle_{,y} + \langle uv \rangle \langle T \rangle_{,x} + \langle v^2 \rangle \langle T \rangle_{,y}$$

$$+ (1 - \alpha_5) \langle \theta u \rangle \langle V \rangle_{,x} - \alpha_5 \langle \theta u \rangle \langle U \rangle_{,y} = 0$$

$$\langle \theta^2 \rangle_{,t} + \langle U \rangle \langle \theta^2 \rangle_{,x} + \langle V \rangle \langle \theta^2 \rangle_{,y} + 2 \langle \theta u \rangle \langle T \rangle_{,x} + 2 \langle \theta v \rangle \langle T \rangle_{,y} = 0$$

$$\langle T \rangle_{,t} + \langle U \rangle \langle T \rangle_{,x} + \langle V \rangle \langle T \rangle_{,y} + \langle \theta u \rangle_{,x} + \langle \theta v \rangle_{,y} = 0$$

As happens in the isothermal case, it occurs that :

**Prop. V.1 :**

System (16) , which may be rewritten in the form :

$$W_{,t} + A_x W_{,x} + A_y W_{,y} = 0$$

is such that :

$$\forall (n_x, n_y) \in R^2 / n_x^2 + n_y^2 = 1, \det(n_x A_x + n_y A_y - \lambda I) = 0 \Rightarrow \lambda \in R$$

if and only if Reynolds stress tensor  $R$  is realisable and if :

$$1 - \alpha_5 \geq 0 \text{ and } : -\alpha_5 \geq 0$$

As underlined in section IV, over-realisability of  $R$  only is needed. However, we may get a better result, setting  $\beta$  to 1, and seeking one dimensional solutions of (16) independent of y-direction (setting :  $W_{,y} = 0$ ), which provides a similar system (see appendix 4) :

$$W_{,t} + A_x W_{,x} = 0 \tag{17}$$

for which we get :

**Prop. V.2 :**

\* System (17) is strictly hyperbolic ; its eigenvalues read:

$$\lambda_1 = \langle U \rangle - (2 \langle u^2 \rangle)^{1/2} \quad \lambda_{2,3} = \langle U \rangle - (\langle u^2 \rangle)^{1/2}$$

$$\lambda_{4,5,6} = \langle U \rangle$$

$$\lambda_{7,8} = \langle U \rangle + (\langle u^2 \rangle)^{1/2} \quad \lambda_9 = \langle U \rangle + (2 \langle u^2 \rangle)^{1/2}$$

\*\* The Riemann problem associated to system (17) and initial conditions :

$$W(x, t = 0) = W_L \quad \text{for } : x < 0$$

$$W(x, t = 0) = W_R \quad \text{for } : x > 0$$

does admit a unique solution as soon as initial values are such that :

$$\langle U \rangle_R - \langle U \rangle_L < 2^{1/2} (\langle u^2 \rangle_R^{1/2} + \langle u^2 \rangle_L^{1/2})$$

and the solution satisfies over-realisability requirement, provided that left and right (initial) states  $W_L$  and  $W_R$  fulfill it.

\* Despite from the fact that the proof requires tedious calculations, it happens to be quite simple (a sketch of the proof is given in appendix 4, and see /13/ for similar proof in the gas dynamics case) but deserves a few comments. First of all, jump conditions pertaining to governing equations of all correlations have been obtained choosing a path to connect two different states on each side of a discontinuity, due to the fact that these equations are not of conservative type. This path is given by (see /6/) :

$$\phi(W_L, W_R, s) = s W_R + (1-s) W_L$$

Unlike in some other models (see /4/, /9/, /16/), pseudo conservative equations may be derived here, while introducing new variables :

$$w_{ij} = \langle U_i \rangle \langle U_j \rangle + R_{ij} \quad (\text{instead of } R_{ij})$$

$$\Xi^2 = \langle \theta^2 \rangle + \langle T \rangle^2 \quad (\text{instead of } \langle \theta^2 \rangle)$$

$$y_i = \langle U_i \rangle \langle T \rangle + X_i \quad (\text{instead of } X_i)$$

and rewriting (16) as follows:

$$\langle U_i \rangle_{,t} + \langle U_j \rangle \langle U_i \rangle_{,j} + (w_{ij} - \langle U_j \rangle \langle U_i \rangle)_{,j} = 0$$

$$(w_{ij})_{,t} + \langle U_k \rangle (w_{ij})_{,k} + (\langle U_j \rangle w_{ik} + \langle U_i \rangle w_{jk} - 2 \langle U_i \rangle \langle U_j \rangle \langle U_k \rangle)_{,k} = 0$$

$$(y_i)_{,t} + \langle U_k \rangle (y_i)_{,k} + (y_k \langle U_i \rangle + w_{ik} \langle T \rangle - 2 \langle T \rangle \langle U_i \rangle \langle U_k \rangle)_{,k} = 0$$

$$\langle \Xi^2 \rangle_{,t} + \langle U_k \rangle \langle \Xi^2 \rangle_{,k} + (2 y_k \langle T \rangle - 2 \langle T \rangle^2 \langle U_k \rangle)_{,k} = 0$$

$$\langle T \rangle_{,t} + \langle U_j \rangle \langle T \rangle_{,j} + (y_j - \langle U_j \rangle \langle T \rangle)_{,j} = 0$$

This allows to introduce new jump conditions, which may be shown to be fully compatible with the ones used in appendix 4 (lemma 4.d).

Another point concerns weak entropy concept introduced in lemma 4.c to select the physically relevant solution to connect two different states on each side of a genuinely non linear field. It must be noted that the entropy function  $\eta$  is of course not strictly convex here since :

$$\eta(W) = \frac{1}{2} \langle U \rangle^2 + \langle u^2 \rangle$$

This however is sufficient to connect states through genuinely non linear fields.

\*\* The proof contains the construction of the solution, which *a posteriori* enables to check that over-realisability constraints are fulfilled :

$$(i) \quad \langle u^2 \rangle(x, t) \geq 0$$

$$(ii) (\langle u^2 \rangle \langle v^2 \rangle - \langle uv \rangle^2)(x, t) \geq 0$$

$$(iii) (\langle u^2 \rangle \langle v^2 \rangle \langle \theta^2 \rangle + 2 \langle uv \rangle \langle \theta u \rangle \langle \theta v \rangle - \langle \theta^2 \rangle \langle uv \rangle^2 - \langle u^2 \rangle \langle \theta v \rangle^2 - \langle v^2 \rangle \langle \theta u \rangle^2)(x, t) \geq 0$$

(which was not quite obvious since the solution is neither  $C^0$  nor  $C^1$ ). This seems to be a rather important result (see section VI for further implications) ; as a matter of fact, it might be argued, looking at the eigenvalues expression, that the only important feature concerns the positivity of normal component  $\langle u^2 \rangle$  ; actually, this is not sufficient. In order to feel convinced of that, just introduce a change of frame due to some (time-space independent) rotation ; hence, define some normalised (statistics independent) vectors  $\mathbf{n}$  and  $\boldsymbol{\tau}$  setting :

$$\mathbf{n} = (n_x, n_y) \quad \text{and} \quad \boldsymbol{\tau} = (-n_y, n_x) / n_x^2 + n_y^2 = 1$$

and also define transformed variables :

$$\langle \mathcal{U} \rangle = n_x \langle U \rangle + n_y \langle V \rangle \quad \text{and} \quad \langle \mathcal{V} \rangle = -n_y \langle U \rangle + n_x \langle V \rangle$$

$$\begin{pmatrix} \langle u'^2 \rangle \\ \langle v'^2 \rangle \\ \langle u'v' \rangle \end{pmatrix} = \begin{pmatrix} n_x^2 & n_y^2 & 2 n_x n_y \\ n_y^2 & n_x^2 & -2 n_x n_y \\ -n_x n_y & n_x n_y & n_x^2 - n_y^2 \end{pmatrix} \begin{pmatrix} \langle u^2 \rangle \\ \langle v^2 \rangle \\ \langle uv \rangle \end{pmatrix}$$

$$\langle \theta' u' \rangle = n_x \langle \theta u \rangle + n_y \langle \theta v \rangle \quad \text{and} \quad \langle \theta' v' \rangle = -n_y \langle \theta u \rangle + n_x \langle \theta v \rangle$$

System (16) is invariant under this frame rotation (see /14/ also) ; thus, the counterpart of the one dimensional Riemann problem associated with (17) in the  $\mathbf{n}$ -direction, is exactly the one examined in previous section, except for the fact that the initial conditions for new normal component :

$$\langle u'^2 \rangle_k = (n_x, n_y) \begin{pmatrix} \langle u^2 \rangle_k & \langle uv \rangle_k \\ \langle uv \rangle_k & \langle v^2 \rangle_k \end{pmatrix} \begin{pmatrix} n_x \\ n_y \end{pmatrix} \quad \text{for} \quad k = L, R$$

should be positive, whatever  $(n_x, n_y)$  stands for, which simply means that the Reynolds stress tensor should be over-realisable. Provided that the fluid is violently stretched so that condition involved in proposition V.2 is violated, some counterpart of gas dynamics vacuum (see /13/) will appear in the solution, which simply means that turbulence will locally vanish.

Some miscellaneous remarks follow. First of all, the  $\langle w^2 \rangle$  component has not been retained in previous analysis to simplify presentation ; anyway, associated governing equation writes :

$$\langle w^2 \rangle_{,t} + \langle U \rangle \langle w^2 \rangle_{,x} + \langle V \rangle \langle w^2 \rangle_{,y} = 0$$

It may be checked that results stated in prop. V.2 remain unchanged. A similar remark holds for turbulent mechanical dissipation  $\varepsilon$  , applying for (12). However, a somewhat annoying point is that the maximum principle for mean temperature (or scalar variance) does not hold through the convection system ; this is briefly discussed at the end of appendix 4.

