THE KATO SQUARE ROOT PROBLEM ON AN ARBITRARY DOMAIN OF R d

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To the memory of Alan McIntosh

ABSTRACT. Let Ω be an open subset of $\mathbb{R}^d$ and $\mathcal{L}_\Omega = -\text{div}A\nabla$ an elliptic operator with bounded measurable and complex coefficients on $L^2(\Omega)$. The operator $\mathcal{L}_\Omega$ is subject to Dirichlet boundary conditions. We solve the Kato square root problem for arbitrary Ω. We prove that $D(\sqrt{\mathcal{L}_\Omega}) = H^1_0(\Omega)$ and there exist a constant $C > 0$ such that

$$C^{-1} \|\nabla u\|_2 \leq \left\| \sqrt{\mathcal{L}_\Omega}\, u \right\|_2 \leq C \|\nabla u\|_2 , \; u \in H^1_0(\Omega).$$

We also allow perturbations by general potentials: for any $V \in L^1_{loc}(\Omega)$ with range contained in a sector of $\mathbb{C}^+$ with angle $\omega_V \in \left[0, \frac{\pi}{2}\right)$ there exists $C_V > 0$ such that

$$C_V^{-1} \left(\left\| |V|^{\frac{1}{2}} u \right\|_2 + \|\nabla u\|_2 \right) \leq \left\| \sqrt{\mathcal{L}_\Omega + V}\, u \right\|_2 \leq C_V \left(\left\| |V|^{\frac{1}{2}} u \right\|_2 + \|\nabla u\|_2 \right)$$

for all $u \in D\left(\sqrt{\mathcal{L}_\Omega + V}\right) = H^1_0(\Omega) \cap D\left(\sqrt{|V|}\right)$. The constant C_V depends on V only through ω_V. In particular, C_V is independent of V for $0 \leq V \in L^1_{loc}(\Omega)$.
We prove similar results for systems.

CONTENTS

1. Introduction and the main results

We consider on $L^2\left(\mathbb{R}^d\right)$ divergence form elliptic operators $\mathcal{L}u = -\mathrm{div}A\nabla u$, where $A = (a_{kl})$ with $a_{kl} \in L^\infty\left(\mathbb{R}^d, \mathbb{C}\right)$ satisfies the usual ellipticity condition

$$\mathrm{Re}\sum_{k,l=1}^{d} a_{kl}(x)\xi_k\overline{\xi_l} \geq \kappa_A|\xi|^2 \tag{1.1}$$

for a.e. $x \in \mathbb{R}^d$ and all $\xi = (\xi_1, ..., \xi_d) \in \mathbb{C}^d$. Here $\kappa_A > 0$ is a constant. Owing to the accretivity of $\mathcal{L}$ one can define its square root $\sqrt{\mathcal{L}}$. A famous problem posed by T. Kato asks whether the domain of $\sqrt{\mathcal{L}}$ coincides with the Sobolev space $H^1\left(\mathbb{R}^d\right)$. The problem was open for decades until it was solved in 2002 by S. Hofmann, M. Lacey and A. McIntosh [17] and P. Auscher, S. Hofmann, M. Lacey, A. McIntosh and Ph. Tchamitchian [3]. We give a rapid review of the history of the problem and refer the reader to [17] and [3] and the references therein for additional information. The first positive answer to the Kato square root problem was given by R. Coifman, A. McIntosh and Y. Meyer [7] in dimension $d = 1$. For higher dimension, R. Coifman, D. Deng and Y. Meyer [8] and E. Fabes, D. Jerison and C. Kenig [12] proved the Kato square root property under the condition that the matrix A is a relatively small perturbation of the identity. A. McIntosh [20] gave a positive answer under the assumption that the coefficients a_{kl} act boundedely on some Sobolev spaces (which requires some regularity on the coefficients). S. Hofmann, M. Lacey and A. McIntosh [17] solved the problem for elliptic operators whose corresponding heat kernel has Gaussian upper bounds (such as the case of bounded measurable and real coefficients for example). The general case of bounded measurable and complex coefficients was solved by P. Auscher, S. Hofmann, M. Lacey, A. McIntosh and Ph. Tchamitchian [3].

Next, we move to the case of elliptic operators on domains. Let Ω be an open subset of $\mathbb{R}^d$ and consider as above an elliptic operator $\mathcal{L}_\Omega$ on $L^2(\Omega)$ that is subject to Dirichlet, Neumann or mixed boundary conditions. Kato's square root problem in this setting becomes whether the domain of $\sqrt{\mathcal{L}_\Omega}$ coincides with the domain of the corresponding sesquilinear form. That is, $H_0^1(\Omega)$, $H^1(\Omega)$ or an appropriate subspace between these two spaces for Dirichlet, Neumann or mixed boundary conditions respectively. P. Auscher and Ph. Tchamitchian [4] proved the Kato square root property for either Dirichlet or Neumann boundary conditions provided Ω is a strongly Lipschitz domain. Based on their abstract approach in [5], A. Axelsson, S. Keith and A. McIntosh dealt in [6] with mixed boundary conditions under the assumption that Ω is a bi-Lipschitz image of a certain smooth domain. The regularity required on Ω was then improved by M. Egert, R. Haller-Dintelmann and P. Tolksdorf [10]. They assume that Ω has an interior corkscrew condition together with the fact that it decomposes into a part D which satisfies the Ahlfors-David condition and $\overline{\partial\Omega \setminus D}$ has local bi-Lipschitz charts. The problem for an arbitrary domain of $\mathbb{R}^d$ is open. One of our main contributions in this paper is to provide a solution to this problem in the case of Dirichlet boundary conditions. See Corollary 1.2 below for the statement and the homogeneous estimate as in the case of the whole space $\mathbb{R}^d$.

There is another motivation for this paper. We deal with the stability of the Kato square root estimate under perturbation by unbounded potential V. A. Axelsson, S. Keith

and A. McIntosh considered non-homogeneous operators on Lipschitz domains with mixed boundary conditions in [6] using the techniques developed in [5]. The potentials that they considered were, however, bounded both from above and below. In [13] and [14], F. Gesztesy, S. Hofmann and R. Nichols studied the domains of square root operators using techniques distinct from those developed in [5]. The aim in [13] and [14] is to prove that the square root property carries over from the homogeneous elliptic operator $\mathcal{L}$ (or systems) with boundary conditions to $\mathcal{L}+V$ for $V \in L^p + L^\infty$ for appropriate $p \geq \frac{d}{2}$. As a result, they deal with mixed boundary conditions if Ω satisfies the assumptions of [10] mentioned above.

In the present paper we deal with general potentials and prove the square root property with homogeneous estimates.

Let $V \in L^1_{loc}(\Omega)$ and suppose that the range of V is contained in the sector

$$S_{\omega_V +} := \{z \in \mathbb{C} \cup \{\infty\} : |\arg(z)| \leq \omega_V \text{ or } z = 0, \infty\}$$

for some $\omega_V \in \left[0, \frac{\pi}{2}\right)$. Define the subspace

$$H_0^{1,V}(\Omega) := H_0^1(\Omega) \cap D\left(|V|^{\frac{1}{2}}\right) := \left\{u \in H^1(\Omega) : |V|^{\frac{1}{2}}\ u \in L^2(\Omega)\right\}. \tag{1.2}$$

We denote by $\mathcal{L}_\Omega + V$ the operator $-\mathrm{div} A\nabla + V$ with Dirichlet boundary conditions. Our main result is the following theorem.

Theorem 1.1. *Let Ω be an open subset of $\mathbb{R}^d$ ($d \geq 1$) and V as above. Then $D\left(\sqrt{\mathcal{L}_\Omega + V}\right) = H_0^{1,V}(\Omega)$ and there exist a constant $C_V > 0$ such that*

$$C_V^{-1}\left(\|\nabla u\| + \left\||V|^{\frac{1}{2}}\ u\right\|\right) \leq \left\|\sqrt{\mathcal{L}_\Omega + V}u\right\| \leq C_V\left(\|\nabla u\| + \left\||V|^{\frac{1}{2}}\ u\right\|\right)$$

for all $u \in H_0^{1,V}(\Omega)$. Moreover, the constant C_V is dependent on the potential only through ω_V.

Here $\|\cdot\|$ is of course the usual norm in $L^2(\Omega)$. In this paper we use $\|\cdot\|$ to denote the norm of the Hilbert space under consideration. In particular, $\|\cdot\|$ is either $\|\cdot\|_{L^2(\Omega)}$ or $\|\cdot\|_{L^2\left(\mathbb{R}^d\right)}$ depending on the context. The notation $\langle\cdot,\cdot\rangle$ will be used to denote the associated inner product.

The previous theorem contains, as a particular case, the solution of the Kato square root problem for elliptic operators with Dirichlet boundary conditions on an arbitrary domain Ω. Simply set $V \equiv 0$ in the above theorem. Due to the importance of this case, it is stated as its own result in the below corollary.

Corollary 1.2. *We have $D\left(\sqrt{\mathcal{L}_\Omega}\right) = H_0^1(\Omega)$ and there exists a constant $C > 0$ such that*

$$C^{-1}\|\nabla u\| \leq \left\|\sqrt{\mathcal{L}_\Omega}u\right\| \leq C\|\nabla u\|$$

for all $u \in H_0^1(\Omega)$.

We also mention that these results remain valid for systems. See Section 6.

Now we explain our strategy of proof. Theorem 1.1 will be proved by first considering the particular case $\Omega = \mathbb{R}^d$. We explain some of the main ideas used to obtain this case.

A few years following the solution to the original Kato square root problem in [3], an alternate method of proof appeared in [5]. This method of proof, in contrast to the original solution, considered first-order operators as opposed to second-order operators. Our solution to the Kato problem with potential for $\Omega = \mathbb{R}^d$ will be based on this method.

Let $\Pi := \Gamma + \Gamma^*$ be a Dirac-type operator on a Hilbert space $\mathcal{H}$ and $\Pi_B := \Gamma + B_1\Gamma^* B_2$ be a perturbation of Π by bounded operators B_1 and B_2. Typically, Π is considered to be a first-order system acting on $\mathcal{H} := L^2\left(\mathbb{R}^d; \mathbb{C}^N\right)$ for some $d,\ N \in \mathbb{N}^*$ and the perturbations B_1 and B_2 are multiplication by matrix-valued functions $B_1,\ B_2 \in L^\infty\left(\mathbb{R}^d; \mathcal{L}\left(\mathbb{C}^N\right)\right)$. In their seminal paper [5], A. Axelsson, S. Keith and A. McIntosh developed a general framework for proving that the perturbed operator Π_B possessed a bounded holomorphic functional calculus. This ultimately amounted to obtaining square function estimates of the form

$$\int_0^\infty \left\| Q_t^B u \right\|^2 \frac{dt}{t} \simeq \|u\|^2, \tag{1.3}$$

where $Q_t^B := t\Pi_B \left(I + t^2 \Pi_B^2\right)^{-1}$ and u is contained in the range $\overline{R\left(\Pi_B\right)}$. They proved that this estimate would follow entirely from a set of simple conditions imposed upon the operators Γ, B_1 and B_2, labelled (H1) - (H8). Then, by checking this list of conditions, the Axelsson-Keith-McIntosh framework, or AKM framework by way of abbreviation, could be used to conclude that the particular selection of operators

$$\Gamma := \begin{pmatrix} 0 & 0 \\ \nabla & 0 \end{pmatrix}, \quad B_1 = I, \quad B_2 = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}, \tag{1.4}$$

defined on $L^2\left(\mathbb{R}^d\right) \oplus L^2\left(\mathbb{R}^d; \mathbb{C}^d\right)$, would satisfy (1.3) and therefore Π_B possess a bounded holomorphic functional calculus. The Kato square root estimate then followed almost trivially from this.

In direct analogy to the potential free case, the Kato problem with potential on $\mathbb{R}^d$ will be solved by constructing appropriate potential dependent Dirac-type operators and demonstrating that they retain a bounded holomorphic functional calculus under perturbation. In particular, this strategy will be applied to the Dirac-type operator

$$\Pi_V := \Gamma_V + \Gamma_V^* := \begin{pmatrix} 0 & 0 & 0 \\ |V|^{\frac{1}{2}} & 0 & 0 \\ \nabla & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & |V|^{\frac{1}{2}} & -\mathrm{div} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.5}$$

defined on $L^2(\mathbb{R}^d) \oplus L^2\left(\mathbb{R}^d\right) \oplus L^2\left(\mathbb{R}^d; \mathbb{C}^d\right)$, under the perturbation

$$B_1 = I, \quad B_2 := \begin{pmatrix} I & 0 & 0 \\ 0 & e^{i\cdot \arg V} & 0 \\ 0 & 0 & A \end{pmatrix}. \tag{1.6}$$

It should be observed that the operator Γ_V will not necessarily satisfy the cancellation and coercivity conditions, (H7) and (H8), of [5] due to the presence of the zero-order potential

term. As such, the original framework developed by Axelsson, Keith and McIntosh cannot be directly applied. The key difficulty in proving our result is then to alter the original framework in order to allow for such operators. The technical challenge presented by the inclusion of the zero-order potential V will be overcome by separating our square function norm into components and demonstrating that the zero-order term will allow for the first two components to be bounded while the third component will be handled by similar arguments as in [5]. We make use of a range of techniques including diagonalisation, a local $T(b)$ type argument and Carleson measure estimates. Our task, comparing with [5] and the other papers mentioned above, is made a bit more complicated by the fact that we keep track of the dependence of the estimates in terms of V in order to have constants which depend only the sectoriality angle ω_V. This dependence only through ω_V is the keystone of our proof for the Kato square root problem on domains.

This strategy does not however allow us to deal directly with general sectorial $V \in L^1_{loc}\left(\mathbb{R}^d\right)$. We first restrict attention to the class $\mathcal{W}_\alpha$ of potentials V for which

$$[V]_\alpha := \sup_{u \in D(-\Delta+|V|)} \frac{\left\| |V|^{\frac{\alpha}{2}} u \right\| + \left\| (-\Delta)^{\frac{\alpha}{2}} u \right\|}{\left\| (-\Delta + |V|)^{\frac{\alpha}{2}} u \right\|} < \infty$$

for some $\alpha \in (1, 2]$. Using the above ideas borrowed from [5] we prove the quadratic estimate

$$\left(C_V \left(1 + [V]_\alpha^2\right)\right)^{-1} \|u\|^2 \leq \int_0^\infty \left\| Q_t^{V,B} u \right\|^2 \frac{dt}{t} \leq C_V \left(1 + [V]_\alpha^2\right) \|u\|^2$$

with $Q_t^{V,B} = t\Pi_{V,B}\left(I + t^2\Pi_{V,B}^2\right)^{-1}$ and $\Pi_{V,B} = \Gamma_V + B_1\Gamma_V^* B_2$. As mentioned above, we pay attention to the constants involved in the estimates in order to have C_V which is independent of α and depends on V only through the angle of sectoriality ω_V. This leads to the existence of a bounded holomorphic functional calculus for the bisectorial operator $\Pi_{V,B}$, which in turn leads to

$$\left(C_V \left(1 + [V]_\alpha^2\right)\right)^{-1} \left(\|\nabla u\| + \left\|V^{\frac{1}{2}} u\right\|\right) \leq \left\| \sqrt{\mathcal{L} + V} u \right\| \leq C_V \left(1 + [V]_\alpha^2\right) \left(\|\nabla u\| + \left\|V^{\frac{1}{2}} u\right\|\right).$$

The rest of the proof of Theorem 1.1 for $\Omega = \mathbb{R}^d$ takes place in two stages. The first consists in removing the dependence of the above estimate on $[V]_\alpha$ by letting $\alpha \to 1$. The second one uses an approximation argument. Here we use some ideas from E.M. Ouhabaz [23] in order to approximate in the resolvent sense $\mathcal{L} + V$ by a sequence $\mathcal{L} + V_n$ with sectorial potentials $V_n \in \mathcal{W}_\alpha$ for some $\alpha \in (1, 2]$.

In order to deal with a general domain Ω in Theorem 1.1 we use two approximation arguments. For smooth Ω, the idea is to approximate $\mathcal{L}_\Omega + V$, as an operator on $L^2(\Omega)$, by the sequence of Schrödinger type operators $\mathcal{L} + V + n\mathbb{1}_{\mathbb{R}^d \setminus \Omega}$ acting on $L^2\left(\mathbb{R}^d\right)$. For general Ω, we approximate $\mathcal{L}_\Omega + V$ by $\mathcal{L}_{\Omega_n} + V$ with an increasing sequence of smooth open sets Ω_n. The control of the constants will be given by the dependence of the constants for $\Omega = \mathbb{R}^d$ on V only through ω_V.

Acknowledgements

Prior to commencing this project, it was suggested by the second named author to Pierre Portal that Kato's estimates for Schrödinger type operators on $\mathbb{R}^d$ with a control on the constant would have interesting applications and, in particular, could be used to solve the Kato square root problem on domains. Very recently, the first named author, a PhD student under the supervision of Pierre Portal, proved the estimates in Theorem 1.1 for $\Omega = \mathbb{R}^d$ for a class of potentials related to the reverse Hölder class. However, at the time the result was proved without the control on the constant required for the applications. These results were subsequently submitted to the journal Mathematische Nachrichten and reviewed. It was only following this that it became clear to us how to obtain the desired control on the constant for general potentials. The present paper greatly improves the first named author's unpublished article and allows us to realize the wish that Kato's square root problem is true on any arbitrary domain. Both authors would like to thank Pierre Portal for numerous very helpful discussions. The first named author would like to thank the anonymous referee of the article submitted to Mathematische Nachrichten for providing such a detailed and thoughtful critique.

