# Persistence of a surface film on a liquid in the presence of disturbance propagation. Journal of Applied Physics, vol. 48, 1, Jan. 1977.

Michel Dudeck, Roger Prud'Homme

# Persistence of a surface film on a liquid in the presence of disturbance propagation

M. Dudeck and R. Prud'Homme

*Laboratoire d'Aérothermique du C.N.R.S et de l'Université Paris VI, 4 ter, route des Gardes F-92190 Meudon, France*

(Received 21 June 1976; in final form 20 September 1976)

An impermeable film (such as oil) can be used to prevent gas entrainment by the free surface of a liquid. This film, lying on the liquid, is modeled as a continuous viscoelastic two-dimensional medium. Cohesion and sufficient thickness of this medium are necessary to ensure effective protection. The authors study the evolution of the system in the case of a progressive surface wave of low amplitude. A theory of linearized disturbances is used to obtain the propagation equation of the wave. The solution of this equation gives the film thickness and the surface tension during the movement and leads to determination of the protection conditions of the liquid.

PACS numbers: 68.25.+j, 46.20.+e, 46.30.Jv, 03.40.Kf

## I. INTRODUCTION

The presence of an impermeable film on the surface of a liquid makes it possible to prevent gas entrainment by the liquid. However, this process requires cohesion of the film, without which the latter would not be impermeable. Hence prior knowledge of the cohesion limits of the film is essential, as a function of external conditions and of the movement of the free surface. Conversely, once the external conditions to which the film is subjected are known, it is important to be in a position to make a suitable choice of the nature of this film.

These considerations led us to investigate factors likely to contribute to a loss in impermeability by breakage of the film during its movement. Reduction of the film thickness can also be considered as a cause of the loss of impermeability.

The present study is concerned with a heavy imcompressible liquid comparable to a perfect fluid. At rest, the free surface is covered by a planar heavy elastic or viscoelastic film of uniform thickness. This film is considered as a plane two-dimensional medium with a certain mass per unit area.

These hypotheses are less restrictive than those employed by Puri[1] in his study of the oscillations of two superposed liquids separated by a membrane assumed to be of uniform mass per unit area and constant tension. When the liquid and film are subjected to small movements, interaction occurs between their movements. We shall always consider the film as a two-dimensional medium and shall assume that it adheres perfectly to the free surface on which it slips with accompanying deformation. In the article by Hansen and Mann,[2] the fluid is assumed to be viscous, and the possibility of film slippage on the surface is discarded.

Since the movements considered are of low amplitude, it is normal to linearize the equations, making it possible to consider the solution of the problem as a sum of monochromatic waves. Hence it is sufficient to analyze the response of the film and the liquid to a stress of given frequency and amplitude.

We shall show that the velocity of the wave depends on the frequency and that damping occurs along the surface if the film is viscoelastic. The dispersion equation of the medium is established, making it possible to relate the velocity of the wave and the damping at this frequency.

The density of the film, and hence its thickness, together with the internal tensions, are calculated as a function of the amplitude of the movement as well as its frequency. It is then possible to determine the nonbreakage limits and hence to resolve the problem which has been posed.

## II. DISPERSION EQUATION

Under the stress of small movements, a given medium can be characterized by the values of disturbances bearing on the different parameters which define it (Fig. 1). Since the medium investigated is

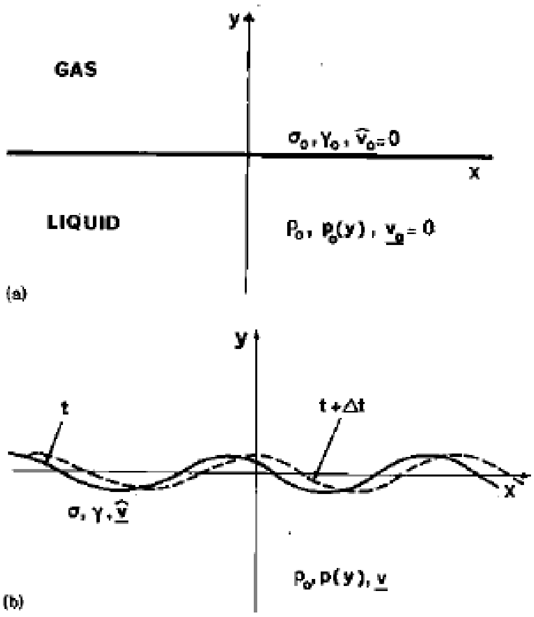


FIG. 1. (a) Reference state; (b) progressive nondamped surface wave.