We now wish to deal with (constant)  $\beta$  non equal to 1, in order to check that previous results pertaining to existence, uniqueness and realisability of weak solutions remain ; to achieve

this, we only examine the pure one dimensional shock tube apparatus, setting initial conditions to :

$$W_L^t = (\langle U \rangle_L, \langle V \rangle_L = 0, \langle u^2 \rangle_L, \langle v^2 \rangle_L = 0, \langle uv \rangle_L = 0, \langle \theta u \rangle_L, \langle \theta v \rangle_L = 0, \langle \theta^2 \rangle_L, \langle T \rangle_L)$$

$$W_R^t = (\langle U \rangle_R, \langle V \rangle_R = 0, \langle u^2 \rangle_R, \langle v^2 \rangle_R = 0, \langle uv \rangle_R = 0, \langle \theta u \rangle_R, \langle \theta v \rangle_R = 0, \langle \theta^2 \rangle_R, \langle T \rangle_R)$$

which enables to get :

**Prop. V.3 :**

The Riemann problem associated to system (17) and initial conditions :

$$W(x, t=0) = W_L \quad \text{for : } x < 0$$

$$W(x, t=0) = W_R \quad \text{for : } x > 0$$

does admit a unique solution as soon as initial conditions are such that :

$$\langle U \rangle_R - \langle U \rangle_L < \frac{2^{1/2}}{\beta^{1/2}} (\langle u^2 \rangle_R^{1/2} + \langle u^2 \rangle_L^{1/2})$$

The solution satisfies over-realisability requirement, provided that initial conditions do.

Main lines of the proof stand in appendix 2. In this case, the entropy-entropy flux pair (see appendix 2, sections c) which is used to select the admissible solution writes :

$$\eta(W) = \frac{2\beta-1}{2} \langle U \rangle^2 + \langle u^2 \rangle ; \quad F_\eta(W) = \frac{2\beta-1}{3} \langle U \rangle^3 + 2\beta \langle u^2 \rangle \langle U \rangle$$

Previous remarks pertaining to the loss of maximum principle for mean temperature and scalar variance, or vacuum occurrence still hold.

## VI Discussion

From now on, we no longer account for thermal effects. In order to solve laminar set (1.1, 1.2), mainly two different numerical approaches are widely used. The first one, which is sometimes referred to as the projection method, has been initially proposed in /3/ and /15/ ; extensions of the original first-order in time scheme to second order in time schemes were recently suggested in /1/and /10/ among others. The basic idea contained in all these schemes is to compute an intermediate velocity field which is then projected onto a divergence free field. The second one, which is somewhat different, treats separately linear and non linear operators, while solving Stokes problem and convection system in two different steps. In /2/, convection effects are accounted for, solving a linear initial value problem :

$$\phi_{,t} + U_i^*(\mathbf{x}) \phi_{,i} = 0 \quad \text{for } (\mathbf{x}, t) \in \Omega_x (t^n, t^n + \Delta t) \quad (18)$$

(where  $\phi$  stands for any velocity component) while using the characteristic method (and appropriate boundary conditions) ; the starred field is chosen to be divergence free vector (at time n for instance). Once this has been computed, a generalised Stokes problem is solved which writes :

$$\begin{aligned} (U_i^{n+1}(\mathbf{x}))_{,i} &= 0 \\ U_i^{n+1}(\mathbf{x}) - \Delta t \left( \frac{-1}{\rho_0} p_{,i}^{n+1}(\mathbf{x}) + \nu (U_i^{n+1}(\mathbf{x}))_{,jj} \right) &= \tilde{U}_i(\mathbf{x}) \end{aligned} \quad (19)$$

where right hand side accounts for velocity field issuing from (18). This step requires to choose pressure and velocity fields in suitable functional spaces, so that the inf-sup condition holds, either in a continuous way or in a discrete sense, when applying for finite element methodology. We refer for instance to /2/ which provides details of the implementation in an industrial CFD code. It should be underlined that convection step is sometimes treated in a non linear way, solving the counterpart of (18), i.e. :

$$U_{i,t} + U_j U_{i,j} = 0 \quad \text{for } (\mathbf{x}, t) \in \Omega_x (t^n, t^n + \Delta t)$$

by means of a Godunov-type scheme (see /1/ for instance).

We turn now to the turbulent system (2.1, 2.2, 2.4), still setting  $\beta_1$  to zero. According to the litterature, a commonly used algorithm seems to apply for the "pseudo-viscosity" method ; this will not be detailed here but it is usually argued that this represents an efficient way to stabilise the whole algorithm, since (2.2) does not contain enough viscosity. Actually, it is implicitly assumed in this kind of approach that the real convection system still writes (as in the purely laminar case) :

$$\phi_{,t} + \langle U_j \rangle \phi_{,j} = 0 \quad \text{for } (\mathbf{x}, t) \in \Omega_x (t^n, t^n + \Delta t)$$

where  $\phi$  stands for any variable among mean velocity or Reynolds stress components. When using finite difference or finite volume formulations, it is also usually wondered whether

Reynolds stress components and mean velocity components should have the same location, and how so-called production terms (namely  $P_{ij}$ ) should be discretized, and so on. However, examining results of previous section suggests that straight extension of laminar algorithms might hold, successively solving a non linear non homogeneous first order differential system :

$$\langle U_i \rangle_{,t} + \langle U_j \rangle \langle U_i \rangle_{,j} + (R_{ij})_{,j} = 0 \quad (20)$$

$$(R_{ij})_{,t} + \langle U_k \rangle (R_{ij})_{,k} + P_{ij} - \Phi_{ij}^r = \Phi_{ij}^{slow}$$

(for  $(\mathbf{x}, t) \in \Omega \mathbf{x} (t^n, t^{n+1})$ ) while applying for Godunov scheme (which requires solving the Riemann problems detailed in prop. V.2 or V.3), and a symmetrical Stokes problem similar to (19) :

$$(\langle U_i \rangle^{n+1}(\mathbf{x}))_{,i} = 0 \quad (21)$$

$$\langle U_i \rangle^{n+1}(\mathbf{x}) - \Delta t \left( \frac{-1}{\rho_0} \langle p \rangle_{,i}^{n+1}(\mathbf{x}) + \nu (\langle U_i \rangle^{n+1}(\mathbf{x}))_{,jj} \right) = \overline{\langle U_i \rangle}(\mathbf{x})$$

In practice, the Godunov solver will act in many regions of the physical domain as a simple upwinding technique, due to the fact that turbulent kinetic energy is usually small compared with the square of the norm of the mean velocity. Constraint on the time stepping which happens to prevent waves interaction, would write :

$$2 \max (|\langle U \rangle| + (2 \langle u^2 \rangle)^{1/2}) \Delta t \leq h$$

Recall that one of the main advantages of exact Godunov solver is that it essentially preserves occurrence of negative values of realisable quantities, which is not the case when approximate Riemann solvers are considered (this is a very well-known feature for "compressible" community, in the laminar case). Moreover, it enables to derive suitable boundary condition treatment. Anyway, it contains one drawback since it requires to solve a non linear scalar equation (see appendix 2, section e) per mesh interface per time step. It must also be emphasised that simple algorithms to account for "slow terms" contribution may be exhibited which again enable to ensure discrete preservation of over-realisability concept.

## **Conclusion**

Intrinsic over-realizability conditions have been recalled here and extrinsic requirements have also been presented, which essentially consists in maximum principle for mean temperature and scalar variance. Some Gaussian second moment closures have been presented and discussed, either considering isothermal or non isothermal flows. These enable to gain strong realizability when dealing with smooth enough solutions and a rather wide class of so-called "slow" contributions which favour return-to-isotropy is exhibited. As in the purely isothermal case, it occurs that there is a strong link between over-realizability constraint and hyperbolicity requirement. While applying for some recent theoretical results (see /6/), the first order non conservative initial-value problems encountered in the whole system have been investigated ; it has been shown that the one dimensional Riemann problem does admit a unique realisable solution. This result might provide a useful (realisable!) numerical tool to investigate incompressible turbulent shear flows, especially when aiming at deriving finite-volume formulations on unstructured meshes (while using colocated arrangement). This of course requires numerical investigation. The loss of the maximum principle for mean temperature and scalar variance is due to the fact that the Reynolds decomposition has been used for instantaneous temperature, which results in the occurrence of a scalar turbulent flux  $X_j$ , together with the fact that no gradient type theory has been retained here to model the latter. It has also been briefly mentioned here that investigation of generalised convection problems might indeed lead to a better understanding of the transition between turbulent and laminar regime.