While the first named author was working on the unpublished article, it was found that Andrew Morris and Andrew Turner from the University of Birmingham had also obtained results on Kato with potential on $\mathbb{R}^d$. After meeting them and discussing their research, it is clear that the two approaches differ in their assumptions and, more substantially, their proofs.

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2. Preliminaries

2.1. **Forms and Operators.** Let Ω be an open subset of $\mathbb{R}^d$. Let A be a $d \times d$-matrix whose coefficients $a_{kl} \in L^\infty(\Omega, \mathbb{C})$ satisfy the ellipticity condition (1.1) on Ω. We define the sesquilinear form

$$\mathfrak{a}_\Omega(u,v) = \int_\Omega \langle A(x)\nabla u, \nabla v\rangle \, dx = \sum_{k,l=1}^d \int_{\mathbb{R}^d} a_{kl} \frac{\partial u}{\partial x_k} \overline{\frac{\partial v}{\partial x_l}} \, dx$$

for u, $v \in H^1_0(\Omega)$. Then $\mathfrak{a}_\Omega$ is a densely defined, sectorial and closed form. Its associated operator $\mathcal{L}_\Omega$ is formally given by

$$\mathcal{L}_\Omega u = -\mathrm{div} A\nabla u$$

and subject to Dirichlet boundary conditions.
For a given potential $V \in L^1_{loc}(\Omega, \mathbb{C})$ with range contained in a sector $S_{\omega_V +}$ of angle $\omega_V \in [0, \frac{\pi}{2})$, we denote by $\mathcal{L}_\Omega + V$ the operator associated with the sectorial and closed form

$$\mathfrak{a}^V_\Omega(u,v) = \mathfrak{a}_\Omega(u,v) + \int_\Omega V u\overline{v} \, dx$$

with domain

$$H_0^{1,V}(\Omega) = \left\{ u \in H_0^1(\Omega), \int_\Omega |V||u|^2\, dx < \infty \right\}.$$

The form $\mathfrak{a}_\Omega^V$ clearly satisfies the Gårding inequality (or coercivity inequality)

$$\operatorname{Re} \mathfrak{a}_\Omega^V(u,u) \geq \kappa_A^V \left(\left\| |V|^{\frac{1}{2}} u \right\|^2 + \|\nabla u\|^2 \right) \tag{2.1}$$

for all $u \in H_0^{1,V}(\Omega)$, where κ_A^V is a positive constant which depends on the potential only through ω_V. Recall that

$$D(\mathcal{L}_\Omega + V) = \left\{ u \in H_0^{1,V}(\Omega) : \exists w \in L^2(\Omega) \ \ s.t.\ \mathfrak{a}_\Omega^V(u,v) = \langle w, v \rangle \ \forall\ v \in H_0^{1,V}(\Omega) \right\}.$$

In the particular case where $\Omega = \mathbb{R}^d$ we drop the subscript Ω in the notation and write $\mathfrak{a}$, $\mathfrak{a}^V$, $\mathcal{L}$ and $\mathcal{L}+V$ for the corresponding forms and operators.

Note that the operators defined above are all maximal accretive and hence we can define their square roots $\sqrt{\mathcal{L}_\Omega}$, $\sqrt{\mathcal{L}_\Omega + V}$, $\sqrt{\mathcal{L}}$ and $\sqrt{\mathcal{L}+V}$.

Remark 2.1. *Note that the results in this paper remain valid under the assumption that the form $\mathfrak{a}_\Omega$ satisfies the corresponding Gårding inequality*

$$\operatorname{Re} \mathfrak{a}_\Omega(u,u) \geq \kappa_A \|\nabla u\|^2$$

for all $u \in D(\mathfrak{a}_\Omega)$, for some $\kappa_A > 0$, instead of the pointwise ellipticity condition (1.1).

2.2. **Holomorphic Functional Calculus for Bisectorial Operators.** Let us briefly recall what it means for a bisectorial operator to possess a bounded holomorphic functional calculus. For a thorough treatment of functional calculus for bisectorial operators, it is recommended that the reader refer to [21], [9], [15], [18] or [1].

For $\mu \in [0,\pi)$, define the open and closed sectors

$$S_{\mu+}^o := \begin{cases} \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \mu\} & \mu \in (0,\pi) \\ (0,\infty) & \mu = 0 \end{cases}$$

and

$$S_{\mu+} := \begin{cases} \{z \in \mathbb{C} \cup \{\infty\} : |\arg(z)| \leq \mu \text{ or } z = 0,\ \infty\} & \mu \in (0,\pi) \\ [0,\infty] & \mu = 0. \end{cases}$$

Then, for $\mu \in \left[0, \frac{\pi}{2}\right)$, define the open and closed bisectors

$$S_\mu^o := \left(S_{\mu+}^o\right) \cup \left(-S_{\mu+}^o\right)$$

and

$$S_\mu := (S_{\mu+}) \cup (-S_{\mu+})$$

respectively. Throughout this section we consider bisectorial operators defined on a Hilbert space $\mathcal{H}$.

Definition 2.2 (Bisectorial Operator)**.** *A linear operator $T : D(T) \subseteq \mathcal{H} \to \mathcal{H}$ is said to be ω-bisectorial for $\omega \in \left[0, \frac{\pi}{2}\right)$ if the spectrum $\sigma(T)$ is contained in the bisector S_ω and if for any $\mu \in \left(\omega, \frac{\pi}{2}\right)$, there exists $C_\mu > 0$ such that the resolvent bound*

$$|\zeta| \left\| (\zeta I - T)^{-1} \right\| \leq C_\mu \tag{2.2}$$

holds for all $\zeta \in \mathbb{C} \setminus S_\mu$. *$T$ is said to be bisectorial if it is ω-bisectorial for some* $\omega \in \left[0, \frac{\pi}{2}\right)$.

An interesting fact concerning bisectorial operators is the following decomposition result.

Proposition 2.3 ([9, Thm. 3.8])**.** *Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a bisectorial operator. Then T is necessarily densely defined and the Hilbert space $\mathcal{H}$ admits the following decomposition*

$$\mathcal{H} = N(T) \oplus \overline{R(T)}.$$

Let T be an ω-bisectorial operator for $\omega \in \left[0, \frac{\pi}{2}\right)$ and $\mu \in \left(\omega, \frac{\pi}{2}\right)$. Define the following algebras of functions

$$H^\infty\left(S_\mu^o\right) := \left\{ f \text{ holomorphic} : \|f\|_\infty := \sup_{z \in S_\mu^o} |f(z)| < \infty \right\}$$

$$H_0^\infty\left(S_\mu^o\right) := \left\{ f \in H^\infty\left(S_\mu^o\right) : \exists\, C,\, \alpha > 0 \text{ s.t. } |f(z)| \leq C \frac{|z|^\alpha}{1 + |z|^{2\alpha}} \ \forall\, z \in S_\mu^o \right\}.$$

For any $f \in H_0^\infty\left(S_\mu^o\right)$, one can define the operator $f(T)$ as follows. For $u \in \mathcal{H}$, define

$$f(T)\, u := \frac{1}{2\pi i} \oint_\gamma f(z) \left(zI - T\right)^{-1} u \, dz,$$

where the curve

$$\gamma := \left\{ \pm r e^{\pm i\nu} : 0 \leq r < \infty \right\}$$

for some $\nu \in (\omega, \mu)$ is traversed anticlockwise. This association is a well-defined algebra homorphism from $H_0^\infty\left(S_\mu^o\right)$ to $\mathcal{L}(\mathcal{H})$.

Definition 2.4. *Let $0 \leq \omega < \mu < \frac{\pi}{2}$. An ω-bisectorial operator $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ is said to have a bounded $H^\infty\left(S_\mu^o\right)$-functional calculus if there exists $c > 0$ such that*

$$\|f(T)\| \leq c \|f\|_\infty \tag{2.3}$$

for all $f \in H_0^\infty\left(S_\mu^o\right)$. T is said to have a bounded holomorphic functional calculus if it has a bounded $H^\infty\left(S_\mu^o\right)$-functional calculus for some μ.

It is a well known fact that if (2.3) holds for all $f \in H_0^\infty\left(S_\mu^o\right)$ then there exists a well-defined extension of the homomorphism $f \mapsto f(T)$ to all of $H^\infty\left(S_\mu^o\right)$. Moreover, $g(T) \in \mathcal{L}(\mathcal{H})$ and (2.3) holds for all $g \in H^\infty\left(S_\mu^o\right)$.

Let $q : S_\mu^o \to \mathbb{C}$ be defined through

$$q(z) := \frac{z}{1 + z^2}, \quad z \in S_\mu^o.$$

For $t > 0$, let q_t denote the function $q_t(z) := q(tz)$ for $z \in S_\mu^o$. Then $q_t \in H_0^\infty\left(S_\mu^o\right)$ for any $t > 0$ and one can define $q_t(T)$.

Definition 2.5 (Square Function Estimates)**.** *A bisectorial operator T on a Hilbert space $\mathcal{H}$ is said to satisfy square function estimates if there exists a constant $C_{SF} > 0$ such that*

$$C_{SF}^{-1} \|u\|^2 \leq \int_0^\infty \|q_t(T)u\|^2 \frac{dt}{t} \leq C_{SF} \|u\|^2 \tag{2.4}$$

for all $u \in \overline{R(T)}$.

Remark 2.6. *The use of the function q in the above definition of square function estimates is somewhat arbitrary. Indeed, it can be swapped with ψ for any $\psi \in H_0^\infty\left(S_\mu^o\right)$ not identically equal to zero on either $S_{\mu+}^o$ or $\left(-S_{\mu+}^o\right)$. This follows from the equivalence of these two norms as stated in* [21] *or* [15, Thm. 7.3.1].

For self-adjoint operators, the following basic result holds.

Proposition 2.7. *Suppose that T is self-adjoint. Then for any $u \in \mathcal{H}$,*

$$\int_0^\infty \|q_t(T)u\|^2 \frac{dt}{t} \leq \frac{1}{2} \|u\|^2 .$$

Equality will hold if $u \in \overline{R(T)}$.

The proof of the following theorem can be found in [9] or [1]. The constant dependence is not explicitly stated in either of these references but it is straightforward to trace through their arguments to obtain the below dependence.

Theorem 2.8. *Let T be an ω-bisectorial operator on $\mathcal{H}$ for $\omega \in \left[0, \frac{\pi}{2}\right)$. Suppose that T satisfies square function estimates with constant $C_{SF} > 0$. Then T must have a bounded $H^\infty\left(S_\mu^o\right)$-functional calculus for any $\mu \in \left(\omega, \frac{\pi}{2}\right)$. In particular, there exists a constant $c > 0$, independent of T, such that*

$$\|f(T)\| \leq c\, C_{SF}\, C_\mu\, \|f\|_\infty$$

for all $f \in H_0^\infty\left(S_\mu^o\right)$, where $C_\mu > 0$ is the constant from the resolvent estimate (2.2).

2.3. **AKM without Cancellation and Coercivity.** The operators that we wish to consider, Γ_V, will satisfy the first six conditions of [5]. However, they will not necessarily satisfy the cancellation condition (H7) and the coercivity condition (H8). It will therefore be fruitful to see what happens to the original AKM framework when the cancellation and coercivity conditions are removed.

Similar to the original result, we begin by assuming that we have operators that satisfy the hypotheses (H1) - (H3) from [5]. Recall these conditions for operators Γ, B_1 and B_2 on a Hilbert space $\mathcal{H}$.

(H1) $\Gamma : D(\Gamma) \to \mathcal{H}$ is a closed, densely defined, nilpotent operator.

(H2) B_1 and B_2 satisfy the accretivity conditions

$$\mathrm{Re}\langle B_1 u, u\rangle \geq \kappa_1 \|u\|^2 \qquad \text{and} \qquad \mathrm{Re}\langle B_2 v, v\rangle \geq \kappa_2 \|v\|^2$$

for all $u \in R(\Gamma^*)$ and $v \in R(\Gamma)$ for some κ_1, $\kappa_2 > 0$.

(H3) The operators Γ and Γ^* satisfy

$$\Gamma^* B_2 B_1 \Gamma^* = 0 \qquad \textit{and} \qquad \Gamma B_1 B_2 \Gamma = 0.$$

In [5] Section 4, the authors assume that they have operators that satisfy the hypotheses (H1) - (H3) and they derive several important operator theoretic consequences from only these three hypotheses. As our operators Γ, B_1 and B_2 also satisfy (H1) - (H3), it follows that the results in [5] Section 4 can be applied for these operators. In the interest of making this article as self-contained as possible, we will now restate any such result that is to be used in this paper.

Proposition 2.9 ([5]). *Define the perturbation dependent operators*

$$\Gamma_B^* := B_1 \Gamma^* B_2, \quad \Gamma_B := B_2^* \Gamma B_1^* \quad \textit{and} \quad \Pi_B := \Gamma + \Gamma_B^*.$$

The Hilbert space $\mathcal{H}$ has the following Hodge decomposition into closed subspaces:

$$\mathcal{H} = N(\Pi_B) \oplus \overline{R(\Gamma_B^*)} \oplus \overline{R(\Gamma)}. \tag{2.5}$$

Moreover, we have $N(\Pi_B) = N(\Gamma_B^*) \cap N(\Gamma)$ and $\overline{R(\Pi_B)} = \overline{R(\Gamma_B^*)} \oplus \overline{R(\Gamma)}$. When $B_1 = B_2 = I$ these decompositions are orthogonal, and in general the decompositions are topological. Similarly, there is also a decomposition

$$\mathcal{H} = N(\Pi_B^*) \oplus \overline{R(\Gamma_B)} \oplus \overline{R(\Gamma^*)}.$$

Proposition 2.10 ([5]). *The perturbed Dirac-type operator Π_B is an ω-bisectorial operator with $\omega := \frac{1}{2}(\omega_1 + \omega_2)$ where*

$$\omega_1 := \sup_{u \in R(\Gamma^*) \setminus \{0\}} |\arg \langle B_1 u, u \rangle| < \frac{\pi}{2}$$

and

$$\omega_2 := \sup_{u \in R(\Gamma) \setminus \{0\}} |\arg \langle B_2 u, u \rangle| < \frac{\pi}{2}.$$

The bisectoriality of Π_B ensures that the following operators will be well-defined.

Definition 2.11. *For $t \in \mathbb{R} \setminus \{0\}$, define the perturbation dependent operators*

$$R_t^B := (I + it\Pi_B)^{-1}, \quad P_t^B := \left(I + t^2 (\Pi_B)^2\right)^{-1},$$

$$Q_t^B := t\Pi_B P_t^B \quad \textit{and} \quad \Theta_t^B := t\Gamma_B^* P_t^B.$$

When there is no perturbation, i.e. when $B_1 = B_2 = I$, the B will dropped from the superscript or subscript. For example, instead of Θ_t^I or Π_I the notation Θ_t and Π will be employed.

Remark 2.12. *An easy consequence of Proposition 2.10 is that the operators R_t^B, P_t^B and Q_t^B are all uniformly $\mathcal{H}$-bounded in t. Furthermore, on taking the Hodge decomposition in Proposition 2.9 into account, it is clear that the operators Θ_t^B will also be uniformly $\mathcal{H}$-bounded in t.*

The next result tells us how the operators Π_B and P_t^B interact with Γ and Γ_B^*.

Lemma 2.13 ([5]). *The following relations are true.*

$$\Pi_B \Gamma u = \Gamma_B^* \Pi_B u \quad \text{for all } u \in D\left(\Gamma_B^* \Pi_B\right),$$
$$\Pi_B \Gamma_B^* u = \Gamma \Pi_B u \quad \text{for all } u \in D\left(\Gamma \Pi_B\right),$$
$$\Gamma P_t^B u = P_t^B \Gamma u \quad \text{for all } u \in D\left(\Gamma\right), \quad \text{and}$$
$$\Gamma_B^* P_t^B u = P_t^B \Gamma_B^* u \quad \text{for all } u \in D\left(\Gamma_B^*\right).$$

The subsequent lemma provides a square function estimate for the unperturbed Dirac-type operator Π. When considering square function estimates for the perturbed operator, there will be several instances where the perturbed case can be reduced with the assistance of this unperturbed estimate. Its proof follows directly from the self-adjointness of the operator Π and Proposition 2.7.

Lemma 2.14 ([5]). *The quadratic estimate*

$$\int_0^\infty \left\|Q_t u\right\|^2 \frac{dt}{t} \leq \frac{1}{2} \left\|u\right\|^2 \tag{2.6}$$

holds for all $u \in \mathcal{H}$. Equality holds on $\overline{R\left(\Pi\right)}$.

The following result will play a crucial role in the reduction of the square function estimate (1.3).

Proposition 2.15 ([5]). *Assume that the estimate*

$$\int_0^\infty \left\|\Theta_t^B P_t u\right\|^2 \frac{dt}{t} \leq c \left\|u\right\|^2 \tag{2.7}$$

holds for all $u \in R\left(\Gamma\right)$ and some constant $c > 0$, together with three similar estimates obtained on replacing $\{\Gamma, B_1, B_2\}$ by $\{\Gamma^*, B_2, B_1\}$, $\{\Gamma^*, B_2^*, B_1^*\}$ and $\{\Gamma, B_1^*, B_2^*\}$. Then Π_B satisfies the quadratic estimate

$$\left(c\,C\right)^{-1} \left\|u\right\|^2 \leq \int_0^\infty \left\|Q_t^B u\right\|^2 \frac{dt}{t} \leq c\,C \left\|u\right\|^2 \tag{2.8}$$

for all $u \in \overline{R\left(\Pi_B\right)}$, for some $C > 0$ entirely dependent on (H1)- (H3).

