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# On a moment problem with holonomic functions

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## Abstract

Many reconstruction algorithms from moments of algebraic data were developed in optimization, analysis or statistics. Lasserre and Putinar proposed an exact reconstruction algorithm for the algebraic support of the Lebesgue measure, or of measures with density equal to the exponential of a known polynomial. Their approach relies on linear recurrences for the moments, obtained using Stokes theorem.

In this article, we extend this study to measures with holonomic densities and support with real algebraic boundary. In the framework of holonomic distributions (i.e. they satisfy a holonomic system of linear partial or ordinary differential equations with polynomial coefficients), an alternate method to creative telescoping is proposed for computing linear recurrences for the moments. When the coefficients of a polynomial vanishing on the support boundary are given as parameters, the obtained recurrences have the advantage of staying linear with respect to them.

This property allows for an efficient reconstruction method. Given a finite number of numerically computed moments for a measure with holonomic density, and assuming a real algebraic boundary for the support, we propose an algorithm for solving the inverse problem of obtaining both the coefficients of a polynomial vanishing on the boundary and those of the polynomials involved in the holonomic operators which annihilate the density.

# 1 Introduction

**Notations.** Let  $n$  be a positive integer for the ambient space  $\mathbb{R}^n$ , whose canonical basis is denoted by  $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ . Let  $\mathbb{K}[x]$  be the ring of polynomials in the variables  $x = (x_1, \dots, x_n)$  over a real finite computable extension of  $\mathbb{Q}$ , and let  $\mathbb{K}[x]_d$  be the vector space of polynomials of total degree at most  $d$ . For every  $d$ , let  $\mathbb{N}_d^n := \{\alpha \in \mathbb{N}^n : |\alpha| \leq d\}$ , where  $|\alpha| = \sum_i \alpha_i$ . In a multivariate setting, we denote  $x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$  and  $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$  for  $\alpha, \beta \in \mathbb{N}^n$ . The derivative  $\frac{\partial p}{\partial x_i}$  is denoted  $p_{x_i}$ . The indicator function of a set  $G$ , is denoted by  $\mathbb{1}_G$ .

The structure of moments of algebraic data is a central question in various reconstruction algorithms, appearing as part of a broad field of inverse problems [9]. We refer to [14] and references therein for various shape reconstruction from their moments of polyhedra [7, 6], planar quadrature domains [4], sublevel sets of homogeneous polynomials [13], together with more applied studies of computerized tomography [16].

In this article, we focus on the structure of moments of holonomic distributions, together with associated inverse problems. It can be seen as a computer algebra-based extension of [14], where the approach was mainly based on techniques recently developed in polynomial optimization [12], which are at the interface between real algebraic geometry, moment problems and polynomial optimization. More precisely, our setting is the following.

**Setting:** Let  $G \subset \mathbb{R}^n$  be a bounded open set of Euclidean space, whose boundary  $\partial G$  is algebraic ( $\partial G$  is contained in the real zero set of finitely many polynomials), and let  $\mu_f = f \mathbb{1}_G dx$  be a measure supported on  $G$ , with a so-called *holonomic weight*  $f$  against Lebesgue volume measure  $dx$  on  $\mathbb{R}^n$ . This means that it satisfies a holonomic system of linear partial or ordinary differential equations with polynomial coefficients (as a generalized function if needed, see Def. 3). Consider also the power moments of  $\mu_f$ :

$$m_\alpha = \int_G x^\alpha d\mu_f(x), \quad \alpha \in \mathbb{N}^n. \quad (1)$$

For instance, the weight  $f(x) = \exp(p(x))$ , with  $p \in \mathbb{R}[x]_d$  is holonomic i.e., it satisfies:

$$\begin{cases} \frac{\partial f}{\partial x_i} - \frac{\partial p}{\partial x_i} f = 0, & i = 0, \dots, n. \end{cases} \quad (2)$$

In [14], the following property is proved, for such an *exponential-polynomial* weight: knowing a priori the coefficients of  $p$ , its degree  $s$  and the degree  $d$  of the variety containing  $\partial G$ , a threshold  $N$  is identified (which depends only on  $d$  and  $s$ ), such that the moments  $m_\alpha$  up to degree  $N$  (i.e.  $\alpha \in \mathbb{N}_N^n$ ) determine

in a constructive and robust manner the coefficients of a polynomial vanishing on  $\partial G$ .

A natural question is whether this result can be generalized, as mentioned in [14]: *the analogy to the well understood moment rigidity of the Gaussian distribution is striking, although the constructive aspects of this finite determinateness remain too theoretical in general.* Motivated by this remark, in this article we revisit and extend this study to related problems, by exploiting holonomicity. In this framework, a first generalization of [14] is to recover the coefficients of both  $g$  and  $p$  in the *exponential-polynomial* case:

**Problem 1** (Exp-Poly Inverse Problem). *Let a measure  $\mu_f = f\mathbb{1}_G dx$ , supported on a compact semi-algebraic set  $G$ , whose algebraic boundary is included in the zero set of a polynomial  $g \in \mathbb{R}[x]_d$ . Let  $f = \exp(p)$ , with  $p \in \mathbb{R}[x]_s$ . Given  $s, d$ , and a finite number of moments  $m_\alpha, |\alpha| \leq N$ , recover the coefficients of both  $g$  and  $p$ .*

More general, the inverse problem for holonomic weights is:

**Problem 2** (General Inverse Problem). *Let a measure  $\mu_f = f\mathbb{1}_G dx$ , supported on a compact semi-algebraic set  $G$ , with holonomic  $f$ . Given a finite number of moments  $m_\alpha, |\alpha| \leq N$ , recover a polynomial  $g \in \mathbb{R}[x]$  vanishing on the algebraic boundary of  $G$  and the coefficients of a holonomic system satisfied by  $f$ .*

Finally, we note the closely related direct problem:

**Problem 3** (General Direct Problem). *Let a measure  $\mu_f = f\mathbb{1}_G dx$ , supported on a compact semi-algebraic set, with given holonomic  $f$ . Find a holonomic system of recurrences for the sequence of moments  $(m_\alpha)$ .*

**Contributions:** We address the above problems in the framework of holonomic distributions, employing well-known algorithmic properties of non-commutative polynomial representation of linear differential operators (see Section 2), as well as a generalized Stokes formula [14]. Firstly, this allows us to solve Problem 1 in Section 4.1: we prove that this reconstruction problem boils down to solving a linear system of  $3d + s - 1$  equations, involving moments up to degree  $|\alpha| \leq 4d + 2(s - 1)$ .

Secondly, as a by-product, an alternate method to creative telescoping is proposed for computing linear recurrences for the moments in Section 3. The advantage is that when the coefficients of  $g$  are given as parameters, the obtained recurrences stay linear with respect to them. However, there is no guarantee that this method provides a holonomic ideal. We could only prove

that it solves Problem 3 (i.e. it provides a holonomic ideal) in the restricted case of exponential-polynomial density and  $g$  nonsingular in  $\mathbb{C}^n$ .

Finally, Problem 2 is solved in Section 4.2: we prove that a holonomic system for  $f$  can be found by solving a finite system of linear equations, but their number cannot be a priori bounded. Once the density is known, the support is reconstructed as solution of a similar linear system, but in this case we provide an explicit uniform bound on the number of required moments.

## 1.1 Related works

**Moment problem:** Concerning the moment problem, let  $\mu$  be a Borel measure on  $\mathbb{R}^n$  with all its moments finite. When  $\mu$  is atomic with finitely many atoms (i.e., when  $\mu = \sum_{k=1}^d \gamma_k \delta_{\xi_k}$ , where  $\delta_{\xi_k}$  is the Dirac measure, for some  $(\xi_k) \subset \mathbb{R}^n$  and some positive weight  $(\gamma_k)$ ), a first classical problem is to retrieve the atoms and the weights of  $\mu$  from some finite truncation of its moment vector  $(m_\alpha)_{\alpha \in \mathbb{R}^n}$ . In [15] a thorough overview of algebraic methods for this problem is given. An important idea consists in computing a sparse polynomial-exponential representation of a multivariate series from its truncated Taylor series, whose coefficients correspond to moments. For instance, for pairwise distinct  $\xi_1, \dots, \xi_d$ , the moment generating series is:

$$\sigma_{\delta_\xi}(y) = \sum m_\alpha \frac{y^\alpha}{\alpha!} = \sum_{k=1}^d \gamma_k \exp(\xi_k y).$$

Such generating functions are also the solutions of systems of partial differential equations with constant coefficients. Hence, the sparse representation of the polynomial-exponential (also known as Prony method) is related to the inverse system of the isolated points of the characteristic variety of this system. Methods to obtain such representation are given in [15]. Also, *flat extension* criteria, like for instance [12, Theorem 3.7], provide purely algebraic methods to reconstruct both the number of atoms, their values and weights function of the rank of the moment matrix.