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### Appendix 1 :

Let us consider the following equation :

$$(Z_{ij})_{,t} + \langle U_k \rangle (Z_{ij})_{,k} = \Phi_{ij} \quad (I)$$

where :

$$\Phi_{ij} = Z_{ik} H_{kj} + Z_{jk} H_{ki} \quad (II)$$

Note that  $Z$  is symmetrical but  $H$  is not necessarily symmetrical.

a) Recall that :

$$\text{For } i=1 \rightarrow 3 : \Delta_1^i = Z_{ii}$$

$$\text{For } i=1 \rightarrow 3 : \Delta_2^i = Z_{\alpha\alpha} Z_{\beta\beta} - (Z_{\alpha\beta})^2 \quad (\alpha \neq \beta ; \alpha \neq i ; \beta \neq i)$$

$$\Delta_3 = \det (Z_{ij})$$

Then (I) enables to rewrite :

$$(\Delta_1^1)_{,t} + \langle U_k \rangle (\Delta_1^1)_{,k} = 2 \Delta_1^1 H_{11} + 2 (Z_{12} H_{21}) + 2 (Z_{13} H_{31}) \quad (1.1)$$

$$(\Delta_2^3)_{,t} + \langle U_k \rangle (\Delta_2^3)_{,k} = 2 \Delta_2^3 (H_{11} + H_{22}) + 2 (Z_{22} Z_{13} - Z_{12} Z_{23}) H_{31} \\ + 2 (Z_{11} Z_{23} - Z_{12} Z_{13}) H_{32} \quad (2.3)$$

$$(\Delta_3)_{,t} + \langle U_k \rangle (\Delta_3)_{,k} = 2 \Delta_3 (H_{11} + H_{22} + H_{33}) \quad (3)$$

Equation (3) shows that  $\Delta_3$  remains positive (see /5/). Hence, noting that :

$$(Z_{12})^2 \leq \Delta_1^1 \Delta_1^2 \\ (Z_{13})^2 \leq \Delta_1^1 \Delta_1^3 \quad (III.1)$$

$$(Z_{22} Z_{13} - Z_{12} Z_{23})^2 \leq \Delta_2^3 \Delta_2^1 \\ (Z_{11} Z_{23} - Z_{12} Z_{13})^2 \leq \Delta_2^3 \Delta_2^2 \quad (III.2)$$

enables to rewrite (1.1) and (2.3) in the following way :

$$(\Delta_1^1)_{,t} + \langle U_k \rangle (\Delta_1^1)_{,k} = (\Delta_1^1)^{1/2} (\Delta_1^1)^{1/2} a_{11} \\ + (\Delta_1^1)^{1/2} (\Delta_1^2)^{1/2} a_{12} + (\Delta_1^1)^{1/2} (\Delta_1^3)^{1/2} a_{13} \quad (1.1)$$

$$(\Delta_2^3)_{,t} + \langle U_k \rangle (\Delta_2^3)_{,k} = (\Delta_2^3)^{1/2} (\Delta_2^3)^{1/2} b_{33} \\ + (\Delta_2^1)^{1/2} (\Delta_2^3)^{1/2} b_{13} + (\Delta_2^2)^{1/2} (\Delta_2^3)^{1/2} b_{23} \quad (2.3)$$

while noting :

$$a_{11} = 2 H_{11}$$

$$a_{12} = 2 Z_{12} (\Delta_1^1)^{-1/2} (\Delta_1^2)^{-1/2} H_{21}$$

$$a_{13} = 2 Z_{13} (\Delta_1^1)^{-1/2} (\Delta_1^3)^{-1/2} H_{31}$$

$$b_{33} = 2 (H_{11} + H_{22})$$

$$b_{13} = 2 (Z_{22} Z_{13} - Z_{12} Z_{23}) (\Delta_2^3)^{-1/2} (\Delta_2^1)^{-1/2} H_{31}$$

$$b_{23} = 2 (Z_{11} Z_{23} - Z_{12} Z_{13}) (\Delta_2^3)^{-1/2} (\Delta_2^2)^{-1/2} H_{32}$$

Moreover, using permutations over 1, 2 and 3 provides :

$$(\Delta_1^i)_{,t} + \langle U_k \rangle (\Delta_1^i)_{,k} = \prod_{j=1}^3 (\Delta_1^i)^{1/2} (\Delta_1^j)^{1/2} a_{ij} \quad (1.i)$$

$$(\Delta_2^i)_{,t} + \langle U_k \rangle (\Delta_2^i)_{,k} = \prod_{j=1}^3 (\Delta_2^i)^{1/2} (\Delta_2^j)^{1/2} b_{ij} \quad (2.i)$$

Obviously, accounting for (III.1) and (III.2) enables to check that all  $a_{ij}$  and  $b_{ij}$  are bounded as soon as all  $H_{ij}$  are bounded. Hence, at some point where  $\Delta_k^i$  vanishes :

$$D_t (\Delta_k^i) = (\Delta_k^i)_{,t} + \langle U_l \rangle (\Delta_k^i)_{,l} = 0 \quad \text{and} \quad : D_t (D_t (\Delta_k^i)) \geq 0$$

b) Let us now set :  $Z = R$  and let us suppose that the formal development  $H(B, R)$  holds :

$$\begin{aligned} H_{ij} = & \beta_0 \delta_{ij} + \beta_1 R_{ij} + \beta_2 R_{ij}^2 + \beta_3 B_{ij} + \beta_4 (B^t)_{ij} + \beta_5 (RB)_{ij} + \beta_6 (BR)_{ij} \\ & + \beta_7 (RB^t)_{ij} + \beta_8 (B^t R)_{ij} + \beta_9 (R^2 B)_{ij} + \beta_{10} (BR^2)_{ij} + \beta_{11} (R^2 B^t)_{ij} + \beta_{12} (B^t R^2)_{ij} \\ & + \beta_{13} (RBR)_{ij} + \beta_{14} (R B^t R)_{ij} \end{aligned}$$

This yields :

$$\begin{aligned} \phi_{ij} = & 2\beta_0 R_{ij} + 2\beta_1 R_{ij}^2 + 2\beta_2 R_{ij}^3 + \beta_3 (RB + B^t R)_{ij} + \beta_4 (BR + RB^t)_{ij} + \beta_5 (R^2 B + B^t R^2)_{ij} \\ & + (\beta_6 + \beta_8)(R(B + B^t)R)_{ij} + \beta_7 (R^2 B^t + B R^2)_{ij} + \beta_9 (R^3 B + B^t R^3)_{ij} \\ & + \beta_{10} (R^2 B^t R + R B R^2)_{ij} + \beta_{11} (R^3 B^t + B R^3)_{ij} + \beta_{12} (R^2 B R + R B^t R^2)_{ij} \\ & + \beta_{13} (R^2 B R + R B^t R^2)_{ij} + \beta_{14} (R^2 B^t R + R B R^2)_{ij} \end{aligned}$$

Cayley-Hamilton identity provides :

$$R_{ij}^3 = I R_{ij}^2 + \frac{1}{2} (II - I^2) R_{ij} + \frac{1}{6} (I^3 + 2 III - 3 (I)(II)) \delta_{ij} \equiv a R_{ij}^2 + b R_{ij} + c \delta_{ij}$$

Hence,  $\Phi_{ij}$  may be rewritten :

$$\begin{aligned} \phi_{ij} = & 2\beta_2 c \delta_{ij} + 2(\beta_0 + \beta_2 b) R_{ij} + 2(\beta_1 + \beta_2 a) R_{ij}^2 + \\ & + c (\beta_9 + \beta_{11}) (B + B^t)_{ij} + (\beta_3 + b \beta_9) (RB + B^t R)_{ij} + (\beta_4 + b \beta_{11}) (BR + RB^t)_{ij} \\ & + (\beta_6 + \beta_8)(R(B + B^t)R)_{ij} + (\beta_7 + a \beta_{11}) (R^2 B^t + B R^2)_{ij} + (\beta_5 + a \beta_9) (B^t R^2 + R^2 B)_{ij} \\ & + \beta_{10} (R^2 B^t R + R B R^2)_{ij} + \beta_{12} (R^2 B R + R B^t R^2)_{ij} \\ & + \beta_{13} (R^2 B R + R B^t R^2)_{ij} + \beta_{14} (R^2 B^t R + R B R^2)_{ij} \end{aligned}$$

Objectivity requirement will be achieved if :

$$\beta_3 = \beta_4 \quad ; \quad \beta_5 = \beta_7 \quad ; \quad \beta_9 = \beta_{11} \quad ; \quad \beta_{10} = \beta_{12} \quad ; \quad \beta_{13} = \beta_{14}$$

Since the following inequalities hold :

$$II \leq I^2 \leq 3 II \quad ; \quad III \leq I^3 \leq 9 III$$

boundedness of  $H_{ij}$  will be achieved if :

$$\beta_0 = \beta_0' (I, II, III) \frac{\epsilon}{I} \quad ; \quad \beta_1 = \beta_1' (I, II, III) \frac{\epsilon}{II} \quad ; \quad \beta_2 = \beta_2' (I, II, III) \frac{\epsilon}{III}$$

$$\beta_3 = \beta_3' (I, II, III) \quad ; \quad \beta_4 = \beta_4' (I, II, III)$$

$$\beta_i = \beta_i' (I, II, III) (I)^{-1} \quad \text{for } i = 5 \text{ to } 8$$

$$\beta_i = \beta_i' (I, II, III) (I)^{-2} \quad \text{for } i = 9 \text{ to } 14$$

provided that  $(\epsilon/I)$  and  $B_{ij}$  remain bounded.