The constant dependence of (2.8) is not explicitly mentioned in Proposition 4.8 of [5], but it is relatively easy to trace through their argument and record where (2.7) is used. The following corollary is proved during the course of the proof of Proposition 4.8 of [5].

Corollary 2.16 (High Frequency Estimate). *For any $u \in R\left(\Gamma\right)$, there exists a constant $c > 0$ for which*

$$\int_0^\infty \left\|\Theta_t^B \left(I - P_t\right) u\right\|^2 \frac{dt}{t} \leq c \left\|u\right\|^2.$$

From this point onwards, it will also be assumed that our operators satisfy the additional hypotheses (H4) - (H6). These hypotheses are stated below for reference.

(H4) The Hilbert space is $\mathcal{H} = L^2\left(\mathbb{R}^d; \mathbb{C}^N\right)$ for some $d,\, N \in \mathbb{N}^*$.

(H5) The operators B_1 and B_2 represent multiplication by matrix-valued functions. That is,
$$B_1(f)(x) = B_1(x)f(x) \qquad \text{and} \qquad B_2(f)(x) = B_2(x)f(x)$$
for all $f \in \mathcal{H}$ and $x \in \mathbb{R}^d$, where $B_1,\, B_2 \in L^\infty\left(\mathbb{R}^d; \mathcal{L}\left(\mathbb{C}^N\right)\right)$.

(H6) For every bounded Lipschitz function $\eta : \mathbb{R}^d \to \mathbb{C}$, we have that $\eta D(\Gamma) \subset D(\Gamma)$ and $\eta D(\Gamma^*) \subset D(\Gamma^*)$. Moreover, the commutators $[\Gamma, \eta I]$ and $[\Gamma^*, \eta I]$ are multiplication operators that satisfy the bound
$$\left|[\Gamma, \eta I](x)\right|,\ \left|[\Gamma^*, \eta I](x)\right| \leq c\left|\nabla\eta(x)\right|$$
for all $x \in \mathbb{R}^d$ and some constant $c > 0$.

In contrast to the original result, our operators will not be assumed to satisfy the cancellation condition (H7) and the coercivity condition (H8). Without these two conditions, many of the results from Section 5 of [5] will fail. One notable exception to this is that the bounded operators associated with our perturbed Dirac-type operator Π_B will satisfy off-diagonal estimates.

Definition 2.17 (Off-Diagonal Bounds)**.** *Define* $\langle x\rangle := 1 + |x|$ *for* $x \in \mathbb{C}$ *and* $\operatorname{dist}(E, F) := \inf\{|x - y| : x \in E, y \in F\}$ *for* $E,\ F \subset \mathbb{R}^d$.

Let $\{U_t\}_{t>0}$ *be a family of operators on* $\mathcal{H} = L^2\left(\mathbb{R}^d; \mathbb{C}^N\right)$. *This collection is said to have off-diagonal bounds of order* $M > 0$ *if there exists* $C_M > 0$ *such that*
$$\|U_t u\|_{L^2(E)} \leq C_M \langle \operatorname{dist}(E, F)/t\rangle^{-M} \|u\| \tag{2.9}$$
whenever $E,\ F \subset \mathbb{R}^d$ *are Borel sets and* $u \in \mathcal{H}$ *satisfies* $\operatorname{supp} u \subset F$.

Proposition 2.18 ([5])**.** *Let* U_t *be given by either* R_t^B, R_{-t}^B, P_t^B, Q_t^B *or* Θ_t^B *for every* $t > 0$. *The collection of operators* $\{U_t\}_{t>0}$ *has off-diagonal bounds of every order* $M > 0$. *Moreover, the constant* C_M *in the estimate* (2.9) *depends only on* M *and the hypotheses (H1) - (H6).*

Introduce the following dyadic decomposition of $\mathbb{R}^d$. Let $\square = \cup_{j=-\infty}^{\infty} \square_{2^j}$ where $\square_t := \{2^j(k + (0,1]^n) : k \in \mathbb{Z}^n\}$ if $2^{j-1} < t \leq 2^j$. Define the averaging operator $A_t : \mathcal{H} \to \mathcal{H}$ through
$$A_t u(x) := \frac{1}{|Q(x,t)|}\int_{Q(x,t)} u(y)\, dy$$
for $x \in \mathbb{R}^d$, $t > 0$ and $u \in \mathcal{H}$, where $Q(x,t)$ is the unique dyadic cube in $\square_t$ that contains the point x.

For an operator family $\{U_t\}_{t>0}$ that satisfies off-diagonal bounds of every order, there exists an extension $U_t : L^\infty\left(\mathbb{R}^d; \mathbb{C}^N\right) \to L^2_{loc}\left(\mathbb{R}^d; \mathbb{C}^N\right)$ for each $t > 0$. This is constructed by defining
$$U_t u(x) := \lim_{r\to\infty} \sum_{\substack{R\in\square_t \\ \operatorname{dist}(Q,R)<r}} U_t\left(\mathbb{1}_R u\right)(x),$$

for $x \in Q \in \square_t$ and $u \in L^\infty\left(\mathbb{R}^d;\mathbb{C}^N\right)$. The convergence of the above limit is guaranteed by the off-diagonal bounds of $\{U_t\}_{t>0}$. Further detail on this construction can be found in [5], [11], [22]. Following [5], the above extension then allows us to introduce the principal part of the operator U_t.

Definition 2.19. *Let $\{U_t\}_{t>0}$ be operators on $\mathcal{H}$ that satisfy off-diagonal bounds of every order. For $t>0$, the principal part of U_t is the operator $\zeta_t : \mathbb{R}^d \to \mathcal{L}\left(\mathbb{C}^N\right)$ defined through*

$$[\zeta_t(x)]\,(w) := (U_t w)\,(x)$$

for each $x \in \mathbb{R}^d$ and $w \in \mathbb{C}^N$.

The following generalisation of Corollary 5.3 of [5] will also be true with an identical proof.

Proposition 2.20. *Let $\{U_t\}_{t>0}$ be operators on $\mathcal{H}$ that satisfy off-diagonal bounds of every order. Let $\zeta_t : \mathbb{R}^d \to \mathcal{L}\left(\mathbb{C}^N\right)$ denote the principal part of the operator U_t. Then there exists $c>0$ such that*

$$\fint_Q |\zeta_t(y)|^2\,dy \leq c$$

for all $Q \in \square_t$. Moreover, the operators $\zeta_t A_t$ are uniformly $\mathcal{H}$-bounded in t.

Finally, the ensuing partial result will also be valid. Its proof follows in an identical manner to the first part of the proof of Proposition 5.5 of [5].

Proposition 2.21. *Let $\{U_t\}_{t>0}$ be operators on $\mathcal{H}$ that satisfy off-diagonal bounds of every order. Let $\zeta_t : \mathbb{R}^d \to \mathcal{L}\left(\mathbb{C}^N\right)$ denote the principal part of U_t. Then there exists $c>0$ such that*

$$\|(U_t - \zeta_t A_t)\,v\| \leq c\,\|t\nabla v\|\,. \tag{2.10}$$

for any $v \in H^1\left(\mathbb{R}^d;\mathbb{C}^N\right) \subset \mathcal{H}$ and $t>0$.

3. Proof of Kato with Potential on $\mathbb{R}^d$

The aim of this section is to prove Theorem 1.1 for $\Omega = \mathbb{R}^d$. As explained in the introduction, the proof is based on the holomorphic functional calculus (equivalently square function estimates) for Dirac-type operators. The proof of one of the square function estimates will be postponed to the next section.

We start by introducing a class of potentials. For $\alpha \in [1,2]$, define $\mathcal{W}_\alpha$ to be the class of all measurable functions $V \in L^1_{loc}\left(\mathbb{R}^d\right)$ for which

$$[V]_\alpha := \sup_{u \in D(-\Delta+|V|)} \frac{\left\||V|^{\frac{\alpha}{2}}\,u\right\| + \left\|(-\Delta)^{\frac{\alpha}{2}}\,u\right\|}{\left\|(-\Delta+|V|)^{\frac{\alpha}{2}}\,u\right\|} < \infty.$$

The supremum is taken over $D\,(-\Delta+|V|)$ but it coincides with the supremum over $D\left((-\Delta+|V|)^{\frac{\alpha}{2}}\right)$ since $D\,(-\Delta+|V|)$ is a core of the operator $(-\Delta+|V|)^{\frac{\alpha}{2}}$.

As will be proved in Lemma 3.9, $\mathcal{W}_\alpha$ is a decreasing class of potentials and $\mathcal{W}_1 = L^1_{loc}\left(\mathbb{R}^d\right)$.

As a first step, we state a weaker version of Theorem 1.1 for $\Omega = \mathbb{R}^d$ for potentials which are in $\mathcal{W}_\alpha$ for some $\alpha \in (1,2]$.

Proposition 3.1. *Let $V \in \mathcal{W}_\alpha$ for some $\alpha \in (1,2]$ and suppose that the range of V is contained in a sector $S_{\omega_V +}$ for some $\omega_V \in \left[0, \frac{\pi}{2}\right)$. Then there exists $C_V > 0$, dependent on the potential only through ω_V, for which*

(3.1)
$$\left(C_V\left(1+[V]_\alpha^2\right)\right)^{-1}\left(\|\nabla u\| + \left\|V^{\frac{1}{2}}u\right\|\right) \leq \left\|\sqrt{\mathcal{L}+V}\,u\right\| \leq C_V\left(1+[V]_\alpha^2\right)\left(\|\nabla u\| + \left\|V^{\frac{1}{2}}u\right\|\right)$$

for all $u \in D(\mathcal{L}+V)$. The constant C_V is independent of $\alpha \in (1,2]$.

Remark 3.2. *In the previous proposition, as well as in the forthcoming results, we emphasize the independence of C_V from α. The reason for this lies in the fact that we will let $\alpha \to 1$ in (3.1) in order to obtain the result for general potentials. Therefore we at least need a constant that does not explode when taking the limit.*

Notation. *For the remainder of this article, the notation $A \lesssim B$ and $A \simeq B$ will be used to denote that there exists a constant $C_V > 0$, independent of α and dependent on the potential only through ω_V, such that $A \leq C_V B$ and $C_V^{-1}B \leq A \leq C_V B$.*

Fix $V \in \mathcal{W}_\alpha$ for some $\alpha \in (1,2]$ with angle of sectoriality $\omega_V \in \left[0, \frac{\pi}{2}\right)$. Define the Hilbert space

$$\mathcal{H} := L^2\left(\mathbb{R}^d; \mathbb{C}^{d+2}\right) = L^2\left(\mathbb{R}^d;\mathbb{C}\right) \oplus L^2\left(\mathbb{R}^d;\mathbb{C}\right) \oplus L^2(\mathbb{R}^d;\mathbb{C}^d).$$

Let $\mathbb{P}_i$ denote the natural projection map onto the subspace corresponding to the ith component of $\mathcal{H}$ for $i = 1$, 2 and 3. The notation $\mathbb{P}_i$ will also be used to denote the projection map onto the ith subspace of $\mathbb{C}^{n+2} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^n$. The ith component of a vector $v \in \mathcal{H}$ will be denoted by v_i so that $v = (v_1, v_2, v_3)$ with v_1, $v_2 \in L^2\left(\mathbb{R}^d\right)$ and $v_3 \in L^2\left(\mathbb{R}^d;\mathbb{C}^d\right)$.

Let Γ_V be the operator on $\mathcal{H}$ defined by

$$\Gamma_V := \begin{pmatrix} 0 & 0 & 0 \\ |V|^{\frac{1}{2}} & 0 & 0 \\ \nabla & 0 & 0 \end{pmatrix}.$$

Also define

$$\Gamma_0 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \nabla & 0 & 0 \end{pmatrix}, \qquad M_V := \begin{pmatrix} 0 & 0 & 0 \\ |V|^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Pi_0 := \Gamma_0 + \Gamma_0^* \quad \textit{and} \quad \Pi_V := \Gamma_V + \Gamma_V^*.$$

Let $B_1,\ B_2 \in L^\infty\left(\mathbb{R}^d; \mathcal{L}\left(\mathbb{C}^{d+2}\right)\right)$ be the matrix-valued multiplication operators

$$B_1 = I, \quad B_2 := \begin{pmatrix} I & 0 & 0 \\ 0 & e^{i\cdot \arg V} & 0 \\ 0 & 0 & A \end{pmatrix}. \tag{3.2}$$

Proposition 3.3. *The family of operators $\{\Gamma_0, B_1, B_2\}$ satisfies the conditions (H1) - (H8) of [5] while $\{\Gamma_V, B_1, B_2\}$ satisfies only (H1) - (H6).*

Proof. Assumptions (H1), (H3), (H4) and (H5) are obvious from the definitions of the operators. (H2) follows from the ellipticity assumption (1.1) for $\{\Gamma_0, B_1, B_2\}$ and from (2.1) for $\{\Gamma_V, B_1, B_2\}$. (H6) for both sets of operators follows from the fact that the order of the operators Γ_0 and Γ_V is less than or equal to one. (H7) for $\{\Gamma_0, B_1, B_2\}$ is given by the homogeneity of Γ_0 and (H8) is given by the boundedness of the Riesz transforms $\partial_j\partial_k\left(-\Delta\right)^{-1}$ on $L^2\left(\mathbb{R}^d\right)$ for $j,\ k = 1, \cdots, d$. □

For reference, the cancellation condition (H7) and the coercivity condition (H8) are given below for the operator Γ_0.

(H7) For any $u \in D\left(\Gamma_0\right)$ and $v \in D\left(\Gamma_0^*\right)$, both compactly supported,

$$\int_{\mathbb{R}^d} \Gamma_0 u = 0 \quad \textit{and} \quad \int_{\mathbb{R}^d} \Gamma_0^* v = 0.$$

(H8) There exists $c > 0$ such that

$$\|\nabla u\| \leq c\ \|\Pi_0 u\|$$

for all $u \in R\left(\Pi_0\right) \cap D\left(\Pi_0\right)$.

Remark 3.4. *Since the operators $\{\Gamma_0, B_1, B_2\}$ satisfy all eight conditions (H1) - (H8) of [5], we can then apply the results from that paper to these operators.*

Definition 3.5. *For $t \in \mathbb{R} \setminus \{0\}$, define the perturbation dependent operators*

$$\Gamma_{V,B}^* := B_1\Gamma_V^* B_2, \quad \Pi_{V,B} := \Gamma_V + \Gamma_{V,B}^*,$$

$$R_t^{V,B} := \left(I + it\Pi_{V,B}\right)^{-1}, \quad P_t^{V,B} := \left(I + t^2\left(\Pi_{V,B}\right)^2\right)^{-1},$$

$$Q_t^{V,B} := t\Pi_{V,B}P_t^{V,B} \quad \textit{and} \quad \Theta_t^{V,B} := t\Gamma_{V,B}^* P_t^{V,B}.$$

When there is no perturbation, i.e. when $B_1 = B_2 = I$, the B will dropped from the superscript or subscript. For example, instead of $\Theta_t^{V,I}$ the notation Θ_t^V will be employed.

The main result that will be used to prove the Kato estimate with potential on $\mathbb{R}^d$ is the following square function estimate.

Theorem 3.6. *Let $\{\Gamma_V, B_1, B_2\}$ be as defined above. There exists a constant $C_V > 0$, dependent on the potential only through ω_V, such that*

$$\int_0^\infty \left\|\Theta_t^{V,B} P_t^V u\right\|^2 \frac{dt}{t} \leq C_V\left(1 + [V]_\alpha^2\right)\|u\|^2 \tag{3.3}$$

for all $u \in R(\Gamma_V)$. *The constant* C_V *is independent of* $\alpha \in (1,2]$.

The proof of this theorem will be reserved for Section 4. Next, we consider an estimate that serves as a dual to (3.3).

Proposition 3.7. *For* $t > 0$, *define the operator*

$$\underline{P}_t^{V,B} := \left(I + t^2 \left(\Gamma_V^* + B_2 \Gamma_V B_1\right)^2\right)^{-1}.$$

The square function estimate

$$\int_0^\infty \left\| \underline{P}_t^{V,B} t B_2 \Gamma_V B_1 P_t^V u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2 \tag{3.4}$$

will hold for all $u \in \mathcal{H}$.

Proof. Since $\{\Gamma_V, B_1, B_2\}$ satisfies (H1) - (H6) it follows that $\{\Gamma_V^*, B_2, B_1\}$ will also satisfy (H1) - (H6). Proposition 2.10 then implies that the operators $\underline{P}_t^{V,B}$ are well-defined and uniformly L^2-bounded (by a constant depending on V only through ω_V). On applying this to the left-hand side of (3.4),

$$\begin{aligned}
\int_0^\infty \left\| \underline{P}_t^{V,B} t B_2 \Gamma_V B_1 P_t^V u \right\|^2 \frac{dt}{t} &\lesssim \int_0^\infty \left\| t B_2 \Gamma_V B_1 P_t^V u \right\|^2 \frac{dt}{t} \\
&\lesssim \int_0^\infty \left\| t \Gamma_V P_t^V u \right\|^2 \frac{dt}{t} \\
&\leq \int_0^\infty \left\| t \Pi_V P_t^V u \right\|^2 \frac{dt}{t} \\
&= \frac{1}{2} \|u\|^2,
\end{aligned}$$

where the inequality $\|\Gamma_V v\| \leq \|\Pi_V v\|$ for $v \in D(\Pi_V)$ follows immediately from the three-by-three matrix form of the operators and Lemma 2.14 was applied to obtain the last line. □

From Theorem 3.6 and the previous proposition, the upper and lower square function estimates for $Q_t^{V,B}$ can be proved.

