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New effective pressure and existence of global strong solution for compressible Navier-Stokes equations with general viscosity coefficient in one dimension

Cosmin Burtea [*], Boris Haspot [†‡]

Abstract

In this paper we prove the existence of global strong solution for the Navier-Stokes equations with general degenerate viscosity coefficients. The cornerstone of the proof is the introduction of a new effective pressure which allows to obtain an Oleinik-type estimate for the so called effective velocity. In our proof we make use of additional regularizing effects on the velocity which requires to extend the technics developed by Hoff for the constant viscosity case.

1 Introduction

We consider the compressible Navier Stokes system in one dimension with $x \in \mathbb{R}$:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x(\mu(\rho)\partial_x u) + \partial_x P(\rho) = 0, \\ (\rho, u)_{t=0} = (\rho_0, u_0). \end{cases} \tag{1.1}$$

Here $u = u(t,x) \in \mathbb{R}$ stands for the velocity field, $\rho = \rho(t,x) \in \mathbb{R}^+$ is the density, $P(\rho) = \rho^\gamma$ is the pressure. We denote by $\mu(\rho)$ the viscosity coefficient of the fluid and (ρ_0, u_0) are the initial data. In the sequel we shall only consider viscosity of the form:

$$\mu(\rho) = \rho^\alpha \tag{1.2}$$

with $\alpha > 0$. This choice is motivated by physical considerations. Indeed it is justified by the derivation of the Navier-Stokes equations from the Boltzmann equation through the Chapman-Enskog expansion to the second order (see [2]), the viscosity coefficient is then a function of the temperature. If we consider the case of isentropic fluids, this dependence is expressed by a dependence on the density function (we refer in particular to [15]). We mention that the case $\mu(\rho) = \rho$ is related to the so called viscous shallow water system. This system with friction has been derived by Gerbeau and Perthame

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in [6] from the Navier-Stokes system with a free moving boundary in the shallow water regime at the first order. This derivation relies on the hydrostatic approximation where the authors follow the role of viscosity and friction on the bottom.
We are now going to rewrite the system (1.1) following the new formulation proposed in [11] (see also [9, 8, 7]), indeed setting:

$$v = u + \frac{\mu(\rho)}{\rho^2}\partial_x \rho, \tag{1.3}$$

called the effective velocity, we can rewrite the system (1.1) as follows:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \rho \partial_t v + \rho u \partial_x v + \partial_x P(\rho) = 0. \end{cases} \tag{1.4}$$

The existence of global *weak* solution has been obtained by Jiu and Xin in [17] for viscosity coefficients verifying (1.2). In passing we point out that a large amount of literature is essentially dedicated to the study of the compressible Navier-Stokes equations with constant viscosity coefficients. In particular the existence of global strong solution with large initial data for initial density far away from the vacuum has been proved for the first time by Kanel [18] (see also [19, 13]). In [16] the authors proved that vacuum states do not arise provided that the initial density is positive almost everywhere. We would like also to mention the results of Hoff in [14] who proved the existence of global weak solution for constant viscosity coefficients with initial density admitting shocks (we refer also to [25, 26, 21]). The author exhibited regularizing effects on the velocity via the use of tricky estimates on the convective derivative:

$$\dot{u} = \partial_t u + u \partial_x u,$$

we will generalize these techniques in the present paper to the case of general viscosity coefficients. In [10], the second author proved also the existence of global weak solution for general viscosity coefficients with initial density admitting shocks and with initial velocity belonging to the set of finite measures. In opposite to [14], the initial data satisfy the BD entropy but not the classical energy, it allows in particular to show some regularizing effects on the density inasmuch as the density becomes instantaneously continuous. It is due to the regularity of the effective velocity v which express the coupling between the velocity and the density.

The problem of existence of global strong solution for system (1.1) with large initial data and with general viscosity coefficients verifying (1.2) is not yet completely solved. Indeed when $\alpha > 1$ it requires conditions of sign on the so called effective flux (see [13, 21]). This quantity represents the force that the fluids exerts on itself and a priori has no reason to be signed. In the following we are going to present the current state of art concerning the existence of global strong solution for system (1.1) with viscosity coefficients verifying (1.2).

It has been first proved by Mellet and Vasseur (see [22]) in the case $0 < \alpha < \frac{1}{2}$. The main argument of their proof consists in using the Bresch-Dejardins entropy (see [1]) in order to estimate the L^∞ norm of $\frac{1}{\rho}$ and using the parabolicity of the momentum equation of (1.1). It is important at this level to point out that the Bresch-Dejardins

entropy gives almost for free the control of $\|\frac{1}{\rho}\|_{L^\infty_{t,x}}$ when $\alpha < \frac{1}{2}$.

In [7], the second author has proved similar results for the case $\frac{1}{2} < \alpha \leq 1$ where he exploited the fact that the effective velocity v satisfies a damped transport equation. It enables to obtain L^∞ estimates for v and using maximum principle to get L^∞ control on $\frac{1}{\rho}$.

More recently Constantin *et al* in [3] have extended the previous results. More precisely, in the range $\alpha \in \left(\frac{1}{2}, 1\right]$ under the condition $\gamma \geq 2\alpha$, the authors obtain global existence of strong solutions for initial data belonging to H^3. They prove that the same result also holds true in the case $\alpha > 1$ with γ belonging to $[\alpha, \alpha+1]$ provided that the initial data satisfy:

$$\partial_x u_0 \leq \rho_0^{\gamma-\alpha}. \tag{1.5}$$

We point out that the condition (1.5) is equivalent to consider a negative effective flux (see for example [13, 21]) at initial time. The main idea of their proof consists in proving via a maximum principle that the effective flux remains negative for all time. This is sufficient to control the L^∞ norm of $\frac{1}{\rho}$.

In the present paper, our goal is double inasmuch as we wish both to show the existence of global strong solution for the case $\alpha > \frac{1}{2}$ without any sign restriction on the initial data and with minimal assumptions in terms of regularity. In [3], Constantin *et al* proved a blow-up criterion for $\alpha > \frac{1}{2}$ which is relied to estimating the $L^\infty_{t,x}$ norm of $\frac{1}{\rho}$. In order to apply this blow-up criterion, we introduce a new effective pressure $y = \frac{\partial_x v}{\rho} + F_2(\rho)$ with $\rho F_2'(\rho) = \frac{F_1(\rho)}{\rho}$ and $F_1(\rho) = \frac{P'(\rho)\rho}{\mu(\rho)}$. We observe then that y satisfies the following equation:

$$\partial_t y + u\partial_x y + F_1(\rho)y - F_1(\rho)F_2(\rho) + F_1'(\rho)\frac{\rho}{\mu(\rho)}(v-u)^2 = 0. \tag{1.6}$$

This last equation enables us to prove that if $y_0 \leq C$ with $C \in \mathbb{R}$ then y remains bounded on the right all along the time which implies in particular that:

$$\partial_x v(t,x) \leq C_1(t) \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.7}$$

with C_1 a continuous increasing function. Using maximum principle for the mass equation of (1.4) allows us to prove that $\frac{1}{\rho}$ is bounded all along the time. In order to show the uniqueness of the solutions, we extend Hoff's techniques to the case of general viscosity coefficients which enables us to prove that $\partial_x u$ belongs to $L^1_{loc}(L^\infty(\mathbb{R}))$. Passing in Lagrangian formulation (see the Appendix and the references therein), we get the uniqueness of the solutions. Finally, we would like to mention that the estimate (1.7) is reminiscent of the so-called Oleinik estimate (see [24, 4]) for scalar conservation law with a flux strictly convex or concave. If we consider the following equation with f regular:

$$\partial_t u + \partial_x f(u) = 0, \; u(0,\cdot) = u_0 \in L^\infty(\mathbb{R}),$$

the Kruzhkov theorem (see [20]) asserts that there exists a unique entropy solution. In addition if f is genuinely non linear, Oleinik has proved the following estimate in the sense of measures for $C > 0$ and for any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$:

$$\partial_x u(t,x) \leq \frac{C}{t}. \tag{1.8}$$

This estimate gives regularizing effects on u since instantaneously $u(t,\cdot)$ with $t>0$ is in $BV_{loc}(\mathbb{R})$. In our case, we have no regularizing effects on v. A possible explanation is the fact that v satisfies a damped transport equation which is in some sense linearly degenerate.

2 Main results

We are now in position to state our main theorem.

Theorem 2.1 *Let $\alpha>\frac{1}{2}$, $\gamma\geq\max(1,\alpha)$, $(\rho_0,\frac{1}{\rho_0})\in(L^\infty(\mathbb{R}))^2$, $(\rho_0-1,u_0)\in(L^2(\mathbb{R}^2))^2$. In addition we assume that $v_0\in L^2(\mathbb{R})$ and that there exists $C\in\mathbb{R}$ such that for any $x>y$ we have:*

$$\frac{v_0(x)-v_0(y)}{x-y}\leq C \tag{2.9}$$

Then there exists a unique global strong solution (ρ,u) for the Navier-Stokes system (1.1) with the following properties. For any given $T>0$, $L>0$ there exist a positive constant $C(T)$, a positive constant $C(T,L)$ depending respectively on T, on T, L and on $\|\rho_0-1\|_{L^2}$, $\|(\rho_0,\frac{1}{\rho_0})\|_{L^\infty}$, $\|u_0\|_{L^2}$, $\|v_0\|_{L^2}$ such that, if $\sigma(t)=\min(1;t)$, then:

$$C(T)^{-1}\leq\rho(T,\cdot)\leq C(T)\quad a.e, \tag{2.10}$$

$$\begin{aligned}&\sup_{0<t\leq T}\big(\|\rho(t,\cdot)-1\|_{L^2}+\|u(t,\cdot)\|_{L^2}+\|\partial_x\rho(t,\cdot)\|_{L^2}+\sigma(t)^{\frac{1}{2}}\|\partial_x u(t,\cdot)\|_{L^2}\\&+\sigma(t)^{\frac{1}{2}}(\|\dot{u}(t,\cdot)\|_{L^2}+\|\partial_x(\rho^\alpha\partial_x u(t,\cdot)-P(\rho)+P(1))\|_{L^2}\big)\leq C(T),\end{aligned} \tag{2.11}$$

$$\int_0^T[\|\partial_x u(t,\cdot)\|_{L^2}^2+\|\partial_x\rho(t,\cdot)\|_{L^2}^2+\sigma(t)\|\dot{u}(t,\cdot)\|_{L^2}^2+\sigma(t)\|\partial_x\dot{u}(t,\cdot)\|_{L^2}^2]dt\leq C(T), \tag{2.12}$$

$$\int_0^T\sigma^{\frac{1}{2}}(\tau)\|\partial_x u(\tau)\|_{L^\infty}^2\,d\tau\leq C(T). \tag{2.13}$$

$$\sup_{0<t\leq T}\sigma(t)^{\frac{1}{2}}\|\partial_x u(t,\cdot)\|_{L^\infty}\leq C(T) \tag{2.14}$$

$$\|v\|_{BV([0,T]\times[-L,L])}\leq C(T,L). \tag{2.15}$$

Furthermore for any $x>y$ and $t\geq 0$, we have almost everywhere:

$$\frac{v(t,x)-v(t,y)}{x-y}\leq C_1(t), \tag{2.16}$$

with C_1 a continuous increasing function.

Remark 1 *It is important to point out that our theorem requires that $\partial_x\rho_0$ belongs to $L^2(\mathbb{R})$. Indeed since $\partial_x\varphi(\rho_0)=v_0-u_0$ with $v_0\in L^2(\mathbb{R})$ and $u_0\in L^2(\mathbb{R})$, it implies that $\partial_x\varphi(\rho_0)\in L^2(\mathbb{R})$. since $\frac{1}{\rho_0}$ is in $L^\infty(\mathbb{R})$, we deduce that $\partial_x\rho_0$ is in $L^2(\mathbb{R})$.*
Furthermore since $\varphi(\rho_0)-1$ is also in $L^2(\mathbb{R})$ using that $\frac{1}{\rho_0}$ and ρ_0 are in $L^\infty(\mathbb{R})$, we deduce that $\varphi(\rho_0)-1$ is in $H^1(\mathbb{R})$. The initial density ρ_0 is then necessary a continuous function which prevents us from considering shock-type initial data.

Remark 2 *We would like to mention that any solution (ρ, u) of system (1.1) in the sense of distributions which verifies the regularity assumptions of Theorem 2.1 is also a strong solution i.e. (ρ, u) satisfy the system (1.1) almost everywhere on $\mathbb{R}^+ \times \mathbb{R}$. Setting $w_1(t,x) = \rho^\alpha \partial_x u(t,x) - P(\rho(t,x)) + P(1)$ the effective flux, we get from (2.10) and (2.11) that for any $t > 0$:*

$$\begin{cases} \sigma(t)^{\frac{1}{2}} \|\partial_x w_1(t,\cdot)\|_{L^2} \leq C(t) \\ \sigma(t)^{\frac{1}{2}} \|\partial_x u(t,\cdot)\|_{L^2} + \|P(\rho)(t,\cdot) - P(1)\|_{L^2} \leq C(t), \end{cases}$$

for C a continuous increasing function. This implies that w_1 belongs to $L^1_{loc}(\mathbb{R}^+, H^1(\mathbb{R}))$. Using now the fact that $(P(\rho) - P(1))$ belongs to $L^\infty_{loc}(H^1(\mathbb{R}))$, we deduce that $\rho^\alpha \partial_x u$ is in $L^1_{loc}(H^1(\mathbb{R}))$. Using (2.10), the fact that $(\frac{1}{\rho^\alpha} - 1)$ belongs to $L^\infty_{loc}(H^1(\mathbb{R}))$ we get using product law in Sobolev spaces that $\partial_x u$ is in $L^1_{loc}(H^1(\mathbb{R}))$. In particular $\partial_{xx} u$ is in $L^1_{loc}(L^2(\mathbb{R}))$. In other words it is easy to observe that each term of (1.1) is in $L^1_{loc}(\mathbb{R}^+ \times \mathbb{R})$ which ensures that (ρ, u) satisfies (1.1) almost everywhere.