## Appendix 2 :

Let us consider the following non viscous problem (P) :

$$\langle U \rangle_{,t} + \langle U \rangle \langle U \rangle_{,x} + \langle u^2 \rangle_{,x} = 0 \quad (\text{P1})$$

$$\langle u^2 \rangle_{,t} + \langle U \rangle \langle u^2 \rangle_{,x} + 2\beta \langle u^2 \rangle \langle U \rangle_{,x} = 0 \quad (\text{P2})$$

$$\langle \theta u \rangle_{,t} + \langle U \rangle \langle \theta u \rangle_{,x} + \beta \langle \theta u \rangle \langle U \rangle_{,x} + \langle u^2 \rangle \langle T \rangle_{,x} = 0 \quad (\text{P3})$$

$$\langle \theta^2 \rangle_{,t} + \langle U \rangle \langle \theta^2 \rangle_{,x} + 2 \langle \theta u \rangle \langle T \rangle_{,x} = 0 \quad (\text{P4})$$

$$\langle T \rangle_{,t} + \langle U \rangle \langle T \rangle_{,x} + \langle \theta u \rangle_{,x} = 0 \quad (\text{P5})$$

and associated viscous perturbed system (Pε) obtained by changing (P1) in (P1ε) :

$$\langle U \rangle_{,t} + \langle U \rangle \langle U \rangle_{,x} + \langle u^2 \rangle_{,x} = \varepsilon \langle U \rangle_{,xx} \quad (\text{P1}\varepsilon)$$

We set :

$$W^t = (\langle U \rangle, \langle u^2 \rangle, \langle \theta u \rangle, \langle \theta^2 \rangle, \langle T \rangle)$$

Thus, (P) may be rewritten in the form :

$$W_{,t} + A(W) W_{,x} = 0$$

Besides, (Pε) writes :

$$W_{,t} + A(W) W_{,x} = \varepsilon (DW_{,x})_{,x}$$

noting :

$$A = \begin{pmatrix} \langle U \rangle & 1 & 0 & 0 & 0 \\ 2\beta \langle u^2 \rangle & \langle U \rangle & 0 & 0 & 0 \\ \beta \langle \theta u \rangle & 0 & \langle U \rangle & 0 & \langle u^2 \rangle \\ 0 & 0 & 0 & \langle U \rangle & 2 \langle \theta u \rangle \\ 0 & 0 & 1 & 0 & \langle U \rangle \end{pmatrix}$$

and :

$$D_{11} = 1 ; D_{ij} = 0 \text{ otherwise.}$$

The following initial conditions will be considered :

$$W(x, t=0) = W_L \quad \text{for : } x < 0$$

$$W(x, t=0) = W_R \quad \text{for : } x > 0$$

Note that β is given by :

$$\beta = 1 - 2\alpha_5 \quad \text{with : } -\alpha_5 \geq 0$$

which provides :  $\beta \geq 1$

a) Lemma 2.a :

System (P) is strictly hyperbolic and is characterized by the following set of eigenvalues :

$$\lambda_1 = \langle U \rangle - (2\beta \langle u^2 \rangle)^{1/2}$$

$$\lambda_2 = \langle U \rangle - (\langle u^2 \rangle)^{1/2}$$

$$\lambda_3 = \langle U \rangle$$

$$\lambda_4 = \langle U \rangle + (\langle u^2 \rangle)^{1/2}$$

$$\lambda_5 = \langle U \rangle + (2\beta \langle u^2 \rangle)^{1/2}$$

and associated right eigenvectors :

$$r_1^t = (1, -(2\beta \langle u^2 \rangle)^{1/2}, \frac{-(2\beta)^{1/2} \beta \langle \theta u \rangle}{(2\beta - 1) \langle u^2 \rangle^{1/2}}, \frac{-(2\beta)^{1/2} \langle \theta u \rangle^2}{(2\beta - 1) \langle u^2 \rangle^{3/2}}, \frac{\beta \langle \theta u \rangle}{(2\beta - 1) \langle u^2 \rangle})$$

$$r_2^t = (0, 0, -\langle u^2 \rangle^{1/2}, \frac{-2 \langle \theta u \rangle}{\langle u^2 \rangle^{1/2}}, 1)$$

$$r_3^t = (0, 0, 0, 1, 0)$$

$$r_4^t = (0, 0, \langle u^2 \rangle^{1/2}, \frac{2 \langle \theta u \rangle}{\langle u^2 \rangle^{1/2}}, 1)$$

$$r_5^t = (1, (2\beta \langle u^2 \rangle)^{1/2}, \frac{(2\beta)^{1/2} \beta \langle \theta u \rangle}{(2\beta - 1) \langle u^2 \rangle^{1/2}}, \frac{(2\beta)^{1/2} \langle \theta u \rangle^2}{(2\beta - 1) \langle u^2 \rangle^{3/2}}, \frac{\beta \langle \theta u \rangle}{(2\beta - 1) \langle u^2 \rangle})$$

From now on, it is assumed that  $\beta$  remains constant.

#### b) Lemma 2.b :

The 1 and 5 characteristic fields are genuinely nonlinear (noted GNL afterwards) and the 2, 3, 4 characteristic fields are linearly degenerate (noted LD afterwards). Moreover, each k-characteristic field does admit four independant Riemann invariants  $\Phi_k^i (i=1 \text{ to } 4)$  which are given by :

1-characteristic family :

$$\phi_1^1(W) = \langle U \rangle + \frac{(2\langle u^2 \rangle)^{1/2}}{\beta} ; \quad \phi_1^2(W) = \langle T \rangle + (2\beta)^{1/2} \frac{\langle \theta u \rangle}{\langle u^2 \rangle^{1/2}}$$

$$\phi_1^3(W) = \langle \theta^2 \rangle - \frac{\langle \theta u \rangle^2}{\langle u^2 \rangle} ; \quad \phi_1^4(W) = \frac{\langle \theta u \rangle}{\langle u^2 \rangle^{\beta/(2\beta-1)}}$$

2-characteristic family :

$$\phi_2^1(W) = \langle U \rangle ; \quad \phi_2^2(W) = \langle u^2 \rangle$$

$$\phi_2^3(W) = \langle \theta^2 \rangle - \frac{\langle \theta u \rangle^2}{\langle u^2 \rangle} ; \quad \phi_2^4(W) = \langle T \rangle + \frac{\langle \theta u \rangle}{\langle u^2 \rangle^{1/2}}$$

3-characteristic family :

$$\phi_3^1(W) = \langle U \rangle ; \quad \phi_3^2(W) = \langle u^2 \rangle ; \quad \phi_3^3(W) = \langle \theta u \rangle ; \quad \phi_3^4(W) = \langle T \rangle$$

4-characteristic family :

$$\phi_4^1(W) = \langle U \rangle ; \quad \phi_4^2(W) = \langle u^2 \rangle$$

$$\phi_4^3(W) = \langle \theta^2 \rangle - \frac{\langle \theta u \rangle^2}{\langle u^2 \rangle} ; \quad \phi_4^4(W) = \langle T \rangle - \frac{\langle \theta u \rangle}{\langle u^2 \rangle^{1/2}}$$

5-characteristic family :

$$\phi_5^1(W) = \langle U \rangle - \frac{(2\langle u^2 \rangle)^{1/2}}{\beta} ; \quad \phi_5^2(W) = \langle T \rangle - (2\beta)^{1/2} \frac{\langle \theta u \rangle}{\langle u^2 \rangle^{1/2}}$$

$$\phi_5^3(W) = \langle \theta^2 \rangle - \frac{\langle \theta u \rangle^2}{\langle u^2 \rangle} ; \quad \phi_5^4(W) = \frac{\langle \theta u \rangle}{\langle u^2 \rangle^{\beta/(2\beta-1)}}$$

Proof :

It is a simple matter to check that :

$$\nabla_W (\lambda_1(W)) \cdot r_1(W) = \nabla_W (\lambda_5(W)) \cdot r_5(W) = 1 + \beta \quad (\text{strictly positive})$$

and :

$$\nabla_W (\lambda_k(W)) \cdot r_k(W) = 0 \quad (\text{for } k = 2, 3, 4)$$

Moreover :

$$\nabla_W (\phi_k^i(W)) \cdot r_k(W) = 0 \quad \text{for given } (k, i)$$

c) Lemma 2.c :

An admissible entropy-flux pair  $(\eta, F_\eta)$  for system (P), consistent with perturbed system (Pε) writes :

$$\eta(W) = \frac{2\beta-1}{2} \langle U \rangle^2 + \langle u^2 \rangle ; \quad F_\eta(W) = \frac{2\beta-1}{3} \langle U \rangle^3 + 2\beta \langle u^2 \rangle \langle U \rangle$$

Proof :

Straightforward computation enables to get :

$$D_W (\eta(W)) \cdot A(W) = D_W (F_\eta(W))$$

noting :

$$D_W (\varphi(W)) = \left( \frac{\partial \varphi}{\partial W_1}, \frac{\partial \varphi}{\partial W_2}, \frac{\partial \varphi}{\partial W_3}, \frac{\partial \varphi}{\partial W_4}, \frac{\partial \varphi}{\partial W_5} \right) \quad \text{whatever } \varphi \text{ stands for.}$$

Even more,  $\eta$  is a (non strictly) convex function of W which is such that :

$$\forall z \in R^5, \quad z^t (\eta_{,WW} D) z = (2\beta-1) (z_1)^2 \geq 0$$

Recall that for regular enough solutions, entropy is governed by equation :

$$\eta_{,t}(W) + (F_\eta(W))_{,x} = 0$$

Let us now introduce the following notations :

$$[\varphi] = \varphi_r - \varphi_l$$

$$\bar{\varphi} = (\varphi_r + \varphi_l) / 2$$

and jump conditions (see /6/) :

$$\sigma [\langle U \rangle] - \overline{\langle U \rangle} [\langle U \rangle] - [\langle u^2 \rangle] = 0$$

$$\sigma [\langle u^2 \rangle] - \overline{\langle U \rangle} [\langle u^2 \rangle] - 2\beta \overline{\langle u^2 \rangle} [\langle U \rangle] = 0$$

$$\sigma [\langle \theta u \rangle] - \overline{\langle U \rangle} [\langle \theta u \rangle] - \beta \overline{\langle \theta u \rangle} [\langle U \rangle] - \overline{\langle u^2 \rangle} [\langle T \rangle] = 0$$

$$\sigma [\langle \theta^2 \rangle] - \overline{\langle U \rangle} [\langle \theta^2 \rangle] - 2 \overline{\langle \theta u \rangle} [\langle T \rangle] = 0$$

$$\sigma [\langle T \rangle] - \overline{\langle U \rangle} [\langle T \rangle] - [\langle \theta u \rangle] = 0$$