All in all, moments of atomic measures satisfy multi-index linear recurrences with constant coefficients [15], which provide another incentive to consider the more general holonomic case. In this sense, these recurrences can be computed by creative telescoping.

**Creative telescoping:** These methods perform integration of functions (with free parameters), in the framework of non-commutative polynomial representation of linear differential operators (see [3, 10, 2] and references therein). In particular, the direct Problem 3 can be solved for instance by the algorithms of Oaku [17]. Based on the D-module theory (see also [5, 19]), one

computes a holonomic system for the definite integral of a holonomic function with parameters over a domain defined by polynomial inequalities. In the algorithms, holonomic distributions are involved, so, a subtle distinction has to be made between the ideal of operators with polynomial coefficients, which correspond to holonomicity, and those with rational coefficients which correspond to so-called D-finiteness. We will come back to this in Section 2.

Also, the Lagrange Identity [8] (see also eq. (12) and Prop. 2), related to integration by parts, will play an important role in our approach. In the one variable case, for a linear differential operator with polynomial coefficients,  $L = c_r \partial_x^r + \dots + c_0$ , its adjoint is defined as  $L^* = (-1)^r \partial_x^r c_r + \dots + c_0$  and the following holds:

$$\varphi L(f) - L^*(\varphi)f = \partial_x(\mathcal{L}_L(f, \varphi)), \quad (3)$$

for any function  $\varphi$  and  $f$ , with an explicit  $\mathcal{L}_L$ .

**Inverse problem in the univariate case:** In [1], the inverse Problem 2 is solved in the univariate case, for piecewise D-finite densities. Specifically,  $\mu_f = \sum_{i=1}^{d-1} \mathbb{1}_{[\xi_i, \xi_{i+1}]} f_i dx$ , for a set of  $d$  unknown points,  $a = \xi_1 < \dots < \xi_d = b$ , with  $[a, b] \subset \mathbb{R}$ , and unknown smooth D-finite functions  $f_i$ . An important observation [1, Thm 2.12] is that the associated distribution  $\sum_{i=1}^{d-1} \mathbb{1}_{[\xi_i, \xi_{i+1}]} f_i$  is annihilated by some holonomic operator  $\hat{L} = g(x)^r L$ , where  $g(x) = \prod_{i=1}^d (x - \xi_i)$  and the operator  $L$  of order  $r$  satisfies  $Lf_i = 0$ .

**Remark 1.** As noted in [1], for general holonomic operators  $L$  with  $r > 1$ , the number  $N$  of required moments, in order to correctly recover the parameters, might depend also on specific coefficients of  $L$ . An example is the  $n$ th Legendre polynomial, whose first  $n$  moments (taken over  $[-1, 1]$ ) vanish, while  $L_n = \partial_x((1 - x^2)\partial_x) + n(n + 1)$ , hence the reconstruction of  $\mu_f$  depends also on the parameter  $n$ , which enters the definition of  $L_n$ . On the contrary, for exponential-polynomial case, we show that  $N$  depends only on the degrees of the polynomials involved.

As discussed above, in the univariate case, the above problems are well tackled in literature, so this article deals with the multivariate case. However, to illustrate the basic ideas, we give two elementary univariate examples of our approach, omitting the technical proofs.

## 1.2 Introductory examples

**Example 1** (Direct problem for erf-like function). *We are interested in computing a recurrence for the moments  $m_i = \int_{-1}^1 x^i e^{-x^2} dx$ . The idea is to include  $\mathbb{1}_{[-1,1]}$  in the integral, and consider the distribution  $u$  corresponding to  $\mathbb{1}_{[-1,1]}(x)e^{-x^2}$ . Although not differentiable as a function,  $u$  satisfies (see Sec. 2.2 for details):*

$$(1 - x^2)(\partial_x + 2x)u = 0.$$

*Integrating for the test function  $x^i$ , using (3) and noticing that its right hand side vanishes after integration, one has:*

$$\int_{-1}^1 e^{-x^2} (\partial_x + 2x)^* ((1 - x^2)x^i) = 0,$$

*which directly provides the recurrence*

$$im_{i-1} - (i + 4)m_{i+1} + 2m_{i+3} = 0.$$

*The extension of this method to the multivariate case is given in Sec. 3.*

**Example 2** (Univariate support and density reconstruction). *Consider the problem of reconstructing the parameters  $\xi_1, \xi_2$  and  $p_2, p_1, p_0$  such that, the first  $N$  moments  $\{m_i, 0 \leq i \leq N\}$  are known:*

$$m_i = \int_{\xi_1}^{\xi_2} x^i e^{p_2 x^2 + p_1 x + p_0} dx. \quad (4)$$

*Like in the previous example,  $u = \mathbb{1}_{[\xi_1, \xi_2]} e^{p_2 x^2 + p_1 x + p_0}$  satisfies:*

$$(x - \xi_1)(x - \xi_2)(\partial_x - 2p_2 x - p_1)u = 0.$$

*Denote by  $\hat{L} := g(x)\partial_x + h(x)$  the operator to be reconstructed such that  $\hat{L}f = 0$ , with  $g(x) = x^2 + g_1 x + g_0$  and  $h(x) = \sum_{i=0}^3 h_i x^i$ . Integrating and using Lagrange identity, one has:*

$$\int_{-\infty}^{\infty} (g(x)\partial_x - h(x))(x^i) u dx = 0. \quad (5)$$

This gives for each  $i \geq 0$ :

$$im_{i+1} + ig_1m_i + ig_0m_{i-1} - h_3m_{i+3} - h_2m_{i+2} - h_1m_{i+1} - h_0m_i = 0. \quad (6)$$

Hence, the coefficients of  $g$  and  $h$  are solution of the above infinite linear system. If  $g$  is recovered,  $p$  (except for  $p_0$  coefficient) could also be recovered from the division  $h/g$ . Finally the constant coefficient  $p_0$  could also be recovered from the equation (4), with  $i = 0$ .

The main question is whether a truncated system (6), which considers only moments up to degree  $N$ , can provide the correct solution for  $g$  and  $h$ . We will address this in Section 4. Specifically, in Thm. 1 we prove a sufficient bound for the case of an  $n$ -variable exponential-polynomial density, together with Algorithm 2 which reconstructs the coefficients. It needs in our case the first  $N = 10$  moments.

## 2 Holonomicity and distributions

For completeness, we start by providing a short reminder on *D-finiteness* versus *holonomicity*. Unlike more classical settings like analytic functions, the distinction between these two very related notions is essential when considering distributions. We refer to [3, 10, 17] for a more comprehensive presentation.

### 2.1 Differential operators and holonomicity

Consider the following rings of linear partial differential operators:

(i) The ring of differential operators with polynomial coefficients (the  $n$ -th Weyl algebra)  $\mathfrak{D}_n := \mathbb{K}[x_1, \dots, x_n] \langle \partial_{x_1}, \dots, \partial_{x_n} \rangle$ , generated by  $\{x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}\}$  and quotiented by the relations:

$$\partial_{x_i}x_j = \begin{cases} x_i\partial_{x_i} + 1, & i = j, \\ x_j\partial_{x_i}, & i \neq j, \end{cases} \quad x_ix_j = x_jx_i, \quad \partial_{x_i}\partial_{x_j} = \partial_{x_j}\partial_{x_i}.$$

We have that  $\{x^\beta \partial_x^\alpha, \alpha, \beta \in \mathbb{N}^n\}$  is a basis of  $\mathfrak{D}_n$  as a  $\mathbb{K}$ -vector space. If  $L = \sum_{\alpha, \beta} c_{\alpha, \beta} x^\beta \partial_x^\alpha$ , its *order* is the largest value of  $|\alpha|$  such that there exists  $\beta$  with  $c_{\alpha, \beta} \neq 0$ .

(ii) The ring of differential operators with rational fraction coefficients (*Ore Algebra*)  $\mathfrak{D}_n^* := \mathbb{K}(x_1, \dots, x_n) \langle \partial_1, \dots, \partial_n \rangle$ , where the commutation rules of  $\mathfrak{D}_n$  are extended by

$$\partial_{x_i}q(x) = q(x)\partial_{x_i} + \frac{\partial q(x)}{\partial x_i}, \quad q(x) \in \mathbb{K}(x_1, \dots, x_n).$$



Differential operators in  $\mathfrak{D}_n$  naturally act on smooth functions via  $\partial_{x_i} f = f_{x_i} := \frac{\partial f}{\partial x_i}$ . The *annihilator*  $\mathfrak{Ann}(f)$  is a left ideal of  $\mathfrak{D}_n$ :

$$\mathfrak{Ann}(f) := \{L \in \mathfrak{D}_n \mid Lf = 0\}.$$

One can also see  $\mathfrak{Ann}(f)$  as a left ideal of  $\mathfrak{D}_n^*$ , and the quotient  $\mathfrak{D}_n^*/\mathfrak{Ann}(f)$  as a  $\mathbb{C}(x_1, \dots, x_n)$ -vector space. A smooth function  $f$  is called *D-finite* if  $\mathfrak{D}_n^*/\mathfrak{Ann}(f)$  has finite dimension. Equivalently, its iterated derivatives  $\{\partial_x^\alpha f, \alpha \in \mathbb{N}^n\}$  form a finite-dimensional vector space over rational fractions.