Remark 3 *Let us point out that compared with [3], we deal with the range $\gamma \geq \max(\alpha, 1)$ whereas in [3] the authors treat the case $\alpha \leq \gamma \leq \alpha + 1$, $\alpha > 1$ provided that $\partial_x u_0 \leq \rho_0^{\gamma - \alpha}$. In a certain sense the method that we developed in our proof unifies the different situations, $\gamma > \alpha + 1$ and $\alpha \leq \gamma < \alpha + 1$. Furthermore we do not require any condition of sign on the initial data.*

Remark 4 *The condition (2.9) is a condition of Oleinik-type which implies that v_0 is in $BV_{loc}(\mathbb{R})$. Indeed we recall that for any $x \in \mathbb{R}$ we have $|x| = (2x)_+ - x$ with $(x)_+ = \max(0, x)$. It yields then that for any interval $[a, b]$ such that $v_0(a)$ and $v_0(b)$ are finite and any increasing subdivision $(x_n)_{n=1,\cdots,N}$ of the interval $[a, b]$ with $N \in \mathbb{N}^*$, we have using (2.9) and taking $x_0 = a$, $x_{N+1} = b$ if $x_1 > a$ and $x_N < b$:*

$$\begin{aligned} &\sum_{i=1}^{N-1} |v_0(x_{i+1}) - v_0(x_i)| \leq \sum_{i=0}^{N} |v_0(x_{i+1}) - v_0(x_i)| \\ &\leq 2 \sum_{i=0}^{N} (v_0(x_{i+1}) - v_0(x_i))_+ + v_0(a) - v_0(b) \\ &\leq 2C \sum_{i=0}^{N} (x_{i+1} - x_i) + v_0(a) - v_0(b) \\ &\leq 2C(b - a) + v_0(a) - v_0(b) \end{aligned}$$

In particular this shows that v_0 is necessary in $L^\infty_{loc}(\mathbb{R})$.
Furthermore (2.16) implies that the Oleinik estimate (2.9) is preserved all along the time. In addition since $x \to v(t,x) - C_1(t)x$ is non-increasing, we deduce that $v(t,\cdot)$ has left and right-hand limits at each points for almost $t \geq 0$.

Remark 5 *Our theorem does not require high regularity assumption on the initial velocity. Indeed, we assume only that u_0 and v_0 are respectively in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}) \cap BV_{loc}(\mathbb{R})$. This is however sufficient in order to ensure uniqueness.*

Remark 6 *We can observe that in the case $\frac{1}{2} < \alpha \leq 1$, our assumption on γ is optimal (from an hyperbolic point of view) since we need only $\gamma \geq 1$. This extends the results of [3, 7].*

Remark 7 *We can observe that (2.13) and (2.14) give a $L^1_{loc}(L^\infty(\mathbb{R}))$ control on $\partial_x u$. In particular, this enables us to define the flow associated to the velocity u (we refer for more details to the Appendix).*

We would like to emphasize that the condition (2.9) is automatically satisfied provided that $\partial_x v_0 \in L^\infty$. A necessary condition for this later condition to hold is to take initial data $\left(\frac{1}{\rho_0} - 1, \rho_0 - 1, u_0\right)$ in the following Sobolev spaces $(H^s(\mathbb{R}))^2 \times H^{s-1}(\mathbb{R})$ with $s_1 > \frac{5}{2}$, . As a by-product of Theorem 2.1 and the Appendix, we establish the following result.

Theorem 2.2 *Consider $\alpha \geq \frac{1}{2}$, $\gamma \geq \max(1, \alpha)$ and*

$$\left(\frac{1}{\rho_0} - 1, \rho_0 - 1, u_0\right) \in (H^s(\mathbb{R}))^2 \times H^{s-1}(\mathbb{R})$$

with $s > \frac{5}{2}$. Then, the compressible Navier-Stokes system (1.1) admits an unique solution, we have

$$(\rho - 1, u) \in C(\mathbb{R}_+, H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})).$$

In the section 3, we prove the Theorem 2.1. An appendix is devoted to the definition and basic properties of the Lagrangian framework, we give also a sketch of the proof of the Theorem 3.3 below.

3 Proof of the Theorem 2.1

A first ingredient is the following blow-up criterion

Theorem 3.3 *Assume that $\alpha > \frac{1}{2}$ and $\gamma \geq \max(\alpha - \frac{1}{2}, 1)$ and let $s \geq 3$ and $(\rho_0 - 1, u_0) \in H^s(\mathbb{R})$. Then there exists $T^* > 0$ such that (ρ, u) is a strong solution on $(0, T^*)$ with:*

$$(\rho - 1) \in C(0, T, H^s(\mathbb{R})),\ u \in C(0, T, H^s(\mathbb{R})) \cap L^2(0, T, H^{s+1}(\mathbb{R})),\ \forall T \in (0, T^*),$$

and for all $t \in (0, T^*)$:

$$\|\frac{1}{\rho}(t, \cdot)\|_{L^\infty} \leq C(t),$$

where $C(t) < +\infty$ if $t \in (0, T^*)$. In addition, if:

$$\sup_{t \in (0, T^*)} \|\frac{1}{\rho}(t, \cdot)\|_{L^\infty} \leq C < +\infty,$$

then the solution can be continued beyond $(0, T^*)$.

The above result says that the only way a regular solution might blow-up is if the L^∞-norm of $1/\rho$ blows-up. Theorem 3.3 is essentially an adaptation to the whole space of the blow-up criterion proved in Constantin *et al* (see Theorem 1.1. from [3]) in the case of the torus. We refer the reader to the Appendix for a sketch of the proof.

The objective of the rest of this section and section 3.1 is to show how to obtain a new bound for the L^∞-norm of $1/\rho$ by analysing a new quantity that we call effective pressure. Consider a pair (ρ_0, u_0) verifying the hypothesis stated in Theorem 2.1 and let us also consider the following sequence:

$$\rho_0^n - 1 = j_n * (\rho_0 - 1) \text{ and } v_0^n = j_n * v_0,$$

with j_n a regularizing kernel, $j_n(y) = nj(ny)$ with $0 \leq j \leq 1$, $\int_{\mathbb{R}} j(y)dy = 1$, $j \in C^\infty(\mathbb{R})$ and supp$j \subset [-2, 2]$. We deduce that $(\rho_0^n - 1, v_0^n)$ belong to all Sobolev spaces $H^s(\mathbb{R})$ with $s \geq 5/2$ and that:

$$0 < c \leq \rho_0^n \leq M < +\infty. \tag{3.17}$$

Also, we consider

$$u_0^n = v_0^n - \partial_x \varphi(\rho_0^n),$$

by composition theorem we know that $\varphi(\rho_0^n) - \varphi(1)$ belongs to $H^k(\mathbb{R})$ for any $k \geq 0$. Then we obtain that $u_0^n \in H^k(\mathbb{R})$ for $k \geq 3$. Finally we have for $x > y$ and using (2.9):

$$\frac{v_0^n(x) - v_0^n(y)}{x - y} = \int_{\mathbb{R}} (\frac{v_0(x-z) - v_0(y-z)}{x-y}) j_n(z)dz \leq C.$$

In particular we deduce that for any $x \in \mathbb{R}$, we have:

$$\partial_x v_0^n(x) \leq C. \tag{3.18}$$

Using the Theorem (3.3), we deduce that there exists a strong solution (ρ_n, u_n) on $(0, T_n^*)$ with $n \in \mathbb{N}$. We are going to prove that $T_n^* = +\infty$ and uniform estimates on $(\rho_n, u_n)_{n\in\mathbb{N}}$ on the time interval $\mathbb{R}^+$. The goal now is to apply the blow-up criterion of the Theorem 3.3. Let us prove that for any $t \in (0, T_n^*)$:

$$\|\frac{1}{\rho_n}(t, \cdot)\|_{L^\infty} \leq C,$$

for any $n \in \mathbb{N}$. Let us recall that there exists $C > 0$ such that for any $t > 0$ we have:

$$\int_{\mathbb{R}} [\rho_n(t,x)|u_n|^2(t,x) + \Pi(\rho_n(t,x)) - \Pi(1)]dx + \int_0^t \int_{\mathbb{R}} \mu(\rho_n(s,x))(\partial_x u_n(s,x))^2 dsdx \leq C, \tag{3.19}$$

and:

$$\int_{\mathbb{R}} [\rho_n(t,x)|v_n|^2(t,x) + \Pi(\rho_n(t,x)) - \Pi(1)]dx + \int_0^t \int_{\mathbb{R}} \frac{\mu(\rho_n)P'(\rho_n)}{\rho_n} |\partial_x \rho_n(s,x)|^2 dsdx \leq C, \tag{3.20}$$

This is due to the fact that it exists $C_1 > 0$ such that:

$$\|v_0^n\|_{L^2(\mathbb{R})} \leq C_1, \ \|\rho_0^n - 1\|_{L^2(\mathbb{R})} \leq C_1 \text{ and } \|\partial_x \rho_0^n\|_{L^2(\mathbb{R})} \leq C_1.$$

Combining (3.19) and (3.20), we deduce that for $C > 0$ large enough we have for any $t \in (0, T_n^*)$:

$$\|\rho_n(t,\cdot) - 1\|_{L_2^\gamma(\mathbb{R})} \leq C, \ \|\sqrt{\rho_n}\partial_x\varphi(\rho_n)\| \leq C. \tag{3.21}$$

We refer to [21] for the definition of Orlicz spaces. Since $\gamma > \alpha + 1$, using (3.21) and the Lemma 3.7 from ([17]) we get for $C > 0$ large enough and independent on n:

$$\|\rho_n\|_{L^\infty([0,T_n^*],L^\infty)} \leq C. \tag{3.22}$$

3.1 New effective pressure y_n and uniform estimates for $\frac{1}{\rho_n}$

We recall now that the effective velocity v_n verifies the momentum equation of the system (1.4), namely:

$$\partial_t v_n + u_n\partial_x v_n + \partial_x F(\rho_n) = 0,$$

with:

$$\partial_x F(\rho_n) = \frac{P'(\rho_n)\rho_n}{\mu(\rho_n)}(v_n - u_n).$$

Let us set now $w_n = \partial_x v_n$, we observe that w_n satisfies the following equation:

$$\partial_t w_n + u_n\partial_x w_n + \partial_x u_n w_n + \frac{P'(\rho_n)\rho_n}{\mu(\rho_n)}w_n - \frac{P'(\rho_n)\rho_n}{\mu(\rho_n)}\partial_x u_n + \partial_x(\frac{P'(\rho_n)\rho_n}{\mu(\rho_n)})(v_n - u_n) = 0.$$

If we set $F_1(\rho) = \frac{P'(\rho)\rho}{\mu(\rho)}$, we have:

$$\partial_t w_n + u_n\partial_x w_n + \partial_x u_n w_n + F_1(\rho_n)w_n - F_1(\rho_n)\partial_x u_n + F_1'(\rho_n)\frac{\rho_n^2}{\mu(\rho_n)}(v_n - u_n)^2 = 0.$$

Let us multiply the previous equation by $\frac{1}{\rho_n}$, we get then:

$$\partial_t(\frac{w_n}{\rho_n}) + u_n\partial_x(\frac{w_n}{\rho_n}) + F_1(\rho_n)\frac{w_n}{\rho_n} - \frac{F_1(\rho_n)}{\rho_n}\partial_x u_n + F_1'(\rho_n)\frac{\rho_n}{\mu(\rho_n)}(v_n - u_n)^2 = 0.$$

We set now $y_n = \frac{w_n}{\rho_n} + F_2(\rho_n)$ with $\rho_n F_2'(\rho_n) = \frac{F_1(\rho_n)}{\rho_n}$, we obtain then:

$$\partial_t y_n + u_n\partial_x y_n + F_1(\rho_n)y_n - F_1(\rho_n)F_2(\rho_n) + F_1'(\rho_n)\frac{\rho_n}{\mu(\rho_n)}(v_n - u_n)^2 = 0. \tag{3.23}$$

We recall now that $P(\rho) = \rho_n^\gamma$, $\mu(\rho_n) = \rho_n^\alpha$ and we get:

$$\begin{cases} F_2(\rho_n) = \dfrac{\gamma}{\gamma - \alpha - 1}\rho_n^{\gamma-\alpha-1} \quad \text{if } \gamma - \alpha - 1 \neq 0 \\ F_2(\rho_n) = \gamma \ln \rho_n \quad \text{if } \gamma = \alpha + 1 \\ F_1(\rho_n) = \gamma\rho_n^{\gamma-\alpha}. \end{cases} \tag{3.24}$$

Now since y_n is continuous on $\mathbb{R}^+ \times \mathbb{R}$ (indeed we recall that the solution (ρ_n, u_n) is regular) and $\lim_{x\to\pm\infty} y_n(t,x) = F_2(1)$, we deduce that $y_n(t,\cdot)$ has a maximum for every $t \geq 0$ and thus the function y_n^M defined as follows:

$$y_M^n(t) = \max_{x\in\mathbb{R}} y_n(t,x).$$

makes sense. Furthermore y_n^M is Lipschitz continuous on any interval $[0,T]$ with $T \in (0,T_n^*)$. Indeed from the triangular inequality we have for $(t_1,t_2) \in (0,T_n^*)$:

$$|y_M^n(t_1) - y_M^n(t_1)| \leq \max_{x\in\mathbb{R}} |y_n(t_1,x) - y_,(t_2,x)| \leq \|\partial_s y_n\|_{L^\infty([t_1,t_2],L^\infty)}|t_1 - t_2|.$$