Theorem 3.8. *The estimate*

$$\left(1 + [V]_\alpha^2\right)^{-1} \|u\|^2 \lesssim \int_0^\infty \left\| Q_t^{V,B} u \right\|^2 \frac{dt}{t} \lesssim \left(1 + [V]_\alpha^2\right) \|u\|^2 \tag{3.5}$$

holds for all $u \in \overline{R(\Pi_V)}$.

Proof. Proposition 2.15 states that in order to prove the square function estimate (3.5), it is sufficient for the estimate (3.3) to be valid for the permutations of operators $\{\Gamma_V, B_1, B_2\}$, $\{\Gamma_V, B_1^*, B_2^*\}$, $\{\Gamma_V^*, B_2, B_1\}$ and $\{\Gamma_V^*, B_2^*, B_1^*\}$. The first permutations $\{\Gamma_V, B_1, B_2\}$ and $\{\Gamma_V, B_1^*, B_2^*\}$ both come under the umbrella of Theorem 3.6. The permutations $\{\Gamma_V^*, B_2, B_1\}$ and $\{\Gamma_V^*, B_2^*, B_1^*\}$ are handled by Proposition 3.7 by observing that $\underline{P}_t^{V,B}$ and $B_2 \Gamma_V B_1$ commute on an appropriate space (cf. Lemma 2.13). □

Proof of Proposition 3.1. From the upper and lower estimates of the previous theorem, Theorem 2.8 then implies that $\Pi_{V,B}$ has a bounded holomorphic functional calculus. Applying this with the functions $f(z) = \frac{z}{\sqrt{z^2}}$ and $g(z) = \frac{\sqrt{z^2}}{z}$ to a vector $(u, 0, 0) \in \mathcal{H}$ with $u \in D\left(\sqrt{\mathcal{L}+V}\right)$ then completes the proof of Proposition 3.1. □

Let us now dispose of the dependence of the Kato estimate on the constant $[V]_\alpha$ through an interpolation argument.

Lemma 3.9. *Suppose that the potential V is in $\mathcal{W}_\alpha$ for some $\alpha \in (1, 2]$. Then $V \in \mathcal{W}_\beta$ for any $\beta \in [1, \alpha]$ with*

$$[V]_\beta \leq 2\left([V]_\alpha\right)^{\frac{\beta-1}{\alpha-1}}. \tag{3.6}$$

Proof. The first observation is that $[V]_1 \leq 2$ for any locally integrable $V : \mathbb{R}^d \to \mathbb{C}$. Indeed, due to positivity of the operator $(-\Delta)$,

$$\begin{aligned}\left\||V|^{\frac{1}{2}} u\right\|^2 &= \langle |V| u, u\rangle \\ &\leq \langle (-\Delta + |V|) u, u\rangle \\ &= \left\|(-\Delta + |V|)^{\frac{1}{2}} u\right\|^2.\end{aligned}$$

Identical reasoning can be applied to obtain the bound

$$\left\|(-\Delta)^{\frac{1}{2}} u\right\| \leq \left\|(-\Delta + |V|)^{\frac{1}{2}} u\right\|.$$

This proves that $[V]_1 \leq 2$ and therefore $\mathcal{W}_1$ is the class of all $V \in L^1_{loc}\left(\mathbb{R}^d\right)$ with no further restriction.

Assume that $V \in \mathcal{W}_\alpha$ for some $\alpha \in (1, 2]$. We use interpolation to prove that $V \in \mathcal{W}_\beta$ for $\beta \in [1, \alpha]$ with the constant given in (3.6). Define the Banach spaces

$$X_1 := \left\{u \in L^2\left(\mathbb{R}^d\right) : \|u\|_{X_1} := \frac{1}{M}\|u\|_2 + \left\|(-\Delta + |V|)^{\frac{1}{2}} u\right\|_2 < \infty\right\}$$

and

$$Y_1 := \left\{u \in L^2\left(\mathbb{R}^d\right) : \|u\|_{Y_1} := \frac{1}{M}\|u\|_2 + \left\|(-\Delta + |V|)^{\frac{\alpha}{2}} u\right\|_2 < \infty\right\},$$

where $M > 0$. The corresponding complex interpolation space is

$$[X_1, Y_1]_\theta := \left\{u \in L^2\left(\mathbb{R}^d\right) : \|u\|_{[X_1,Y_1]_\theta} := \frac{1}{M}\|u\|_2 + \left\|(-\Delta + |V|)^{\frac{1}{2}+\left(\frac{\alpha-1}{2}\right)\theta} u\right\|_2 < \infty\right\},$$

for $0 \leq \theta \leq 1$. Also set

$$X_2 = Y_2 = L^2\left(\mathbb{R}^d\right).$$

For z in the strip

$$S := \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\},$$

define the operator T_z on $X_1 \cap Y_1$ through

$$T_z u := |V|^{\frac{1}{2}+\left(\frac{\alpha-1}{2}\right)z} u,$$

for $u \in X_1 \cap Y_1$. Then, if $t \in \mathbb{R}$,

$$\begin{aligned}\|T_{it}u\|_{X_2} &= \|T_{it}u\|_{L^2} \\ &= \left\| |V|^{\frac{1}{2}+\left(\frac{\alpha-1}{2}\right)it} u \right\|_{L^2} \\ &= \left\| |V|^{\frac{1}{2}} u \right\|_{L^2} \\ &\leq \left\| (-\Delta + |V|)^{\frac{1}{2}} u \right\|_{L^2} \\ &\leq \|u\|_{X_1}\end{aligned}$$

for $u \in X_1 \cap Y_1$. This implies that T_{it} can be extended to a linear operator on X_1 with

$$\|T_{it}\|_{\mathcal{L}(X_1,X_2)} \leq 1.$$

Similarly,

$$\begin{aligned}\|T_{1+it}u\|_{Y_2} &= \|T_{1+it}u\|_{L^2} \\ &= \left\| |V|^{\frac{\alpha}{2}} u \right\|_{L^2} \\ &\leq [V]_\alpha \left\| (-\Delta + |V|)^{\frac{\alpha}{2}} u \right\|_{L^2} \\ &\leq [V]_\alpha \|u\|_{Y_1}\end{aligned}$$

for $u \in X_1 \cap Y_1$. This implies that T_{1+it} extends to a linear operator on Y_1 with

$$\|T_{1+it}\|_{\mathcal{L}(Y_1,Y_2)} \leq [V]_\alpha .$$

Hence,

$$\|T_\theta u\|_{[X_2,Y_2]_\theta} \leq 1^{1-\theta} [V]_\alpha^\theta \|u\|_{[X_1,Y_1]_\theta} \tag{3.7}$$

for any $0 \leq \theta \leq 1$. Setting $\theta = \frac{\beta-1}{\alpha-1}$ in (3.7) then gives

$$\left\| |V|^{\frac{\beta}{2}} u \right\| \leq [V]_\alpha^{\left(\frac{\beta-1}{\alpha-1}\right)} \left(\frac{1}{M} \|u\| + \left\| (-\Delta + |V|)^{\frac{\beta}{2}} u \right\| \right).$$

Let $M \to \infty$ to obtain

$$\left\| |V|^{\frac{\beta}{2}} u \right\| \leq [V]_\alpha^{\left(\frac{\beta-1}{\alpha-1}\right)} \left\| (-\Delta + |V|)^{\frac{\beta}{2}} u \right\|.$$

A similar interpolation argument can be applied to obtain the bound

$$\left\| (-\Delta)^{\frac{\beta}{2}} u \right\| \leq [V]_\alpha^{\left(\frac{\beta-1}{\alpha-1}\right)} \left\| (-\Delta + |V|)^{\frac{\beta}{2}} u \right\|$$

for any $u \in D(-\Delta + |V|)$. It proceeds identically but it should be noted that the boundedness of the imaginary powers of $(-\Delta)$ must be used when evaluating the endpoints of the interpolation argument. □

It is now possible to get rid of the dependence of (3.1) on the constant $[V]_\alpha$. Since $V \in \mathcal{W}_\alpha$, it follows from the previous lemma that $[V]_\beta < \infty$ for all $\beta \in [1, \alpha]$. This implies that (3.1) is valid for all $\beta \in [1, \alpha]$,

(3.8)
$$\left(C_V\left(1+[V]_\beta^2\right)\right)^{-1}\left(\left\||V|^{\frac{1}{2}}u\right\|+\|\nabla u\|\right) \leq \left\|\sqrt{\mathcal{L}+V}u\right\| \leq C_V\left(1+[V]_\beta^2\right)\left(\left\||V|^{\frac{1}{2}}u\right\|+\|\nabla u\|\right).$$

Moreover, the previous lemma also allows us to deduce that

$$\lim_{\beta\to 1^+}[V]_\beta \leq 2.$$

Sending $\beta \to 1^+$ in (3.8) leads to

$$C_V^{-1}\left(\left\||V|^{\frac{1}{2}}u\right\|+\|\nabla u\|\right) \leq \left\|\sqrt{\mathcal{L}+V}u\right\| \leq C_V\left(\left\||V|^{\frac{1}{2}}u\right\|+\|\nabla u\|\right). \tag{3.9}$$

In order to conclude our proof of Theorem 1.1 for $\Omega = \mathbb{R}^d$ we have to extend (3.9) to all sectorial $V \in L^1_{loc}\left(\mathbb{R}^d\right)$. This will be achieved by an approximation argument. In order to do so, we need to approximate in the strong resolvent sense $\mathcal{L}+V$ by a sequence $\mathcal{L}+V_n$ with $V_n \in \mathcal{W}_\alpha$ and apply (3.9). We shall borrow some ideas from [23].

Let $V \in L^1_{loc}\left(\mathbb{R}^d\right)$ with angle of sectoriality $\omega_V \in [0, \frac{\pi}{2})$. For each m, $n \in \|^*$ with $m \leq n$ set

$$V_{n,m} := (\operatorname{Re} V)\,\mathbb{1}_{|V|\leq n} + i\,(\operatorname{Im} V)^+\,\mathbb{1}_{|V|\leq n} - i\,(\operatorname{Im} V)^-\,\mathbb{1}_{|V|\leq m}.$$

It is clear that for $m \leq n$ the potential $V_{n,m} \in L^\infty\left(\mathbb{R}^d\right)$ will have angle of sectoriality at most ω_V. Obviously, for each $\epsilon > 0$ we have $V_{n,m}+\epsilon \in \mathcal{W}_2$ and so (3.9) implies

$$C_V^{-1}\left(\left\||V_{n,m}|^{\frac{1}{2}}u\right\|+\|\nabla u\|\right) \leq \left\|\sqrt{\mathcal{L}+V_{n,m}+\epsilon}\,u\right\| \leq C_V\left(\left\||V_{n,m}+\epsilon|^{\frac{1}{2}}u\right\|+\|\nabla u\|\right). \tag{3.10}$$

In particular, the operators $|V_{n,m}|^{\frac{1}{2}}\left(\mathcal{L}+V_{n,m}+\epsilon\right)^{-\frac{1}{2}}$ and $\nabla\left(\mathcal{L}+V_{n,m}+\epsilon\right)^{-\frac{1}{2}}$ are uniformly L^2-bounded in n, m and ϵ for $m \leq n$. Our aim is to take the limit as $n \to \infty$ and $m \to \infty$. We deal with $\nabla\left(\mathcal{L}+V_{n,m}+\epsilon\right)^{-\frac{1}{2}}$. The arguments are similar for $|V_{n,m}|^{\frac{1}{2}}\left(\mathcal{L}+V_{n,m}+\epsilon\right)^{-\frac{1}{2}}$. For a given $f \in L^2\left(\mathbb{R}^d\right)$, the sequence $\left(\left(\mathcal{L}+V_{n,m}+\epsilon\right)^{-\frac{1}{2}}f\right)_n$ is bounded in $H^1\left(\mathbb{R}^d\right)$. Hence, after extraction of a subsequence, we may assume that this sequence converges weakly in $H^1\left(\mathbb{R}^d\right)$ as $n \to \infty$. We prove that $\left(\mathcal{L}+V_{n,m}+\epsilon\right)_n$ converges to $\left(\mathcal{L}+V_m+\epsilon\right)$ in the strong resolvent sense. Here

$$V_m := \operatorname{Re} V + i\,(\operatorname{Im} V)^+ - i\,(\operatorname{Im} V)^-\,\mathbb{1}_{|V|\leq m}.$$

Once this is proved we obtain

$$\left\|\nabla\left(\mathcal{L}+V_m+\epsilon\right)^{-\frac{1}{2}}f\right\| \leq \liminf_n \left\|\nabla\left(\mathcal{L}+V_{n,m}+\epsilon\right)^{-\frac{1}{2}}f\right\|$$

which will imply that $\left(\nabla\left(\mathcal{L}+V_m+\epsilon\right)^{-\frac{1}{2}}\right)_m$ is uniformly L^2-bounded in m and ϵ. We then repeat the same argument by letting $m \to \infty$ and obtain

$$\left(\left\||V|^{\frac{1}{2}}u\right\|+\|\nabla u\|\right) \leq C_V\left\|\sqrt{\mathcal{L}+V+\epsilon}\,u\right\|. \tag{3.11}$$

We let $\epsilon \to 0$ and obtain the first estimate in Theorem 1.1. The second estimate is the obtained from the first one by a well known duality argument as follows

$$\begin{aligned}
\left\| \sqrt{\mathcal{L}+V}\, u \right\| &= \sup \left\{ \left| \langle (\mathcal{L}+V)\, u, v\rangle \right|, \ \left\| \sqrt{(\mathcal{L}+V)^*} v \right\| \leq 1 \right\} \\
&= \sup \left\{ \left| \int_{\mathbb{R}^d} A\nabla u \overline{\nabla v} + Vu\overline{v} \right|, \ \left\| \sqrt{(\mathcal{L}+V)^*} v \right\| \leq 1 \right\} \\
&\lesssim \sup \left\{ \|\nabla u\| \, \|\nabla v\| + \left\| |V|^{\frac{1}{2}} u \right\| \left\| |V|^{\frac{1}{2}} v \right\|, \ \left\| \sqrt{(\mathcal{L}+V)^*} v \right\| \leq 1 \right\} \\
&\lesssim \sup \left\{ \left[\|\nabla u\| + \left\| |V|^{\frac{1}{2}} u \right\| \right] \left[\|\nabla v\| + \left\| |V|^{\frac{1}{2}} v \right\| \right], \ \left\| \sqrt{(\mathcal{L}+V)^*} v \right\| \leq 1 \right\} \\
&\lesssim \|\nabla u\| + \left\| |V|^{\frac{1}{2}} u \right\|,
\end{aligned}$$

where in the last inequality we use the lower estimate (we just proved) of Theorem 1.1 for the adjoint operator $(\mathcal{L}+V)^*$.

Our final task is to prove the above claims on the convergence in the strong resolvent sense. Recall that any sesquilinear form $\mathfrak{E}$ on a complex Hilbert space can be written as

$$\mathfrak{E} = \mathfrak{b} + i\mathfrak{c} \tag{3.12}$$

where $\mathfrak{b} = \frac{1}{2}(\mathfrak{E} + \mathfrak{E}^*)$ (called the real part of $\mathfrak{E}$) and $\mathfrak{c} = \frac{1}{2i}(\mathfrak{E} - \mathfrak{E}^*)$ (the imaginary part). We apply this decomposition to the sesquilinear forms $\mathfrak{a}^V$ and $\mathfrak{a}^{V_{n,m}+\epsilon}$ (for $m \leq n$)

$$\mathfrak{a}^{V+\epsilon} = \mathfrak{b} + i\mathfrak{c} \quad \text{and} \quad \mathfrak{a}^{V_{n,m}+\epsilon} = \mathfrak{b}^{n,m} + i\mathfrak{c}^{n,m}.$$

We introduce the forms

$$\mathfrak{a}^{V+\epsilon}(z) = \mathfrak{b} + z\mathfrak{c} \quad \text{and} \quad \mathfrak{a}^{V_{n,m}+\epsilon}(z) = \mathfrak{b}^{n,m} + z\mathfrak{c}^{n,m}$$

for $z \in St_\delta := \{z \in \mathbb{C}, \, |\mathrm{Im}\,(z)| < \delta\}$. The sectoriality of $V_{n,m} + \epsilon$ implies that for $\delta > 0$ small enough, each form $\mathfrak{a}^{V_{n,m}+\epsilon}(z)$ and $\mathfrak{a}^{V+\epsilon}(z)$ is sectorial for every $z \in St_\delta$. On the other hand, these forms are of type (a) in the sense of [19, p. 393] and hence the corresponding resolvents are holomorphic in the strip St_δ and uniformly L^2-bounded. Observe that for $z \in [0, \delta)$ the forms $\mathfrak{a}^{V_{n,m}+\epsilon}(z)$ are symmetric and the sequence is increasing in the sense that

$$\mathfrak{a}^{V_{n,m}+\epsilon}(z)(u,u) \leq \mathfrak{a}^{V_{n+1,m}+\epsilon}(z)(u,u).$$

When $n \to \infty$, $\mathfrak{a}^{V_{n,m}+\epsilon}(z)(u,u)$ converges to the form $\mathfrak{a}^{V_m+\epsilon}(z)(u,u)$. The monotone convergence theorem for non-decreasing forms [19, p. 461] implies the convergence in the strong resolvent sense of $\mathfrak{a}^{V_{n,m}+\epsilon}(z)$ to $\mathfrak{a}^{V_m+\epsilon}(z)$ as $n \to \infty$ for all $z \in [0, \delta)$. We conclude by Vitali's theorem (see [16, Thm. 3.14.1] or [2, p. 458]) that the strong convergence holds for all z in the strip and in particular for $z = i$. This proves the strong convergence in the resolvent sense of $\mathcal{L} + V_{n,m} + \epsilon$ to $\mathcal{L} + V_m + \epsilon$. We repeat the same argument either by taking $z \in (-\delta, 0]$ and use convergence of non-decreasing symmetric forms as above or $z \in [0, \delta)$ and then use convergence of non-increasing symmetric forms. This concludes our proof of Theorem 1.1 for $\Omega = \mathbb{R}^d$.