In other cases, when  $f$  is a “generalized function”, for instance a distribution,  $\mathfrak{Ann}(f)$  can only be seen as a left ideal of  $\mathfrak{D}_n$  and  $\mathfrak{D}_n/\mathfrak{Ann}(f)$  as a  $\mathbb{C}$ -vector space. For example, the univariate Dirac distribution, defined by  $\langle \delta, f \rangle = f(0)$ , is annihilated (as a distribution) by  $x$ , since  $\langle x\delta, f \rangle = \langle \delta, xf \rangle = 0$ . However, a left ideal of  $\mathfrak{D}_1^*$  containing  $x$  is necessarily  $\mathfrak{D}_1^*$ , but 1 annihilating  $\delta$  would imply  $\delta = 0$ . In that setting, the relevant notion is *holonomicity*.

**Definition 1.** Let  $\mathfrak{I}$  be a left ideal of  $\mathfrak{D}_n$ . For  $L \in \mathfrak{D}_n$ , let  $[L]_{\mathfrak{I}}$  denote the class of  $L$  in the quotient  $\mathfrak{D}_n/\mathfrak{I}$ . For  $s \geq 0$ , define

$$\left(\mathfrak{D}_n/\mathfrak{I}\right)_s = \text{Span}_{\mathbb{K}} \{[x^\beta \partial_x^\alpha]_{\mathfrak{I}}, |\alpha| + |\beta| \leq s\}.$$

Then there exists a polynomial  $b(s) \in \mathbb{K}[s]$  such that  $\dim_{\mathbb{K}}(\mathfrak{D}_n/\mathfrak{I})_s = b(s)$  for  $s$  large enough. The degree of  $b(s)$  is called the *Bernstein dimension* of  $\mathfrak{D}_n/\mathfrak{I}$ . The left ideal  $\mathfrak{I}$  is called *holonomic* if the Bernstein dimension of  $\mathfrak{D}_n/\mathfrak{I}$  is equal to  $n$ .

The well-known notion of Gröbner bases was generalized to this non-commutative setting (see for example [5, 3, 10] and references therein). This is the building block of efficient closure operations for D-finite [3, 10] or holonomic [5, 19, 17] objects, thus allowing for their algorithmic treatment.

Similarly,  $\mathfrak{R}_n := \mathbb{K}[\alpha_1, \dots, \alpha_n] \langle S_{\alpha_1}, \dots, S_{\alpha_n} \rangle$  is the set of difference operators with polynomial coefficients in  $\alpha$ , acting on sequences  $u = (u(\gamma_1, \dots, \gamma_n))_{\gamma \in \mathbb{N}^n}$  via

$$\begin{aligned} (\alpha_i u)(\gamma_1, \dots, \gamma_n) &= \gamma_i u(\gamma_1, \dots, \gamma_n), & \text{and} \\ (S_{\alpha_i} u)(\gamma_1, \dots, \gamma_n) &= u(\gamma_1, \dots, \gamma_i + 1, \dots, \gamma_n), & \gamma \in \mathbb{N}^n. \end{aligned}$$

The annihilator  $\mathfrak{Ann}(u) = \{R \in \mathfrak{R}_n \mid R u = 0\}$  is the set of recurrence relations satisfied by  $u$ , which is holonomic when its generating series is holonomic [3].

## 2.2 Holonomic distributions

Introduced by Schwartz [18], *distributions* generalize functions and measures. A minimal introduction to this topic is provided below.

**Definition 2** (Test functions and distributions). *Let  $\mathcal{E} = \mathcal{C}^\infty(\mathbb{R}^n)$  be the set of smooth functions over  $\mathbb{R}^n$ , equipped with the compact-open topology:  $\varphi_k \rightarrow \varphi$  in  $\mathcal{E}$  if  $\partial_x^\alpha \varphi_k$  converges uniformly to  $\partial_x^\alpha \varphi$  over every compact set, for each  $\alpha \in \mathbb{N}^n$ .*

*Its topological dual  $\mathcal{E}'$  is the set of compactly supported distributions (or simply distributions in this article) i.e. linear forms  $T : \mathcal{E} \rightarrow \mathbb{R}$  such that:*

- *There exists a minimal compact set  $K \subseteq \mathbb{R}^n$  (the support of  $T$ ) such that  $\langle T, \varphi \rangle = 0$  whenever  $\varphi$  vanishes over  $K$ .*
- *$\langle T, \varphi_k \rangle \rightarrow 0$  whenever  $\varphi_k \rightarrow 0$  in  $\mathcal{E}$ .*

*$\mathcal{E}'$  has a canonical  $\mathfrak{D}_n$ -module structure:*

$$\langle LT, \varphi \rangle := \langle T, L^* \varphi \rangle, \quad L \in \mathfrak{D}_n, T \in \mathcal{E}', \varphi \in \mathcal{E}, \quad (7)$$

where the adjoint operator  $L^*$  is defined by

$$x_i^* = x_i, \quad \partial_{x_i}^* = -\partial_{x_i}, \quad \text{and} \quad (L_1 L_2)^* = L_2^* L_1^*.$$

**Definition 3** (Holonomic distribution). *A distribution  $T \in \mathcal{E}'$  is holonomic if its annihilator is a holonomic ideal of  $\mathfrak{D}_n$ :*

$$\mathfrak{Ann}(T) := \{L \in \mathfrak{D}_n \mid LT = 0 \text{ as a distribution}\}.$$

A measure supported on a set  $G$ , with density  $f \in \mathcal{E}$ , is represented by the distribution  $f \mathbb{1}_G$ , with  $\langle f \mathbb{1}_G, \varphi \rangle = \int_G \varphi(x) f(x) dx$ .

We make the following assumption on  $G \subseteq \mathbb{R}^n$ :

**Assumption 1.**  *$G$  is a compact  $n$ -dimensional semi-algebraic set. In particular, the following holds:*

(1)  *$G$  is an  $n$ -dimensional compact manifold such that its boundary can be decomposed as  $\partial G = Z \cup Z'$ , with  $Z$  a finite union of  $(n-1)$ -dimensional manifolds and  $Z'$  a negligible set w.r.t the  $(n-1)$ -dimensional Hausdorff measure.*

(2) *the ideal of polynomials vanishing over  $\partial G$  is radical and principal i.e., generated by a single square-free polynomial  $g$ . In particular, the family  $\{g, g_{x_1}, \dots, g_{x_n}\}$  is coprime, implying that the set of singular points  $\{x \mid g(x) = 0 \text{ and } \nabla g(x) = 0\}$  is negligible in  $\partial G$ .*

## 2.3 Moments of a distribution

**Definition 4** (Moments of a compactly supported distribution). *The moments of a distribution  $T \in \mathcal{E}'$  are:*

$$m_\alpha(T) := \langle T, x^\alpha \rangle, \quad \alpha \in \mathbb{N}^n. \quad (8)$$

Note that if  $T = f\mathbb{1}_G$  with  $G$  compact and  $f \in \mathcal{E}$ , then  $m_\alpha(f\mathbb{1}_G)$  coincides with the moments defined in eq. (1).

A convenient way to deal with moments of a distribution is the *Fourier transform* (also called *characteristic function*).