According to Rademacher theorem, y_n^M is differentiable almost everywhere on $[0,T_n^*)$. Furthermore there exists for each $t \in [0,T_n^*)$ a point x_t^n such that:

$$y_M^n(t) = y(t,x_t^n).$$

We are going to verify now that for almost all $t \in (0,T_n^*)$ we have $(y_M^n)'(t) = \partial_t y_n(t,x_t^n)$. Indeed we have:

$$\begin{aligned}(y_M^n)'(t) &= \lim_{h\to 0^+} \frac{y_M^n(t+h) - y_M^n(t)}{h} = \lim_{h\to 0^+} \frac{y_n(t+h,x_{t+h}^n) - y^n(t,x_t^n)}{h} \\ &\geq \lim_{h\to 0^+} \frac{y_n(t+h,x_t^n) - y^n(t,x_t^n)}{h} = \partial_t y_n(t,x_t^n).\end{aligned}$$

Similarly, we have:

$$\begin{aligned}(y_M^n)'(t) &= \lim_{h\to 0^+} \frac{y_M^n(t) - y_M^n(t-h)}{h} = \lim_{h\to 0^+} \frac{y_n(t,x_t^n) - y^n(t-h,x_{t-h}^n)}{h} \\ &\leq \lim_{h\to 0^+} \frac{y_n(t,x_t^n) - y^n(t-h,x_t^n)}{h} = \partial_t y_n(t,x_t^n).\end{aligned}$$

We deduce from (3.23) using the fact that $\partial_x y_n(t,x_t^n) = 0$ since $y_n(t,\cdot)$ reaches its maximum in x_t^n and that for all $t \in (0,T_n^*)$ we have:

$$\partial_t y_M^n(t) + F_1(\rho_n)(t,x_t^n)y_M^n(t) = F_1(\rho_n)F_2(\rho_n)(t,x_t^n) - F_1'(\rho_n)\frac{\rho_n}{\mu(\rho_n)}(v_n - u_n)^2(t,x_t^n). \tag{3.25}$$

Basic computations give now:

$$\begin{cases} F_1(\rho)F_2(\rho) = \dfrac{\gamma^2}{\gamma-\alpha-1}\rho^{2\gamma-2\alpha-1} \quad \text{if} \quad \gamma \neq \alpha+1 \\ F_1(\rho)F_2(\rho) = \gamma^2 \ln\rho\rho^{\gamma-\alpha} \quad \text{if} \quad \gamma = \alpha+1 \\ \dfrac{F_1'(\rho)\rho}{\mu(\rho)} = \gamma(\gamma-\alpha)\rho^{\gamma-2\alpha} \\ F_1(\rho) = \gamma\rho^{\gamma-\alpha}. \end{cases} \tag{3.26}$$

We recall that we have $\gamma \geq \alpha$ such that using (3.25) and (3.26) we get that for $\gamma \neq \alpha+1$:

$$\partial_t y_M^n(t) + F_1(\rho_n)(t,x_t^n)y_M^n(t) \leq \max(0, \frac{\gamma^2}{\gamma-\alpha-1})\|\rho_n(t,\cdot)\|_{L^\infty}^{2\gamma-2\alpha-1}. \tag{3.27}$$

From (3.26) and (3.27), we get that for any $t \in [0,T_n^*)$ one has with $C_\gamma = \max(0, \frac{\gamma^2}{\gamma-\alpha-1})$:

$$\partial_t(y_M^n(t)e^{\gamma\int_0^t \rho_n^{\gamma-\alpha}(s,x_s^n)ds}) \leq C_\gamma\|\rho_n(t,\cdot)\|_{L^\infty}^{2\gamma-2\alpha-1}e^{\gamma\int_0^t \rho_n^{\gamma-\alpha}(s,x_s^n)ds}. \tag{3.28}$$

It yields for any $t \in (0, T_n^*)$:

$$\begin{aligned} y_M^n(t) \leq & e^{-\gamma \int_0^t \rho_n^{\gamma-\alpha}(s,x_s^n)ds} y_M^n(0) \\ & + C_\gamma \int_0^t \|\rho_n(t,\cdot)\|_{L^\infty}^{2\gamma-2\alpha-1} e^{-\gamma \int_s^t \rho_n^{\gamma-\alpha}(s',x_s^n)ds'} ds. \end{aligned} \tag{3.29}$$

From (3.18) and (3.17), we deduce that for any $n \in \mathbb{N}$ we have for any $x \in \mathbb{R}$ and $\gamma \neq \alpha + 1$:

$$y^n(0,x) \leq \frac{\max(0,C)}{c} + \frac{\gamma}{\gamma - \alpha - 1} M^{\gamma-\alpha-1} = C_1. \tag{3.30}$$

We obtain now from (3.29), (3.30) and since ρ_n is positive:

$$y_M^n(t) \leq C_1 + C_\gamma \int_0^t \|\rho_n(t,\cdot)\|_{L^\infty}^{2\gamma-2\alpha-1} ds. \tag{3.31}$$

Combining (3.22) and (3.31), we deduce that for any $t \in (0, T_n^*)$ we have:

$$y_M^n(t) \leq C(t), \tag{3.32}$$

with C a continuous function on $\mathbb{R}^+$ when $\gamma \neq \alpha + 1$. From (3.32), it yields for any $t \in (0, T_n^*)$ and $x \in \mathbb{R}$ when $\gamma \neq \alpha + 1$:

$$\frac{\partial_x v_n(t,x)}{\rho_n(t,x)} \leq C(t) + \frac{\gamma}{\alpha + 1 - \gamma} \rho_n^{\gamma-\alpha-1}(t,x), \tag{3.33}$$

with C a continuous function on $\mathbb{R}^+$. Next we recall that we have:

$$\partial_t(\frac{1}{\rho_n}) + u_n \partial_x(\frac{1}{\rho_n}) - \frac{1}{\rho_n}\partial_x u_n = 0.$$

We can rewrite the equation as follows:

$$\partial_t(\frac{1}{\rho_n}) + u_n \partial_x(\frac{1}{\rho_n}) - \frac{1}{\rho_n}\partial_x v_n - \frac{\mu(\rho_n)}{\rho_n}\partial_{xx}(\frac{1}{\rho_n}) - \frac{1}{\rho_n}\partial_x \mu(\rho_n)\partial_x(\frac{1}{\rho_n}) = 0.$$

Using again a maximum principle and following the same arguments as previously, we set now:

$$z_n(t) = \sup_{x \in \mathbb{R}} \frac{1}{\rho_n}(t,x) = \frac{1}{\rho_n}(t, x_t^n).$$

We have then:

$$\partial_t z_n(t) = \frac{\mu(\rho_n)}{\rho_n}\partial_{xx}(\frac{1}{\rho_n})(t, x_t^n) + \frac{1}{\rho_n}\partial_x v_n(t, x_t^n).$$

From (3.33) and since $\partial_{xx}(\frac{1}{\rho_n})(t, x_t^n) \leq 0$ (indeed x_t^n is a point where $\frac{1}{\rho_n}$ reaches its maximum) we deduce that:

$$\begin{aligned} \partial_t z_n(t) &\leq C(t) + \frac{\gamma}{\alpha + 1 - \gamma} \rho_n^{\gamma-\alpha-1}(t, x_t^n) \\ &\leq C(t) + \frac{\gamma}{\alpha + 1 - \gamma} z_n(t)^{\alpha+1-\gamma}. \end{aligned} \tag{3.34}$$

Using Gronwall lemma, it implies that there exists a continuous function C_2 on $\mathbb{R}^+$ such that for any $t \in (0, T_n^*)$ we have:

$$z_n(t) \leq C_2(t).$$

This implies that for any $t \in (0, T_n^*)$ we get:

$$\|\frac{1}{\rho_n}(t,\cdot)\|_{L^\infty} \leq C_2(t). \tag{3.35}$$

Combining the blow-up criterion in Theorem 3.3 and (3.35), we obtain that $T_n^* = +\infty$ and for any $t > 0$:

$$\|\frac{1}{\rho_n}(t,\cdot)\|_{L^\infty} \leq C_2(t), \tag{3.36}$$

with C_2 a continuous function on $\mathbb{R}^+$. From (3.33), (3.22) and (3.36), we get again for any $t \in (0, T_n^*)$ and $x \in \mathbb{R}$ when $\gamma \neq \alpha + 1$:

$$\partial_x v_n(t,x) \leq C_1(t), \tag{3.37}$$

with C_1 a continuous increasing function. We can easily prove similar results for $\gamma = \alpha+1$.

3.2 Estimates à la Hoff

In the sequel for simplifying the notation we drop the index n. Introducing the convective derivative

$$\dot{u} = \partial_t u + u\partial_x u,$$

we rewrite the momentum equation as

$$\rho\dot{u} - \partial_x\left(\rho^\alpha u_x\right) + \partial_x \rho^\gamma = 0.$$

Let us observe that:

$$\begin{aligned} -\int_{\mathbb{R}} \partial_x\left(\rho^\alpha \partial_x u\right)\partial_t u &= \int_{\mathbb{R}} \rho^\alpha \partial_x u \partial^2_{xt} u = \frac{1}{2}\int_{\mathbb{R}} \rho^\alpha \partial_t\left((\partial_x u)^2\right) \\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \rho^\alpha\left(\partial_x u\right)^2 - \frac{1}{2}\int_{\mathbb{R}} \partial_t \rho^\alpha (\partial_x u)^2. \end{aligned} \tag{3.38}$$

Next, we see that:

$$\begin{aligned} -\int_{\mathbb{R}} \partial_x\left(\rho^\alpha \partial_x u\right) u\partial_x u &= -\int_{\mathbb{R}} u\partial_x \rho^\alpha (\partial_x u)^2 - \int_{\mathbb{R}} \rho^\alpha u \partial^2_{xx} u \partial_x u \\ &= -\int_{\mathbb{R}} u\partial_x \rho^\alpha (\partial_x u)^2 + \frac{1}{2}\int_{\mathbb{R}} \partial_x\left(u\rho^\alpha\right)(\partial_x u)^2 \\ &= -\int_{\mathbb{R}} u\partial_x \rho^\alpha (\partial_x u)^2 + \frac{1}{2}\int_{\mathbb{R}} \rho^\alpha (\partial_x u)^3 + \frac{1}{2}\int_{\mathbb{R}} u\partial_x \rho^\alpha (\partial_x u)^2 \\ &= -\frac{1}{2}\int_{\mathbb{R}} u\partial_x \rho^\alpha (\partial_x u)^2 + \frac{1}{2}\int_{\mathbb{R}} \rho^\alpha (\partial_x u)^3. \end{aligned}$$

Thus, we gather that:

$$-\int_{\mathbb{R}} \partial_x\left(\rho^\alpha \partial_x u\right)\dot{u} = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \rho^\alpha\left(\partial_x u\right)^2 - \frac{1}{2}\int_{\mathbb{R}} \partial_t \rho^\alpha (\partial_x u)^2 - \frac{1}{2}\int_{\mathbb{R}} u\partial_x \rho^\alpha (\partial_x u)^2 + \frac{1}{2}\int_{\mathbb{R}} \rho^\alpha (\partial_x u)^3$$

$$= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \rho^{\alpha}\left(\partial_x u\right)^2 + \frac{1+\alpha}{2}\int_{\mathbb{R}} \rho^{\alpha}(\partial_x u)^3.$$

Moreover, we see that:

$$\begin{aligned}\int_{\mathbb{R}} \partial_x \rho^{\gamma}\left(\partial_t u + u\partial_x u\right) &= -\int_{\mathbb{R}} \rho^{\gamma}\partial_{tx} u + \int_{\mathbb{R}} u\partial_x \rho^{\gamma}\partial_x u \\ &= -\frac{d}{dt}\int_{\mathbb{R}} \rho^{\gamma}\partial_x u + \int_{\mathbb{R}} \partial_t \rho^{\gamma}\partial_x u + \int_{\mathbb{R}} u\partial_x \rho^{\gamma}\partial_x u \\ &= -\frac{d}{dt}\int_{\mathbb{R}} \rho^{\gamma}\partial_x u - \gamma\int_{\mathbb{R}} \rho^{\gamma}(\partial_x u)^2.\end{aligned}$$

Multiplying the momentum equation with $\dot{u}$ yields:

$$\int_{\mathbb{R}} \rho\dot{u}^2 + \frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}} \rho^{\alpha}\left(\partial_x u\right)^2 - \int_{\mathbb{R}} \rho^{\gamma}\partial_x u\right\} = -\frac{1+\alpha}{2}\int_{\mathbb{R}} \rho^{\alpha}(\partial_x u)^3 + \gamma\int_{\mathbb{R}} \rho^{\gamma}(\partial_x u)^2. \quad (3.39)$$

Let us multiply the previous estimate by $\sigma\left(t\right) = \min(1,t)$ and integrate in time on $[0,t]$ with $t > 0$, we have then:

$$\begin{aligned}&\frac{\sigma\left(t\right)}{2}\int_{\mathbb{R}} \rho^{\alpha}\left(t\right)\left(\partial_x u\right)^2\left(t\right) + \int_0^t\int_{\mathbb{R}} \sigma\rho\dot{u}^2 \\ &= \sigma\left(t\right)\int_{\mathbb{R}} \left(\rho^{\gamma}-1\right)\partial_x u + \int_0^{\min\{1,t\}}\int_{\mathbb{R}} \left[\frac{1}{2}\rho^{\alpha}\left(\partial_x u\right)^2 - \left(\rho^{\gamma}-1\right)\partial_x u\right] \\ &- \frac{1+\alpha}{2}\int_0^t\int_{\mathbb{R}} \sigma\rho^{\alpha}(\partial_x u)^3 + \gamma\int_0^t\int_{\mathbb{R}} \sigma\rho^{\gamma}(\partial_x u)^2.\end{aligned}$$

Let us denote by:

$$A\left(\rho,u\right)\left(t\right) = \frac{\sigma\left(t\right)}{2}\int_{\mathbb{R}} \rho^{\alpha}\left(t\right)\left(\partial_x u\right)^2\left(t\right) + \int_0^t\int_{\mathbb{R}} \sigma\rho\dot{u}^2.$$