$\sigma$  designates the speed of the travelling discontinuity. Moreover, we note :

$$\gamma = \frac{2\beta - 1}{\beta}$$

Taking into account the previous jump conditions enables to state :

d) Lemma 2.d :

\* States connected through the 1-shock family are such that :

$$\begin{aligned} \langle u^2 \rangle_r &= z \langle u^2 \rangle_l \quad \text{with } : z > 1 \\ \langle U \rangle_r &= \langle U \rangle_l + \frac{(1-z)}{(\beta(1+z))^{1/2}} \langle u^2 \rangle_l^{1/2} \\ \langle \theta u \rangle_r &= \langle \theta u \rangle_l \frac{(\gamma+1)z + \gamma - 1}{(\gamma-1)z + \gamma + 1} \\ \langle \theta^2 \rangle_r &= \langle \theta^2 \rangle_l + \frac{(\langle \theta u \rangle_r^2 - \langle \theta u \rangle_l^2)}{\beta(z+1)\langle u^2 \rangle_l} \\ \langle T \rangle_r &= \langle T \rangle_l - \frac{(\langle \theta u \rangle_r - \langle \theta u \rangle_l)}{(\beta(z+1)\langle u^2 \rangle_l)^{1/2}} \end{aligned}$$

\*\* States connected through the 5-shock family are such that :

$$\begin{aligned} \langle u^2 \rangle_r &= z \langle u^2 \rangle_l \quad \text{with } : z < 1 \\ \langle U \rangle_r - \langle U \rangle_l &= \frac{(z-1)}{(\beta(1+z))^{1/2}} \langle u^2 \rangle_l^{1/2} \\ \langle \theta u \rangle_r &= \langle \theta u \rangle_l \frac{(\gamma+1)z + \gamma - 1}{(\gamma-1)z + \gamma + 1} \\ \langle \theta^2 \rangle_r &= \langle \theta^2 \rangle_l + \frac{(\langle \theta u \rangle_r^2 - \langle \theta u \rangle_l^2)}{\beta(z+1)\langle u^2 \rangle_l} \\ \langle T \rangle_r &= \langle T \rangle_l + \frac{(\langle \theta u \rangle_r - \langle \theta u \rangle_l)}{(\beta(z+1)\langle u^2 \rangle_l)^{1/2}} \end{aligned}$$

\*\*\* States on each side of any linearly degenerate field ( $k = 2$  or  $3$  or  $4$ ) may be connected in a similar way, either using previous jump conditions (see above) or using Riemann invariant approach.

The proof is obvious and thus omitted. It must be emphasized that the physically relevant solution has been obtained applying for entropy concept (see lemma 2.c). Note also that the "limit case  $\beta=1$ " may be included in the previous parametrization. All these features enable to derive :

e) Proposition 2 :

\* The Riemann problem associated to system (P) and initial conditions :

$$W(x, t=0) = W_L \quad \text{for } : x < 0$$

$$W(x, t=0) = W_R \quad \text{for } : x > 0$$

does admit a unique solution as soon as initial values are close enough to each other in the following sense

$$\langle U \rangle_R - \langle U \rangle_L < \frac{2^{1/2}}{\beta^{1/2}} (\langle u^2 \rangle_R^{1/2} + \langle u^2 \rangle_L^{1/2})$$

:



\*\*Moreover, the solution expending in the (x,t) plane fullfills over-realisability requirement.

Proof:

\* It must be emphasized that the dynamical part of the unknowns, say  $\langle U \rangle$  and  $\langle u^2 \rangle$ , may be solved independently. We only focus on this subset here and define :

$$h_1(z) = \frac{1-z}{(2(1+z))^{1/2}} \text{ if } :z > 1 \quad ; \quad h_1(z) = 1 - z^{1/2} \text{ if } :z \leq 1$$

$$h_2(z) = \frac{z-1}{(2z(1+z))^{1/2}} \text{ if } :z < 1 \quad ; \quad h_2(z) = 1 - z^{-1/2} \text{ if } :z \geq 1$$

Obviously,  $h_2$  (resp.  $h_1$ ) is an increasing (resp. decreasing) function.

Left and right states may then be connected writing :

$$\langle u^2 \rangle_R = z_1 z_2 \langle u^2 \rangle_L \quad (1)$$

$$\langle U \rangle_R = \langle U \rangle_L + h_2(z_2) \left(\frac{2}{\beta}\right)^{1/2} \langle u^2 \rangle_R^{1/2} + \left(\frac{2}{\beta}\right)^{1/2} \langle u^2 \rangle_L^{1/2} h_1(z_1) \quad (2)$$

Eliminating  $z_1$  from (1) enables to write :

$$z_1 = z_1(z_2) \text{ with } :z_1'(z_2) < 0$$

Hence, (2) may be rewritten below :

$$\Psi(z_2) = h_2(z_2) \left(\frac{2}{\beta}\right)^{1/2} \langle u^2 \rangle_R^{1/2} + \left(\frac{2}{\beta}\right)^{1/2} \langle u^2 \rangle_L^{1/2} h_1(z_1(z_2)) = \langle U \rangle_R - \langle U \rangle_L \quad (2')$$

Evenmore,  $\Psi$  is an increasing function which is such that :

$$\Psi(0), +\infty(0) = -\infty, \quad \left(\frac{2}{\beta}\right)^{1/2} (\langle u^2 \rangle_R^{1/2} + \langle u^2 \rangle_L^{1/2})$$

Eventually, (2') will admit a unique solution as soon as above mentionned condition holds. As soon as  $z_1$  and  $z_2$  have been computed solving nonlinear system ((1), (2)), other state components referring to  $\langle \theta u \rangle$ ,  $\langle \theta^2 \rangle$ ,  $\langle T \rangle$  may be derived in an explicit way.

\*\* Obviously, parametrization provides positive values for  $\langle u^2 \rangle$  variable. Moreover, provided that the 1-family (respectively the 5-family) occurs to be a rarefaction wave, then the following Riemann invariant :

$$\phi_1^3(W) = \phi_2^3(W) = \phi_4^3(W) = \phi_5^3(W) = \langle \theta^2 \rangle - \frac{\langle \theta u \rangle^2}{\langle u^2 \rangle}$$

remains constant on the left side (respectively on the right side) of the 3-family.

Provided that the 1-family occurs to be a shock wave, then  $\langle \theta^2 \rangle - \frac{\langle \theta u \rangle^2}{\langle u^2 \rangle}$  will strictly increase through the shock (from left to right) ; in a similar way, provided that the 5-family occurs to be a shock

wave,  $\langle \theta^2 \rangle - \frac{\langle \theta u \rangle^2}{\langle u^2 \rangle}$  will strictly decrease through the shock (still from left to right). This eventually means that first and second fundamental minors of the correlation tensor remain positive in the (x,t)

plane. It should be mentionned here that in the particular case " $\beta = 1$ ",  $\langle \theta^2 \rangle - \frac{\langle \theta u \rangle^2}{\langle u^2 \rangle}$  remains constant through genuinely non linear fields.

### Appendix 3 :

Let us consider the following system :

$$(R_{ij})_{,t} = \Phi_{ij} \quad (1)$$

where :

$$\Phi_{ij} = a(I, II, III) \frac{\mathcal{E}}{I} R_{ij} + b(I, II, III) \frac{\mathcal{E}}{II} R_{ij}^2$$

The Reynolds stress tensor  $R$  ( in  $R_{n \times n}$  ) is characterised by the initial realisable conditions :

$$R_{ij}(0) = (Q(0) D(0) Q^{-1}(0))_{ij}$$

$D(0)$  stands for the diagonal matrix of (positive)  $R$  -eigenvalues. Furthermore, it is assumed that  $a$  and  $b$  design bounded functions of unknowns such that :

$$b_{min} \leq b(I, II, III) \leq b_{max} < 0$$

and that the following requirement pertaining to turbulent time scale holds :

$$0 < \left(\frac{\mathcal{E}}{I}\right)_{min} \leq \frac{\mathcal{E}}{I} \leq \left(\frac{\mathcal{E}}{I}\right)_{max}$$

(1) admits solutions of the form :

$$R_{ij}(t) = (Q(0) D(t) Q^{-1}(0))_{ij}$$

The two and three dimensional cases must be examined separately.

\* In the two dimensional case ( $n = 2$ ), eigenvalues will fulfil :

$$\lambda_{i,t} = \frac{\mathcal{E}}{I} \lambda_i \left( a + \frac{b I \lambda_i}{II} \right) \quad \text{for } : i = 1, 2 \quad (1)$$

and a straightforward measure of anisotropy amount is given by :

$$\varphi(D) = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1^2 + \lambda_2^2} = 1 - \frac{2 \lambda_1 \lambda_2}{\lambda_1^2 + \lambda_2^2} \text{ which belongs to } [0, 1]$$

$\varphi$  variations will obey :

$$\varphi_{,t} = \left(\frac{\mathcal{E}}{I}\right) \left(\frac{I^2}{II}\right) b(I, II, III) \varphi (1 - \varphi) \quad (2)$$

Recall here that :  $I = \lambda_1 + \lambda_2$  ;  $II = \lambda_1^2 + \lambda_2^2$  ;  $III = \lambda_1^3 + \lambda_2^3$

Hence, noting that :  $II \leq I^2 \leq 2 II$

(2) enables to get a lower bound and upper bound for  $\varphi$  :

$$\frac{K_0 \exp \left( 2 \left(\frac{\mathcal{E}}{I}\right)_{max} b_{min} t \right)}{1 + K_0 \exp \left( 2 \left(\frac{\mathcal{E}}{I}\right)_{max} b_{min} t \right)} \leq \varphi(t) \leq \frac{K_0 \exp \left( \left(\frac{\mathcal{E}}{I}\right)_{min} b_{max} t \right)}{1 + K_0 \exp \left( \left(\frac{\mathcal{E}}{I}\right)_{min} b_{max} t \right)}$$

noting :

$$K_0 = \frac{\varphi(0)}{1 - \varphi(0)}$$

which guarantees return to isotropy process.