**Definition 5.** *The Fourier transform of a distribution  $T \in \mathcal{E}'$  is the analytic function  $\mathcal{F}\{T\}$  of  $z = (z_1, \dots, z_n) \in \mathbb{R}^n$  defined by:*

$$\mathcal{F}\{T\}(z) = \sum_{\alpha \in \mathbb{N}^n} m_\alpha(T) \frac{(-\mathbf{i}z)^\alpha}{\alpha!} = \langle T, e^{-\mathbf{i}x \cdot z} \rangle, \quad z \in \mathbb{R}^n.$$

**Proposition 1.** *Let  $T \in \mathcal{E}'$  and  $L = \sum_{\beta} q_\beta(x) \partial_x^\beta \in \mathfrak{D}_n$ .*

(i) *The Fourier transform of  $LT$  is related to that of  $T$  by*

$$\mathcal{F}\{LT\} = L^{\mathcal{F}} \mathcal{F}\{T\}, \quad \text{with} \\ L^{\mathcal{F}} = L \left[ \begin{array}{c} x_i \mapsto \mathbf{i} \partial_{z_i} \\ \partial_{x_i} \mapsto \mathbf{i} z_i \end{array} \right] = \sum_{\beta} q_\beta(\mathbf{i} \partial_z) (\mathbf{i} z)^\beta. \quad (9)$$

(ii) *The moments of  $LT$  are related to those of  $T$  by*

$$(m_\alpha(LT)) = L^{\mathcal{M}}(m_\alpha(T)), \quad \text{with} \\ L^{\mathcal{M}} = L \left[ \begin{array}{c} x_i \mapsto S_{\alpha_i} \\ \partial_{x_i} \mapsto -\alpha_i S_{\alpha_i}^{-1} \end{array} \right] = \sum_{\beta} (-1)^{|\beta|} q_\beta(S_\alpha) \left( \prod_{i=1}^n (\alpha_i S_{\alpha_i}^{-1})^{\beta_i} \right), \quad (10)$$

*Proof.* Similar to [15, Sec. 5.1.] (see A.1 for completeness).  $\square$

**Proposition 2.** *Let  $T \in \mathcal{E}'$ . An operator  $L \in \mathfrak{D}_n$  satisfies*

$$\langle T, L^* x^\alpha \rangle = 0, \quad \text{for all } \alpha \in \mathbb{N}^n, \quad (11)$$

*if and only if  $L \in \mathfrak{Ann}(T)$ .*

*Proof.* By the injectivity of the Fourier transform on compactly supported distributions [18].  $\square$

### 3 Direct problem for moments

As mentioned in the introduction, the direct Problem 3 can be solved using an algorithm presented in [17]. However, one may ask whether the simple roadmap of Example 1 can be generalized to the multivariate case and provide a more efficient method. For that, firstly, Lagrange identity in the multivariate setting is needed:

**Lemma 1** (Lagrange identity). *For  $f, g \in \mathcal{E}$  and  $L \in \mathfrak{D}_n$  of order  $r$ , there exists a vector field  $\mathcal{L}_L(f, g) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , called bilinear concomitant, depending on  $L$  and linear in  $f$  and  $g$ , such that:*

$$(Lf)g - f(L^*g) = \nabla \cdot \mathcal{L}_L(f, g). \quad (12)$$

Each component of  $\mathcal{L}_L(f, g)$  can be written

$$\mathcal{L}_{L,i}(f, g) = \sum_{|\alpha|+|\beta| \leq r-1} c_{L,i,\alpha,\beta}(x) (\partial_x^\alpha f) (\partial_x^\beta g), \quad i \in [1 \dots n]. \quad (13)$$

with coefficients  $c_{L,i,\alpha,\beta}(x) \in \mathbb{K}[x]$  depending on  $L$ .

Secondly, the action of differential operators on compactly supported distributions of the form  $f\mathbb{1}_G$  is provided:

**Proposition 3.** *Let  $G$  as in Assumption 1,  $f \in \mathcal{E}$  and  $L \in \mathfrak{D}_n$ . Then the distribution  $L(f\mathbb{1}_G)$  admits the following expression:*

$$\langle L(f\mathbb{1}_G), \varphi \rangle = \int_G \varphi(Lf) dx - \int_{\partial G} \mathcal{L}_L(f, \varphi) \cdot \mathbf{n} dS, \quad (14)$$

where  $\mathbf{n}$  and  $dS$  respectively denote the normal vector and the  $(n - 1)$ -dimensional Hausdorff measure on  $\partial G$ .

*Proof.* Integrating Lagrange's identity (12) with  $g = \varphi$  and using the divergence theorem, we have:

$$\int_G \varphi(Lf) dx - \int_G (L^*\varphi)f dx = \int_{\partial G} \mathcal{L}_L(f, \varphi) \cdot \mathbf{n} dS. \quad (15)$$

Following [14], the divergence theorem is a consequence of Stokes' theorem when  $\partial G$  is smooth, or of a generalization by Whitney [20, Theorem 14A] when  $G$  satisfies Assumption 1.(1).  $\square$

Finally, the following proposition provides differential equations for measures supported on semi-algebraic sets.

**Proposition 4.** *Let  $G$  and  $g$  as in Assumption 1,  $f \in \mathcal{E}$ ,  $L \in \mathfrak{Ann}(f)$  of order  $r$ . Then  $g^r L \in \mathfrak{Ann}(f \mathbb{1}_G)$ .*

*Proof.* Using Prop. 3, one needs to prove that  $\int_G g^r \varphi(Lf) dx$  and  $\int_{\partial G} \mathcal{L}_L(f, g^r \varphi) \cdot \mathbf{n} \, dS$  are zero. The first one is trivial since  $L \in \mathfrak{Ann}(f)$ . For the second,  $\mathcal{L}_L(f, g^r \varphi)$  involves derivatives  $\partial_x^\alpha(g^r \varphi)$  with  $|\alpha| < r$  (Lemma 1), so it vanishes over  $\partial G$ .  $\square$

Hence, Proposition 4 gives an easy way to construct operators in  $\mathfrak{Ann}(f \mathbb{1}_G)$  from operators in  $\mathfrak{Ann}(f)$ . Indeed, given a Gröbner basis  $\{L_1, \dots, L_k\}$  of  $\mathfrak{Ann}(f)$ , and  $g \in \mathbb{R}[x]$  vanishing over  $\partial G$ , each operator  $g^{r_i} L_i$  (with  $r_i$  the order of  $L_i$ ) annihilates  $f \mathbb{1}_G$  as a distribution. Therefore, each operator  $R_i := (g^{r_i} L_i)^\mathcal{M}$  gives a valid recurrence for the sequence of moments  $(m_\alpha)$ .

However, from the fact that  $f$  is holonomic one can not directly guarantee that the ideal generated by  $\{g^{r_1} L_1, \dots, g^{r_k} L_k\}$  is holonomic. Similarly, we are not able to prove (or refute) that  $\{R_1, \dots, R_k\}$  is holonomic in general. Nevertheless, one can apply a Gröbner basis algorithm to it, which will possibly terminate and return such a basis. This heuristic is proposed in Algorithm 1. We prove that this algorithm terminates, in the particular case

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**Algorithm 1** RECURRENCESMOMENTS( $n, g, \{L_1, \dots, L_k\}$  )

---

**Input:** Gröbner basis  $\{L_1, \dots, L_k\}$  for  $\mathfrak{Ann}(f)$ ,  $g$ .

**Output:** Gröbner basis for  $\mathfrak{Ann}(m_\alpha)$ .

---

- 1:  $R_i \leftarrow (g^{r_i} L_i)^\mathcal{M}$ , as in (10), with  $r_i$  the order of  $L_i$ , for  $i \in [1..k]$
  - 2: **return** GröbnerBasis( $\{R_1, \dots, R_k\}, \mathfrak{R}_n$ )
- 

of an exponential-polynomial density (including the Lebesgue measure), and a smooth boundary, extending [17, Prop. 4].

**Proposition 5.** *Let  $f(x) = e^{p(x)}$  with  $p \in \mathbb{R}[x]_s$ , and  $g \in \mathbb{R}[x]_d$  vanishing over  $\partial G$ . Suppose moreover that  $g$  is nonsingular in  $\mathbb{C}^n$ , that is, there exists no  $x \in \mathbb{C}^n$  such that  $g(x) = 0$  and  $\nabla g(x) = 0$ .*

(i) *The operators*

$$L_i = g(\partial_{x_i} - p_{x_i}), \quad i \in [1..n],$$

*are generators of an holonomic ideal  $\mathfrak{I}$  contained in  $\mathfrak{Ann}(f \mathbb{1}_G)$ .*

(ii) *The operators  $L_i^\mathcal{F}$  ( $i \in [1..n]$ ) span a holonomic ideal  $\mathfrak{I}^\mathcal{F}$  contained in  $\mathfrak{Ann}(\mathcal{F}\{f \mathbb{1}_G\})$ .*

*Proof.* For (i), first note that the operators  $L_i$  also generate

$$\begin{aligned} L_{ij} &:= (\partial_{x_j} - p_{x_j})L_i - (\partial_{x_i} - p_{x_i})L_j \\ &= g_{x_j}(\partial_{x_i} - p_{x_i}) - g_{x_i}(\partial_{x_j} - p_{x_j}), \quad 1 \leq i < j \leq n. \end{aligned} \quad (16)$$

Holonomicity is proved via the *characteristic variety*, as for instance in [17]. For  $L = \sum_{|\alpha| \leq r} q_\alpha(x) \partial_x^\alpha$  of order  $r$ , define its principal symbol  $\sigma(L)(x, \xi) = \sum_{|\alpha|=r} q_\alpha(x) \xi^\alpha$  for  $(x, \xi) \in \mathbb{C}^{2n}$ . Then for a left ideal  $\mathfrak{J}$ ,  $\text{Char}(\mathfrak{J}) = \{(x, \xi) \in \mathbb{C}^{2n} \mid \sigma(L)(x, \xi) = 0 \ \forall L \in \mathfrak{J} \setminus \{0\}\}$ . With these notations,  $\mathfrak{J}$  is holonomic if and only if all the components of  $\text{Char}(\mathfrak{J})$  are of dimension at most  $n$ . In our case,

$$\sigma(L_i)(x, \xi) = g(x) \xi_i, \quad \text{and} \quad \sigma(L_{ij})(x, \xi) = g_{x_j}(x) \xi_i - g_{x_i}(x) \xi_j.$$

Hence, if  $(x, \xi) \in \text{Char}(\mathfrak{J})$ , then either  $g(x) \neq 0$ , implying  $\xi = 0$ , or  $g(x) = 0$ . In the latter case,  $\nabla g(x) \neq 0$  (since  $g$  is nonsingular) and hence there exists  $\lambda \in \mathbb{C}$  s.t.  $\xi = \lambda \nabla g(x)$ . In both cases, the corresponding components of  $\text{Char}(\mathfrak{J})$  have dimension  $n$ .