Let us observe that using (3.19), (3.22) and (3.36) we have:

$$\begin{aligned}\sigma\left(t\right)\int_{\mathbb{R}} \left(\rho^{\gamma}-1\right)\partial_x u &\leq \sqrt{\sigma(t)}\left\|\frac{\rho^{\gamma}-1}{\rho^{\frac{\alpha}{2}}}\right\|_{L_t^{\infty}L^2}\left(\int_{\mathbb{R}} \sigma\left(t\right)\rho^{\alpha}\left(t\right)\left(\partial_x u\right)^2\left(t\right)\right)^{\frac{1}{2}} \\ &\leq C\left(t\right)\left\|\frac{\rho^{\gamma}-1}{\rho^{\frac{\alpha}{2}}}\right\|^2_{L_t^{\infty}L^2} + \frac{1}{4}\int_{\mathbb{R}} \sigma\left(t\right)\rho^{\alpha}\left(t\right)\left(\partial_x u\right)^2\left(t\right) \\ &\leq C_1\left(t\right) + \frac{1}{4}\int_{\mathbb{R}} \sigma\left(t\right)\rho^{\alpha}\left(t\right)\left(\partial_x u\right)^2\left(t\right),\end{aligned} \quad (3.40)$$

with C and C_1 continuous on $\mathbb{R}^+$. Next, we see that owing to the estimate (3.19), (3.22) and (3.36), we have that:

$$\int_0^{\min\{1,t\}}\int_{\mathbb{R}} \left[\frac{1}{2}\rho^{\alpha}\left(\partial_x u\right)^2 - \left(\rho^{\gamma}-1\right)\partial_x u\right] + \gamma\int_0^t\int_{\mathbb{R}} \sigma\rho^{\gamma}(\partial_x u)^2 \leq C_2\left(t\right), \quad (3.41)$$

with C_2 a continuous function on $\mathbb{R}^+$. Combining (3.39), (3.40) and (3.41) , we thus get for all $t \geq 0$:

$$\begin{aligned} A(\rho,u)(t) &\leq C(t) + \frac{1}{4}\int_{\mathbb{R}} \sigma(t)\,\rho^{\alpha}(t)\,(\partial_x u)^2(t) - \frac{1+\alpha}{2}\int_0^t\int_{\mathbb{R}} \sigma\rho^{\alpha}(\partial_x u)^3 \\ &\leq C_3(t) + \frac{1}{2}A(\rho,u)(t) - \frac{1+\alpha}{2}\int_0^t\int_{\mathbb{R}} \sigma\rho^{\alpha}(\partial_x u)^3 \end{aligned}$$

with C_3 a continuous fonction on $\mathbb{R}^+$. Consequently it yields:

$$A(\rho,u)(t) \leq C(t) + (1+\alpha)\int_0^t\int_{\mathbb{R}} \sigma\rho^{\alpha}(\partial_x u)^3$$

which also implies that (C can be choseen to be increasing in t):

$$\sup_{\tau\in[0,t]} A(\rho,u)(\tau) \leq C(t) + (1+\alpha)\int_0^t\int_{\mathbb{R}} \sigma\rho^{\alpha}(\partial_x u)^3 \tag{3.42}$$

Let us observe that for all $\varepsilon > 0$ we have using Gagliardo-Nirenberg inequality (3.19) and (3.22):

$$\begin{aligned} &\int_0^t \sigma^{\frac{1}{2}}(\tau)\,\|(\rho^{\alpha}\partial_x u - \rho^{\gamma})(\tau)\|_{L^\infty}^2 \leq 2\int_0^t \sigma^{\frac{1}{2}}(\tau)\,\|(\rho^{\alpha}\partial_x u - (\rho^{\gamma}-1))(\tau)\|_{L^\infty}^2 + 2t \\ &\leq 2\int_0^t \sigma^{\frac{1}{2}}(\tau)\,\|(\rho^{\alpha}\partial_x u - (\rho^{\gamma}-1))(\tau)\|_{L^2}\,\|\partial_x(\rho^{\alpha}\partial_x u - \rho^{\gamma})(\tau)\|_{L^2} + 2t \\ &\leq C_\varepsilon\int_0^t \|(\rho^{\alpha}\partial_x u - (\rho^{\gamma}-1))(\tau)\|_{L^2}^2 + \varepsilon\int_0^t \sigma(\tau)\,\|\partial_x(\rho^{\alpha}\partial_x u - \rho^{\gamma})(\tau)\|_{L^2}^2 + 2t \\ &\leq C_\varepsilon\int_0^t \|(\rho^{\alpha}\partial_x u - (\rho^{\gamma}-1))(\tau)\|_{L^2}^2 + \varepsilon\int_0^t \sigma(\tau)\,\|\rho\dot{u}(\tau)\|_{L^2}^2 + 2t \\ &\leq C(t,\varepsilon) + \varepsilon\|\rho\|_{L^\infty([0,t],L^\infty)}^{\frac{1}{2}} A(\rho,u)(t) \qquad (3.43) \\ &\leq C(t,\varepsilon) + \varepsilon C_0 A(\rho,u)(t), \qquad (3.44) \end{aligned}$$

with C a continuous function on $\mathbb{R}^+$. We are going now to estimate the last term of (3.42) and using (3.19), (3.22), (3.36) and (3.44) with $\varepsilon = 1/(2(1+\alpha)C_0)$ we obtain that:

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}} \sigma\rho^{\alpha}(\partial_x u)^3 = \int_0^t\int_{\mathbb{R}} \sigma(\partial_x u)^2(\rho^{\alpha}\partial_x u - \rho^{\gamma}) + \int_0^t\int_{\mathbb{R}} \sigma\rho^{\gamma}(\partial_x u)^2 \\ &\leq \int_0^t \left(\sigma^{\frac{1}{4}}\,\|(\rho^{\alpha}\partial_x u - \rho^{\gamma})(\tau)\|_{L^\infty}\,\sigma^{\frac{3}{4}}\int_{\mathbb{R}} (\partial_x u)^2(\tau)\right) d\tau + \int_0^t\int_{\mathbb{R}} \sigma\rho^{\gamma}(\partial_x u)^2 \\ &\leq C(t) + \int_0^t \sigma^{\frac{1}{2}}(\tau)\,\|(\rho^{\alpha}\partial_x u - \rho^{\gamma})(\tau)\|_{L^\infty}^2 + \int_0^t \sigma^{\frac{3}{2}}(\tau)\left(\int_{\mathbb{R}} (\partial_x u)^2(\tau)\,dx\right)^2 \\ &\leq C(t) + \frac{1}{2(1+\alpha)}A(\rho,u)(t) + \int_0^t \left\|\frac{1}{\rho(\tau)}\right\|_{L^\infty}^{2\alpha} \sigma^{\frac{3}{2}}(\tau)\,(\int_{\mathbb{R}} \rho^{\alpha}(\partial_x u)^2(\tau)\,dx)^2 \\ &\leq C(t) + \frac{1}{2(1+\alpha)}A(\rho,u)(t) + C_1(t)\int_0^t \sigma(\tau)\int_{\mathbb{R}} \rho^{\alpha}(\partial_x u)^2(\tau)\,dx\int_{\mathbb{R}} (\rho^{\alpha}\partial_x u)^2(\tau)\,dx \end{aligned}$$

$$\leq C(t) + \frac{1}{2(1+\alpha)} A(\rho,u)(t) + 2C_1(t) \int_0^t A(\rho,u)(\tau) \int_{\mathbb{R}} (\rho^\alpha \partial_x u)^2 (\tau)\, d\tau, \tag{3.45}$$

with C and C_1 continuous increasing functions. Finally, putting together (3.42) and (3.45) we get that

$$\sup_{\tau\in[0,t]} A(\rho,u)(\tau) \leq C_2(t) + C_2(t) \int_0^t A(\rho,u)(\tau) \int_{\mathbb{R}} (\rho^\alpha \partial_x u)^2 (\tau)\, d\tau,$$

with C_2 an increasing continuous function. Using Gronwall's lemma and (3.19) leads to

$$\sup_{\tau\in[0,t]} A(\rho,u)(\tau) \leq C(t), \tag{3.46}$$

with C an increasing continuous function. The control over $A(\rho,u)$ and (3.44) yields

$$\int_0^t \sigma^{\frac{1}{2}}(\tau) \left\|(\rho^\alpha \partial_x u - \rho^\gamma)(\tau)\right\|_{L^\infty}^2 d\tau \leq C(t),$$

and consequently we get using in addition (3.22):

$$\int_0^t \sigma^{\frac{1}{2}}(\tau) \left\|\partial_x u(\tau)\right\|_{L^\infty}^2 d\tau \leq C(t). \tag{3.47}$$

The last inequality also provides an estimate in $L_t^1(L^\infty)$ of $\partial_x u$ for any $t > 0$ using Cauchy-Schwarz inequality:

$$\int_0^t \left\|\partial_x u(\tau)\right\|_{L^\infty} d\tau \leq \left(\int_0^t \sigma^{-\frac{1}{2}}(\tau)\, d\tau\right)^{\frac{1}{2}} \left(\int_0^t \sigma^{\frac{1}{2}}(\tau) \left\|\partial_x u(\tau)\right\|_{L^\infty}^2\right)^{\frac{1}{2}} \leq C(t).$$

Next, we aim at obtaining estimate for the L^2-norm of $\partial_x \dot{u}$. This will be useful in order to recover regularity properties of u. The idea is to apply the operator $\partial_t + u\partial_x$ to the velocity's equation:

$$(\partial_t + u\partial_x)(\rho\dot{u}) - (\partial_t + u\partial_x)\partial_x(\rho^\alpha u_x) + (\partial_t P(\rho) + u\partial_x P(\rho)) = 0$$

and to test it with $\min\{1,t\}\dot{u}$. We begin by observing that

$$\int_{\mathbb{R}} (\rho\dot{u})_t\, \dot{u} = \int_{\mathbb{R}} \rho_t \dot{u}^2 + \frac{1}{2}\int_{\mathbb{R}} \rho \frac{d\dot{u}^2}{dt} = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \rho\dot{u}^2 + \frac{1}{2}\int_{\mathbb{R}} \rho_t \dot{u}^2.$$

We remark that:

$$\int_{\mathbb{R}} u\partial_x(\rho\dot{u})\, \dot{u} = -\int_{\mathbb{R}} \rho\dot{u}\partial_x(u\dot{u}) = -\int_{\mathbb{R}} \partial_x u \rho \dot{u}^2 + \frac{1}{2}\int_{\mathbb{R}} (\rho u)_x\, \dot{u}^2.$$

Summing the above two relations yields:

$$\int_{\mathbb{R}} (\partial_t + u\partial_x)(\rho\dot{u})\dot{u} = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \rho\dot{u}^2 - \int_{\mathbb{R}} \partial_x u \rho \dot{u}^2. \tag{3.48}$$

Next, we take a look at the second term:

$$-\int_{\mathbb{R}}(\partial_t+u\partial_x)\partial_x\left(\rho^\alpha\partial_x u\right)\dot{u}=\int_{\mathbb{R}}\partial_t\rho^\alpha\partial_x u\partial_x\dot{u}+\int_{\mathbb{R}}\rho^\alpha\partial_x u_t\partial_x\dot{u}+\int_{\mathbb{R}}\partial_x(\rho^\alpha\partial_x u)\partial_x(u\dot{u}) \tag{3.49}$$

Let us treat separately the last term appearing in the above inequality :

$$\begin{aligned}
&\int_{\mathbb{R}}\partial_x(\rho^\alpha\partial_x u)\partial_x(u\dot{u})\\
&=\int_{\mathbb{R}}\partial_x\rho^\alpha(\partial_x u)^2\dot{u}+\int_{\mathbb{R}}u\partial_x\rho^\alpha\partial_x u\partial_x\dot{u}+\int_{\mathbb{R}}\rho^\alpha\partial^2_{xx}u\partial_x u\dot{u}+\int_{\mathbb{R}}\rho^\alpha u\partial^2_{xx}u\partial_x\dot{u}\\
&=\int_{\mathbb{R}}\partial_x\rho^\alpha(\partial_x u)^2\dot{u}+\int_{\mathbb{R}}u\partial_x\rho^\alpha\partial_x u\partial_x\dot{u}-\frac{1}{2}\int_{\mathbb{R}}(\partial_x u)^2\partial_x(\rho^\alpha\dot{u})+\int_{\mathbb{R}}\rho^\alpha\partial_x(u\partial_x u)\partial_x\dot{u}-\int_{\mathbb{R}}(\partial_x u)^2\rho^\alpha\partial_x\dot{u}\\
&=\frac{1}{2}\int_{\mathbb{R}}\partial_x\rho^\alpha(\partial_x u)^2\dot{u}+\int_{\mathbb{R}}u\partial_x\rho^\alpha\partial_x u\partial_x\dot{u}-\frac{3}{2}\int_{\mathbb{R}}(\partial_x u)^2\rho^\alpha\partial_x\dot{u}+\int_{\mathbb{R}}\rho^\alpha\partial_x(u\partial_x u)\partial_x\dot{u}
\end{aligned} \tag{3.50}$$

Combining the two identities (3.49) and (3.50) we get that

$$\begin{aligned}
&-\int_{\mathbb{R}}(\partial_t+u\partial_x)\partial_x\left(\rho^\alpha\partial_x u\right)\dot{u}=\int_{\mathbb{R}}\partial_t\rho^\alpha\partial_x u\partial_x\dot{u}+\int_{\mathbb{R}}u\partial_x\rho^\alpha\partial_x u\partial_x\dot{u}\\
&+\int_{\mathbb{R}}\rho^\alpha\partial_x u_t\partial_x\dot{u}+\int_{\mathbb{R}}\rho^\alpha\partial_x(u\partial_x u)\partial_x\dot{u}-\frac{3}{2}\int_{\mathbb{R}}(\partial_x u)^2\rho^\alpha\partial_x\dot{u}+\frac{1}{2}\int_{\mathbb{R}}\partial_x\rho^\alpha(\partial_x u)^2\dot{u}\\
&=-\alpha\int_{\mathbb{R}}\rho^\alpha(\partial_x u)^2\partial_x\dot{u}+\int_{\mathbb{R}}\rho^\alpha(\partial_x\dot{u})^2-\frac{3}{2}\int_{\mathbb{R}}(\partial_x u)^2\rho^\alpha\partial_x\dot{u}+\frac{1}{2}\int_{\mathbb{R}}\partial_x\rho^\alpha(\partial_x u)^2\dot{u}\\
&=\int_{\mathbb{R}}\rho^\alpha(\partial_x\dot{u})^2-\left(\alpha+\frac{3}{2}\right)\int_{\mathbb{R}}\rho^\alpha(\partial_x u)^2\partial_x\dot{u}+\frac{1}{2}\int_{\mathbb{R}}\partial_x\rho^\alpha(\partial_x u)^2\dot{u}.
\end{aligned} \tag{3.51}$$