4. Square Function Estimates

In this section a proof of Theorem 3.6 will be provided. The first part of the proof consists in showing that the operators P_t^V can effectively be diagonalised when estimating square function norms from above.

Throughout this section we assume that V is sectorial and belongs to $\mathcal{W}_\alpha$ for some $\alpha \in (1,2]$. Recall that $\lesssim$ means that the implicit constant depends on V only through its angle of sectoriality ω_V and is independent of $\alpha \in (1,2]$.

4.1. Diagonalisation of the P_t^V Operators.

Define, for $t>0$, the bounded operator $\mathcal{P}_t^V : \mathcal{H} \to \mathcal{H}$ through

$$\mathcal{P}_t^V := \begin{pmatrix} \left(I + t^2\left(-\Delta + |V|\right)\right)^{-1} & 0 & 0 \\ 0 & \left(I + t^2|V|\right)^{-1} & 0 \\ 0 & 0 & \left(I - t^2\nabla\mathrm{div}\right)^{-1} \end{pmatrix}.$$

Observe that since the operators $(-\Delta + |V|)$, $|V|$ and $(-\nabla\mathrm{div})$ are all self-adjoint, it follows from Proposition 2.7 that square function estimates hold for each of these operators with constant bounded by 1. Therefore each of these operators possess a bounded holomorphic functional calculus with constant bounded by 1 and thus the operators $\mathcal{P}_t^V$ are uniformly L^2-bounded by 1. The following theorem will be proved.

Theorem 4.1. *The estimate*

$$\int_0^\infty \left\| \left(\mathcal{P}_t^V - P_t^V\right) u \right\|^2 \frac{dt}{t} \lesssim \left(1 + [V]_\alpha^2\right) \|u\|^2 \tag{4.1}$$

holds for all $u \in R\left(\Gamma_V\right)$.

Such a diagonalisation will aid us tremendously in bounding our main square function estimate (3.3). This theorem will be proved by inspecting each component separately.

Remark 4.2. *It is easy to see that the diagonalisation estimate (4.1) is trivially satisfied on the first component for any* $u \in \mathcal{H}$ *since* $\mathbb{P}_1 \mathcal{P}_t^V = \mathbb{P}_1 P_t^V$.

Proposition 4.3. *For any* $u \in \mathcal{H}$,

$$\int_0^\infty \left\| \mathcal{P}_t^V \left(P_t^V - I\right) u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2 . \tag{4.2}$$

Proof. The estimate is trivially satisfied on $N(\Pi_V)$ since $\left(P_t^V - I\right) u = 0$ for any $u \in N(\Pi_V)$ and $t>0$. Suppose that $u \in \overline{R\left(\Pi_V\right)}$. Using a Schur argument identical to Proposition 5.7 of [5], the proof of (4.2) for $u \in \overline{R\left(\Pi_V\right)}$ can be reduced to the statement

$$\left\| \mathcal{P}_t^V \left(P_t^V - I\right) Q_s^V \right\| \lesssim \min\left\{ \frac{t}{s}, \frac{s}{t} \right\}^{\frac{1}{2}}, \qquad t,\, s > 0. \tag{4.3}$$

First assume that $t \le s$. On noting that $\left(P_t^V - I\right) Q_s^V = \frac{t}{s} Q_t^V \left(P_s^V - I\right)$ we obtain

$$\left\| \mathcal{P}_t^V \left(P_t^V - I\right) Q_s^V \right\| \le \left\| \left(P_t^V - I\right) Q_s^V \right\| = \frac{t}{s} \left\| Q_t^V \left(P_s^V - I\right) \right\| \lesssim \frac{t}{s}. \tag{4.4}$$

Next, suppose that $t > s$. Then the equality $P_t^V Q_s^V = \frac{s}{t} Q_t^V P_s^V$ gives

$$\left\| \mathcal{P}_t^V \left(P_t^V - I \right) Q_s^V \right\| \leq \left\| P_t^V Q_s^V \right\| + \left\| \mathcal{P}_t^V Q_s^V \right\| \lesssim \frac{s}{t} + \left\| \mathcal{P}_t^V Q_s^V \right\|.$$

The term $\mathcal{P}_t^V Q_s^V$ will be considered component-wise. For the first component, recall that $\mathbb{P}_1 \mathcal{P}_t^V = \mathbb{P}_1 P_t^V$ and observe

$$\begin{aligned} \left\| \mathbb{P}_1 \mathcal{P}_t^V Q_s^V \right\| &= \left\| \mathbb{P}_1 P_t^V s \Pi_V P_s^V \right\| \\ &= \frac{s}{t} \left\| \mathbb{P}_1 P_t^V t \Pi_V P_s^V \right\| \\ &= \frac{s}{t} \left\| \mathbb{P}_1 Q_t^V P_s^V \right\| \\ &\lesssim \frac{s}{t}. \end{aligned}$$

For the second component, note that

$$\mathbb{P}_2 \mathcal{P}_t^V = \left(I + t^2 |V| \right)^{-1} \mathbb{P}_2 \quad \text{and} \quad \mathbb{P}_2 \Pi_V = |V|^{\frac{1}{2}} \mathbb{P}_1.$$

This gives

$$\begin{aligned} \left\| \mathbb{P}_2 \mathcal{P}_t^V Q_s^V \right\| &= s \left\| \left(I + t^2 |V| \right)^{-1} |V|^{\frac{1}{2}} \mathbb{P}_1 P_s^V \right\| \\ &\lesssim \frac{s}{t}. \end{aligned}$$

Lastly, for the third component, we have

$$\mathbb{P}_3 \mathcal{P}_t^V = \mathbb{P}_3 P_t^0 = P_t^0 \mathbb{P}_3,$$

leading to

$$\begin{aligned} \left\| \mathbb{P}_3 \mathcal{P}_t^V Q_s^V \right\| &= \left\| P_t^0 \mathbb{P}_3 s \Pi_V P_s^V \right\| \\ &= \left\| P_t^0 \mathbb{P}_3 s \Pi_0 P_s^V \right\| \\ &= \frac{s}{t} \left\| \mathbb{P}_3 P_t^0 t \Pi_0 P_s^V \right\| \\ &= \frac{s}{t} \left\| \mathbb{P}_3 Q_t^0 P_s^V \right\| \\ &\lesssim \frac{s}{t}. \end{aligned}$$

Putting everything together gives (4.3). □

Proposition 4.4. *The estimate*

$$\int_0^\infty \left\| \mathbb{P}_2 \left(I - \mathcal{P}_t^V \right) P_t^V u \right\|^2 \frac{dt}{t} \lesssim [V]_\alpha^2 \|u\|^2$$

holds for any $u \in R(\Gamma_V)$.

Proof. Observe that

$$\begin{aligned}\left\|\mathbb{P}_2\left(I-\mathcal{P}_t^V\right)P_t^V u\right\| &= \left\|t^2\left|V\right|\left(1+t^2\left|V\right|\right)^{-1}\mathbb{P}_2 P_t^V u\right\| \\ &\lesssim \left\|t^{\alpha-1}\left|V\right|^{\frac{\alpha-1}{2}}\mathbb{P}_2 P_t^V u\right\|.\end{aligned}$$

As $u\in R\left(\Gamma_V\right)$ and P_t^V commutes with Γ_V through Lemma 2.13, there must exist some $v=(v_1,0,0)\in D\left(\Gamma_V\right)$ for which $P_t^V u=\Gamma_V(v_1,0,0)$. This then gives

$$\begin{aligned}\left\|\mathbb{P}_2\left(I-\mathcal{P}_t^V\right)P_t^V u\right\| &\lesssim \left\|t^{\alpha-1}V^{\frac{\alpha}{2}}v_1\right\| \\ &\leq [V]_\alpha\left\|t^{\alpha-1}\left(-\Delta+\left|V\right|\right)^{\frac{\alpha}{2}}v_1\right\| \\ &= [V]_\alpha\left\|t^{\alpha-1}\left|\Pi_V\right|^\alpha v\right\| \\ &\lesssim [V]_\alpha\left\|t^{\alpha-1}\left|\Pi_V\right|^{\alpha-1}\Pi_V v\right\| \\ &= [V]_\alpha\left\|t^{\alpha-1}\left|\Pi_V\right|^{\alpha-1}P_t^V u\right\|,\end{aligned}$$

where $\left|\Pi_V\right|:=\sqrt{\Pi_V^2}$ and in the fourth line we applied the bounded holomorphic functional calculus of the operator Π_V. Therefore

$$\begin{aligned}\int_0^\infty\left\|\mathbb{P}_2\left(I-\mathcal{P}_t^V\right)P_t^V u\right\|^2\frac{dt}{t} &\lesssim [V]_\alpha^2\int_0^\infty\left\|t^{\alpha-1}\left|\Pi_V\right|^{\alpha-1}P_t^V u\right\|^2\frac{dt}{t} \\ &\lesssim [V]_\alpha^2\left\|u\right\|^2,\end{aligned}$$

where we used the fact that Π_V satisfies quadratic estimates and Remark 2.6 in the last line. □

Proposition 4.5. *The estimate*

$$\int_0^\infty\left\|\mathbb{P}_3\left(I-\mathcal{P}_t^V\right)P_t^V u\right\|^2\frac{dt}{t}\lesssim [V]_\alpha^2\left\|u\right\|^2 \tag{4.5}$$

holds for all $u\in R\left(\Gamma_V\right)$.

Proof. First note that the integrand of (4.5) can be re-written as

$$\mathbb{P}_3\left(I-\mathcal{P}_t^V\right)P_t^V u=\mathbb{P}_3\left(I-P_t^0\right)P_t^V u=\left(I-P_t^0\right)\mathbb{P}_3 P_t^V u.$$

It then follows from the bounded holomorphic functional calculus of Π_0 that

$$\left\|\left(I-P_t^0\right)\mathbb{P}_3 P_t^V u\right\|\lesssim\left\|t^{\alpha-1}\left|\Pi_0\right|^{\alpha-1}\mathbb{P}_3 P_t^V u\right\|=\left\|\mathbb{P}_3 t^{\alpha-1}\left|\Pi_0\right|^{\alpha-1}P_t^V u\right\|.$$

On recalling that $u=\Gamma_V v$ for some $v=(v_1,0,0)\in D\left(\Gamma_V\right)$ and that Γ_V commutes with P_t^V,

$$\begin{aligned}\int_0^\infty\left\|\mathbb{P}_3 t^{\alpha-1}\left|\Pi_0\right|^{\alpha-1}P_t^V u\right\|^2\frac{dt}{t} &= \int_0^\infty\left\|\mathbb{P}_3 t^{\alpha-1}\left|\Pi_0\right|^{\alpha-1}\Gamma_V P_t^V v\right\|^2\frac{dt}{t} \\ &= \int_0^\infty\left\|\mathbb{P}_3 t^{\alpha-1}\left|\Pi_0\right|^{\alpha-1}\Pi_0 P_t^V v\right\|^2\frac{dt}{t}.\end{aligned}$$

On exploiting the bounded holomorphic functional calculus of the operator Π_0 once more,

$$\int_0^\infty \left\| \mathbb{P}_3 t^{\alpha-1} \left|\Pi_0\right|^{\alpha-1} \Pi_0 P_t^V v \right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\| t^{\alpha-1} \left|\Pi_0\right|^{\alpha} P_t^V v \right\|^2 \frac{dt}{t}.$$

Observe that since $P_t^V v$ is non-zero only in the first entry,

$$\begin{aligned} \left\| \left|\Pi_0\right|^{\alpha} P_t^V v \right\| &= \left\| (-\Delta)^{\frac{\alpha}{2}} \mathbb{P}_1 P_t^V v \right\| \\ &\leq [V]_\alpha \left\| (|V| - \Delta)^{\frac{\alpha}{2}} \mathbb{P}_1 P_t^V v \right\| \\ &= [V]_\alpha \left\| \left|\Pi_V\right|^{\alpha} P_t^V v \right\|. \end{aligned}$$

The bounded holomorphic functional calculus of the operator Π_V then leads to

$$\begin{aligned} [V]_\alpha^2 \int_0^\infty \left\| t^{\alpha-1} \left|\Pi_V\right|^{\alpha} P_t^V v \right\|^2 \frac{dt}{t} &\lesssim [V]_\alpha^2 \int_0^\infty \left\| t^{\alpha-1} \left|\Pi_V\right|^{\alpha-1} \Pi_V P_t^V v \right\|^2 \frac{dt}{t} \\ &= [V]_\alpha^2 \int_0^\infty \left\| t^{\alpha-1} \left|\Pi_V\right|^{\alpha-1} P_t^V u \right\|^2 \frac{dt}{t} \\ &\lesssim [V]_\alpha^2 \left\| u \right\|^2, \end{aligned}$$

where we used the fact that Π_V satisfies quadratic estimates in the final line. □

Proof of Theorem 4.1. We combine Propositions 4.3, 4.4 and 4.5 together and note that $\mathbb{P}_1 P_t^V u = 0$ for $u \in R\left(\Gamma_V\right)$ to obtain Theorem 4.1. □

Continuing with the proof of Theorem 3.6, split our main square function into components

$$\int_0^\infty \left\| \Theta_t^{V,B} P_t^V u \right\|^2 \frac{dt}{t} \lesssim \sum_{i=1,2,3} \int_0^\infty \left\| \Theta_t^{V,B} \mathbb{P}_i P_t^V u \right\|^2 \frac{dt}{t}.$$

Notice that the first component vanishes since $\mathbb{P}_1 P_t^V u = 0$ for $u \in R\left(\Gamma_V\right)$. For the second component, the triangle inequality gives

$$\int_0^\infty \left\| \Theta_t^{V,B} \mathbb{P}_2 P_t^V u \right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\| \Theta_t^{V,B} \mathbb{P}_2 \left(\mathcal{P}_t^V - P_t^V \right) u \right\|^2 \frac{dt}{t} + \int_0^\infty \left\| \Theta_t^{V,B} \mathbb{P}_2 \mathcal{P}_t^V u \right\|^2 \frac{dt}{t}.$$

The uniform L^2-boundedness of the operators $\Theta_t^{V,B}$ together with Theorem 4.1 give

$$\int_0^\infty \left\| \Theta_t^{V,B} \mathbb{P}_2 \left(\mathcal{P}_t^V - P_t^V \right) u \right\|^2 \frac{dt}{t} \lesssim \left(1 + [V]_\alpha^2 \right) \left\| u \right\|^2.$$

For the second term, observe that $\mathbb{P}_2 \mathcal{P}_t^V u \in D\left(\Gamma_{V,B}^*\right)$. Lemma 2.13 together with the uniform L^2 boundedness of the operators $P_t^{V,B}$ gives

$$\begin{aligned} \int_0^\infty \left\| \Theta_t^{V,B} \mathbb{P}_2 \mathcal{P}_t^V u \right\|^2 \frac{dt}{t} &= \int_0^\infty \left\| P_t^{V,B} t \Gamma_{V,B}^* \mathbb{P}_2 \mathcal{P}_t^V u \right\|^2 \frac{dt}{t} \\ &\lesssim \int_0^\infty \left\| t \Gamma_{V,B}^* \mathbb{P}_2 \mathcal{P}_t^V u \right\|^2 \frac{dt}{t} \\ &= \int_0^\infty \left\| t \left|V\right|^{\frac{1}{2}} e^{i \arg V} \left(1 + t^2 \left|V\right| \right)^{-1} \mathbb{P}_2 u \right\|^2 \frac{dt}{t} \\ &\lesssim \left\| u \right\|^2, \end{aligned}$$

where the last line follows from the fact that the multiplication operator $|V|^{\frac{1}{2}}$ is self-adjoint and therefore satisfies square function esimates with constant independent of V and α by Proposition 2.7. This reduces our theorem to proving boundedness of the third component.

4.2. **The Third Component.** This section is dedicated to bounding the third component of our square function norm thus completing the proof of Theorem 3.6. Specifically, it will be proved that

$$\int_0^\infty \left\|\Theta_t^{V,B}\mathbb{P}_3 P_t^V u\right\|^2 \frac{dt}{t} \lesssim \left(1+[V]_\alpha^2\right)\|u\|^2 \tag{4.6}$$

holds for any $u \in R(\Gamma_V)$. A similar argument to that of [5] will be used but one will need to keep track of the effect of the projection $\mathbb{P}_3$ and the dependence of the constants on V and α.