\*\* In the three dimensional case ( $n = 3$ ), eigenvalues will be governed by equation below :

$$\lambda_{i,t} = \frac{\varepsilon}{I} \lambda_i \left( a + \frac{b I \lambda_i}{II} \right) \quad \text{for } i = 1, 3 \quad (1'')$$

and  $\varphi$  (which belongs to  $[0, 1]$ ) now writes :

$$\varphi(D) = \frac{(\lambda_1 - \lambda_2)^2 + (\lambda_1 - \lambda_3)^2 + (\lambda_3 - \lambda_2)^2}{2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)} = 1 - \frac{\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_3 \lambda_2}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$$

Recall here that :  $I = \lambda_1 + \lambda_2 + \lambda_3$  ;  $II = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$  ;  $III = \lambda_1^3 + \lambda_2^3 + \lambda_3^3$

which allows to derive :

$$\varphi_{,t} = \left( \frac{\varepsilon}{I} \right) \left( \frac{I^2}{II} \right) b(I, II, III) \frac{((I)(III) - II^2)}{II^2}$$

Now :

$$(I)(III) - II^2 = \lambda_1 \lambda_2 (\lambda_1 - \lambda_2)^2 + \lambda_1 \lambda_3 (\lambda_1 - \lambda_3)^2 + \lambda_3 \lambda_2 (\lambda_3 - \lambda_2)^2$$

Thus :

$$0 \leq \frac{(I)(III) - II^2}{II^2} \leq 2 \varphi (1 - \varphi)$$

The whole only provides :

$$\frac{K_0 \exp \left( 6 \left( \frac{\varepsilon}{I} \right)_{max} b_{min} t \right)}{1 + K_0 \exp \left( 6 \left( \frac{\varepsilon}{I} \right)_{max} b_{min} t \right)} \leq \varphi(t) \leq 1$$

However, return to isotropy process should at least occur in one "plane" ((e1, e2) or (e1, e3) or (e2, e3), where e<sub>i</sub> 's stand for right eigenvectors of R) ; this may be seen defining :

$$\varphi_{ij}(D) = \frac{(\lambda_i - \lambda_j)^2}{\lambda_i^2 + \lambda_j^2} = 1 - \frac{2 \lambda_i \lambda_j}{\lambda_i^2 + \lambda_j^2} \quad \text{for } i \neq j \quad (\text{and } i \text{ greater than } j)$$

$\varphi_{ij}$  will be such that :

$$\varphi_{ij,t} = \left( \frac{\varepsilon}{I} \right) \left( \frac{I^2}{II} - \frac{\lambda_k I}{II} \right) b(I, II, III) \varphi_{ij} (1 - \varphi_{ij}) \quad (\text{with } k \text{ non equal to } i \text{ and } k \text{ non equal to } j)$$

This provides:

$${}_{3 \geq i > j \geq 1} \frac{\varphi_{ij,t}}{\varphi_{ij} (1 - \varphi_{ij})} = \left( \frac{\varepsilon}{I} \right) \left( \frac{2I^2}{II} \right) b(I, II, III)$$

which ensures that :

$${}_{3 \geq i > j \geq 1} \frac{\pi}{(1 - \varphi_{ij}(t))} \leq {}_{3 \geq i > j \geq 1} \frac{\pi}{(1 - \varphi_{ij}(0))} \exp \left( 2 \left( \frac{\varepsilon}{I} \right)_{min} b_{max} t \right)$$

or :

$${}_{3 \geq i > j \geq 1} \pi \varphi_{ij}(t) \leq {}_{3 \geq i > j \geq 1} \frac{\pi}{(1 - \varphi_{ij}(0))} \exp \left( 2 \left( \frac{\varepsilon}{I} \right)_{min} b_{max} t \right)$$

\*\*\* Indeed, we may go further on in the three dimensional case, retaining third order contribution :

$$\Phi_{ij} = a(I, II, III) \frac{\varepsilon}{I} R_{ij} + b(I, II, III) \frac{\varepsilon}{II} R_{ij}^2 + c(I, II, III) \frac{\varepsilon}{III} R_{ij}^3$$

assuming in addition that c stands for some dimensionless negative bounded function.

Using similar definition :

$$\varphi_{ij}(D) = \frac{(\lambda_i - \lambda_j)^2}{\lambda_i^2 + \lambda_j^2} = 1 - \frac{2\lambda_i\lambda_j}{\lambda_i^2 + \lambda_j^2} \text{ for } i \neq j$$

yields :

$$\frac{\varphi_{ij,t}}{\varphi_{ij}(1 - \varphi_{ij})} = \left(\frac{\varepsilon}{I}\right) \left(I \left(\frac{I - \lambda_k}{II}\right) b(I, II, III) + I \frac{(I - \lambda_k)^2}{III} c(I, II, III)\right)$$

(with k non equal to i and k non equal to j), which also provides :

$$\pi_{3 \geq i > j \geq 1} \varphi_{ij}(t) \leq \pi_{3 \geq i > j \geq 1} \frac{\varphi_{ij}(0)}{(1 - \varphi_{ij}(0))} \exp\left(2 \left(\frac{\varepsilon}{I}\right)_{min} b_{max} t\right)$$

Return to isotropy will only be ensured in one plane at least.

#### Appendix 4 :

Let us consider the following non viscous problem (P) :

$$\langle U \rangle_{,t} + \langle U \rangle \langle U \rangle_{,x} + \langle u^2 \rangle_{,x} = 0 \quad (\text{P1})$$

$$\langle V \rangle_{,t} + \langle U \rangle \langle V \rangle_{,x} + \langle uv \rangle_{,x} = 0 \quad (\text{P2})$$

$$\langle u^2 \rangle_{,t} + \langle U \rangle \langle u^2 \rangle_{,x} + 2 \langle u^2 \rangle \langle U \rangle_{,x} = 0 \quad (\text{P3})$$

$$\langle v^2 \rangle_{,t} + \langle U \rangle \langle v^2 \rangle_{,x} + 2 \langle uv \rangle \langle V \rangle_{,x} = 0 \quad (\text{P4})$$

$$\langle uv \rangle_{,t} + \langle U \rangle \langle uv \rangle_{,x} + \langle u^2 \rangle \langle V \rangle_{,x} + \langle uv \rangle \langle U \rangle_{,x} = 0 \quad (\text{P5})$$

$$\langle \theta u \rangle_{,t} + \langle U \rangle \langle \theta u \rangle_{,x} + \langle \theta u \rangle \langle U \rangle_{,x} + \langle u^2 \rangle \langle T \rangle_{,x} = 0 \quad (\text{P6})$$

$$\langle \theta v \rangle_{,t} + \langle U \rangle \langle \theta v \rangle_{,x} + \langle \theta u \rangle \langle V \rangle_{,x} + \langle uv \rangle \langle T \rangle_{,x} = 0 \quad (\text{P7})$$

$$\langle \theta^2 \rangle_{,t} + \langle U \rangle \langle \theta^2 \rangle_{,x} + 2 \langle \theta u \rangle \langle T \rangle_{,x} = 0 \quad (\text{P8})$$

$$\langle T \rangle_{,t} + \langle U \rangle \langle T \rangle_{,x} + \langle \theta u \rangle_{,x} = 0 \quad (\text{P9})$$

and associated viscous perturbed system (P $\epsilon$ ) obtained by changing (P1) and (P2) in :

$$\langle U \rangle_{,t} + \langle U \rangle \langle U \rangle_{,x} + \langle u^2 \rangle_{,x} = \epsilon \langle U \rangle_{,xx} \quad (\text{P1}\epsilon)$$

$$\langle V \rangle_{,t} + \langle U \rangle \langle V \rangle_{,x} + \langle uv \rangle_{,x} = \epsilon \langle V \rangle_{,xx} \quad (\text{P2}\epsilon)$$

We set :

$$W^t = (\langle U \rangle, \langle V \rangle, \langle u^2 \rangle, \langle v^2 \rangle, \langle uv \rangle, \langle \theta u \rangle, \langle \theta v \rangle, \langle \theta^2 \rangle, \langle T \rangle)$$

Thus, (P) may be rewritten in the form :

$$W_{,t} + A(W) W_{,x} = 0$$

Besides, (P $\epsilon$ ) writes :

$$W_{,t} + A(W) W_{,x} = \epsilon (DW_{,x})_{,x}$$

noting :

$$D_{11} = 1 ; D_{22} = 1 ; D_{ij} = 0 \text{ otherwise.}$$

The following initial conditions are considered :

$$W(x, t=0) = W_L \quad \text{for : } x < 0$$

$$W(x, t=0) = W_R \quad \text{for : } x > 0$$

a) Lemma 4.a :