For (ii), since the Fourier transform maps  $x_i$  to  $\mathbf{i} \partial_{z_i}$  and  $\partial_{x_i}$  to  $\mathbf{i} z_i$ , it is clear that  $\mathfrak{J}$  is holonomic if and only if  $\mathfrak{J}^{\mathcal{F}}$  is holonomic.  $\square$

Interesting enough, for the examples we tried for an exponential-polynomial density, Algorithm 1 always terminated, even when the boundary was not smooth (see Example 3). Also, it was faster than "classical" creative telescoping, which firstly constructs a Gröbner basis for  $f \mathbb{1}_G$  and then applies Takayama algorithm [17]. Further investigation is needed to provide a comparison in this case.

However, having a Gröbner basis is not mandatory for the reconstruction problem addressed in the next section. The recurrences obtained as above turn out to be sufficient and constitute the basic brick of our reconstruction method.

## 4 Reconstruction methods

Given some moments  $m_\alpha(f \mathbb{1}_G)$  associated to a measure of unknown D-finite density  $f \in \mathcal{E}$  and unknown compact algebraic support  $G$ , our goal is to reconstruct a polynomial  $\tilde{g}$  vanishing on the boundary  $\partial G$  of  $G$  and operators  $\tilde{L} \in \mathfrak{Ann}(f)$ .

**The general approach** is the following:

- Take an *ansatz*  $L' = \sum_{\beta \in A} q_\beta(x) \partial_x^\beta$ , for a specified finite set  $A \subset \mathbb{N}^n$  and polynomials  $q_\beta(x)$  with specified degrees  $d_\beta$ .
- Let  $R = L'^{\#}$ . Solve a finite-dimensional linear system in the unknown coefficients of the polynomials  $q_\beta$ :

$$(Rm(f\mathbb{1}_G))_\alpha = 0, \quad |\alpha| \leq N. \quad (17)$$

This requires the knowledge of moments  $m_\alpha(f\mathbb{1}_G)$  with  $|\alpha| \leq N + \max_{\beta \in A} \{d_\beta - |\beta|\}$  (see eq. (10)).

- From the solution  $L'$  of (17), extract a polynomial  $g$  vanishing on  $\partial G$  and an operator  $\tilde{L} \in \mathfrak{Ann}(f)$ .

Note that the solution of (17) corresponds to a truncation of the infinite system (11), since  $\langle f\mathbb{1}_G, L'^* x^\alpha \rangle = 0$ , for  $|\alpha| \leq N$ . Hence one is interested in obtaining bounds  $\hat{N}$  on  $N$ , such that any solution of (17) is also solution of (11). Such an *a priori* uniform bound depending only on  $A$  and  $d_\beta$  does not exist in general, cf. Remark 1.

Another issue is that  $L'$  may not be factorized as  $\tilde{g}(x)^r \tilde{L}$  with  $\tilde{g}$  vanishing on  $\partial G$  and  $\tilde{L}f = 0$ . See for instance the operator in (16).

In Section 4.1, we solve both issues when  $f$  is exponential-polynomial and give the associated algorithm. Then, in Section 4.2, we address the general holonomic case in two steps: firstly, for recovering the density, we prove that  $N$  is finite, but no *a priori* bound for it is known; secondly, once the density is known, a stronger result is proved for the support reconstruction, since an explicit uniform bound on the number of required moments is given.

In the algorithms proposed below, “exact computations” are assumed, that is, both the polynomial coefficients and the given moments  $m_\alpha$  lie in a computable finite extension of  $\mathbb{Q}$ . The practical case of approximately known numerical moments is briefly analyzed in Section 5.

## 4.1 Exponential-polynomial densities

Let  $f(x) = \exp(p(x))$  with  $\deg p = s$ , together with  $G$  and  $g$  as in Assumption 1,  $\deg g = d$ . Then  $f$  is annihilated by  $L_i = \partial_{x_i} - p_{x_i}$  for  $i \in [1..n]$ . Algorithm 2 follows the general approach above, with ansatz  $L'_i = h_0 \partial_{x_i} - h_i$  ( $i \in [1..n]$ ) for unknown polynomials  $h_0, \dots, h_n$  where  $\deg h_0 \leq d$  and  $\deg h_i \leq d + s - 1$  for  $i \in [1..n]$ . Theorem 1 establishes its correctness, with an explicit bound  $\hat{N}$ .

---

**Algorithm 2** RECONSTRUCTEXPOLY( $n, d, s, N, (m_\alpha)_{|\alpha| \leq N+d+s-1}$ )

---

**Input:**  $n \geq 2$ , degrees  $d, s \geq 0$ , moments  $m_\alpha$  for  $|\alpha| \leq N + d + s - 1$ .

**Output:**  $\tilde{g}, \tilde{p} \in \mathbb{K}[x]$  with  $\deg(\tilde{g}) \leq d$  and  $\deg(\tilde{p}) \leq s$ .

---

▷ Find  $L'_i \in \mathfrak{Ann}(f \mathbb{1}_G)$

1:  $h_0 \leftarrow \sum_{|\gamma| \leq d} h_{0\gamma} x^\gamma$  and  $h_i \leftarrow \sum_{|\gamma| \leq d+s-1} h_{i\gamma} x^\gamma$  for  $i \in [1..n]$ ,  
with symbolic coefficients  $h_{i\gamma}$

2:  $L'_i \leftarrow h_0 \partial_{x_i} - h_i$  for  $i \in [1..n]$

3: Find a nontrivial solution  $\{h_{i\gamma}\}$  of the linear system:

$$(L'_i m)_\alpha = 0, \quad i \in [1..n], \quad |\alpha| \leq N$$

▷ Reconstruct  $\tilde{g}$  and  $\tilde{p}$

4:  $\tilde{g} \leftarrow h_0$  and  $\tilde{p}_i \leftarrow h_i / \tilde{g}$  for  $i \in [1..n]$

5:  $\tilde{p} \leftarrow \sum_{i=1}^n \int_0^{x_i} \tilde{p}_i(0, \dots, 0, t_i, x_{i+1}, \dots, x_n) dt_i$

6: **return**  $(\tilde{g}, \tilde{p})$

---

**Theorem 1.** Let  $f(x) = \exp(p(x))$  with  $\deg p = s$ , and  $G, g$  with  $\deg g = d$  as in Assumption 1. If  $N \geq \hat{N} = 3d + s - 1$ , then RECONSTRUCTEXPOLY( $n, d, s, N, (m_\alpha)$ ) returns  $\tilde{g} = \lambda g$  with  $\lambda \in \mathbb{K}^*$ , and  $\tilde{p} = p - p(0)$ . This requires moments up to degree  $4d + 2(s - 1)$ .

Moreover, if  $g \geq 0$  over  $G$ ,  $\hat{N}$  can be only  $2d + s - 1$ , requiring moments up to degree  $3d + 2(s - 1)$ .

**Remark 2.** This method cannot reconstruct the constant coefficient of  $p$ , which is the scaling factor of the density. In case of a probability measure over  $\mathbb{R}^n$ , this coefficient is uniquely recovered by imposing  $\int_{\mathbb{R}^n} \exp(\tilde{p}(x)) dx = 1$ . Otherwise, one can compute  $p(0) = \log(m_0 / \int_G \exp(\tilde{p}(x)) dx)$ , for example.

*Proof.* First,  $\{h_0 \leftarrow g, h_i \leftarrow gp_{x_i}, i \in [1..n]\}$  is a solution of the linear system in line 3. Hence, one can always get a solution with  $h_0 \neq 0$ . Then  $\langle L'_i(f \mathbb{1}_G), \varphi \rangle = 0$  for all  $i \in [1..n]$  and  $\varphi \in \mathbb{K}[x]_N$ . Using Proposition 3, this expands to:

$$\int_G (h_0 p_{x_i} - h_i) \varphi f dx + \int_{\partial G} h_0 \varphi f \mathbf{e}_i \cdot \mathbf{n} dS = 0. \quad (18)$$

With  $\varphi = (h_0 p_{x_i} - h_i) g^2$  of degree at most  $d + (s - 1) + 2d \leq N$ , the second integral is zero since  $g$  vanishes over  $\partial G$ . Hence the first integral is zero too. Therefore, its integrand  $(h_0 p_{x_i} - h_i)^2 g^2$  is zero almost everywhere over  $G$ . Since  $G$  has nonempty interior and  $f > 0$ ,  $g \neq 0$ , this necessarily implies  $h_i = h_0 p_{x_i}$  for all  $i \in [1..n]$ .