4.2.1. $T(1)$**-Reduction.** Our first step towards a $T(1)$-reduction is to use the splitting

$$\int_0^\infty \left\|\Theta_t^{V,B}\mathbb{P}_3 P_t^V u\right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\|\Theta_t^{V,B}\mathbb{P}_3\left(P_t^V-\mathcal{P}_t^V\right)u\right\|^2 \frac{dt}{t} + \int_0^\infty \left\|\Theta_t^{V,B}\mathbb{P}_3\mathcal{P}_t^V u\right\|^2 \frac{dt}{t}.$$

The uniform L^2-boundedness of the operators $\Theta_t^{V,B}$ and Theorem 4.1 can be applied to the first term to obtain

$$\int_0^\infty \left\|\Theta_t^{V,B}\mathbb{P}_3\left(\mathcal{P}_t^V-P_t^V\right)u\right\|^2 \frac{dt}{t} \lesssim \left(1+[V]_\alpha^2\right)\|u\|^2.$$

On recalling that $\mathbb{P}_3\mathcal{P}_t^V = \mathbb{P}_3 P_t^0$, this reduces the task of proving our square function estimate to obtaining the bound

$$\int_0^\infty \left\|\Theta_t^{V,B}\mathbb{P}_3 P_t^0 u\right\|^2 \frac{dt}{t} \lesssim \left(1+[V]_\alpha^2\right)\|u\|^2.$$

Introduce the notation $\tilde{\Theta}_t^{V,B}$ to denote the operators $\tilde{\Theta}_t^{V,B} := \Theta_t^{V,B}\mathbb{P}_3$. Let $\gamma_t^{V,B}$ and $\tilde{\gamma}_t^{V,B}$ denote the principal parts of the operators $\Theta_t^{V,B}$ and $\tilde{\Theta}_t^{V,B}$ respectively. That is, they are the multiplication operators defined through

$$\gamma_t^{V,B}(x)w := \Theta_t^{V,B}(w)(x) \quad \textit{and} \quad \tilde{\gamma}_t^{V,B}(x)(w) := \left(\Theta_t^{V,B}\mathbb{P}_3\right)(w)(x),$$

for $w \in \mathbb{C}^{d+2}$ and $x \in \mathbb{R}^d$. Evidently we must have $\tilde{\gamma}_t^{V,B}(x)w = \gamma_t^{V,B}(x)\mathbb{P}_3 w$.

Our square function norm can be reduced to this principal part by applying the splitting

$$\int_0^\infty \left\|\tilde{\Theta}_t^{V,B}P_t^0 u\right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\|\left(\tilde{\Theta}_t^{V,B}-\tilde{\gamma}_t^{V,B}A_t\right)P_t^0 u\right\|^2 \frac{dt}{t} + \int_0^\infty \left\|\tilde{\gamma}_t^{V,B}A_t P_t^0 u\right\|^2 \frac{dt}{t}. \tag{4.7}$$

Since the operator $\Theta_t^{V,B}$ satisfies the conditions of Proposition 2.21, it follows that

$$\begin{aligned}\int_0^\infty \left\|\left(\tilde{\Theta}_t^{V,B} - \tilde{\gamma}_t^{V,B} A_t\right) P_t^0 u\right\|^2 \frac{dt}{t} &= \int_0^\infty \left\|\left(\Theta_t^{V,B} - \gamma_t^{V,B} A_t\right) \mathbb{P}_3 P_t^0 u\right\|^2 \frac{dt}{t} \\ &\lesssim \int_0^\infty \left\|t\nabla \mathbb{P}_3 P_t^0 u\right\|^2 \frac{dt}{t} \\ &\lesssim \int_0^\infty \left\|t\Pi_0 P_t^0 u\right\|^2 \frac{dt}{t} \\ &= \frac{1}{2}\|u\|^2,\end{aligned}$$

where the estimate $\|\nabla \mathbb{P}_3 P_t^0 u\| \lesssim \|\Pi_0 P_t^0 u\|$ follows from (H8) for the operator Γ_0. It should be noted that in order to use (H8) we had to use the fact that $u = \Gamma_V v$ for some $v \in D(\Gamma_V)$ and therefore

$$\mathbb{P}_3 P_t^0 u = P_t^0 \mathbb{P}_3 \Gamma_V v = P_t^0 \mathbb{P}_3 \Gamma_0 v = \Gamma_0 P_t^0 v \in R(\Gamma_0).$$

Our theorem has thus been reduced to a proof of the following square function estimate

$$\int_0^\infty \left\|\tilde{\gamma}_t^{V,B} A_t P_t^0 u\right\|^2 \frac{dt}{t} \lesssim \left(1 + [V]_\alpha^2\right)\|u\|^2.$$

The triangle inequality leads to

$$\int_0^\infty \left\|\tilde{\gamma}_t^{V,B} A_t P_t^0 u\right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\|\tilde{\gamma}_t^{V,B} A_t \left(P_t^0 - I\right) u\right\|^2 \frac{dt}{t} + \int_0^\infty \left\|\tilde{\gamma}_t^{V,B} A_t u\right\|^2 \frac{dt}{t}. \tag{4.8}$$

Proposition 2.20 states that the uniform estimate $\left\|\tilde{\gamma}_t^{V,B} A_t\right\| \lesssim 1$ is true for all $t > 0$. Furthermore, notice that $A_t^2 = A_t$ and $\mathbb{P}_3 A_t = A_t \mathbb{P}_3$ for all $t > 0$. These facts combine together to produce

$$\begin{aligned}\int_0^\infty \left\|\tilde{\gamma}_t^{V,B} A_t \left(P_t^0 - I\right) u\right\|^2 \frac{dt}{t} &= \int_0^\infty \left\|\gamma_t^{V,B} A_t \mathbb{P}_3 A_t \left(P_t^0 - I\right) u\right\|^2 \frac{dt}{t} \\ &\lesssim \int_0^\infty \left\|\mathbb{P}_3 A_t \left(P_t^0 - I\right) u\right\|^2 \frac{dt}{t}.\end{aligned}$$

According to the argument from Proposition 5.7 of [5], this final term can be bounded by

$$\int_0^\infty \left\|A_t \left(P_t^0 - I\right) u\right\|^2 \frac{dt}{t} \lesssim \|u\|^2,$$

since $\{\Gamma_0, B_1, B_2\}$ by hypothesis satisfies (H1) - (H8).

Recall the definition of the Carleson norm for a measure ν on $\mathbb{R}_+^{d+1} := \mathbb{R}^d \times [0,\infty)$,

$$\|\nu\|_{\mathcal{C}} := \sup_{Q \in \square} \frac{\nu(R_Q)}{|Q|},$$

where $R_Q := Q \times [0, l(Q))$. For the second term in (4.8), apply Carleson's theorem ([26, p. 59]) to obtain

$$\int_0^\infty \left\|\tilde{\gamma}_t^{V,B} A_t u\right\|^2 \frac{dt}{t} \lesssim \|\mu_{V,B}\|_{\mathcal{C}} \|u\|^2,$$

where $\mu_{V,B}$ is the measure on $\mathbb{R}^{d+1}_+$ defined through

$$d\mu_{V,B}(x,t) := \left|\tilde{\gamma}_t^{V,B}(x)\right|^2 \frac{dx\,dt}{t}, \quad x \in \mathbb{R}^d,\ t>0.$$

The proof of Theorem 3.6 has thus been reduced to showing that the measure $\mu_{V,B}$ is a Carleson measure with constant smaller than a multiple of $\left(1+[V]_\alpha^2\right)$.

4.2.2. **Carleson Measure Estimate.** Our goal now is to prove the following Carleson measure estimate,

$$\sup_{Q\in\square}\frac{1}{|Q|}\int_0^{l(Q)}\int_Q \left|\tilde{\gamma}_t^{V,B}(x)\right|^2\frac{dx\,dt}{t} \lesssim \left(1+[V]_\alpha^2\right)<\infty. \tag{4.9}$$

Let $\mathcal{L}_3$ denote the subspace

$$\mathcal{L}_3 := \left\{\nu\in\mathcal{L}\left(\mathbb{C}^{d+2}\right)\setminus\{0\} : \nu\mathbb{P}_3=\nu\right\}. \tag{4.10}$$

By construction, we have $\tilde{\gamma}_t^{V,B}(x)\in\mathcal{L}_3$ for any $t>0$ and $x\in\mathbb{R}^d$ since

$$\begin{aligned}\tilde{\gamma}_t^{V,B}(x)\mathbb{P}_3 w &= \left(\Theta_t^{V,B}\mathbb{P}_3\right)\left(\mathbb{P}_3 w\right)(x)\\ &= \left(\Theta_t^{V,B}\mathbb{P}_3\right)(w)(x)\\ &= \tilde{\gamma}_t^{V,B}(x)(w).\end{aligned} \tag{4.11}$$

Let $\sigma>0$ be a constant to be determined at a later time. Let $\mathcal{V}$ be a finite set consisting of $\nu\in\mathcal{L}_3$ with $|\nu|=1$ such that $\cup_{\nu\in\mathcal{V}}K_\nu=\mathcal{L}_3\setminus\{0\}$, where

$$K_\nu := \left\{\nu'\in\mathcal{L}_3\setminus\{0\} : \left|\frac{\nu'}{|\nu'|}-\nu\right|\le\sigma\right\}.$$

Then, in order to prove our Carleson measure estimate (4.9), it is sufficient to fix $\nu\in\mathcal{V}$ and prove that

$$\sup_{Q\in\square}\frac{1}{|Q|}\int\int_{\substack{(x,t)\in R_Q\\ \tilde{\gamma}_t^{V,B}(x)\in K_\nu}}\left|\tilde{\gamma}_t^{V,B}(x)\right|^2\frac{dx\,dt}{t}\lesssim 1+[V]_\alpha^2<\infty. \tag{4.12}$$

The John-Nirenberg lemma for Carleson measures, as applied in [5] and [3], can then be used to reduce the proof of our theorem to the following proposition.

Proposition 4.6. *There exists $\beta>0$ and $\sigma>0$ that will satisfy the following conditions. For every $\nu\in\mathcal{V}$ and $Q\in\square$, there is a collection $\{Q_k\}_k\subset\square$ of disjoint subcubes of Q such that $E_{Q,\nu}=Q\setminus\cup_k Q_k$ satisfies $|E_{Q,\nu}|>\beta|Q|$ and such that*

$$\sup_{Q\in\square}\frac{1}{|Q|}\int\int_{\substack{(x,t)\in E^*_{Q,\nu}\\ \tilde{\gamma}_t^{V,B}(x)\in K_\nu}}\left|\tilde{\gamma}_t^{V,B}(x)\right|^2\frac{dx\,dt}{t}\lesssim 1+[V]_\alpha^2<\infty, \tag{4.13}$$

where $E^*_{Q,\nu}:=R_Q\setminus\cup_k R_{Q_k}$. Moreover, β and σ are independent of α and dependent on V only through ω_V.

For now, fix $\nu \in \mathcal{V}$ and $Q \in \square$. Let w^ν, $\hat{w}^\nu \in \mathbb{C}^{d+2}$ with $|\hat{w}^\nu| = |w^\nu| = 1$ and $\nu^*(\hat{w}^\nu) = w^\nu$. To simplify notation, when superfluous, this dependence will be kept implicit by defining $w := w^\nu$ and $\hat{w} := \hat{w}^\nu$. Notice that since ν satisfies $\nu = \nu\mathbb{P}_3$, w must satisfy $\mathbb{P}_3 w = w$.

For $\epsilon > 0$ the function $f^w_{Q,\epsilon}$ can be defined in an identical manner to [5]. Specifically, let $\eta_Q : \mathbb{R}^d \to [0,1]$ be a smooth cutoff function equal to 1 on $2Q$, with support in $4Q$ and with $\|\nabla \eta_Q\|_\infty \leq \frac{1}{l}$, where $l := l(Q)$. Then define $w_Q := \eta_Q w$ and

$$f^w_{Q,\epsilon} := w_Q - \epsilon li \Gamma_V \left(I + \epsilon li \Pi_{V,B}\right)^{-1} w_Q = \left(I + \epsilon li \Gamma^*_{V,B}\right)\left(I + \epsilon li \Pi_{V,B}\right)^{-1} w_Q.$$

Lemma 4.7. *There exists a constant $C > 0$ that satisfies $\left\| f^w_{Q,\epsilon} \right\| \leq C\,|Q|^{\frac{1}{2}}$,*

$$\int\!\!\int_{R_Q} \left|\Theta^{V,B}_t f^w_{Q,\epsilon}(x)\right|^2 \frac{dx\,dt}{t} \leq C\frac{|Q|}{\epsilon^2}, \text{ and} \tag{4.14}$$

$$\left| \fint_Q \mathbb{P}_3 f^w_{Q,\epsilon} - w \right| \leq C\,\epsilon^{\frac{1}{2}}, \tag{4.15}$$

for any $\epsilon > 0$. Moreover, C is independent of Q, ν, ϵ and α and is dependent on V only through ω_V.

Proof. The first two parts of this lemma follow in an identical manner to [5, Lem. 5.10]. For the third part, recall that w is zero in the first two components. This gives

$$\begin{aligned}\left| \fint_Q \mathbb{P}_3 f^w_{Q,\epsilon} - w \right|^2 &= \left| \fint_Q \mathbb{P}_3 \epsilon li \Gamma_V \left(I + \epsilon li \Pi_{V,B}\right)^{-1} w_Q \right|^2 \\ &= \left| \fint_Q \epsilon li \Gamma_0 \left(I + \epsilon li \Pi_{V,B}\right)^{-1} w_Q \right|^2.\end{aligned}$$

At this point, apply Lemma 5.6 of [5] to the operator $\Upsilon = \Gamma_0$ to obtain

$$\begin{aligned}\left| \fint_Q \epsilon li \Gamma_0 \left(I + \epsilon li \Pi_{V,B}\right)^{-1} w_Q \right|^2 &\lesssim \frac{(\epsilon l)^2}{l} \left(\fint_Q \left| \left(I + \epsilon li \Pi_{V,B}\right)^{-1} w_Q \right|^2 \right)^{\frac{1}{2}} \\ &\qquad \cdot \left(\fint_Q \left| \Gamma_0 \left(I + \epsilon li \Pi_{V,B}\right)^{-1} w_Q \right|^2 \right)^{\frac{1}{2}} \\ &\lesssim \epsilon \left(\fint_Q \left| \epsilon li \Gamma_0 \left(I + \epsilon li \Pi_{V,B}\right)^{-1} w_Q \right|^2 \right)^{\frac{1}{2}} \\ &\leq \epsilon \left(\fint_Q \left| \epsilon li \Gamma_V \left(I + \epsilon li \Pi_{V,B}\right)^{-1} w_Q \right|^2 \right)^{\frac{1}{2}} \\ &\lesssim \epsilon,\end{aligned}$$

where the inequality $\|\Gamma_0 v\| \leq \|\Gamma_V v\|$ for $v \in D(\Gamma_V)$ follows trivially from the matrix form of Γ_0 and Γ_V. ∎

From this point forward, with C as in Lemma 4.7, set $\epsilon := \frac{1}{4C^2}$ and introduce the notation $f^w_Q := f^w_{Q,\epsilon}$. With this choice of ϵ it must be true that

$$\left| \fint_Q \mathbb{P}_3 f^w_Q - w \right| \leq \frac{1}{2}.$$

That is,

$$\begin{aligned} 1 - 2\mathrm{Re}\left\langle \fint_Q \mathbb{P}_3 f_Q^w, w \right\rangle &= |w|^2 - 2\mathrm{Re}\left\langle \fint_Q \mathbb{P}_3 f_Q^w, w \right\rangle \\ &\leq \left| \fint_Q \mathbb{P}_3 f_Q^w - w \right|^2 \\ &\leq \frac{1}{4}. \end{aligned}$$

On rearranging we find that

$$\mathrm{Re}\left\langle \fint_Q \mathbb{P}_3 f_Q^w, w \right\rangle \geq \frac{1}{4}. \tag{4.16}$$

In this context, Lemma 5.11 of [5] will take on the below form.

Lemma 4.8. *There exists β, c_1, $c_2 > 0$ and a collection $\{Q_k\}$ of dyadic subcubes of Q such that $|E_{Q,\nu}| > \beta |Q|$ and such that*

$$\mathrm{Re}\left\langle w, \fint_{Q'} \mathbb{P}_3 f_Q^w \right\rangle \geq c_1 \quad \textit{and} \quad \fint_{Q'} \left| \mathbb{P}_3 f_Q^w \right| \leq c_2$$

for all dyadic subcubes $Q' \in \square$ of Q which satisfy $R_{Q'} \cap E_{Q,\nu}^* \neq \emptyset$. Moreover, β, c_1 and c_2 are independent of Q, σ, ν and α and dependent on V only through ω_V.

The proof of this statement follows in an identical manner to the argument in [5]. If we set $\sigma = \frac{c_1}{2c_2}$, then the following pointwise estimate can be deduced.