planar and two-dimensional, we can define the free surface by

$$y = \epsilon f(x, t), \tag{1}$$

where $y$ is the ascending vertical axis, $\epsilon$ is chosen such that $\epsilon/l$ is small in relation to unity, and $l$ is a given reference length.

$$\sigma_{1,t} + \sigma_0 \hat{u}_{1,x} + [\rho_0(v_1 - f_{,t})]_-^+ = 0 \quad \text{at } y = \epsilon f \tag{2}$$

is the equation of mass conservation of the interface[3] dealing with disturbances $\sigma_1$, $u_1$, and $v_1$. We have $\sigma = \sigma_0 + \epsilon\sigma_1 + O(\epsilon^2)$, the mass per unit area; $u = \epsilon u_1 + O(\epsilon^2)$, the horizontal surface velocity; $v = \epsilon v_1 + O(\epsilon^2)$, the vertical velocity of the fluid; $\rho$ is the density of the fluid. $[\ ]_-^+$ denotes change of the corresponding term between the upper and lower media on either side of the film (in principle, the upper fluid is a gas of negligible mass).

If the film is impermeable, then necessarily

$$v_1 = f_{,t}, \tag{3}$$

and the condition of adherence of the film to the surface also gives

$$\hat{v}_1 = f_{,t}. \tag{4}$$

The impulsion balance equation of the film is written

$$\sigma_0 \hat{u}_{1,t} - \gamma_{1,x} - f_{,x}[p_0]_-^+ = 0, \quad y = \epsilon f(x, t) \tag{5}$$

$$\sigma_0 \hat{v}_{1,t} - \gamma_0 f_{,xx} + [p_1]_-^+ = -\sigma_1 g - \rho g f, \quad y = \epsilon f(x, t), \tag{6}$$

where $p$ is the pressure of the fluid, $\gamma$ is the surface tension of the film, $\hat{v}$ is its vertical velocity, and $g$ is the gravitational acceleration.

If we assume that the tension $\gamma_0$ is null in the reference state (rest), we obtain, for the Bernoulli equation[4,5] of the fluid,

$$\Phi_{1,t} = -gf + \frac{\sigma_0}{\rho_0 g} + f_{,tt} + \frac{\sigma_1}{\rho_0}, \quad y = \epsilon f(x, t), \tag{7}$$

with the velocity potential $\Phi = \epsilon\Phi_1 + \cdots$ such that

$$u_1 = \Phi_{1,x}, \quad v_1 = \Phi_{1,y}. \tag{8}$$

Knowledge of the function $\Phi_1$ such that

$$\Phi_{1,xx} + \Phi_{1,yy} = 0 \tag{9}$$

enables determination of the fluid movement between the surface and the bottom, assumed to be of infinite depth ($y \to -\infty$). In order to resolve the problem, it is assumed that Eq. (7) becomes, for $y \geq \epsilon f$,

$$\Phi_{1,t} = -gP(y)\left(f + \frac{\sigma_0}{\rho_0 g} f_{,tt} + \frac{\sigma_1}{\rho_0}\right). \tag{10}$$

Condition (9) gives rise to two separate equations at $y$ and at $(x, t)$. Their resolution in the complex plane gives

$$\Phi_1 = \frac{gd}{i\omega} \exp[i(K\bar{z} - \omega t)], \tag{11}$$

where $d$ is a constant of integration, $\bar{z} = x - iy$, $\omega$ is the angular frequency, and $K = k + i\alpha$ is the complex wave number.