Remark 8 *The last term of the above identity, $\frac{1}{2}\int_{\mathbb{R}}\partial_x\rho^\alpha(\partial_x u)^2\dot{u}$ will be apear with sign minus in the next identity*

Let us observe that

$$\begin{aligned}
&\int_{\mathbb{R}}(\partial_x\rho_t^\gamma+u\partial^2_{xx}\rho^\gamma)\dot{u}=-\int_{\mathbb{R}}\rho_t^\gamma\partial_x\dot{u}+\int_{\mathbb{R}}u\partial^2_{xx}\rho^\gamma\dot{u}\\
&=\int_{\mathbb{R}}u\partial_x\rho^\gamma\partial_x\dot{u}+\gamma\int_{\mathbb{R}}\rho^\gamma\partial_x u\partial_x\dot{u}+\int_{\mathbb{R}}u\partial^2_{xx}\rho^\gamma\dot{u}\\
&=-\int_{\mathbb{R}}\partial_x u\partial_x\rho^\gamma\dot{u}+\gamma\int_{\mathbb{R}}\rho^\gamma\partial_x u\partial_x\dot{u}\\
&=\int_{\mathbb{R}}\partial_x u\rho\dot{u}^2-\int_{\mathbb{R}}\partial_x u\partial_x\left(\rho^\alpha\partial_x u\right)\dot{u}+\gamma\int_{\mathbb{R}}\rho^\gamma\partial_x u\partial_x\dot{u},\\
&=\int_{\mathbb{R}}\partial_x u\rho\dot{u}^2+\int_{\mathbb{R}}\rho^\alpha\partial_x u\partial_x\left(\dot{u}\partial_x u\right)+\gamma\int_{\mathbb{R}}\rho^\gamma\partial_x u\partial_x\dot{u},\\
&=\int_{\mathbb{R}}\partial_x u\rho\dot{u}^2+\int_{\mathbb{R}}\rho^\alpha(\partial_x u)^2\partial_x\dot{u}+\int_{\mathbb{R}}\dot{u}\rho^\alpha\partial_x u\partial^2_{xx}u+\gamma\int_{\mathbb{R}}\rho^\gamma\partial_x u\partial_x\dot{u},
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \partial_x u \rho \dot{u}^2 + \int_{\mathbb{R}} \rho^\alpha (\partial_x u)^2 \partial_x \dot{u} - \frac{1}{2} \int_{\mathbb{R}} \partial_x (\dot{u} \rho^\alpha)(\partial_x u)^2 + \gamma \int_{\mathbb{R}} \rho^\gamma \partial_x u \partial_x \dot{u}, \\
&= \int_{\mathbb{R}} \partial_x u \rho \dot{u}^2 + \frac{1}{2} \int_{\mathbb{R}} \rho^\alpha (\partial_x u)^2 \partial_x \dot{u} - \frac{1}{2} \int_{\mathbb{R}} \dot{u} \partial_x \rho^\alpha (\partial_x u)^2 + \gamma \int_{\mathbb{R}} \rho^\gamma \partial_x u \partial_x \dot{u}, \qquad (3.52)
\end{aligned}
$$

where we have used the equation of the velocity to replace

$$
-\partial_x \rho^\gamma = \rho \dot{u} - \partial_x (\rho^\alpha \partial_x u).
$$

We sum up the relations (3.48), (3.51) and (3.52) in order to obtain that:

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \rho \dot{u}^2 + \int_{\mathbb{R}} \rho^\alpha (\partial_x \dot{u})^2 = (\alpha + 1) \int_{\mathbb{R}} \rho^\alpha (\partial_x u)^2 \partial_x \dot{u} - \gamma \int_{\mathbb{R}} \rho^\gamma \partial_x u \partial_x \dot{u}.
$$

Multiplying with $\sigma(t)$ and integrating in time on $[0, t]$ with $t > 0$ leads to:

$$
\begin{aligned}
B(\rho, u)(t) &= \frac{1}{2} \int_{\mathbb{R}} \sigma(t) \rho \dot{u}^2(t) + \int_0^t \int_{\mathbb{R}} \sigma(t) \rho^\alpha (\partial_x \dot{u})^2 \\
&= \int_0^{\min(1,t)} \int_{\mathbb{R}} \rho \dot{u}^2 + (\alpha + 1) \int_0^t \int_{\mathbb{R}} \sigma \rho^\alpha (\partial_x u)^2 \partial_x \dot{u} - \gamma \int_0^t \int_{\mathbb{R}} \sigma \rho^\gamma \partial_x u \partial_x \dot{u}. \qquad (3.53)
\end{aligned}
$$

Obviously using (3.46) we have that,

$$
\int_0^{\min(1,t)} \int_{\mathbb{R}} \rho \dot{u}^2 \leq A(\rho, u)(1) \leq C. \qquad (3.54)
$$

for all $t > 0$. Next, we infer using (3.22) that:

$$
\begin{aligned}
\gamma \int_0^t \int_{\mathbb{R}} \sigma \rho^\gamma \partial_x u \partial_x \dot{u} &\leq \gamma \left\| \rho^{\gamma - \alpha} \right\|_{L_t^\infty L^\infty} \left(\int_0^t \int_{\mathbb{R}} \rho^\alpha (\partial_x u)^2 \right)^{\frac{1}{2}} \left(\int_0^t \int_{\mathbb{R}} \sigma^2 \rho^\alpha (\partial_x \dot{u})^2 \right)^{\frac{1}{2}} \\
&\leq C(t) + \frac{1}{4} B(\rho, u)(t), \qquad (3.55)
\end{aligned}
$$

with C a continuous increasing function. Finally, using again (3.46), (3.19) and (3.36), we get:

$$
\begin{aligned}
(\alpha + 1) \int_0^t \int_{\mathbb{R}} \sigma \rho^\alpha (\partial_x u)^2 \partial_x \dot{u} &\leq \frac{1}{4} \int_0^t \int_{\mathbb{R}} \sigma \rho^\alpha (\partial_x \dot{u})^2 + (\alpha + 1)^2 \int_0^t \int_{\mathbb{R}} \sigma \rho^\alpha (\partial_x u)^4 \\
&\leq \frac{1}{4} B(\rho, u)(t) + (\alpha + 1)^2 \left\| \frac{1}{\rho} \right\|_{L_t^\infty (L^\infty)}^{2\alpha} \int_0^t \int_{\mathbb{R}} \sigma \rho^{3\alpha} (\partial_x u)^4 \\
&\leq \frac{1}{4} B(\rho, u)(t) + C(t) \int_0^t \sigma \left\| \rho^\alpha \partial_x u \right\|_{L^\infty}^2 \int_{\mathbb{R}} \rho^\alpha (\partial_x u)^2 \\
&\leq \frac{1}{4} B(\rho, u)(t) + C(t) \sup_{\tau \in [0,t]} \sigma(\tau) \left\| (\rho^\alpha \partial_x u)(\tau) \right\|_{L^\infty}^2.
\end{aligned}
$$

Let us observe that for all $t > 0$ we have using Gagliardo-Nirenberg inequality, (3.22) and (3.36):

$$
\sigma(t) \left\| \rho^\alpha \partial_x u(t) \right\|_{L^\infty}^2 \leq 2\sigma \left\| (\rho^\alpha \partial_x u - (\rho^\gamma - 1))(t) \right\|_{L^\infty}^2 + 2 \left\| (\rho^\gamma - 1)(t) \right\|_{L^\infty}^2
$$

$$
\begin{aligned}
&\leq 2\sigma \left\|\left(\rho^{\alpha}\partial_x u-\left(\rho^{\gamma}-1\right)\right)(t)\right\|_{L^2}\left\|\partial_x\left(\rho^{\alpha}\partial_x u-\left(\rho^{\gamma}-1\right)\right)(t)\right\|_{L^2}+C(t)\\
&\leq 2\sigma\left(\left\|\rho^{\alpha}\partial_x u\right\|_{L^2}+C(t)\left\|\rho-1\right\|_{L^2}\right)\left\|\rho\dot{u}\right\|_{L^2}+C(t)\\
&\leq C(t)\left(\sigma^{\frac{1}{2}}\left\|\rho^{\alpha}\partial_x u\right\|_{L^2}+C(t)\right)\sigma^{\frac{1}{2}}\left\|\rho^{\frac{1}{2}}\dot{u}\right\|_{L^2}+C(t)\\
&\leq C(t)\left(A^{\frac{1}{2}}(\rho,u)(t)+C(t)\right)B^{\frac{1}{2}}(\rho,u)(t)+C(t). \qquad (3.56)
\end{aligned}
$$

Thus, we get from (3.46) and Young inequality:

$$
\begin{aligned}
(\alpha+1)\int_0^t\int_{\mathbb{R}}\sigma\rho^{\alpha}(\partial_x u)^2\partial_x\dot{u} &\leq \frac{1}{4}B(\rho,u)(t)+C(t)\left(A^{\frac{1}{2}}(\rho,u)(t)+C(t)\right)B^{\frac{1}{2}}(\rho,u)(t)+C(t)\\
&\leq C(t)+\frac{1}{2}B(\rho,u)(t). \qquad (3.57)
\end{aligned}
$$

Gathering (3.54), (3.55) and (3.57) yields the fact that B is also bounded:

$$
B(\rho,u)(t)\leq C(t), \qquad (3.58)
$$

with C a continuous increasing function. The control over $\left\|\frac{1}{\rho}\right\|_{L^\infty}$, $A(\rho,u)$ and $B(\rho,u)$ gives us, via the estimate (3.56) the following

$$
\sigma(t)^{\frac{1}{2}}\left\|\partial_x u(t)\right\|_{L^\infty}\leq C(t), \qquad (3.59)
$$

for any $t\geq 0$.

3.3 Uniform BV-estimates for the effective velocities v_n

Owing to the estimate (3.36) and (3.19) we recover the following estimates:

$$
\left\|\partial_x u_n\right\|_{L^2_t(L^2)}\leq\left\|\frac{1}{\rho_n}\right\|^{\frac{\alpha}{2}}_{L^\infty_t(L^\infty)}\leq C(t), \quad \left\|\sqrt{\rho_n}u_n\right\|_{L^2_t(L^2)}\leq C_1(t),
$$

where C, C_1 are increasing continuous functions. From Sobolev embedding, we get that for any $t>0$, there exists $C(t)$ such that

$$
\left\|u_n\right\|_{L^1_t(L^\infty)}\leq C(t). \qquad (3.60)
$$

Let us introduce the flow of u_n i.e.

$$
X_n(t,x)=x+\int_0^t u_n(\tau,X_n(\tau,x))\,d\tau. \qquad (3.61)
$$

We immediately get that:

$$
-|x|-C(t)\leq\left|X_n^{\pm 1}(t,x)\right|\leq|x|+C(t),
$$

which implies that for any $L>0$ the segment

$$
\left[X_n^{-1}(t,-L),X_n^{-1}(t,L)\right]\leq\left[-L-C(t),L+C(t)\right].
$$

This information is usefull in order to show that we can propagate the L^∞_{loc} norm of v_n. Indeed, let us recall that:

$$\partial_t v_n + u_n \partial_x v_n + \frac{P'(\rho_n)\rho_n^2}{\mu(\rho_n)} \frac{\mu(\rho_n)}{\rho_n^2} \partial_x \rho_n = 0,$$

rewrites as

$$\partial_t v_n + u_n \partial_x v_n + \frac{P'(\rho_n)\rho_n^2}{\mu(\rho_n)} (v_n - u_n) = 0. \tag{3.62}$$

Passing into Lagrangian coordinates (see Appendix) i.e.

$$(\widetilde{v}_n, \widetilde{u}_n, \widetilde{\rho}_n)(t,x) = (v_n, u_n, \rho_n)(t, X_n(t,x)),$$

we see that (3.62) rewrites as:

$$\partial_t \widetilde{v}_n + \frac{P'(\widetilde{\rho}_n)\widetilde{\rho_n}^2}{\mu(\widetilde{\rho}_n)} \widetilde{v}_n = \frac{P'(\widetilde{\rho}_n)\widetilde{\rho_n}^2}{\mu(\widetilde{\rho}_n)} \widetilde{u}_n. \tag{3.63}$$

The last relation implies using (3.22) and (3.36):

$$\begin{aligned}
|\widetilde{v}_n(t,x)| &\leq \left| v_{0n}(x) \exp\left(-\int_0^t \frac{P'(\widetilde{\rho}_n(\tau,x))\widetilde{\rho}_n^2(\tau,x)}{\mu(\widetilde{\rho}_n(\tau,x))} d\tau \right) \right| \\
&+ \left| \int_0^t \exp\left(-\int_s^t \frac{P'(\widetilde{\rho}_n(\tau,x)\widetilde{\rho}_n^2(\tau,x))}{\mu(\widetilde{\rho}_n(\tau,x))} d\tau \right) \frac{P'(\widetilde{\rho}_n(s,x))\widetilde{\rho}_n^2(s,x)}{\mu(\widetilde{\rho}_n(s,x))} \widetilde{u}_n(s,x)\, ds \right| \\
&\leq C(t) \left(|v_{0n}(x)| + \int_0^t \|u_n(s)\|_{L^\infty} ds \right). \\
&\leq C(t)(1 + |v_{0n}(x)|),
\end{aligned}$$

and consequently for any $t > 0$, $x \in \mathbb{R}$:

$$|v_n(t,x)| \leq C(t)\left(1 + |v_{0n}\left(X_n^{-1}(t,x)\right)|\right).$$

Thus, we see that:

$$\|v_n(t)\|_{L^\infty([-L,L])} \leq C(t)\left(1 + \|v_{0n}\|_{L^\infty([-L-C(t),L+C(t)])}\right). \tag{3.64}$$