Lemma 4.9. *If $(x,t) \in E_{Q,\nu}^*$ and $\tilde{\gamma}_t^{V,B}(x) \in K_\nu$ then*

$$\left| \tilde{\gamma}_t^{V,B}(x) \left(A_t f_Q^w(x) \right) \right| \geq \frac{1}{2} c_1 \left| \tilde{\gamma}_t^{V,B}(x) \right|. \tag{4.17}$$

Proof. First observe that

$$\begin{aligned} \left| \nu \left(A_t f_Q^w(x) \right) \right| &\geq \mathrm{Re}\left\langle \hat{w}, \nu \left(A_t f_Q^w(x) \right) \right\rangle \\ &= \mathrm{Re}\left\langle w, A_t f_Q^w(x) \right\rangle \\ &= \mathrm{Re}\left\langle w, A_t \mathbb{P}_3 f_Q^w(x) \right\rangle \\ &\geq c_1. \end{aligned}$$

Then

$$\begin{aligned} \left| \frac{\tilde{\gamma}_t^{V,B}(x)}{\left| \tilde{\gamma}_t^{V,B}(x) \right|} \left(A_t f_Q^w(x) \right) \right| &= \left| \frac{\tilde{\gamma}_t^{V,B}(x)}{\left| \tilde{\gamma}_t^{V,B}(x) \right|} \left(A_t \mathbb{P}_3 f_Q^w(x) \right) \right| \\ &\geq \left| \nu \left(A_t f_Q^w(x) \right) \right| - \left| \frac{\tilde{\gamma}_t^{V,B}(x)}{\left| \tilde{\gamma}_t^{V,B}(x) \right|} - \nu \right| \left| A_t \mathbb{P}_3 f_Q^w(x) \right| \\ &\geq c_1 - \sigma c_2 \\ &= \frac{1}{2} c_1. \end{aligned}$$

□

Proof of Proposition 4.6. From the pointwise bound of the previous lemma,

$$\int\int_{\substack{(x,t)\in E^*_{Q,\nu}\\ \tilde{\gamma}_t^{V,B}(x)\in K_\nu}} \left|\tilde{\gamma}_t^{V,B}(x)\right|^2 \frac{dx\,dt}{t} \lesssim \int\int_{R_Q} \left|\tilde{\gamma}_t^{V,B}(x) A_t f_Q^w(x)\right|^2 \frac{dx\,dt}{t}.$$

At this stage we can begin to unravel our square function norm,

$$\begin{aligned}\int\int_{R_Q} \left|\tilde{\gamma}_t^{V,B}(x) A_t f_Q^w(x)\right|^2 \frac{dx\,dt}{t} \lesssim \int\int_{R_Q} \left|\Theta_t^{V,B} f_Q^w(x) - \tilde{\gamma}_t^{V,B}(x) A_t f_Q^w(x)\right|^2 \frac{dx\,dt}{t}\\ + \int\int_{R_Q} \left|\Theta_t^{V,B} f_Q^w(x)\right|^2 \frac{dx\,dt}{t}.\end{aligned} \tag{4.18}$$

The second part of Lemma 4.7 states that the final term in the above estimate will be bounded from above by a multiple of $|Q|$. This reduces the task of proving the proposition to bounding the first term of the above splitting. Recall that f_Q^w can be expressed in the form

$$f_Q^w := w_Q - u_Q^w,$$

where $u_Q^w \in R(\Gamma_V)$ is given by

$$u_Q^w := \epsilon l i \Gamma_V \left(I + \epsilon l i \Pi_{V,B}\right)^{-1} w_Q.$$

An application of the triangle inequality then leads to

$$\begin{aligned}&\int\int_{R_Q} \left|\Theta_t^{V,B} f_Q^w(x) - \tilde{\gamma}_t^{V,B}(x) A_t f_Q^w(x)\right|^2 \frac{dx\,dt}{t}\\ &\lesssim \int\int_{R_Q} \left|\Theta_t^{V,B} w_Q(x) - \tilde{\gamma}_t^{V,B}(x) A_t w_Q(x)\right|^2 \frac{dx\,dt}{t}\\ &\quad + \int\int_{R_Q} \left|\Theta_t^{V,B} u_Q^w(x) - \tilde{\gamma}_t^{V,B}(x) A_t u_Q^w(x)\right|^2 \frac{dx\,dt}{t}.\end{aligned} \tag{4.19}$$

On noticing that for every $x \in Q$ and $0 < t < l(Q)$

$$\begin{aligned}\Theta_t^{V,B} w_Q(x) - \tilde{\gamma}_t^{V,B}(x) A_t w_Q(x) &= \Theta_t^{V,B} w_Q(x) - \Theta_t^{V,B}\left(A_t w_Q(x)\right)(x)\\ &= \Theta_t^{V,B}\left(\left(\eta_Q - 1\right) w\right)(x),\end{aligned}$$

it is clear that the first term in (4.19) can be handled in an identical manner as in the proof of Proposition 5.9 from [5]. Specifically, since $\left(\operatorname{supp}\left(\eta_Q - 1\right) w\right) \cap 2Q = \emptyset$, the off-diagonal estimates of the operator $\Theta_t^{V,B}$ lead to

$$\int_Q \left|\Theta_t^{V,B}\left(\left(\eta_Q - 1\right) w\right)(x)\right|^2 dx \lesssim \frac{t\,|Q|}{l},$$

which implies that

$$\int\int_{R_Q} \left|\Theta_t^{V,B} w_Q(x) - \tilde{\gamma}_t^{V,B}(x) A_t w_Q(x)\right|^2 \frac{dx\,dt}{t} \lesssim |Q|.$$

As for the second term in (4.19),

$$\begin{aligned}\int\int_{R_Q}\left|\Theta_t^{V,B}u_Q^w-\tilde{\gamma}_t^{V,B}(x)A_tu_Q^w(x)\right|^2\frac{dx\,dt}{t}&\\ \lesssim\int\int_{R_Q}\left|\Theta_t^{V,B}\left(I-P_t^V\right)u_Q^w(x)\right|^2\frac{dx\,dt}{t}&\\ +\int\int_{R_Q}\left|\Theta_t^{V,B}P_t^Vu_Q^w(x)-\tilde{\gamma}_t^{V,B}(x)A_tu_Q^w(x)\right|^2\frac{dx\,dt}{t}.&\end{aligned}\tag{4.20}$$

Since $u_Q^w\in R\left(\Gamma_V\right)$, Corollary 2.16 gives

$$\begin{aligned}\int\int_{R_Q}\left|\Theta_t^{V,B}\left(I-P_t^V\right)u_Q^w\right|^2\frac{dx\,dt}{t}&\lesssim\left\|u_Q^w\right\|^2\\ &\lesssim|Q|.\end{aligned}$$

For the remaining term in (4.20) we use the triangle inequality and the definition $\tilde{\Theta}_t^{V,B}:=\Theta_t^{V,B}\mathbb{P}_3$ to write

$$\begin{aligned}\int\int_{R_Q}\left|\Theta_t^{V,B}P_t^Vu_Q^w(x)-\tilde{\gamma}_t^{V,B}(x)A_tu_Q^w(x)\right|^2\frac{dx\,dt}{t}&\\ \lesssim\int\int_{R_Q}\left|\Theta_t^{V,B}\left(I-\mathbb{P}_3\right)P_t^Vu_Q^w\right|^2\frac{dx\,dt}{t}&\\ +\int\int_{R_Q}\left|\tilde{\Theta}_t^{V,B}P_t^Vu_Q^w(x)-\tilde{\gamma}_t^{V,B}(x)A_tu_Q^w(x)\right|^2\frac{dx\,dt}{t}.&\end{aligned}\tag{4.21}$$

Since we have already proved the boundedness of the first and second components,

$$\begin{aligned}\int\int_{R_Q}\left|\Theta_t^{V,B}\left(I-\mathbb{P}_3\right)P_t^Vu_Q^w\right|^2\frac{dx\,dt}{t}&\lesssim\left(1+[V]_\alpha^2\right)\left\|u_Q^w\right\|^2\\ &\lesssim\left(1+[V]_\alpha^2\right)|Q|.\end{aligned}$$

For the second term in (4.21),

$$\begin{aligned}\int\int_{R_Q}\left|\tilde{\Theta}_t^{V,B}P_t^Vu_Q^w(x)-\tilde{\gamma}_t^{V,B}(x)A_tu_Q^w(x)\right|^2\frac{dx\,dt}{t}&\\ \lesssim\int\int_{R_Q}\left|\tilde{\Theta}_t^{V,B}P_t^Vu_Q^w(x)-\tilde{\gamma}_t^{V,B}(x)A_tP_t^Vu_Q^w(x)\right|^2\frac{dx\,dt}{t}&\\ +\int\int_{R_Q}\left|\gamma_t^{V,B}(x)\mathbb{P}_3\left(A_tP_t^V-A_t\right)u_Q^w(x)\right|^2\frac{dx\,dt}{t},&\end{aligned}\tag{4.22}$$

where we have used (4.11). To bound the first term on the right-hand side of the above estimate notice that

$$\tilde{\Theta}_t^{V,B}P_t^Vu_Q^w(x)-\tilde{\gamma}_t^{V,B}(x)A_tP_t^Vu_Q^w(x)=\left(\Theta_t^{V,B}-\gamma_t^{V,B}A_t\right)\mathbb{P}_3P_t^Vu_Q^w(x).$$

Theorem 4.1 allows us to diagonalise our P_t^V operators in the first term of (4.22) to get

$$\int_0^{l(Q)} \left\| \left(\Theta_t^{V,B} - \gamma_t^{V,B} A_t \right) \mathbb{P}_3 P_t^V u_Q^w \right\|_{L^2(Q)}^2 \frac{dt}{t} \lesssim \left(1 + [V]_\alpha^2 \right) |Q| + \int_0^\infty \left\| \left(\Theta_t^{V,B} - \gamma_t^{V,B} A_t \right) \mathbb{P}_3 \mathcal{P}_t^V u_Q^w \right\|^2 \frac{dt}{t}.$$

From Proposition 2.21 we know that

$$\begin{aligned} \int_0^\infty \left\| \left(\Theta_t^{V,B} - \gamma_t^{V,B} A_t \right) \mathbb{P}_3 \mathcal{P}_t^V u_Q^w \right\|^2 \frac{dt}{t} &\lesssim \int_0^\infty \left\| t \nabla \mathbb{P}_3 \mathcal{P}_t^V u_Q^w \right\|^2 \frac{dt}{t} \\ &= \int_0^\infty \left\| t \nabla \mathbb{P}_3 P_t^0 u_Q^w \right\|^2 \frac{dt}{t} \\ &\lesssim \int_0^\infty \left\| t \Pi_0 P_t^0 u_Q^w \right\|^2 \frac{dt}{t} \\ &= \int_0^\infty \left\| Q_t^0 u_Q^w \right\|^2 \frac{dt}{t} \\ &\lesssim |Q|, \end{aligned}$$

where in the third line we applied (H8) for the operators $\{\Gamma_0, B_1, B_2\}$. It remains to bound the second term in (4.22),

$$\int \int_{R_Q} \left| \gamma_t^{V,B}(x) \mathbb{P}_3 A_t \left(P_t^V - I \right) u_Q^w(x) \right|^2 \frac{dx\, dt}{t} = \int \int_{R_Q} \left| \gamma_t^{V,B} A_t \mathbb{P}_3 A_t \left(P_t^V - I \right) u_Q^w(x) \right|^2 \frac{dx\, dt}{t}$$

On noting the uniform L^2-boundedness of the $\gamma_t^{V,B} A_t$ operators and applying the triangle inequality,

$$\begin{aligned} \int \int_{R_Q} \left| \gamma_t^{V,B} A_t \mathbb{P}_3 A_t \left(P_t^V - I \right) u_Q^w(x) \right|^2 \frac{dx\, dt}{t} &\lesssim \int_0^\infty \int_{\mathbb{R}^d} \left| \mathbb{P}_3 A_t \left(P_t^V - I \right) u_Q^w(x) \right|^2 \frac{dx\, dt}{t} \\ &\lesssim \int_0^\infty \int_{\mathbb{R}^d} \left| \mathbb{P}_3 A_t \left(P_t^V - \mathcal{P}_t^V \right) u_Q^w(x) \right|^2 \frac{dx\, dt}{t} \\ &\quad + \int_0^\infty \int_{\mathbb{R}^d} \left| \mathbb{P}_3 A_t \left(\mathcal{P}_t^V - I \right) u_Q^w(x) \right|^2 \frac{dx\, dt}{t}. \end{aligned}$$

Applying Theorem 4.1 and recalling that $\mathbb{P}_3 \mathcal{P}_t^V = \mathbb{P}_3 P_t^0$,

$$\int \int_{R_Q} \left| \gamma_t^{V,B} A_t \mathbb{P}_3 A_t \left(P_t^V - I \right) u_Q^w(x) \right|^2 \frac{dx\, dt}{t} \lesssim \left(1 + [V]_\alpha^2 \right) \|u\|^2 + \int_0^\infty \int_{\mathbb{R}^d} \left| \mathbb{P}_3 A_t \left(P_t^0 - I \right) u_Q^w(x) \right|^2 \frac{dx\, dt}{t}.$$

From the proof of Proposition 5.7 of [5] we know that

$$\int_0^\infty \int_{\mathbb{R}^d} \left| A_t \left(P_t^0 - I \right) u_Q^w(x) \right|^2 \frac{dx\, dt}{t} \lesssim |Q|,$$

allowing us to finally conclude our proof.

5. Proof of Kato on Domains

In this section, we give the proof of the full statement of Theorem 1.1, the Kato square root property with potential on an arbitrary domain. Recall that $\mathcal{L}_\Omega = -\mathrm{div} A\nabla$ is an elliptic operator with bounded measurable and complex coefficients on $L^2(\Omega)$ and subject to Dirichlet boundary conditions. Given $V \in L^1_{loc}(\Omega)$ with range contained in a sector $S_{\omega_V +} \subset \mathbb{C}^+$ for some $\omega_V \in \left[0, \frac{\pi}{2}\right)$, recall that $\mathcal{L}_\Omega + V$ is the maximal accretive operator defined via the sesquilinear form $\mathfrak{a}^V_\Omega$ as in Section 2.1.

Proof of Theorem 1.1. First, we extend the coefficients a_{kl} to $\mathbb{R}^d$ in such a way that the extension satisfies again the ellipticity condition (1.1) on $\mathbb{R}^d$. One may take for example δ_{kl} on $\mathbb{R}^d \setminus \Omega$. For simplicity we denote this extension by a_{kl} as well. We extend V to all of $\mathbb{R}^d$ by setting it equal to zero on $\mathbb{R}^d \setminus \Omega$. The extension will also be denoted by V. We define $\mathcal{L} + V = -\mathrm{div} A\nabla + V$ on $L^2\left(\mathbb{R}^d\right)$. We consider the sequence of operators

$$\mathcal{L}_k := \mathcal{L} + V_k := \mathcal{L} + V + \epsilon + k\mathbb{1}_{\mathbb{R}^d \setminus \Omega}.$$

Each $\mathcal{L}_k$ is defined by the corresponding sesquilinear form $\mathfrak{a}^{V_k}$. We apply Theorem 1.1 for the already proved case $\Omega = \mathbb{R}^d$ to $\mathcal{L}_k$ and find $C_V > 0$ such that

$$C_V^{-1}\left(\left\||V_k|^{\frac{1}{2}} u\right\| + \|\nabla u\|\right) \leq \left\|\sqrt{\mathcal{L}_k}\, u\right\| \leq C_V\left(\left\||V_k|^{\frac{1}{2}} u\right\| + \|\nabla u\|\right). \tag{5.1}$$

The constant C_V depends on V only through ω_V since $V + k\mathbb{1}_{\mathbb{R}^d \setminus \Omega}$ is sectorial with angle at most ω_V. In particular, C_V is independent of k. Now the idea is to take the limit as $k \to \infty$. In order to do so, we argue similarly to the end of the proof of Theorem 1.1 for the case $\Omega = \mathbb{R}^d$. We consider first the case of domains with smooth boundary.

- *Step 1: smooth domain.* We assume that each connected component of Ω has smooth boundary (C^1 for example). It follows from (5.1) that the operators $\nabla \mathcal{L}_k^{-\frac{1}{2}}$ and $|V|^{\frac{1}{2}} \mathcal{L}_k^{-\frac{1}{2}}$ are uniformly L^2-bounded. We deal with $\nabla \mathcal{L}_k^{-\frac{1}{2}}$ only since the arguments for $|V|^{\frac{1}{2}} \mathcal{L}_k^{-\frac{1}{2}}$ are similar. For $f \in L^2\left(\mathbb{R}^d\right)$

$$\left\|\nabla \mathcal{L}_k^{-\frac{1}{2}} f\right\| \leq C_V \|f\|. \tag{5.2}$$

On the other hand since

$$\mathrm{Re}\,\langle \mathcal{L}_k u, u\rangle \geq \epsilon \|u\|^2$$

it follows that

$$\left\|\left(1 + t^2 \mathcal{L}_k\right)^{-1} f\right\| \leq \frac{t^{-2}}{t^{-2} + \epsilon} \|f\|.$$

In particular, the sequence $\left(\mathcal{L}_k^{-\frac{1}{2}} f\right)_k$ is L^2-bounded. This and (5.2) show that $\left(\mathcal{L}_k^{-\frac{1}{2}} f\right)_k$ is bounded in $H^1\left(\mathbb{R}^d\right)$. Hence, after extracting a subsequence we may assume that $\left(\mathcal{L}_k^{-\frac{1}{2}} f\right)_k$ converges weakly in $H^1\left(\mathbb{R}^d\right)$ to some Ψ.

Let $f \in L^2(\Omega)$ and set $\tilde{f}$ to be the extension of f to $\mathbb{R}^d$ obtained by setting it equal to zero outside of Ω. We claim that:

the sequence $\left(\mathcal{L}_k^{-\frac{1}{2}}\tilde{f}\right)_k$ *converges in* $L^2\left(\mathbb{R}^d\right)$ *(as* $k\to\infty$*) to* $\left(\mathcal{L}_\Omega+V+\epsilon\right)^{-\frac{1}{2}}f$ *on* Ω *and* 0 *outside* Ω.