This gives the velocity field of the fluid. If we assume that the modulus of the function $f$ is equal to unity for $x = 0$ and $t = 0$, we have

$$f = \exp[i(Kx - \omega t)] = \exp(-\alpha x)\exp[i(kx - \omega t)]. \tag{12}$$

Condition (3) then gives

$$d = \omega^2/gK. \tag{13}$$

With a law of behavior of the viscoelastic type film,[2]

$$\gamma_1 = -\frac{E}{\sigma_0}\sigma_1 - \frac{E_v}{\sigma_0}\sigma_{1,t}, \tag{14}$$

one obtains, after elimination of velocity $u_1$ between Eqs. (2) and (5) the following result:

$$\sigma_{1,tt} - \frac{E}{\sigma_0}\sigma_{1,xx} - \frac{E_v}{\sigma_0}\sigma_{1,txx} - \sigma_0 g f_{,xx} = 0. \tag{15}$$

Since Eqs. (7) and (11) also give us, in view of Eq. (13),

$$\sigma_1 = \rho_0 d \exp[i(Kx - \omega t)] - \rho_0 f - \frac{\sigma_0}{g} f_{,tt}, \tag{16}$$

we therefore have the following dispersion equation in $K$:

$$\left(1 - c_E^2\frac{K^2}{\omega^2} + ic_v^2\frac{K^2}{\omega^2}\frac{\omega}{\omega_\sigma}\right)\left(1 - \frac{\omega^2}{\omega_\sigma^2} - \frac{\omega^2}{\omega_\sigma}\frac{1}{Kc_\sigma}\right) + c_v^2\frac{K^2}{\omega^2} = 0, \tag{17}$$

where

$$c_E^2 = \frac{E}{\sigma_0}, \quad c_v^2 = \frac{E_v}{\sigma_0}\omega_\sigma,$$

$$c_\sigma^2 = \frac{\sigma_0 g}{\rho_0}, \quad \omega_\sigma^2 = \frac{\rho_0 g}{\sigma_0}, \quad \omega_\sigma c_\sigma = g. \tag{18}$$

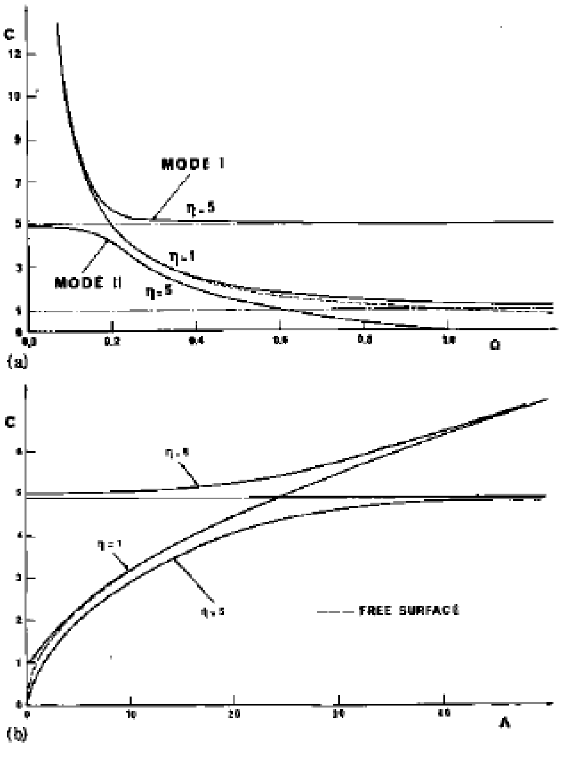


FIG. 2. (a) Wave velocity as a function of frequency; (b) wave velocity as a function of wavelength.

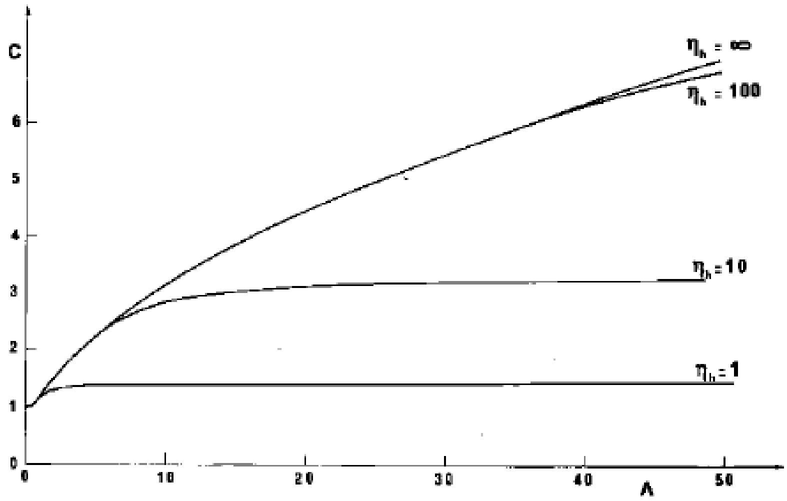


FIG. 3. Influence of depth on wave propagation.