In addition $(v_0^n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty_{loc}(\mathbb{R})$. Indeed since v_0 is in $L^\infty_{loc}(\mathbb{R})$ (see the Remark 4), we have for any $x \in [-L,L]$ and any $n \in \mathbb{N}$:

$$|v_0^n(x)| \leq \int_{-1}^1 j(y) v_0(x - \frac{y}{n}) dy \leq \|v_0\|_{L^\infty([-L-1,L+1])}. \tag{3.65}$$

This piece of information along with the estimate:

$$\partial_x v_n(t,x) \leq C(t)$$

ensures that v^n is uniformly bounded in $L^\infty([0,T]; BV_{loc}(\mathbb{R}))$. Ideed, the function

$$w_n(t,x) = v_n(t,x) - C(t)\, x$$

being nonincreasing, it holds using (3.64) that:

$$TV_{[-L,L]}w_n(t,\cdot) = v_n(t,-L) - v_n(t,L) + 2C(t)L \leq C(t)\left(L + \|v_{0n}\|_{L^\infty([-L-C(t),L+C(t)])}\right).$$

Owing to the fact that

$$v_n = w_n(t,x) + C(t)x$$

we get that:

$$TV_{[-L,L]}v_n(t,\cdot) \leq C(t)\left(L + \|v_{0n}\|_{L^\infty([-L-C(t),L+C(t)])}\right). \tag{3.66}$$

From (3.64), (3.65), (3.66), we get :

$$\|v_n(t)\|_{BV([-L,L])} \leq C(T,L). \tag{3.67}$$

Owing to (3.22), (3.36), (3.60), (3.64) and (3.63) we get that:

$$\|\partial_t \widetilde{v}_n\|_{L^1([0,T]\times[-L,L])} \leq C(T,L). \tag{3.68}$$

Next, fix $\phi \in C_b([0,T]\times[-L,L])$ with:

$$\|\phi\|_{L^\infty([0,T]\times[-L,L])} \leq 1$$

and let us write that:

$$\begin{aligned}
&\int_0^T \int_{-L}^L \phi(t,x)\,\partial_t v_n(t,x)\,dxdt \\
&= \int_0^T \left(\int_{X_n(t,-L)}^{X_n(t,L)} \widetilde{\phi}(t,x)\,\widetilde{\partial_t v_n}(t,x)\,\frac{\widetilde{\rho}_n(t,x)}{\rho_{0n(x)}}dx\right)dt \\
&= \int_0^T \left(\int_{X_n(t,-L)}^{X_n(t,L)} \widetilde{\phi}(t,x)\,\partial_t \widetilde{v}_n(t,x)\,\frac{\widetilde{\rho}_n(t,x)}{\rho_{0n(x)}}dx\right)dt \\
&- \int_0^T \left(\int_{X_n(t,-L)}^{X_n(t,L)} \widetilde{\phi}(t,x)\,\widetilde{u}_n(t,x)\,\widetilde{\partial_x v_n}((t,x))\,\frac{\widetilde{\rho}_n(t,x)}{\rho_{0n(x)}}dx\right)dt \\
&= \int_0^T \left(\int_{X_n(t,-L)}^{X_n(t,L)} \widetilde{\phi}(t,x)\,\partial_t \widetilde{v}_n(t,x)\,\frac{\widetilde{\rho}_n(t,x)}{\rho_{0n(x)}}dx\right)dt - \int_0^T \int_{-L}^L \phi(t,x)\,u_n(t,x)\,\partial_x v_n((t,x))\,dxdt.
\end{aligned}$$

Owing to (3.22), (3.36), (3.60), (3.67) and (3.68) and using the fact that ϕu_n belongs to $L^1([0,T],C^0(\mathbb{R}))$, we conclude that

$$\left|\int_0^T \int_{-L}^L \phi(t,x)\,\partial_t v_n(t,x)\,dxdt\right| \leq C(T,L). \tag{3.69}$$

Combining (3.69) and (3.66) gives us for any $T>0$, $L>0$:

$$\|v_n\|_{BV([0,T]\times[-L,L])} \leq C(T,L). \tag{3.70}$$

3.4 Compactness

We recall the previous estimates that we have obtained, for every $T > 0$ we have for C a continuous increasing function independent on n and any $n \in \mathbb{N}$:

$$C(T)^{-1} \leq \rho_n(T,\cdot) \leq C(T), \tag{3.71}$$

$$\begin{aligned} &\sup_{0<t\leq T} \big(\|\rho_n(t,\cdot)-1\|_{L^2} + \|u_n(t,\cdot)\|_{L^2} + \|\partial_x\rho_n(t,\cdot)\|_{L^2} + \sigma(t)^{\frac{1}{2}}\|\partial_x u_n(t,\cdot)\|_{L^2} \\ &+ \sigma(t)^{\frac{1}{2}}(\|\dot{u_n}(t,\cdot)\|_{L^2} + \|\partial_x(\rho_n^\alpha\partial_x u_n(t,\cdot) - P(\rho_n) + P(1))\|_{L^2}\big) \leq C(T), \end{aligned} \tag{3.72}$$

$$\int_0^T [\|\partial_x u_n(t,\cdot)\|_{L^2}^2 + \|\partial_x\rho_n(t,\cdot)\|_{L^2}^2 + \sigma(t)\|\dot{u_n}(t,\cdot)\|_{L^2}^2 + \sigma(t)\|\partial_x\dot{u_n}(t,\cdot)\|_{L^2}^2]dt \leq C(T), \tag{3.73}$$

$$\int_0^T \sigma^{\frac{1}{2}}(\tau)\,\|\partial_x u_n(\tau)\|_{L^\infty}^2\,d\tau \leq C(T). \tag{3.74}$$

$$\sup_{0<t\leq T} \sigma(t)^{\frac{1}{2}}\|\partial_x u_n(t,\cdot)\|_{L^\infty} \leq C(T). \tag{3.75}$$

Using classical arguments (see [17, 23]), we prove that up to a subsequence, $(\rho_n, u_n)_{n\in\mathbb{N}}$ converges in the sense of distributions to (ρ, u), a global weak solution of (1.1). Furthermore the limit functions ρ, u inherit all the bounds (3.71), (3.72), (3.73), (3.74), (2.15) and (3.75) via Fatou type-lemmas for the weak topology.
We wish now to prove (2.16), to do this we are going to prove that up to a subsequence $(v_n)_{n\in\mathbb{N}}$ converges almost everywhere to v on $\mathbb{R}^+\times\mathbb{R}$. This is a direct consequence of the estimate (3.70), indeed since $(v_n)_{n\in\mathbb{N}}$ is uniformly bounded in $BV_{loc}((0,T)\times\mathbb{R})$ for any $T > 0$, we deduce that up to a subsequence $(v_n)_{n\in\mathbb{N}}$ converges to v in $L^1_{loc}((0,T)\times\mathbb{R})$. In particular up to a subsequence $(v_n)_{n\in\mathbb{N}}$ converges almost everywhere to v in $(0,+\infty)\times\mathbb{R}$. Using now (3.37) and the fact that v_n converges almost everywhere to v on $\mathbb{R}^+\times\mathbb{R}$ implies (2.16) since for all $x > y$ and $t > 0$ we have:

$$\frac{v_n(t,x)-v_n(t,y)}{x-y} = \frac{1}{x-y}\int_y^x \partial_z v_n(t,z)dz \leq C(t),$$

with C a continuous function on $\mathbb{R}^+$. It concludes the proof of (2.16).

3.5 Uniqueness

Consider two solutions (ρ_i, u_i) ($i \in \overline{1,2}$) verifying the estimates (2.10)-(2.14) and generated by the same initial data:

$$\begin{cases} \partial_t\rho_i + \partial_x(\rho_i u_i) = 0, \\ \partial_t(\rho_i u_i) + \partial_x\left(\rho_i u_i^2\right) - \partial_x(\mu(\rho_i)\partial_x u_i) + \partial_x p_i = 0, \\ \left(\rho_{i|t=0}, u_{i|t=0}\right) = (\rho_0, u_0). \end{cases} \tag{3.76}$$

We define now the flows generated by u_i

$$X_i(t,x) = x + \int_0^t u_i(\tau, X(\tau,x))\,d\tau$$

and denoting with tildes the functions

$$\widetilde{v}_i\left(t,x\right)=v_i\left(t,X_i\left(t,x\right)\right)$$

for $v\in\{\rho,u\}$. We get that (according to the results from the Appendix):

$$\begin{cases}\partial_t\left(\dfrac{\partial X^i}{\partial x}\widetilde{\rho}_i\right)=0,\\ \rho_0\partial_t\widetilde{u}_i-\partial_x\left(\dfrac{\widetilde{\rho}_i\mu(\widetilde{\rho}_i)}{\rho_0}\,\partial_x\widetilde{u}_i\right)+\partial_xP(\widetilde{\rho}_i)=0,\\ X_i(t,x)=x+\displaystyle\int_0^t\widetilde{u}_i\left(\tau,x\right)d\tau.\end{cases}\tag{3.77}$$

for $i=1,2$. Setting $\delta\widetilde{u}=\widetilde{u}_1-\widetilde{u}_2$, by difference we have that:

$$\rho_0\partial_t\delta\widetilde{u}-\partial\left(\frac{\widetilde{\rho}_1\mu(\widetilde{\rho}_1)}{\rho_0}\partial_x\delta\widetilde{u}\right)=\partial_xG_1+\partial_xG_2,\tag{3.78}$$

where

$$\begin{cases}G_1=P\left(\dfrac{\rho_0}{1+\int_0^t\partial_x[\widetilde{u}_2]}\right)-P\left(\dfrac{\rho_0}{1+\int_0^t\partial_x[\widetilde{u}_1]}\right),\\ G_2=\left(\dfrac{\widetilde{\rho}_1\mu(\widetilde{\rho}_1)}{\rho_0}-\dfrac{\widetilde{\rho}_2\mu(\widetilde{\rho}_2)}{\rho_0}\right)\partial_x\widetilde{u}_2.\end{cases}$$

We multiply (3.78) by $\delta\widetilde{u}$, integrate it over $\mathbb{R}$ and by obvious manipulation we get for $t>0$:

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}\rho_0(x)(\delta\widetilde{u})^2(t,x)dx+\frac{1}{2}\inf_{s\in]0,t],x}\frac{\widetilde{\rho}_1\left(s,x\right)\mu(\widetilde{\rho}_1\left(s,x\right))}{\rho_0\left(x\right)}\int_0^t\int_{\mathbb{R}}\left(\partial_x(\delta\widetilde{u}(s,x))\right)^2ds\,dx\\ \leq C(t)[\int_0^t\int_{\mathbb{R}}(G_1)^2+\int_0^t\int_{\mathbb{R}}(G_2)^2],\end{aligned}\tag{3.79}$$

with C a continuous increasing function. In the following we will estimate G_1 and G_2. First, we we get using (4.94) for $t>0$ and $x\in\mathbb{R}$:

$$\begin{aligned}\delta\widetilde{\rho}\left(t,x\right)=\delta\rho_1-\delta\rho_2=\frac{\rho_0\left(x\right)}{1+\int_0^t\partial_x\widetilde{u}_1}-\frac{\rho_0\left(x\right)}{1+\int_0^t\partial_x\widetilde{u}_2}=\frac{-\rho_0\left(x\right)\int_0^t\partial_x\delta\widetilde{u}\left(\tau,x\right)d\tau}{\partial_xX^1(t,x)\partial_xX^2(t,x)}\\ |\delta\widetilde{\rho}\left(t,x\right)|\leq\sqrt{t}C\left(t\right)\left(\int_0^t\left|\partial_x\delta\widetilde{u}\left(\tau,x\right)\right|^2d\tau\right)^{\frac{1}{2}}\end{aligned}\tag{3.80}$$

and consequently using (4.92) we get:

$$G_1\left(t,x\right)\leq\sup_{s\in[1/C(T),C(T)]}P'\left(s\right)\frac{\rho_0\left(x\right)\left|\int_0^t\partial_x\delta\widetilde{u}\left(\tau,x\right)d\tau\right|}{|\partial_xX^1(t,x)\partial_xX^2(t,x)|}\leq\sqrt{t}C\left(t\right)\left(\int_0^t\left|\partial_x\delta\widetilde{u}\left(\tau,x\right)\right|^2d\tau\right)^{\frac{1}{2}},$$

with C a continuous increasing function. It implies that

$$\int_0^t\int_{\mathbb{R}}(G_1)^2\left(s,x\right)ds\,dx\leq t^{\frac{3}{2}}C\left(t\right)\int_0^t\int_{\mathbb{R}}(\partial_x\delta\widetilde{u})^2(s,x)dsdx.\tag{3.81}$$

Let us turn our attention towards G_2. We first write that for any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, we have:

$$\begin{aligned} G_2(t,x) &= \left(\frac{\widetilde{\rho}_1 \mu(\widetilde{\rho}_1)}{\rho_0} - \frac{\widetilde{\rho}_2 \mu(\widetilde{\rho}_2)}{\rho_0} \right) \partial_x \widetilde{u}_2(t,x) \\ &= \frac{1}{\rho_0(x)} (\mu)' \left(\theta_{t,x} \widetilde{\rho}_1(t,x) + (1-\theta_{t,x}) \widetilde{\rho}_2(t,x) \right) \delta\widetilde{\rho}(t,x) \partial_x \widetilde{u}_2(t,x). \end{aligned}$$