Suppose for a moment that we have proved this claim. Then $\Psi=\left(\mathcal{L}_\Omega+V+\epsilon\right)^{-\frac{1}{2}}f$ on Ω and 0 outside Ω. In addition, the above weak convergence implies

$$\left\|\nabla\left(\mathcal{L}_\Omega+V+\epsilon\right)^{-\frac{1}{2}}f\right\|_{L^2(\Omega)}\leq\liminf\left\|\nabla\left(\mathcal{L}_k^{-\frac{1}{2}}\tilde{f}\right)^{-\frac{1}{2}}\tilde{f}\right\|_{L^2(\mathbb{R}^d)}.$$

Inserting (5.2) yields the following estimate on $L^2(\Omega)$

$$\left\|\nabla\left(\mathcal{L}_\Omega+V+\epsilon\right)^{-\frac{1}{2}}f\right\|\leq C_V\left\|f\right\|. \tag{5.3}$$

We let $\epsilon\to 0$ and obtain

$$\left\|\nabla\left(\mathcal{L}_\Omega+V\right)^{-\frac{1}{2}}f\right\|\leq C_V\left\|f\right\|. \tag{5.4}$$

This together with the same estimate for $|V|^{\frac{1}{2}}\left(\mathcal{L}_\Omega+V\right)^{-\frac{1}{2}}$ imply the lower estimate in Theorem 1.1. The upper estimate follows by a duality argument as we did in the case of $\Omega=\mathbb{R}^d$.

It remains to prove the above claim. We apply the decomposition (3.12) to $\mathfrak{a}_\Omega^{V+\epsilon}$ and $\mathfrak{a}^{V_k+\epsilon}$ (and noting that the imaginary part of the latter form is independent of k)

$$\mathfrak{a}_\Omega^{V+\epsilon}=\mathfrak{b}+i\mathfrak{c}\quad\text{and}\quad\mathfrak{a}^{V_k+\epsilon}=\mathfrak{b}_k+i\mathfrak{c}.$$

We introduce the forms

$$\mathfrak{a}_\Omega^{V_k+\epsilon}(z):=\mathfrak{b}_k+z\mathfrak{c}$$

for z in the strip $St'_\delta=\{z\in\mathbb{C},|\mathrm{Re}\,z|<\delta\}$. It follows from ellipticity that for $\delta>0$ small enough, the forms are closed (with domain $H^{1,V}\left(\mathbb{R}^d\right)$). These forms are a family of type (a) in the sense of [19, p. 393] and hence the corresponding resolvents are holomorphic in the strip. For $z\in(-\delta,\delta)$ the forms $\mathfrak{a}^{V_k+\epsilon}(z)$ are symmetric and the sequence is non-decreasing. When $k\to\infty$, $\mathfrak{a}^{V_k+\epsilon}(z)(u,u)$ converges to the form

$$\mathfrak{a}_\Omega^{V+\epsilon}(z)(u,u)=\mathfrak{b}(u,u)+z\mathfrak{c}(u,u)$$

with domain

$$D(\mathfrak{a}_\Omega^{V_k+\epsilon}(z))=\left\{u\in H^{1,V}\left(\mathbb{R}^d\right),\ u=0\text{ a.e. on }\mathbb{R}^d\setminus\Omega\right\}.$$

It is well known that for smooth Ω (or star shaped) the set

$$\left\{u\in H^1\left(\mathbb{R}^d\right),\ u=0\text{ a.e. on }\mathbb{R}^d\setminus\Omega\right\}$$

coincides with $H_0^1(\Omega)$. We apply the monotone convergence theorem for the symmetric forms (cf. [25, Thm. 4.1]) to obtain that the corresponding resolvents of $\mathfrak{a}^{V_k+\epsilon}(z)$ converge strongly on $L^2\left(\mathbb{R}^d\right)$ to the resolvent of the form $\mathfrak{a}_\Omega^{V+\epsilon}(z)$. We apply again Vitali's theorem to obtain this strong convergence when $z=i$. This proves the claim.

- *Step 2: General domain.* Let now Ω be an arbitrary open subset of $\mathbb{R}^d$. We take a sequence Ω_n of smooth subdomains of Ω such that $\Omega_n\subset\Omega_{n+1}$ and $\Omega=\cup_n\Omega_n$. For each n, we have $\mathcal{L}_n:=\mathcal{L}_{\Omega_n}+V+\epsilon$ as an operator on $L^2(\Omega_n)$. We denote by $\mathfrak{a}_n$ the corresponding

sesquilinear forms. We keep the notation $\mathfrak{a}_\Omega^{V+\epsilon}$ for the sesquilinear form of $\mathcal{L}_\Omega + V + \epsilon$. For each n, the form $\mathfrak{a}_n$ is seen as a form on $L^2(\Omega)$ with (non-dense) domain $H_0^{1,V}(\Omega_n)$. By (5.3) we have

$$\left\| \nabla \mathcal{L}_n^{-\frac{1}{2}} f \right\|_{L^2(\Omega_n)} \leq C_V \|f\|_{L^2(\Omega_n)}. \tag{5.5}$$

for all $f \in L^2(\Omega_n)$. The same estimate holds for $|V|^{\frac{1}{2}} \mathcal{L}_n^{-\frac{1}{2}} f$. We can then argue exactly as in Step 1. What remains to prove is the strong convergence of the resolvent sense of $\mathfrak{a}_n$ to the resolvent of $\mathfrak{a}_\Omega^{V+\epsilon}$. For this, we decompose as above the forms $\mathfrak{a}_n = \mathfrak{b}_n + i\mathfrak{c}_n$ and introduce $\mathfrak{a}_n(z) = \mathfrak{b}_n + z\mathfrak{c}$ for z in a strip St'_δ for $\delta > 0$ and small enough. We observe that for real z in the strip, the symmetric forms $\mathfrak{a}_n(z)$ satisfy $\mathfrak{a}_{n+1} \leq \mathfrak{a}_n$ since $H_0^1(\Omega_n) \subset H_0^1(\Omega_{n+1})$. We apply the monotone convergence theorem for symmetric forms (see again [25, Thm. 4.1]) and obtain the strong convergence of the resolvents. We appeal again to Vitali's theorem as above. □

Remark 5.1. *Our Kato estimates on L^2 extend to L^p in the following way. Suppose that the semigroup $e^{-t(\mathcal{L}_\Omega + V)}$ is uniformly bounded on L^{p_0} for some $p_0 \in [1, 2)$. Then for all $p \in (p_0, 2]$*

$$\left(\left\| |V|^{\frac{1}{2}} u \right\|_p + \|\nabla u\|_p \right) \leq C \left\| \sqrt{\mathcal{L}_\Omega + V}\, u \right\|_p.$$

together with a weak type (p_0, p_0) estimate for the end-point. In particular, if the coefficients a_{kl} are real-valued then this holds with $p_0 = 1$. The extension from L^2 to L^p uses $L^p - L^q$ off-diagonal estimates (or Gaussian estimates for real-valued coefficients). We refer to [3] or [24] and the references therein. We may also add terms of order one to the differential operator.

6. Kato for Systems

In this section we describe how our proof for Kato with potential on an arbitrary domain carries over to systems. Once more, let $\Omega \subset \mathbb{R}^d$ be an arbitrary domain. Fix $m \geq 1$ and let $A \in L^\infty\left(\Omega; \mathcal{L}\left(\mathbb{C}^d \otimes \mathbb{C}^m\right)\right)$ be uniformly elliptic. That is, there exists a constant $\kappa_A > 0$ such that

$$\operatorname{Re} \langle A(x)\xi, \xi \rangle \geq \kappa_A |\xi|^2 \tag{6.1}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{C}^d \otimes \mathbb{C}^m$. Let $V : \Omega \to \mathcal{L}(\mathbb{C}^m)$ be a matrix-valued function with coefficients in $L^1_{loc}(\Omega)$. Suppose that the numerical range of $V(x)$ is contained in $S_{\omega_V +}$ for some $\omega_V \in \left[0, \frac{\pi}{2}\right)$ for a.e. $x \in \mathbb{R}^d$.

One can then define sesquilinear forms $\mathfrak{a}_\Omega$ and $\mathfrak{a}_\Omega^V$ with affilitated maximal accretive operators $\mathcal{L}_\Omega$ and $\mathcal{L}_\Omega + V$ as in Section 2.1. The notation $\mathfrak{a}$, $\mathfrak{a}^V, \mathcal{L}$ and $\mathcal{L} + V$ will once again be used to denote the case $\Omega = \mathbb{R}^d$. The sectoriality of the potential and the ellipticity condition (6.1) imply that the form $\mathfrak{a}_\Omega^V$ satisfies the corresponding Gårding inequality

$$\operatorname{Re} \mathfrak{a}_\Omega^V(u, u) \geq \kappa_A^V \left(\left\| |V|^{\frac{1}{2}} u \right\|^2 + \|\nabla u\|^2 \right)$$

for all $u \in H_0^{1,V}(\Omega;\mathbb{C}^m) := H_0^1(\Omega;\mathbb{C}^m) \cap D\left(|V|^{\frac{1}{2}}\right)$, where $\kappa_A^V > 0$ depends on V only through ω_V.

The following system version of Theorem 1.1 will be proved.

Theorem 6.1. *Let $A \in L^\infty\left(\Omega;\mathcal{L}\left(\mathbb{C}^d \otimes \mathbb{C}^m\right)\right)$ and $V : \Omega \to \mathcal{L}(\mathbb{C}^m)$ be as defined above. Assume that $V(x)$ is a normal matrix for almost every $x \in \Omega$. We have $D\left(\sqrt{\mathcal{L}_\Omega + V}\right) = H_0^{1,V}(\Omega;\mathbb{C}^m)$ and there exists a constant $C_V > 0$ such that*

$$C_V^{-1}\left(\|\nabla u\| + \left\||V|^{\frac{1}{2}} u\right\|\right) \leq \left\|\sqrt{\mathcal{L}_\Omega + V}\, u\right\| \leq C_V\left(\|\nabla u\| + \left\||V|^{\frac{1}{2}} u\right\|\right)$$

for all $u \in H_0^{1,V}(\Omega;\mathbb{C}^m)$. Moreover, the constant C_V depends on the potential only through ω_V.

Proof for $\Omega = \mathbb{R}^d$. As in the scalar case, we start by considering $\Omega = \mathbb{R}^d$. Fix V satisfying the hypotheses of the theorem. As $V(x)$ is normal for a.e. $x \in \mathbb{R}^d$, it follows from the polar decomposition theorem that there must exist some $U : \mathbb{R}^d \to \mathcal{L}(\mathbb{C}^m)$, with $U(x)$ unitary for all $x \in \mathbb{R}^d$, such that

$$V(x) = U(x)\,|V(x)|$$

for a.e. $x \in \mathbb{R}^d$, where $|V(x)| := \sqrt{V(x)^* V(x)}$. For $V(x)$ normal, it is well known that the matrices $U(x)$ and $|V(x)|$ commute. Therefore V can be decomposed as

$$V(x) = |V(x)|^{\frac{1}{2}}\, U(x)\, |V(x)|^{\frac{1}{2}} \tag{6.2}$$

for a.e. $x \in \mathbb{R}^d$. Consider the Hilbert space

$$\mathcal{H} := L^2\left(\mathbb{R}^d;\mathbb{C}^m\right) \oplus L^2\left(\mathbb{R}^d;\mathbb{C}^m\right) \oplus L^2\left(\mathbb{R}^d;\mathbb{C}^d \otimes \mathbb{C}^m\right).$$

Define the operators $\{\Gamma_V, B_1, B_2\}$ on $\mathcal{H}$ through

$$\Gamma_V := \begin{pmatrix} 0 & 0 & 0 \\ |V|^{\frac{1}{2}} & 0 & 0 \\ \nabla & 0 & 0 \end{pmatrix}, \quad B_1 := I \quad \text{and} \quad B_2 := \begin{pmatrix} I & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & A \end{pmatrix}.$$

Let Π_V, $\Pi_{V,B}$ and $Q_{V,B}$ be defined as in the scalar case. It follows from (6.2) that the operator $\Pi_{V,B}^2$ will be of the form

$$\Pi_{V,B}^2 = \begin{pmatrix} \mathcal{L} + V & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Let $[V]_\alpha$ denote the system analogue of the quantity defined in Section 3 for $\alpha \in [1,2]$. The proof of the following theorem is identical to that of the scalar case $m = 1$.

Theorem 6.2. *Suppose that $V(x)$ is normal for almost every $x \in \mathbb{R}^d$ and $[V]_\alpha < \infty$ for some $\alpha \in (1,2]$. The estimate*

$$\left(C_V\left(1 + [V]_\alpha^2\right)\right)^{-1} \|u\|^2 \leq \int_0^\infty \left\|Q_t^{V,B} u\right\|^2 \frac{dt}{t} \leq C_V\left(1 + [V]_\alpha^2\right)\|u\|^2$$

holds for all $u \in \overline{R(\Pi_V)}$. *Moreover, the constant* C_V *is independent of* α *and depends on* V *only through* ω_V.

The above estimate implies that the operator $\Pi_{V,B}$ has a bounded holomorphic functional calculus. This can be applied with the functions $f(z) = \frac{z}{\sqrt{z^2}}$ and $g(z) = \frac{\sqrt{z^2}}{z}$ to the vector $(u, 0, 0) \in \mathcal{H}$ with $u \in D(\mathcal{L} + V)$ to obtain the Kato estimate

$$\left(C_V\left(1 + [V]_\alpha^2\right)\right)^{-1}\left(\|\nabla u\| + \left\||V|^{\frac{1}{2}}u\right\|\right) \leq \left\|\sqrt{\mathcal{L} + V}u\right\| \leq C_V\left(1 + [V]_\alpha^2\right)\left(\|\nabla u\| + \left\||V|^{\frac{1}{2}}u\right\|\right),$$

provided that $[V]_\alpha < \infty$ for some $\alpha \in (1, 2]$. An interpolation argument identical to that of Section 3 will then allow us to dispose of the dependence of this estimate on the quantity $[V]_\alpha$, giving us Theorem 6.1 for when $\Omega = \mathbb{R}^d$ and $[V]_\alpha < \infty$ for some $\alpha \in (1, 2]$.

For a general potential V, define

$$V_{n,m}(x) := (\operatorname{Re} V(x))\,\mathbb{1}_{\|V(x)\| \leq n} + i\,(\operatorname{Im} V(x))^+\,\mathbb{1}_{\|V(x)\| \leq n} - i\,(\operatorname{Im} V(x))^-\,\mathbb{1}_{\|V(x)\| \leq m}$$

for all $x \in \mathbb{R}^n$ and $m, n \in \|^*$ with $m \leq n$. In the above definition, $\|V(x)\|$ denotes the matrix norm of $V(x)$, $\operatorname{Re} V(x) := \frac{V(x)+V(x)^*}{2}$, $\operatorname{Im} V(x) := \frac{V(x)-V(x)^*}{2i}$ and M^+ and M^- denote the positive and negative parts of a self-adjoint matrix M. Observe that $V_{n,m}(x)$ is a normal matrix for a.e. $x \in \mathbb{R}^d$ and all m, n with $m \leq n$. This is trivial for $x \in \mathbb{R}^d$ with either $\|V(x)\| \leq m$ or $\|V(x)\| > n$. For $x \in \mathbb{R}^d$ with $m < \|V(x)\| \leq n$, this can be derived from the fact that a matrix M is normal if and only if its real and imaginary parts commute. Note that for this case we have

$$V_{n,m}(x) = \operatorname{Re} V(x) + i\,(\operatorname{Im} V(x))^+.$$

Since $(\operatorname{Im} V(x))^+ = \frac{\sqrt{(\operatorname{Im} V(x))^2} + \operatorname{Im} V(x)}{2}$, it follows that the real part of $V_{n,m}(x)$ commutes with its imaginary part and therefore $V_{n,m}(x)$ is normal.

The potentials $V_{n,m} + \epsilon$ will also clearly be normal for $\epsilon > 0$ and a.e. $x \in \mathbb{R}^d$. On noting that $[V_{n,m} + \epsilon]_2 < \infty$, we obtain the estimate

$$C_V\left(\left\||V_{n,m} + \epsilon|^{\frac{1}{2}}u\right\| + \|\nabla u\|\right) \leq \left\|\sqrt{\mathcal{L} + V_{n,m} + \epsilon}\,u\right\| \leq C_V\left(\left\||V_{n,m} + \epsilon|^{\frac{1}{2}}u\right\| + \|\nabla u\|\right),$$

where the constant in the above estimate only depends on V through ω_V and is independent of ϵ, m and n. Applying the limiting argument from Section 3 verbatim then allows us to conclude the proof of Theorem 6.1 for the case $\Omega = \mathbb{R}^d$. □

Proof for General Domains. Start by considering smooth $\Omega \subset \mathbb{R}^d$. Fix V satisfying the hypotheses of the theorem. Extend V to all of $\mathbb{R}^d$ by setting it equal to zero on $\mathbb{R}^d \setminus \Omega$. Then consider the potentials

$$V_k := V + \epsilon + k \cdot \mathrm{Id} \cdot \mathbb{1}_{\mathbb{R}^d \setminus \Omega}$$

for $\epsilon > 0$ and $k \in \|^*$, where Id denotes the identity matrix acting on $\mathbb{C}^m$. It is obvious that V_k is normal for a.e. $x \in \mathbb{R}^d$.

Theorem 6.1 for the case $\Omega = \mathbb{R}^d$ then gives the Kato estimate for V_k with constant independent of k, ϵ and V. The approximation argument from Section 5 can then be applied verbatim to obtain Theorem 6.1 for smooth Ω. Arbitrary domains are then obtained using the second approximation argument of Section 5. □

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