System (P) is strictly hyperbolic and is characterized by the following set of eigenvalues :

$$\lambda_1 = \langle U \rangle - (2 \langle u^2 \rangle)^{1/2}$$

$$\lambda_{2,3} = \langle U \rangle - (\langle u^2 \rangle)^{1/2}$$

$$\lambda_{4,5,6} = \langle U \rangle$$

$$\lambda_{7,8} = \langle U \rangle + (\langle u^2 \rangle)^{1/2}$$

$$\lambda_9 = \langle U \rangle + (2 \langle u^2 \rangle)^{1/2}$$

and associated right eigenvectors :

$$\begin{aligned}
r_1^t &= (1, \frac{\langle uv \rangle}{\langle u^2 \rangle}, -(2 \langle u^2 \rangle)^{1/2}, \frac{-(2)^{1/2} \langle uv \rangle^2}{\langle u^2 \rangle^{3/2}}, \frac{-(2)^{1/2} \langle uv \rangle}{\langle u^2 \rangle^{1/2}}, \frac{-(2)^{1/2} \langle \theta u \rangle}{\langle u^2 \rangle^{1/2}} \dots \\
&\quad , \frac{-(2)^{1/2} \langle uv \rangle \langle \theta u \rangle}{\langle u^2 \rangle^{3/2}}, \frac{-(2)^{1/2} \langle \theta u \rangle^2}{\langle u^2 \rangle^{3/2}}, \frac{\langle \theta u \rangle}{\langle u^2 \rangle}) \\
r_2^t &= (0, 0, 0, 0, 0, -\langle u^2 \rangle^{1/2}, \frac{-\langle uv \rangle}{\langle u^2 \rangle^{1/2}}, \frac{-2 \langle \theta u \rangle}{\langle u^2 \rangle^{1/2}}, 1) \\
r_3^t &= (0, 1, 0, \frac{-2 \langle uv \rangle}{\langle u^2 \rangle^{1/2}}, -\langle u^2 \rangle^{1/2}, 0, \frac{-\langle \theta u \rangle}{\langle u^2 \rangle^{1/2}}, 0, 0) \\
r_4^t &= (0, 0, 0, 1, 0, 0, 0, 0, 0) \\
r_5^t &= (0, 0, 0, 0, 0, 0, 1, 0, 0) \\
r_6^t &= (0, 0, 0, 0, 0, 0, 0, 1, 0) \\
r_7^t &= (0, 1, 0, \frac{2 \langle uv \rangle}{\langle u^2 \rangle^{1/2}}, \langle u^2 \rangle^{1/2}, 0, \frac{\langle \theta u \rangle}{\langle u^2 \rangle^{1/2}}, 0, 0) \\
r_8^t &= (0, 0, 0, 0, 0, \langle u^2 \rangle^{1/2}, \frac{\langle uv \rangle}{\langle u^2 \rangle^{1/2}}, \frac{2 \langle \theta u \rangle}{\langle u^2 \rangle^{1/2}}, 1) \\
r_9^t &= (1, \frac{\langle uv \rangle}{\langle u^2 \rangle}, (2 \langle u^2 \rangle)^{1/2}, \frac{(2)^{1/2} \langle uv \rangle^2}{\langle u^2 \rangle^{3/2}}, \frac{(2)^{1/2} \langle uv \rangle}{\langle u^2 \rangle^{1/2}}, \frac{(2)^{1/2} \langle \theta u \rangle}{\langle u^2 \rangle^{1/2}} \dots \\
&\quad , \frac{(2)^{1/2} \langle uv \rangle \langle \theta u \rangle}{\langle u^2 \rangle^{3/2}}, \frac{(2)^{1/2} \langle \theta u \rangle^2}{\langle u^2 \rangle^{3/2}}, \frac{\langle \theta u \rangle}{\langle u^2 \rangle})
\end{aligned}$$

**b) Lemma 4.b :**

The 1 and 9 characteristic fields are genuinely nonlinear and the 2, 3, 4, 5, 6, 7, 8 characteristic fields are linearly degenerate. Moreover, each k-characteristic field does admit eight independent Riemann invariants  $\Phi_k^i$  (i=1 to 8) which are given by :

1-characteristic family :

$$\begin{aligned}
\phi_1^1(W) &= \langle U \rangle + (2 \langle u^2 \rangle)^{1/2} ; \phi_1^2(W) = \langle V \rangle + (2)^{1/2} \langle uv \rangle (\langle u^2 \rangle)^{-1/2} \\
\phi_1^3(W) &= \langle v^2 \rangle - \frac{\langle uv \rangle^2}{\langle u^2 \rangle} ; \phi_1^4(W) = \frac{\langle uv \rangle}{\langle u^2 \rangle} ; \phi_1^5(W) = \frac{\delta_3}{\langle u^2 \rangle} \\
\phi_1^6(W) &= \langle \theta^2 \rangle - \frac{\langle \theta u \rangle^2}{\langle u^2 \rangle} ; \phi_1^7(W) = \frac{\langle \theta u \rangle}{\langle u^2 \rangle} ; \phi_1^8(W) = \langle T \rangle + (2)^{1/2} \frac{\langle \theta u \rangle}{\langle u^2 \rangle^{1/2}}
\end{aligned}$$

2-characteristic family :

$$\begin{aligned}
\phi_2^1(W) &= \langle U \rangle ; \phi_2^2(W) = \langle u^2 \rangle ; \phi_2^3(W) = \langle v^2 \rangle ; \phi_2^4(W) = \langle uv \rangle ; \phi_2^5(W) = \langle V \rangle \\
\phi_2^6(W) &= \langle \theta^2 \rangle - \frac{\langle \theta u \rangle^2}{\langle u^2 \rangle} ; \phi_2^7(W) = \delta_3 ; \phi_2^8(W) = \langle T \rangle + \frac{\langle \theta u \rangle}{\langle u^2 \rangle^{1/2}}
\end{aligned}$$

3-characteristic family :

$$\begin{aligned}
\phi_3^1(W) &= \langle U \rangle ; \phi_3^2(W) = \langle u^2 \rangle ; \phi_3^3(W) = \langle V \rangle + \langle uv \rangle (\langle u^2 \rangle)^{-1/2} \\
\phi_3^4(W) &= \langle v^2 \rangle - \frac{\langle uv \rangle^2}{\langle u^2 \rangle} ; \phi_3^5(W) = \langle \theta^2 \rangle - \frac{\langle \theta u \rangle^2}{\langle u^2 \rangle} ; \phi_3^6(W) = \frac{\langle \theta u \rangle}{\langle u^2 \rangle} \\
\phi_3^7(W) &= \delta_3 ; \phi_3^8(W) = \langle T \rangle
\end{aligned}$$

4-characteristic family :

$$\phi_5^1(W) = \langle U \rangle ; \phi_5^2(W) = \langle u^2 \rangle ; \phi_5^3(W) = \langle uv \rangle ; \phi_5^4(W) = \langle V \rangle$$

$$\phi_5^5(W) = \langle \theta u \rangle ; \phi_5^6(W) = \langle \theta v \rangle ; \phi_5^7(W) = \langle \theta^2 \rangle ; \phi_5^8(W) = \langle T \rangle ;$$

5-characteristic family :

$$\phi_4^1(W) = \langle U \rangle ; \phi_4^2(W) = \langle u^2 \rangle ; \phi_4^3(W) = \langle v^2 \rangle ; \phi_4^4(W) = \langle uv \rangle ; \phi_4^5(W) = \langle V \rangle$$

$$\phi_4^6(W) = \langle \theta u \rangle ; \phi_4^7(W) = \langle \theta^2 \rangle ; \phi_4^8(W) = \langle T \rangle ;$$

6-characteristic family :

$$\phi_6^1(W) = \langle U \rangle ; \phi_6^2(W) = \langle u^2 \rangle ; \phi_6^3(W) = \langle uv \rangle ; \phi_6^4(W) = \langle V \rangle$$

$$\phi_6^5(W) = \langle v^2 \rangle ; \phi_6^6(W) = \langle \theta u \rangle ; \phi_6^7(W) = \langle \theta v \rangle ; \phi_6^8(W) = \langle T \rangle ;$$

7-characteristic family :

$$\phi_7^1(W) = \langle U \rangle ; \phi_7^2(W) = \langle u^2 \rangle ; \phi_7^3(W) = \langle V \rangle - \langle uv \rangle (\langle u^2 \rangle)^{-1/2}$$

$$\phi_7^4(W) = \langle v^2 \rangle - \frac{\langle uv \rangle^2}{\langle u^2 \rangle} ; \phi_7^5(W) = \langle \theta^2 \rangle - \frac{\langle \theta u \rangle^2}{\langle u^2 \rangle} ; \phi_7^6(W) = \frac{\langle \theta u \rangle}{\langle u^2 \rangle}$$

$$\phi_7^7(W) = \delta_3 ; \phi_7^8(W) = \langle T \rangle$$

8-characteristic family :

$$\phi_8^1(W) = \langle U \rangle ; \phi_8^2(W) = \langle u^2 \rangle ; \phi_8^3(W) = \langle v^2 \rangle ; \phi_8^4(W) = \langle uv \rangle ; \phi_8^5(W) = \langle V \rangle$$

$$\phi_8^6(W) = \langle \theta^2 \rangle - \frac{\langle \theta u \rangle^2}{\langle u^2 \rangle} ; \phi_8^7(W) = \delta_3 ; \phi_8^8(W) = \langle T \rangle - \frac{\langle \theta u \rangle}{\langle u^2 \rangle^{1/2}}$$

9-characteristic family :

$$\phi_9^1(W) = \langle U \rangle - (2 \langle u^2 \rangle)^{1/2} ; \phi_9^2(W) = \langle V \rangle - (2)^{1/2} \langle uv \rangle (\langle u^2 \rangle)^{-1/2}$$

$$\phi_9^3(W) = \langle v^2 \rangle - \frac{\langle uv \rangle^2}{\langle u^2 \rangle} ; \phi_9^4(W) = \frac{\langle uv \rangle}{\langle u^2 \rangle} ; \phi_9^5(W) = \frac{\delta_3}{\langle u^2 \rangle}$$

$$\phi_9^6(W) = \langle \theta^2 \rangle - \frac{\langle \theta u \rangle^2}{\langle u^2 \rangle} ; \phi_9^7(W) = \frac{\langle \theta u \rangle}{\langle u^2 \rangle} ; \phi_9^8(W) = \langle T \rangle - (2)^{1/2} \frac{\langle \theta u \rangle}{\langle u^2 \rangle^{1/2}}$$