Thus, we get using (2.10), (3.80) that for $t > 0$:

$$\begin{aligned} &|G_2(t,x)|^2 \leq C(t) \left(t \left| \partial_x \widetilde{u}_2(t,x) \right|^2 \right) \left(\int_0^t \left| \partial_x \delta\widetilde{u}(\tau,x) \right|^2 d\tau \right) \\ &\leq C(t) \left(\left(\sigma^{\frac{1}{2}}(t) \mathbf{1}_{[0,1]}(t) + t \mathbf{1}_{[1,\infty)}(t) \right) \left\| \partial_x \widetilde{u}_2(t) \right\|_{L^\infty} \right)^2 \left(\int_0^t \left| \partial_x \delta\widetilde{u}(\tau,x) \right|^2 d\tau \right) \end{aligned}$$

such that by integration and using (2.14), (4.93) and (2.10) we have:

$$\int_{\mathbb{R}} \left| G_2(t,x)\, dx \right|^2 \leq C(t) \int_0^t \int_{\mathbb{R}} [\partial_x(\delta\widetilde{u}(s,x))]^2 dx\, ds. \tag{3.82}$$

Putting together the inequalities (3.79), (3.81), (3.82) and integrating in time, we get that for $t > 0$:

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}} \rho_0(x) (\delta\widetilde{u})^2(t,x) dx + \frac{1}{2} \inf_{s \in]0,t], x} \frac{\widetilde{\rho}_1(s,x) \mu(\widetilde{\rho}_1(s,x))}{\rho_0(x)} \int_0^t \int_{\mathbb{R}} \left(\partial_x(\delta\widetilde{u}(s,x)) \right)^2 ds\, dx \\ \leq t C_1(t) \int_0^t \int_{\mathbb{R}} [\partial_x(\delta\widetilde{u}(s,x))]^2 dx\, ds, \end{aligned} \tag{3.83}$$

with C_1 a continuous increasing function. Taking $T_0 > 0$ small enough, we have using a bootstrap argument for any $t \in [0, T_0]$:

$$\frac{1}{2} \int_0^1 \rho_0 (\delta\widetilde{u})^2 + \frac{1}{4} \inf_{t,x} \frac{\widetilde{\rho}_1(t,x) \mu(\widetilde{\rho}_1(t,x))}{\rho_0(x)} \int_0^t \int_0^1 \left(\partial_x(\delta\widetilde{u}) \right)^2 \leq 0 \ \forall t \in [0, T_0].$$

Thus, we get a local uniqueness property. Reiterating this process gives us the uniqueness of the two solutions on their whole domain of definition.

4 Appendix

In this appendix, we gather a few useful facts regarding the $1D$ Navier-Stokes equations in Lagrangian coordinates. The results belong to the mathematical folklore and can be found in, by now classical papers devoted to the $1D$ Navier-Stokes system, see [19], [25], [26]. The Lagrangian framework offers an elegant method of obtaining apriori estimates (for example on the L^∞ norm of ρ) either uniqueness of solutions (see the relatively recent paper [5]).

Let us first derive the Lagrangian formulation of the Navier-Stokes system. We will

supose that we are give $(\rho, u) \in L^\infty([0,\infty)\times\mathbb{R}) \times L^\infty\left(L^2(\mathbb{R})\right) \cap L^2\left(\dot{H}(\mathbb{R})\right)$ a solution of the Navier-Stokes system

$$\begin{cases} \rho_t + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x\left(\rho u^2\right) - \partial_x(\mu(\rho)\partial_x u) + \partial_x P(\rho) = 0, \\ \left(\rho_{|t=0}, u_{|t=0}\right) = (\rho_0, u_0). \end{cases} \tag{4.84}$$

First, we recall the definition of the flow of u.

Proposition 4.1 *Consider $T > 0$ and $u \in L^2((0,T); L^\infty(\mathbb{R}))$ with $\partial_x u \in L^1((0,T); L^\infty(\mathbb{R}))$. Then, for any $x \in \mathbb{R}$ there exists a unique solution $X(\cdot, x) : [0,\infty) \to \mathbb{R}$ of*

$$\begin{cases} X(t,x) = x + \displaystyle\int_0^t u(t, X(t,x)), \\ X(0,x) = x. \end{cases} \tag{4.85}$$

Moreover $X(t,x)$ verifies the following properties:

- *$X \in BV_{loc}([0,T]\times\mathbb{R})$ for any $T > 0$. In addition, for all $t \geq 0$ and for almost all $x \in \mathbb{R}$*
$$\partial_x X(t,x) = \exp\left(\int_0^t \partial_x u(\tau, X(\tau,x))\, d\tau\right)$$
- *For each $t > 0$, $X(t,\cdot)$ is a homeorphism from $\mathbb{R}$ to $\mathbb{R}$.*
- *We have that $\partial_t X, \partial_t X \in L^2_t(L^\infty_x)$ and $\partial_x X, \partial_x X^{-1} \in L^\infty_t(L^\infty_x)$*

Notation 1 *For any function $v : [0,\infty)\times\mathbb{R} \to \mathbb{R}$, we denote by $\widetilde{v}$ the function defined as:*

$$\widetilde{v}(t,x) := v(t, X(t,x))$$

We note that

$$X(t,x) = x + \int_0^t u(\tau, X(\tau,x)) = x + \int_0^t \widetilde{u}(\tau,x)\, d\tau.$$

and thus

$$\frac{\partial X}{\partial x}(t,x) = 1 + \int_0^t \partial_x \widetilde{u}(\tau,x)\, d\tau.$$

Owing to

$$\partial_x \widetilde{v}(t,x) = \widetilde{\partial_x v}(t,x)\frac{\partial X}{\partial x}(t,x) \tag{4.86}$$

we obtain that

$$\widetilde{\partial_x v}(t,x) = \frac{\partial X}{\partial x}(t,x)^{-1}\partial_x \widetilde{v}(t,x) = \frac{1}{1 + \int_0^t \partial_x \widetilde{u}(\tau,x)\, d\tau}\partial_x \widetilde{v}(t,x). \tag{4.87}$$

Let us investigate the first equation of (4.84). For any $\psi \in \mathcal{D}((0,T)\times\mathbb{R})$ we have that :

$$\int_0^T \int_{\mathbb{R}} \rho\psi_t + \rho u \partial_x \psi = 0.$$

Owing to the fact that ρ, $\rho u \in L^2_T\left(L^2_{loc}\right)$ the set of test functions can be enlarged to $\psi \in C^0\left((0,T)\times\mathbb{R}\right)$ (continious functions vanishing at the boundary) with $\psi_t, \partial_x\psi \in L^2_T\left(L^2_{loc}\right)$. In view of the regularity properties of $X(t,x)$ it follows that for any $\psi \in \mathcal{D}\left((0,T)\times\mathbb{R}\right)$, $\psi\circ X^{-1}$ can be used as a test function. Using this along with the fact that $X(t,x)$ is a homeomorphism for all t, we write that

$$\begin{aligned}
0 &= \int_0^T\int_{\mathbb{R}} \rho(\partial_t\psi\circ X^{-1}) + \rho u\partial_x(\psi\circ X^{-1})dxdt \\
&= \int_0^T\int_{\mathbb{R}} \widetilde{\rho}\left(\partial_t(\widetilde{\psi\circ X}^{-1}) + \widetilde{u}\partial_x(\widetilde{\psi\circ X}^{-1})\right)\partial_x X(t,x)\,dxdt \\
&= \int_0^T\int_{\mathbb{R}} \widetilde{\rho}\partial_x X\partial_t\psi
\end{aligned}$$

witch translates into

$$\frac{d}{dt}\left(\frac{\partial X}{\partial x}\widetilde{\rho}\right) = 0. \tag{4.88}$$

Prooceding in a symilar manner, we get that the velocity's equation rewrites as

$$\rho_0(x)\,\partial_t\widetilde{u} - \partial_x\left(\left(\frac{\partial X}{\partial x}\right)^{-1}\mu(\widetilde{\rho})\,\partial_x\widetilde{u}\right) + \partial_x P(\widetilde{\rho}) = 0. \tag{4.89}$$

Putting together equations, (4.88) and (4.89) we deduce that the system (4.84) can be writen in lagrangian coordinates as:

$$\begin{cases}
\dfrac{d}{dt}\left(\dfrac{\partial X}{\partial x}\widetilde{\rho}\right) = 0, \\
\rho_0(x)\,\partial_t\widetilde{u} - \partial_x\left(\left(\dfrac{\partial X}{\partial x}\right)^{-1}\mu(\widetilde{\rho})\,\partial_x\,\widetilde{u}\right) + \partial_x P(\widetilde{\rho}) = 0, \\
X(t,x) = x + \displaystyle\int_0^t \widetilde{u}(\tau,x)\,d\tau,
\end{cases} \tag{4.90}$$

or, equivalently

$$\begin{cases}
\dfrac{d}{dt}\left(\dfrac{\partial X}{\partial x}\widetilde{\rho}\right) = 0, \\
\rho_0(x)\,\partial_t\widetilde{u} - \partial_x\left(\dfrac{\widetilde{\rho}\mu(\widetilde{\rho})}{\rho_0}\,\partial_x\,\widetilde{u}\right) + \partial_x P(\widetilde{\rho}) = 0, \\
X(t,x) = x + \displaystyle\int_0^t \widetilde{u}(\tau,x)\,d\tau,
\end{cases} \tag{4.91}$$

Let us close this appendix observing that if we dispose of an inequality of the following type (it is the case in our case, see (2.10)):

$$C(t)^{-1} \le \widetilde{\rho}(t,x) \le C(t) \tag{4.92}$$

then one may obtain from (4.91) that

$$C(t)^{-1}\inf\rho_0 \le \frac{\partial X}{\partial x}(t,x) \le C(t)\sup\rho_0, \tag{4.93}$$

along with

$$\frac{C(t)}{\inf\rho_0} \ge \left(\frac{\partial X}{\partial x}(t,x)\right)^{-1} = \frac{\widetilde{\rho}(t,x)}{\rho_0(x)} \ge \frac{C(t)^{-1}}{\sup\rho_0}. \tag{4.94}$$

Sketch of the proof of the Theorem 3.3

In this section, we are just giving a sketch of the proof of the blow-up criterion. The part concerning the existence of strong solution in finite time is classical. We begin by observing that the Navier-Stokes system can be writen under the following form:

$$\begin{cases} \partial_t u + 2u\partial_x u - \partial_x \left(\rho^{\alpha-1}\partial_x u\right) = v\partial_x u - \gamma\rho^{\gamma-\alpha}\left(v-u\right), \\ \partial_t v + u\partial_x v = -\gamma\rho^{\gamma-\alpha+1}\left(v-u\right) \end{cases} \tag{4.95}$$

Let us recall a classical product law in Sobolev spaces along with the Kato-Ponce comutator estimate

Lemma 1 (Kato-Ponce) *The following estimates holds true for $s>0$ with $\mathcal{F}\Lambda_s f(\xi) = |\xi|^s \mathcal{F}f(\xi)$ for f a temperated distribution:*

$$\|\Lambda_s\left(fg\right)\|_{L^2} \leq \|f\|_{L^\infty}\|\Lambda_s g\|_{L^2} + \|g\|_{L^\infty}\|\Lambda_s f\|_{L^2} \tag{4.96}$$

$$\|\Lambda_s\left(f\partial_x g\right) - f\Lambda_s\partial_x g\|_{L^2} \leq C\left(\|\partial_x f\|_{L^\infty}\|\Lambda_s g\|_{L^2} + \|\Lambda_s f\|_{L^2}\|\partial_x g\|_{L^\infty}\right) \tag{4.97}$$

In the sequel we wish to describe how to preserve all along the time the H^s norm of u and $\rho-1$ for $s>\frac{3}{2}$.
We rewrite the system (4.95) as

$$\begin{cases} \partial_t\Lambda_s u + 2u\partial_x\Lambda_s u - \partial_x\left(\rho^{\alpha-1}\partial_x\Lambda_s u\right) = \Lambda_s\left(v\partial_x u\right) - \gamma\Lambda_s\left(\rho^{\gamma-\alpha}\left(v-u\right)\right) \\ +2\left[\Lambda_s, u\right]\partial_x u + \partial_x\left(\left[\rho^{\alpha-1},\Lambda_s\right]\partial_x u\right), \\ \partial_t\Lambda_s v + u\partial_x\Lambda_s v = -\gamma\Lambda_s\left(\rho^{\gamma-\alpha+1}\left(v-u\right)\right) + \left[\Lambda_s, u\right]\partial_x v \end{cases} \tag{4.98}$$

Multiply the first equation with $\Lambda_s u$ and integrate over $\mathbb{R}$, we get that:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}|\Lambda_s u|^2 + \int_{\mathbb{R}}\rho^{\alpha-1}|\partial_x\Lambda_s u|^2 = \int_{\mathbb{R}}\partial_x u\,|\Lambda_s u|^2 + \int_{\mathbb{R}}\Lambda_s\left(v\partial_x u\right)\Lambda_s u \\ &-\gamma\int_{\mathbb{R}}\Lambda_s\left(\rho^{\gamma-\alpha}\left(v-u\right)\right)\Lambda_s u + \int_{\mathbb{R}}2\left[\Lambda_s, u\right]\partial_x u\Lambda_s u + \int_{\mathbb{R}}\partial_x\left(\left[\rho^{\alpha-1},\Lambda_s\right]\partial_x u\right)\Lambda_s u. \end{aligned} \tag{4.99}$$

Multipliyng the second equation of (4.98) with $\Lambda_s v$ we obtain that:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}|\Lambda_s v|^2 = \frac{1}{2}\int_{\mathbb{R}}\partial_x u\,|\Lambda_s v|^2 - \gamma\int_{\mathbb{R}}\Lambda_s\left(\rho^{\gamma-\alpha+1}\left(v-u\right)\right)\Lambda_s v + \int_{\mathbb{R}}\left[\Lambda_s, u\right]\partial_x v\Lambda_s v. \tag{4.100}$$