This complex equation, after separation of the real and imaginary portions, makes it possible to calculate the wave velocity $c(\omega)$ and damping $\alpha(\omega)$ such that

$$\frac{K}{\omega}=\frac{1}{c}+i\frac{\alpha}{\omega}. \tag{19}$$

For a purely elastic film, $c_v$ is null and it can be shown that the wave is not damped along $x$ so that

$$\left(1-\frac{c_E^2}{c^2}\right)\left(1-\frac{\omega^2}{\omega_c^2}-\frac{\omega}{\omega_\sigma}\frac{c}{c_\sigma}\right)+\frac{c_E^2}{c^2}=0. \tag{20}$$

For $c_E=0$, one obtains the result of the free surface

$$\frac{c_\sigma}{c}=\frac{\omega}{\omega_\sigma}, \tag{21}$$

or

$$\omega c=g. \tag{22}$$

Let us set

$$\eta=\frac{c_E}{c_\sigma}, \quad \eta_v=\frac{c_v}{c_\sigma}, \quad \Omega=\frac{\omega}{\omega_\sigma}, \quad C=\frac{c}{c_\sigma}, \tag{23}$$

and introduce the wavelength $\lambda$ and the corresponding reduced variable

$$\Lambda=\frac{\lambda}{\lambda_\sigma}=\frac{C}{\Omega}. \tag{24}$$

$C$ and $\Lambda$ verify an equation which, in the case of the elastic film, is written

$$(\Lambda+1)C^4-[\Lambda^2+\eta^2(\Lambda+1)]C^2+\Lambda^2(\eta^2-1)=0. \tag{25}$$

In reduced variables, the velocity is represented in Figs. 2(a) and (2b) as a function of frequency and wavelength. Two modes appear, corresponding to the interaction of the conditions specific to a liquid with a free surface, with the vibration of a plane film. The coupling occurs through the contact condition of the film and by the effect on the liquid of a variable mass per unit area. For large wavelengths, mode I tends to the solution associated with a liquid with a free surface ($\eta=0$, $C^2=\Lambda$); for low wavelengths, mode II tends to the solution associated with the vibration of a plane film. The velocity region lying between $c_\sigma(\eta^2-1)^{1/2}$ and $c_\sigma\eta$ is not included. As the elasticity of the film increases, the solution obtained approaches that corresponding to a liquid with a free surface; two states appear beyond the specific value $\eta=1$. In the first mode, the velocity of the film is greater than that of the liquid for $y\to 0^-$, and the directions of these two velocities are identical. In the second mode, the velocities have opposite direction and the modulus of the film velocity is greater or lower than that of the liquid near the surface if the phase velocity is greater or lower than the velocity $C_1$ ($C_1=3.4$ for $\eta=5$).

The slip velocity is expressed as

$$\frac{\bar{u}_1-\bar{\bar{u}}_1}{\omega_\sigma}=\Omega-\frac{C^2}{C^2-\eta^2}. \tag{26}$$

The mass per unit area of the film is expressed as

$$\sigma=\sigma_0+\epsilon\sigma_1+\cdots=\sigma_0+\frac{\epsilon\rho_0}{C^2-\eta^2}\exp[i(kx-\omega t)]+\cdots. \tag{27}$$

For the first mode, the maximum value of the mass per unit area is reached at the highest points of the film ($y=\epsilon$) and the minimum value at the lowest points ($y=-\epsilon$). The effect is reversed for the second mode.

In many practical cases, it is important to know the effect of depth $h$ of the liquid. This depth, assumed to be constant, acts through the coefficient

$$\eta_h=\frac{c_h}{c_\sigma}=\frac{\omega_\sigma h}{c_\sigma}.$$

Without going into detail concerning the mathematical formulation in the case of finite depth, we present here the results concerning the elastic film itself, obtained from the dispersion equation which is written, in this case,

$$(1-\eta^2C^{-2})[-\Omega^2-\Omega C\coth(\eta_h\Omega/C)+1]+C^{-2}=0. \tag{28}$$

The solutions are shown in Fig. 3.