Now, the first integral in (18) being always zero for all polynomial  $\varphi$  with  $\deg \varphi \leq N$ , so is the second. Noticing that  $\mathbf{e}_i \cdot \mathbf{n} = g_{x_i} / \|\nabla g\|$  when  $\nabla g \neq 0$ , and by taking  $\varphi = h_0 g_{x_i}$  of degree at most  $2d - 1 \leq N$ , we have

$$\int_{\partial G} (h_0 p_{x_i})^2 \frac{f}{\|\nabla g\|} dS = 0. \quad (19)$$

By summing this equality for  $i \in [1..n]$ , we get that  $h_0 \|\nabla g\|$  vanishes over  $\partial G$ . Since by Assumption 1.(2),  $\{x \in \partial G \mid \nabla g(x) = 0\}$  is negligible in  $\partial G$ , we have that  $h_0$  (of degree at most  $d$ ) vanishes over  $\partial G$ , whence  $h_0 = \lambda g$  since  $g$  is square-free.

Finally,  $\tilde{p} = p - p(0)$  is reconstructed from  $p_{x_i} = \tilde{p}_i$  in line 5.

For the case where  $g \geq 0$  over  $\partial G$ , the first step of the proof still holds with  $\varphi = (h_0 p_{x_i} - h_i)g$ , of degree  $2d + s - 1$ , in (18).  $\square$

## 4.2 Holonomic densities

For higher order holonomic operators, the proof of Thm. 1 cannot be generalized: the key argument for deducing a uniform bound  $\hat{N}$  was to write in (18),  $\int_G \varphi(L'f)dx$  as  $\int_G h\varphi f dx$ , with  $h \in \mathbb{K}[x]$ .

Instead, we proceed in two steps. Firstly in Section 4.2.1, a holonomic system for  $f$  is reconstructed, but it requires a finite number  $N$  of linear equations, which cannot be *a priori* bounded. Secondly, the support is reconstructed Section 4.2.2.

### 4.2.1 Reconstructing the density

Algorithm 3 produces a holonomic ideal  $\mathfrak{J} \subseteq \mathfrak{Ann}(f)$  spanned by a *rectangular system*  $\{L_1, \dots, L_n\}$ , that is  $L_i \in \mathfrak{Ann}(f) \cap \mathbb{K}[x]\langle \partial_{x_i} \rangle$  only involves derivatives w.r.t  $x_i$ . For that, it is sufficient to find operators annihilating  $f \mathbb{1}_G$ .

**Proposition 6.** *Let  $f$  analytic over  $G$  satisfying Assumption 1. Then  $\mathfrak{Ann}(f \mathbb{1}_G) \subseteq \mathfrak{Ann}(f)$ .*

*Proof.* Let  $L' \in \mathfrak{Ann}(f \mathbb{1}_G)$  of order  $r$ . Prop. 3 with  $\varphi = g^{2r}(L'f)$  gives:

$$\int_G g^{2r}(L'f)^2 dx = 0.$$

This implies that the analytic function  $g^r(L'f)$  vanishes over  $G$  of nonempty interior, hence is 0. Since  $g \neq 0$ ,  $L'f = 0$ .  $\square$

Theorem 2 guarantees that Algorithm 3 always returns an  $L \in \mathfrak{Ann}(f)$  for  $N$  large enough.

---

**Algorithm 3** RECONSTRUCTDENSITY( $n, i, r, s, N, (m_\alpha)_{|\alpha| \leq N+s}$ )

---

**Input:**  $n \geq 2, i \in [1 \dots n]$ , order  $r$ , maximum degree  $s$ , moments  $m_\alpha$  for  $|\alpha| \leq N+s$ .

**Output:**  $\tilde{L} = \sum_{j=0}^r \tilde{q}_j(x) \partial_{x_j}^j$  with  $\deg(\tilde{q}_j) \leq s$ .

---

▷ Find  $L' \in \mathfrak{Ann}(f \mathbb{1}_G) \cap \mathbb{K}[x] \langle \partial_{x_i} \rangle$

1:  $h_j \leftarrow \sum_{|\gamma| \leq s} h_{j\gamma} x^\gamma$  for  $j \in [0 \dots r]$  with symbolic coefficients  $h_{j\gamma}$

2:  $L' \leftarrow \sum_{j=0}^r h_j(x) \partial_{x_j}^j$

3: Find a nontrivial solution  $\{h_{j\gamma}\}$  of the linear system:

$$(L'^{\mathcal{M}} m)_\alpha = 0, \quad |\alpha| \leq N$$

▷ Extract minimal  $L \in \mathfrak{Ann}(f) \cap \mathbb{K}[x] \langle \partial_{x_i} \rangle$

4:  $\ell \leftarrow \text{GCD}(h_0, \dots, h_r)$  and  $\tilde{q}_j \leftarrow h_j / \ell$  for  $j \in [1 \dots n]$ .

5: **return**  $\tilde{L} = \sum_{j=0}^r \tilde{q}_j(x) \partial_{x_j}^j$

---

**Theorem 2.** Let  $i \in [1 \dots n]$ ,  $f$  analytic,  $G, g \in \mathbb{K}[x]_d$  satisfying Assumption 1, and  $L = \sum_{j=0}^r q_j(x) \partial_{x_j}^j \in \mathfrak{Ann}(f) \cap \mathbb{K}[x] \langle \partial_{x_i} \rangle$  of minimal order  $r$ , with  $q_r$  of minimal degree. Then, Algorithm RECONSTRUCTDENSITY( $n, i, r, s, N, (m_\alpha)$ ) returns  $\tilde{L} = \lambda L$  with  $\lambda \in \mathbb{K}^*$  for  $s \geq dr + \max\{\deg(q_j)\}$  and  $N$  large enough.

*Proof.* The linear system line 3 always has  $g^r L$  as solution, by Proposition 4. Now let  $K_N$  denote the kernel of this system, that is  $L' \in K_N$  iff  $\langle L'(f \mathbb{1}_G), x^\alpha \rangle = 0$  for all  $|\alpha| \leq N$ . The infinite inclusion chain of finite-dimensional linear subspaces  $\dots \subseteq K_N \subseteq K_{N+1} \subseteq \dots$  is necessarily stationary. So for  $N$  large enough,  $L' \in K_N$  implies  $\langle L'(f \mathbb{1}_G), x^\alpha \rangle = 0$  for all  $\alpha$  and hence  $L' \in \mathfrak{Ann}(f \mathbb{1}_G)$  by Proposition 2. Finally,  $L' \in \mathfrak{Ann}(f)$  by Proposition 6

The coefficients  $\{\tilde{q}_0, \dots, \tilde{q}_r\}$  of the returned operator  $\tilde{L} = \sum \tilde{q}_j \partial_{x_j}^j$  form a coprime family (line 4). This is also true for  $\{q_0, \dots, q_r\}$  by minimality of  $\deg(q_r)$ . By minimality of  $r$ , we have  $\tilde{q}_r L - q_r \tilde{L} = 0$ , that is  $\tilde{q}_r q_j = q_r \tilde{q}_j$  for all  $j$ . Since  $\mathbb{K}[x]$  has the unique factorization property, there exists  $\lambda \in \mathbb{K}$  s.t.  $\tilde{q}_r = \lambda q_r$ , yielding  $\tilde{L} = \lambda L$ .  $\square$

### 4.2.2 Reconstructing the support

From now on, we assume that a rectangular system  $\{L_1, \dots, L_n\}$  for the density  $f$  is known, and that  $L_i$  have same order  $r$ .<sup>1</sup> Let:

$$L_i = \sum_{j=0}^r q_{i,j}(x) \partial_{x_i}^j \in \mathfrak{Ann}(f) \cap \mathbb{K}[x] \langle \partial_{x_i} \rangle, \quad i \in [1..n].$$

The next assumption is crucial for support reconstruction. Roughly speaking, the differential system must not be singular over the Zariski closure of  $\partial G$ , except for a zero-measure set.

**Assumption 2.** *The pair  $\{g, q_{i,r}\}$  is coprime for each  $i \in [1..n]$ .*

Thm. 3 proves that Algorithm 4 is correct.

---

**Algorithm 4** RECONSTRUCTSUPPORT( $n, d, r, \{L_i\}_{i=1}^n, N, (m_\alpha)$ )

---

**Input:**  $n \geq 2$ , degree  $d$ , order  $r$ , rectangular system  $\{L_1, \dots, L_n\}$  of order  $r$ , moments  $m_\alpha$  for  $|\alpha| \leq N + dr + \max_{i,j} \{\deg(q_{i,j}) - j\}$ .