It has been set here :

$$\delta_3 = \langle \theta^2 \rangle \langle u^2 \rangle \langle v^2 \rangle + 2 \langle \theta u \rangle \langle \theta v \rangle \langle uv \rangle - \langle \theta^2 \rangle \langle uv \rangle^2 - \langle u^2 \rangle \langle \theta v \rangle^2 - \langle v^2 \rangle \langle \theta u \rangle^2$$

Proof :

It may be checked that :

$$\nabla_W (\lambda_1(W)) \cdot r_1(W) = \nabla_W (\lambda_9(W)) \cdot r_9(W) = 2$$

and :

$$\nabla_W (\lambda_k(W)) \cdot r_k(W) = 0 \quad (\text{for } k = 2 \text{ to } 8)$$

Moreover :

$$\nabla_W (\phi_k^i(W)) \cdot r_k(W) = 0 \text{ for given } (k, i)$$

c) Lemma 4.c :

An admissible entropy-flux pair  $(\eta, F_\eta)$  for system (P), consistent with perturbed system (Pe) writes :

$$\eta(W) = \frac{1}{2} \langle U \rangle^2 + \langle u^2 \rangle ; F_\eta(W) = \frac{1}{3} \langle U \rangle^3 + 2 \langle u^2 \rangle \langle U \rangle$$

Proof:

Again,  $\eta$  is a (non strictly) convex function of  $W$  which is such that :

$$\forall z \in R^9, z^t (\eta_{,WW} D) z = (z_1)^2 \geq 0$$

and :

$$D_W (\eta (W)) \cdot A (W) = D_W (F_\eta (W))$$

which implies that, for regular enough solutions :

$$\eta_{,t} (W) + (F_\eta (W))_{,x} = 0$$

Using previously defined notations enables to write approximate jump conditions :

$$\sigma [\langle U \rangle] - \overline{\langle U \rangle} [\langle U \rangle] - [\langle u^2 \rangle] = 0$$

$$\sigma [\langle V \rangle] - \overline{\langle U \rangle} [\langle V \rangle] - [\langle uv \rangle] = 0$$

$$\sigma [\langle u^2 \rangle] - \overline{\langle U \rangle} [\langle u^2 \rangle] - 2 \overline{\langle u^2 \rangle} [\langle U \rangle] = 0$$

$$\sigma [\langle v^2 \rangle] - \overline{\langle U \rangle} [\langle v^2 \rangle] - 2 \overline{\langle uv \rangle} [\langle V \rangle] = 0$$

$$\sigma [\langle uv \rangle] - \overline{\langle U \rangle} [\langle uv \rangle] - \overline{\langle uv \rangle} [\langle U \rangle] - \overline{\langle u^2 \rangle} [\langle V \rangle] = 0$$

$$\sigma [\langle \theta u \rangle] - \overline{\langle U \rangle} [\langle \theta u \rangle] - \overline{\langle \theta u \rangle} [\langle U \rangle] - \overline{\langle u^2 \rangle} [\langle T \rangle] = 0$$

$$\sigma [\langle \theta v \rangle] - \overline{\langle U \rangle} [\langle \theta v \rangle] - \overline{\langle \theta u \rangle} [\langle V \rangle] - \overline{\langle uv \rangle} [\langle T \rangle] = 0$$

$$\sigma [\langle \theta^2 \rangle] - \overline{\langle U \rangle} [\langle \theta^2 \rangle] - 2 \overline{\langle \theta u \rangle} [\langle T \rangle] = 0$$

$$\sigma [\langle T \rangle] - \overline{\langle U \rangle} [\langle T \rangle] - [\langle \theta u \rangle] = 0$$

Now, it may be stated that :

d) Lemma 4.d :

\* States connected through the 1-shock family are such that :

$$\langle u^2 \rangle_r = z \langle u^2 \rangle_l \quad \text{with } : z > 1$$

$$\langle v^2 \rangle_r = \langle v^2 \rangle_l + \frac{(z-1)}{\langle u^2 \rangle_l} \langle uv \rangle_l^2$$

$$\langle uv \rangle_r = z \langle uv \rangle_l$$

$$\langle U \rangle_r = \langle U \rangle_l + \frac{(1-z)}{(1+z)^{1/2}} \langle u^2 \rangle_l^{1/2}$$

$$\langle V \rangle_r = \langle V \rangle_l + \frac{(1-z)}{(1+z)^{1/2}} \frac{\langle uv \rangle_l}{\langle u^2 \rangle_l^{1/2}}$$

$$\langle \theta u \rangle_r = z \langle \theta u \rangle_l$$

$$\langle \theta v \rangle_r = \langle \theta v \rangle_l + (z-1) \langle \theta u \rangle_l \frac{\langle uv \rangle_l}{\langle u^2 \rangle_l}$$

$$\langle \theta^2 \rangle_r = \langle \theta^2 \rangle_l + (z-1) \frac{\langle \theta u \rangle_l^2}{\langle u^2 \rangle_l}$$

$$\langle T \rangle_r = \langle T \rangle_l - \frac{(z-1) \langle \theta u \rangle_l}{(z+1)^{1/2} \langle u^2 \rangle_l^{1/2}}$$

\*\* States connected through the 9-shock family are such that :

$$\langle u^2 \rangle_r = z \langle u^2 \rangle_l \quad \text{with } : z < 1$$



$$\begin{aligned} \langle v^2 \rangle_r &= \langle v^2 \rangle_l + \frac{(z-1)}{\langle u^2 \rangle_l} \langle uv \rangle_l^2 \\ \langle uv \rangle_r &= z \langle uv \rangle_l \\ \langle U \rangle_r &= \langle U \rangle_l + \frac{(z-1)}{(1+z)^{1/2}} \langle u^2 \rangle_l^{1/2} \\ \langle V \rangle_r &= \langle V \rangle_l + \frac{(z-1)}{(1+z)^{1/2}} \frac{\langle uv \rangle_l}{\langle u^2 \rangle_l^{1/2}} \\ \langle \theta u \rangle_r &= z \langle \theta u \rangle_l \\ \langle \theta v \rangle_r &= \langle \theta v \rangle_l + (z-1) \langle \theta u \rangle_l \frac{\langle uv \rangle_l}{\langle u^2 \rangle_l} \\ \langle \theta^2 \rangle_r &= \langle \theta^2 \rangle_l + (z-1) \frac{\langle \theta u \rangle_l^2}{\langle u^2 \rangle_l} \\ \langle T \rangle_r &= \langle T \rangle_l + \frac{(z-1) \langle \theta u \rangle_l}{(z+1)^{1/2} \langle u^2 \rangle_l^{1/2}} \end{aligned}$$

\*\*\* States on each side of any linearly degenerate field ( $k = 2$  to  $8$ ) may be connected in a similar way, either using previous jump conditions (see above) or using Riemann invariant approach.

The proof is straightforward and omitted. Here again, the physically relevant solution has been obtained applying for entropy concept (see above). Eventually, we get :

e) Proposition 4 :

\* The Riemann problem associated to system (P) and initial conditions :

$$W(x, t=0) = W_L \quad \text{for } : x < 0$$

$$W(x, t=0) = W_R \quad \text{for } : x > 0$$

does admit a unique solution as soon as initial values are close enough to each other in the following sense :

$$\langle U \rangle_R - \langle U \rangle_L < (2\langle u^2 \rangle_R)^{1/2} + (2\langle u^2 \rangle_L)^{1/2}$$

\*\*Moreover, the solution expanding in the (x,t) plane fulfils over-realisability requirement.

The proof is very similar to the one given in appendix 2 and thus omitted. One only needs to have a glance at the parametrization given above for either GNL or LD fields.

f) We have more than that in fact, since the solution is only governed by the dynamical part of the initial conditions. Let us have a look at the 1-characteristic field for instance. We imagine first that the solution of the RP is such that the 1-characteristic field occurs to be a shock ; hence, we get :

$$\langle T \rangle_1 = \langle T \rangle_L - \frac{(z_1 - 1) \langle \theta u \rangle_L}{(z_1 + 1)^{1/2} \langle u^2 \rangle_L^{1/2}} \quad \text{with } : z_1 > 1$$

Since the solution of the RP does not depend on the initial values  $\langle T \rangle_R$ ,  $\langle T \rangle_L$  and  $\langle \theta u \rangle_L$ , we may choose them such that :

$$\langle T \rangle_R > \langle T \rangle_L \quad \text{and } : \langle \theta u \rangle_L > 0$$

Consequently, we get :

$$\langle T \rangle_1 < \min ( \langle T \rangle_R , \langle T \rangle_L )$$

which means that the maximum principle for  $\langle T \rangle$  variable no longer holds. If we turn now to the case where 1-characteristic field occurs to be a rarefaction wave, we get :

$$\langle T \rangle_1 = \langle T \rangle_L + (2)^{1/2} \frac{\langle \theta u \rangle_L}{\langle u^2 \rangle_L} \langle u^2 \rangle_L^{1/2} (1 - z_1^{1/2}) \quad \text{with } : z_1 \leq 1$$

Thus, if  $\langle T \rangle_R$  ,  $\langle T \rangle_L$  and  $\langle \theta u \rangle_L$  , are such that :

$$\langle T \rangle_R < \langle T \rangle_L \quad \text{and } : \langle \theta u \rangle_L > 0$$

intermediate state will be such that :

$$\langle T \rangle_1 > \max ( \langle T \rangle_R , \langle T \rangle_L )$$

which confirms result stated above when dealing with shocks.