For a given wavelength, the presence of a bottom at a finite distance reduces the propagation velocity associated with each of the branches of the curve. In the case of the free surface, the effect is identical.

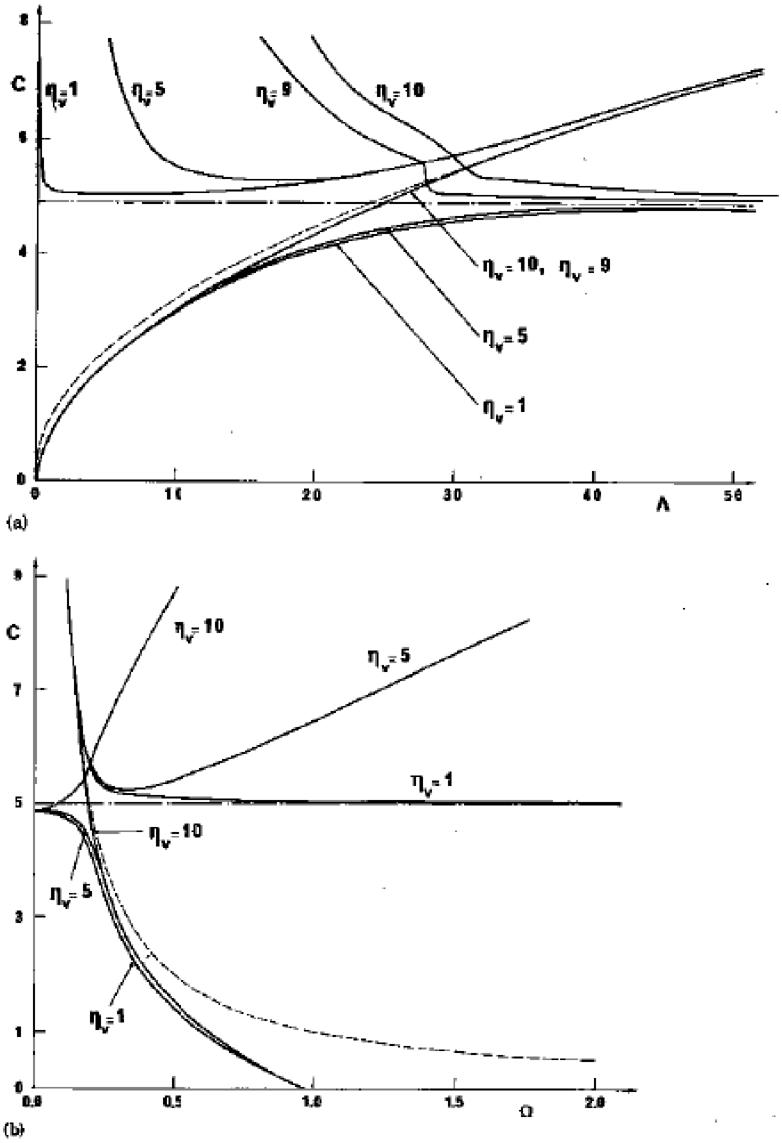


FIG. 4. (a) Wave velocity in the case of a viscoelastic film as a function of wavelength. Solid curve: $\eta = 5$; broken curve: free surface; (b) wave velocity in the case of a viscoelastic film as a function of frequency. Solid curve: $\eta = 5$; broken curve: free surface.

The separation of real and imaginary portions of Eq. (17) for a viscoelastic film gives

$$(1-\eta^2C^{-2}+\eta^2A^2-2\eta_v^2\Omega C^{-1}A)\left(-\Omega^2-\frac{\Omega C^{-1}}{C^{-2}+A^2}+1\right)$$
$$-(-2\eta^2C^{-1}A+\eta_v^2\Omega C^{-2}-\Omega\eta_v^2A^2)\frac{\Omega A}{C^{-2}+A^2}+C^{-2}-A^2=0, \tag{29}$$

$$(1-\eta^2C^{-2}+\eta^2A^2-2\eta_v^2\Omega C^{-1}A)\frac{\Omega A}{C^{-2}+A^2}+2C^{-2}A$$
$$+(-2\eta^2C^{-1}A+\eta_v^2\Omega C^{-2}-\Omega\eta_v^2A^2)\left(-\Omega^2-\frac{\Omega C^{-1}}{C^{-2}+A^2}+1\right)=0, \tag{30}$$

where the reduced damping $A$ is given by

$$A=c_0\alpha/\omega. \tag{28}$$

This system of two equations with three variables $C$, $A$, and $\Omega$ is resolved numerically, as the elimination of $A$ between the two equations leads to an equation linking $C$ and $\Omega$ (or $C$ and $A$) and involving high powers of these variables.