**Output:** polynomial  $\tilde{g}(x) \in \mathbb{K}[x]_d$  vanishing over  $\partial G$ .

---

1:  $h \leftarrow \sum_{|\gamma| \leq dr} h_\gamma x^\gamma$  with symbolic coefficients  $h_\gamma$

2: Find a nontrivial solution  $\{h_\gamma\}$  of the linear system:

$$\left( (hL_i)^{\mathcal{M}} m \right)_\alpha = 0, \quad |\alpha| \leq N, i \in [1..n]$$

3:  $\tilde{g} \leftarrow h / \text{GCD}(h, h_{x_1}, \dots, h_{x_n})$

4: **return**  $\tilde{g}$

---

**Theorem 3.** *Let analytic  $f$  annihilated by the order  $r$  rectangular system  $\{L_1, \dots, L_n\}$ , and  $G$  as in Assumption 1 with  $g \in \mathbb{K}[x]$  of degree  $d$ . Assume also Assumption 2. Then, for  $N \geq \hat{N} := (2r - 1)d + (d - 1)b + s$ , with  $b = r \bmod 2$  and  $s = \max\{q_{i,r}\}$ , RECONSTRUCTSUPPORT( $n, d, r, \{L_i\}, N, (m_\alpha)$ ) returns  $\tilde{g} = \lambda g$  with  $\lambda \in \mathbb{K}^*$ .*

*In particular, this proves that when the density is known, the support can be reconstructed using moments up to degree  $(3r - 1)d + (d - 1)b + s + \max_{i,j} \{\deg(q_{i,j}) - j\}$ .*

---

<sup>1</sup>Indeed, if  $L_i$  has order  $r_i < r$ , then it is replaced by  $\partial_{x_i}^{r-r_i} L_i$ , which has order  $r$  and the same leading polynomial coefficient.

*Proof.* First  $h = g^r$  satisfies the linear system line 2 since  $g^r L_i \in \mathfrak{Ann}(f\mathbb{1}_G)$  by Prop. 4. Let  $h$  be any nontrivial solution, then  $\langle hL_i(f\mathbb{1}_G), \varphi \rangle = 0$  for all  $i \in [1..n]$  and  $\varphi \in \mathbb{K}[x]_N$ . Using Prop. 3 combined with  $L_i f = 0$ , we get

$$\int_{\partial G} \mathcal{L}_{L_i}(f, h\varphi) \cdot \mathbf{n} \, dS = 0, \quad i \in [1..n], \varphi \in \mathbb{K}[x]_N.$$

Since  $L_i$  involves derivatives only in  $x_i$ , we have  $\mathcal{L}_{L_i}(f, h\varphi) = \mathcal{L}_{L_{i,i}}(f, h\varphi)\mathbf{e}_i$ , with the Lagrange bilinear concomitant [8]:

$$\begin{aligned} \mathcal{L}_{L_{i,i}}(f, h\varphi) &= f [q_{i,1}h\varphi - \partial_{x_i}(q_{i,2}h\varphi) + \cdots + (-1)^{r-1}\partial_{x_i}^{r-1}(q_{i,r}h\varphi)] \\ &\quad + \partial_{x_i}(f) [q_{i,2}h\varphi - \partial_{x_i}(q_{i,3}h\varphi) + \cdots + (-1)^{r-2}\partial_{x_i}^{r-2}(q_{i,r}h\varphi)] \\ &\quad + \cdots \\ &\quad + \partial_{x_i}^{r-1}(f) q_{i,r}h\varphi. \end{aligned} \tag{20}$$

We prove  $h = \lambda g^r$  for some  $\lambda \in \mathbb{K}^*$  by induction for  $k$  from 0 to  $r$ , showing  $h = g^k h_k$  with  $h_k \in \mathbb{R}[x]_{(r-k)d}$ . Of course this is true for  $k = 0$  with  $h_0 = h$ . Now suppose that  $h = g^k h_k$  for some  $k < r$ . Then let

$$\varphi = q_{i,r} h_k g^{r-1-k} g_{x_i}^b \in \mathbb{K}[x]_{(2r-2k-1)d+(d-1)b+s} \subseteq \mathbb{K}[x]_N,$$

Since  $h\varphi$  is a multiple of  $g^{r-1}$ , all the terms in (20) are multiples of  $g$  (hence they vanish over  $\partial G$ ), except for the derivative of order  $r - 1$ , which we can write as

$$\partial_{x_i}^{r-1}(q_{i,r}h\varphi) = (r-1)! g_{x_i}^{r-1+b} q_{i,r}^2 h_k^2 + \ell(x)g(x), \quad \ell(x) \in \mathbb{K}[x].$$

Therefore, integrating  $\mathcal{L}_{L_{i,i}}(f, h\varphi) \mathbf{e}_i \cdot \mathbf{n} \, dS$  over  $\partial G$  gives

$$\int_{\partial G} \left( g_{x_i}^{\frac{r+b}{2}} q_{i,r} h_k \right)^2 \frac{f}{\|\nabla g\|} \, dS = 0,$$

implying that the squared polynomial in the integrand vanishes over  $\partial G$ , hence is a multiple of  $g$ . But  $g$  and  $q_{i,r}$  are coprime by Assumption 2, so that  $g$  divides  $h_k g_{x_i}$ , for all  $i \in [1..n]$ . Finally, since  $\{g, g_{x_1}, \dots, g_{x_n}\}$  is a coprime family,  $g$  divides  $h_k$ , giving  $h_k = gh_{k+1}$ .

Now that  $h = \lambda g^r$ ,  $\text{GCD}(h, h_{x_1}, \dots, h_{x_n}) = g^{r-1}$  (again since  $\{g, g_{x_1}, \dots, g_{x_n}\}$  is coprime), so  $\tilde{g} = \lambda g$ .  $\square$

## 5 Examples and Conclusion

We now exemplify our methods in the two dimensional case, with respect to Lebesgue and restricted Gaussian measures<sup>2</sup>. The implementation uses

<sup>2</sup>The corresponding code will be available at <http://homepages.laas.fr/fbrehard/HolonomicMomentProblem>

OreAlgebra and OreGroebnerBasis routines from the HolonomicFunctions library [11]. The exactly computed moments  $m_{ij}$  (obtained from the recurrences given by Algorithm 1 together with closed-form initial conditions, when possible) are truncated to  $\tilde{m}_{ij}$ , s.t.  $\lfloor -\log_{10} \frac{m_{i,j} - \tilde{m}_{i,j}}{m_{i,j}} \rfloor = \varepsilon$  i.e.,  $\varepsilon$  represents the number of correct digits of  $\tilde{m}_{ij}$ .

In a second time, Algorithm 2 solves the inverse problem given the approximate  $\tilde{m}_{ij}$ . For numerically solving the resulting overdetermined systems of linear equations, we employ a Least Mean Squares method of Mathematica.

**Example 3** (Algebraic Support, Lebesgue measure). *Consider the moments  $m_{ij} = \int_G x^i y^j dx dy$ , with respect to the Lebesgue measure, with  $G$  depicted with the checkered pattern in Figure 1.*

(i) Direct problem: *Given  $g = (x^2 + y^2 - 1)(x^2 + y^2 - 9)(x^2 + (y - 2)^2 - 1)((x - 2)^2 + y^2 - 1)$ , which vanishes on  $\partial G$ , and  $\mathfrak{Ann}\{1\} = \{\partial_x, \partial_y\}$ , Algorithm 1 returns a Gröbner basis with 9 generators and with 36 monomials under the staircase:  $\{S_i^k S_j^l, k, l \in \mathbb{N}, k + l \leq 7\}$ .*

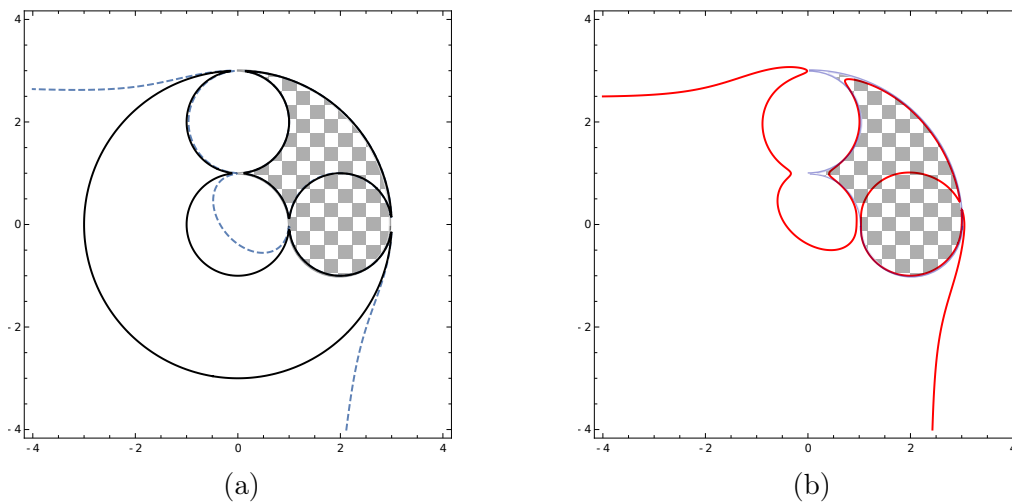


Figure 1: (a)  $G$  in checkered pattern, together with  $\partial G$  in black. For  $\varepsilon > 4$ : reconstructed and original boundary cannot be distinguished at this scale; in dashed-blue,  $\varepsilon = 4$ , while in red (b)  $\varepsilon = 2$ .