If we add up (4.99) and (4.100), it yields that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left\{|\Lambda_s u|^2 + |\Lambda_s v|^2\right\} + \int_{\mathbb{R}}\rho^{\alpha-1}|\partial_x\Lambda_s u|^2 \\ &= \int_{\mathbb{R}}\partial_x u\,|\Lambda_s u|^2 + \int_{\mathbb{R}}\Lambda_s\left(v\partial_x u\right)\Lambda_s u - \gamma\int_{\mathbb{R}}\Lambda_s\left(\rho^{\gamma-\alpha}\left(v-u\right)\right)\Lambda_s u \\ &+ \int_{\mathbb{R}}2\left[\Lambda_s, u\right]\partial_x u\Lambda_s u + \int_{\mathbb{R}}\partial_x\left(\left[\rho^{\alpha-1},\Lambda_s\right]\partial_x u\right)\Lambda_s u \\ &\frac{1}{2}\int_{\mathbb{R}}\partial_x u\,|\Lambda_s v|^2 - \gamma\int_{\mathbb{R}}\Lambda_s\left(\rho^{\gamma-\alpha+1}\left(v-u\right)\right)\Lambda_s v + \int_{\mathbb{R}}\left[\Lambda_s, u\right]\partial_x v\Lambda_s v \end{aligned} \tag{4.101}$$

In the following lines, we analyse the different terms appearing in the left hand side of (4.101). The first two terms are treated in the following manner using Lemma 1:

$$\begin{aligned}
&\int_{\mathbb{R}} \partial_x u \left|\Lambda_s u\right|^2 + \int_{\mathbb{R}} \Lambda_s \left(v \partial_x u\right) \Lambda_s u \\
&\lesssim \|\partial_x u\|_{L^\infty} \|\Lambda_s u\|_{L^2}^2 + \|\partial_x u\|_{L^\infty} \|\Lambda_s v\|_{L^2} \|\Lambda_s u\|_{L^2} + \|v\|_{L^\infty} \|\partial_x \Lambda_s u\|_{L^2} \|\Lambda_s u\|_{L^2} \\
&\lesssim \|\partial_x u\|_{L^\infty} \|\Lambda_s u\|_{L^2}^2 + \|\partial_x u\|_{L^\infty} \|\Lambda_s v\|_{L^2} \|\Lambda_s u\|_{L^2} + \|v\|_{L^\infty} \left\|\rho^{1-\alpha}\right\|_{L^\infty}^{\frac{1}{2}} \left\|\rho^{\alpha-1} \partial_x \Lambda_s u\right\|_{L^2} \|\Lambda_s u\|_{L^2} \\
&\leq C \|\partial_x u\|_{L^\infty} \|\Lambda_s u\|_{L^2}^2 + C \|\partial_x u\|_{L^\infty} \|\Lambda_s v\|_{L^2} \|\Lambda_s u\|_{L^2} + C \|v\|_{L^\infty}^2 \left\|\rho^{1-\alpha}\right\|_{L^\infty} \|\Lambda_s u\|_{L^2}^2 \\
&\qquad + \frac{1}{8} \left\|\rho^{\frac{\alpha-1}{2}} \partial_x \Lambda_s u\right\|_{L^2}^2 .
\end{aligned} \tag{4.102}$$

The third term can be treated as follows:

$$\begin{aligned}
\int_{\mathbb{R}} \Lambda_s \left(\rho^{\gamma-\alpha} (v-u)\right) \Lambda_s u \lesssim \big(&\left\|\rho^{\gamma-\alpha}\right\|_{L^\infty} \left(\|\Lambda_s v\|_{L^2} + \|\Lambda_s u\|_{L^2}\right) \|\Lambda_s u\|_{L^2} \\
&+ \|(v-u)\|_{L^\infty} \left\|\Lambda_s \left(\rho^{\gamma-\alpha} - 1\right)\right\|_{L^2} \big) \|\Lambda_s u\|_{L^2} .
\end{aligned} \tag{4.103}$$

We have for the fourth term using Lemma 1:

$$\int_{\mathbb{R}} 2 \left[\Lambda_s, u\right] \partial_x u \Lambda_s u \lesssim \|\partial_x u\|_{L^\infty} \|\Lambda_s u\|_{L^2}^2 \tag{4.104}$$

The fifth term :

$$\begin{aligned}
&\int_{\mathbb{R}} \partial_x \left(\left[\rho^{\alpha-1}, \Lambda_s\right] \partial_x u\right) \Lambda_s u \\
&\leq C \left\|\left[\rho^{\alpha-1}, \Lambda_s\right] \partial_x u\right\|_{L^2} \|\partial_x \Lambda_s u\|_{L^2} \\
&\leq C \left\|\rho^{\alpha-1}\right\|_{L^\infty} \left(\left\|\partial_x \rho^{\alpha-1}\right\|_{L^\infty} \|\Lambda_s u\|_{L^2} + \|\partial_x u\|_{L^\infty} \left\|\Lambda_s \left(\rho^{\alpha-1} - 1\right)\right\|_{L^2}\right)^2 + \frac{1}{8} \left\|\rho^{\frac{\alpha-1}{2}} \partial_x \Lambda_s u\right\|_{L^2}^2 \\
&\leq C \left\|\rho^{\alpha-1}\right\|_{L^\infty} \left(\|v-u\|_{L^\infty} \|\Lambda_s u\|_{L^2} + \|\partial_x u\|_{L^\infty} \left\|\Lambda_s \left(\rho^{\alpha-1} - 1\right)\right\|_{L^2}\right)^2 + \frac{1}{8} \left\|\rho^{\frac{\alpha-1}{2}} \partial_x \Lambda_s u\right\|_{L^2}^2 .
\end{aligned} \tag{4.105}$$

We skip the sixth term. Seventh term :

$$\begin{aligned}
&\int_{\mathbb{R}} \Lambda_s \left(\rho^{\gamma-\alpha+1} (v-u)\right) \Lambda_s v \lesssim \|\Lambda_s v\|_{L^2} \|\Lambda_s u\|_{L^2} + \|\Lambda_s v\|_{L^2}^2 \\
&+ \left(\left\|\rho^{\gamma-\alpha+1} - 1\right\|_{L^\infty} \|\Lambda_s (v-u)\|_{L^2} + \|v-u\|_{L^\infty} \left(\left\|\Lambda_s \left(\rho^{\gamma-\alpha+1} - 1\right)\right\|_{L^2}\right)\right) \|\Lambda_s v\|_{L^2}
\end{aligned} \tag{4.106}$$

Last term :

$$\int_{\mathbb{R}} \left[\Lambda_s, u\right] \partial_x v \Lambda_s v \lesssim \left(\|\partial_x u\|_{L^\infty} \|\Lambda_s v\|_{L^2} + \|\partial_x v\|_{L^\infty} \|\Lambda_s u\|_{L^2}\right) \|\Lambda_s v\|_{L^{\acute{e}}} \tag{4.107}$$

Let us observe that in the estimates (4.103) (4.105) and (4.106) we have to treat the H^s-norm of $\rho^{\gamma-\alpha}$,$\rho^{\alpha-1}$ and $\rho^{\gamma-\alpha+1}$ respectively. This is the objective of the following lines. For each β, we may write that

$$\partial_t \rho^\beta + u \partial_x \rho^\beta = -\beta \rho^\beta \partial_x u.$$

and consequently

$$\partial_t \Lambda_s \left(\rho^\beta - 1\right) + u\partial_x \Lambda_s \left(\rho^\beta - 1\right) = -\beta \Lambda_s \left(\rho^\beta \partial_x u\right) + [\Lambda_s, u]\, \partial_x \left(\rho^\beta - 1\right).$$

We get that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \left|\Lambda_s \left(\rho^\beta - 1\right)\right|^2 \leq (\frac{1}{2} + \beta) \left\|\partial_x u\right\|_{L^\infty} \left\|\Lambda_s \left(\rho^\beta - 1\right)\right\|_{L^2}^2 \\
&+\beta \left(\left\|\partial_x u\right\|_{L^\infty} \left\|\Lambda_s \left(\rho^\beta - 1\right)\right\|_{L^2}^2 + \left\|\left(\rho^\beta - 1\right)\right\|_{L^\infty} \left\|\Lambda_s \partial_x u\right\|_{L^2}\right) \left\|\Lambda_s \left(\rho^\beta - 1\right)\right\|_{L^2} \\
&+ \left(\left\|\partial_x u\right\|_{L^\infty} \left\|\Lambda_s \left(\rho^\beta - 1\right)\right\|_{L^2} + \left\|\partial_x \rho^\beta\right\|_{L^\infty} \left\|\Lambda_s u\right\|_{L^2}\right) \left\|\Lambda_s \left(\rho^\beta - 1\right)\right\|_{L^2} \\
&\leq C_\varepsilon \left(\left\|\partial_x u\right\|_{L^\infty} + \left\|\rho^{1-\alpha}\right\|_{L^\infty}^2 \left\|\left(\rho^\beta - 1\right)\right\|_{L^\infty}^2 + \left\|v - u\right\|_{L^\infty}^2\right) \left\|\Lambda_s \left(\rho^\beta - 1\right)\right\|_{L^2}^2 \\
&+ \left\|\Lambda_s u\right\|_{L^2}^2 + \varepsilon \left\|\rho^{\frac{\alpha-1}{2}} \partial_x \Lambda_s u\right\|_{L^2}^2
\end{aligned} \tag{4.108}$$

Thus putting togheter the estimates (4.102),(4.103),(4.104),(4.105),(4.106),(4.107) and (4.108) for $\beta = \alpha - 1$, $\gamma - \alpha$, $\gamma - \alpha + 1$ we get that

$$\begin{aligned}
&\int_{\mathbb{R}} \left\{|\Lambda_s u|^2 + |\Lambda_s v|^2 + \left|\Lambda_s \left(\rho^{\alpha-1} - 1\right)\right|^2 + \left|\Lambda_s \left(\rho^{\gamma-\alpha} - 1\right)\right|^2 + \left|\Lambda_s \left(\rho^{\gamma-\alpha+1} - 1\right)\right|^2\right\} (t,x) dx \\
&+ \int_0^t \int_{\mathbb{R}} \rho^{\alpha-1} \left|\partial_x \Lambda_s u\right|^2 (s,x) ds dx \\
&\leq C\left(u_0, \rho_0\right) \exp\left(\int_0^t \left(1 + \left\|\left(\rho, \frac{1}{\rho}\right)\right\|_{L^\infty}\right)^\delta \left(1 + \left\|(u, v, \partial_x u, \partial_x v)\right\|_{L^\infty}^2\right)\right)
\end{aligned} \tag{4.109}$$

with δ depending on α and γ. We mention also that $C\left(u_0, \rho_0\right)$ depends on $\|u_0\|_{H^s}$, $\|\rho_0 - 1\|_{H^s}$, $\|\rho_0\|_{L^\infty}$ and $\|\frac{1}{\rho_0}\|_{L^\infty}$. Next, let us analyse in more detail the equation of v :

$$\partial_t v + u\partial_x v = -\gamma \rho^{\gamma-\alpha+1} (v - u)$$

We get that

$$\|v\|_{L_t^\infty(L_x^\infty)} \leq \|v_0\|_{L_x^\infty} + \left\|\rho^{\gamma-\alpha+1}\right\|_{L_t^\infty(L_x^\infty)} \int_0^t \|u(s,\cdot)\|_{L^\infty} ds.$$

Moreover, writing the equation of $\partial_x v$ we see that

$$(\partial_x v)_t + u\partial_x (\partial_x v) + \left(\partial_x u + \gamma \rho^{\gamma-\alpha}\right) \partial_x v = \gamma \rho^{\gamma-\alpha} \partial_x u - \gamma (\gamma - \alpha) \rho^{\gamma-2\alpha+2} (v - u)^2$$

From which we deduce that

$$\|\partial_x v\|_{L_t^\infty(L_x^\infty)} \lesssim \|\partial_x v_0\|_{L_x^\infty} + \psi\left(\left(1 + \left\|\left(\rho, \frac{1}{\rho}\right)\right\|_{L_t^\infty(L^\infty)}\right)^{\delta_1} \left(1 + \int_0^t \|(u, \partial_x u)\|_{L^\infty}\right)\right)$$

with $\psi(r) = r \exp r$ and δ_1 depending on γ and α. Moreover, the Bresch-Desjardins entropy allows a uniform control on $\|\rho\|_{L_t^\infty(L_x^\infty)}$. Let us denote by:

$$\widetilde{A}(\rho, u)(t) = \int_{\mathbb{R}} \rho^\alpha (t) (\partial_x u)^2 (t) + \int_0^t \int_{\mathbb{R}} \rho \dot{u}^2.$$

Using the same techniques as in the section on the Hoff estimates, we may show that

$$\widetilde{A}\left(\rho,u\right)(t) \leq C_0 \exp\left(t\left(1+\left\|\frac{1}{\rho}\right\|_{L_t^\infty(L^\infty)}\right)^{\delta_2}\right)$$

which, in turn, ensures a control on $\|\partial_x u\|_{L_t^2(L^\infty)}$ provided that we control $\|\frac{1}{\rho}\|_{L_t^\infty(L^\infty)}$. To summarize:

- The Bresch-Desjardins entropy provides control on $\|\rho\|_{L_t^\infty(L_x^\infty)}$ for any $t>0$,
- $\|(v,\partial_x v)\|_{L_t^\infty(L^\infty)}$ is controlled by $\|(u,\partial_x u)\|_{L^1(L^\infty)}$ and $\left\|\left(\rho,\frac{1}{\rho}\right)\right\|_{L_t^\infty(L^\infty)}$,
- The Hoff-type estimates ensure that $\|\partial_x u\|_{L_t^2(L_x^\infty)}$ is controlled by $\left\|\left(\rho,\frac{1}{\rho}\right)\right\|_{L_t^\infty(L^\infty)}$
- Using the basic energy estimate we obtain that $\|u\|_{L_t^2(L_x^\infty)}$ is controlled by $\left\|\left(\rho,\frac{1}{\rho}\right)\right\|_{L_t^\infty(L^\infty)}$.

Taking into account the estimate (4.109) we get that for any $T>0$ and any $s>\frac{3}{2}$ the H^s-Sobolev norm of $(u,v,\rho-1)$ is uniformly controlled by $\left\|\frac{1}{\rho}\right\|_{L_t^\infty(L^\infty)}$.

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