(ii) Inverse problem: *Suppose now given a finite number of numerically computed moments  $\tilde{m}_{ij}$  of the Lebesgue measure with unknown support  $G$ . The goal is to reconstruct  $g = \sum_{i+j \leq 8} g_{ij} x^i y^j$  which vanishes on  $\partial G$ . The results of Algorithm 2 called with parameters  $(2, 8, 0, 22, (\tilde{m}_{ij})_{|i+j| \leq 29})$  are depicted in Figure 1: the reconstructed boundary cannot be distinguished from the exact*

at the drawing scale, when the moments  $\tilde{m}_{ij}$  are given with more than 4 correct digits. When  $2 \leq \varepsilon \leq 4$ , the actual geometric boundary of  $G$ , can still be very well reconstructed, although the algebraic boundary is degraded.

**Example 4** (Algebraic Support, Gaussian measure). Consider the moments  $m_{ij} = \int_G x^i y^j \exp(p(x, y)) dx dy$ . In Figure 2(a),  $G$  is checkered and the level curves of  $\exp(p(x, y))$  are in dashed.

(i) Direct problem: Given  $g = (x^2 - 9/10)^2 + (x^2 - 11/10)^2 - 1$ , which vanishes on  $\partial G$ , and  $f = \exp(-x^2 + xy - y^2/2)$ , with  $\mathfrak{Ann}\{f\} = \{\partial_x + 2x - 1, \partial_y + y - 1\}$ , apply Algorithm 1 to compute a Gröbner basis for the sequence of moments  $m_{ij}$ . In the same setting as above, a Gröbner basis with 5 generators and with 28 monomials under the staircase is obtained.

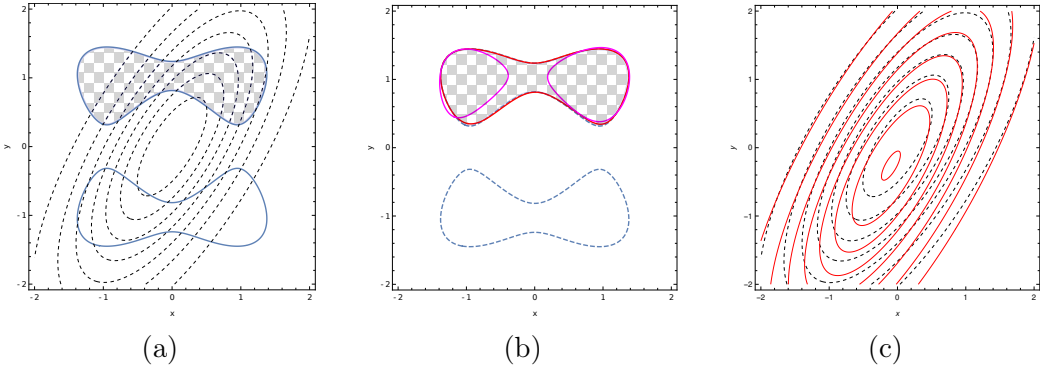


Figure 2: (a)  $G$  in checkered pattern, exact Gaussian level curves in dashed-black,  $\partial G$  in blue; (b) When  $\varepsilon > 8$ : reconstructed and original boundary (in dashed blue) cannot be distinguished at this scale; in red,  $\varepsilon = 6$ , and in magenta for  $\varepsilon = 4$  digits. (c) Reconstructed Gaussian level curves in red when  $\varepsilon = 8$ ; when  $\varepsilon > 8$ , the level curves of exact and reconstructed coincide at this scale.

(ii) Inverse problem: Suppose now given a finite number of numerically computed moments  $\tilde{m}_{ij}$ , with unknown support  $G$  and unknown Gaussian weight. The goal is to reconstruct  $g = \sum_{i+j \leq 4} g_{ij} x^i y^j$  which vanishes on  $\partial G$ , as well as  $p = \sum_{i+j \leq 2} p_{ij} x^i y^j$ . Algorithm 2 called with parameters  $(2, 4, 2, 14, (\tilde{m}_{ij})_{|i+j| \leq 18})$  provides the reconstructed  $g$ , as depicted in Figure 2(b): the reconstructed boundary cannot be distinguished from the exact at the drawing scale, when  $\varepsilon > 8$ . When  $4 \leq \varepsilon \leq 8$ , the actual geometric boundary of  $G$ , can still be very well reconstructed. Concerning the Gaussian weight, the situation is similar, cf. Figure 2(c).

The examples above are purely academic and even if the proposed method is very robust in these cases, further investigation is needed for the efficient

numerical implementation of the provided algorithms, in practical higher-dimensional applications.

On the theoretical side, this article provides further insight on the question raised in [14] regarding the *finite determinateness of a measure*. To sum up, provided Assumption 1 and 2 hold, for a measure with compact algebraic support  $G$ , with  $g \in \mathbb{R}[x]_d$  vanishing on  $\partial G$  and known holonomic density  $f$ , the moments up to degree  $N$  (which only depends on  $d$  and the order of a rectangular differential system which annihilates  $f$ ) determine in a constructive and robust manner the coefficients of  $g$ . Thus, this determines in turn all the other moments. When both the density and the support are unknown, a uniform bound  $N$  does not exist in general. We provided in this article the solution for the special case of unknown *exponential-polynomial* density.

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# Appendix

## A.1 Complementary proofs

*Proof of Proposition 1.* To prove (i), we use  $\partial_{x_i} e^{-\mathbf{i}x \cdot z} = -\mathbf{i} z_i e^{-\mathbf{i}x \cdot z}$  and  $\partial_{z_i} e^{-\mathbf{i}x \cdot z} = -\mathbf{i} x_i e^{-\mathbf{i}x \cdot z}$ :

$$\begin{aligned}
\mathcal{F}\{LT\} &= \langle T, L^*(e^{-\mathbf{i}x \cdot z}) \rangle \\
&= \langle T, \sum_{\beta} (-1)^{|\beta|} \partial_x^\beta (q_\beta(x) e^{-\mathbf{i}x \cdot z}) \rangle \quad \text{by definition,} \\
&= \langle T, \sum_{\beta} (-1)^{|\beta|} q_\beta(\mathbf{i} \partial_z) \partial_x^\beta (e^{-\mathbf{i}x \cdot z}) \rangle \\
&= \langle T, \sum_{\beta} (-1)^{|\beta|} q_\beta(\mathbf{i} \partial_z) (-\mathbf{i} z)^\beta e^{-\mathbf{i}x \cdot z} \rangle \\
&= \langle T, L^{\mathcal{F}} e^{-\mathbf{i}x \cdot z} \rangle = L^{\mathcal{F}} \mathcal{F}\{T\}.
\end{aligned}$$

The last equality holds since for any  $\mathcal{C}^\infty$  function  $f(x, z)$ ,

$$\begin{aligned}
\langle T, z_i f(x, z) \rangle &= z_i \langle T, f(x, z) \rangle, \quad \text{and} \\
\langle T, \partial_{z_i} f(x, z) \rangle &= \left\langle T, \lim_{h \rightarrow 0} \frac{f(x, z + h \mathbf{e}_i) - f(x, z)}{h} \right\rangle \\
&= \lim_{h \rightarrow 0} \frac{1}{h} (\langle T, f(x, z + h \mathbf{e}_i) \rangle - \langle T, f(x, z) \rangle) \\
&= \partial_{z_i} \langle T, f(x, z) \rangle,
\end{aligned}$$

where the commutation of the limit symbol comes from the fact that  $f_{z,h} : x \mapsto \frac{f(x, z + h \mathbf{e}_i) - f(x, z)}{h}$  converges to  $f_z : x \mapsto \partial_{z_i} f(x, z)$  for the compact-open topology of  $\mathcal{E}$ .

To prove (ii), one just need to notice that  $L^{\mathcal{M}}$  is obtained from  $L^{\mathcal{F}}$  using

$$z_i \mapsto \mathbf{i} \alpha_i S_{\alpha_i}^{-1}, \quad \text{and} \quad \partial_{z_i} \mapsto -\mathbf{i} S_{\alpha_i}.$$

□