The solutions of this system were determined by an elasticity coefficient $\eta$ equal to 5 and for a positive phase velocity. For a slightly viscoelastic film, only the behavior at low wavelengths of the first mode differs from that of the elastic film, by an increase in the phase velocity. This effect becomes more pronounced as the viscoelasticity of the film increases. Beyond a certain value of the viscoelasticity coefficient $\eta_v$, an exchange occurs between the asymptotic branches, and no region is excluded for the phase velocity. In reduced variables, the phase velocity and damping coefficient are represented as a function of wavelength and frequency (Figs. 4 and 5). The evolution of the behavior of

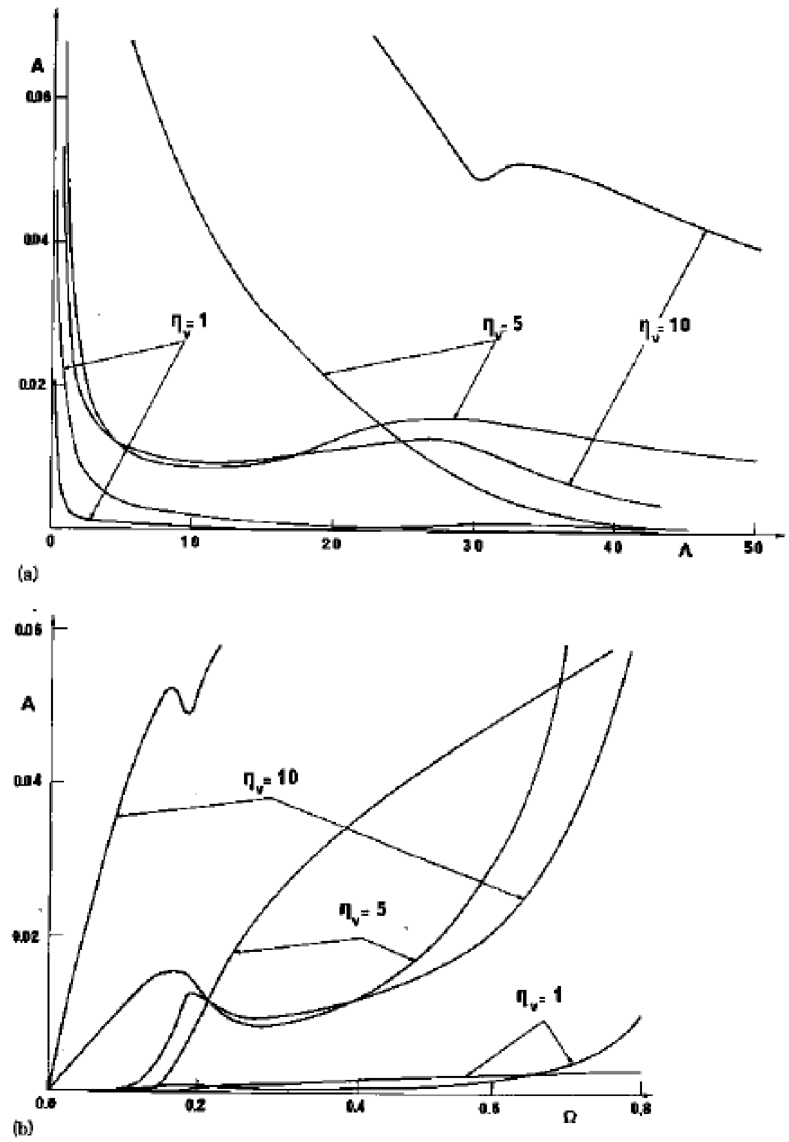


FIG. 5. (a) Viscoelastic damping coefficient as a function of wavelength ($\eta = 5$); (b) viscoelastic damping coefficient as a function of frequency ($\eta = 5$).

phase velocity as a function of wavelength is shown in the neighborhood of the singular point in Fig. 6.

## III. INTERNAL TENSIONS AND NONBREAKAGE CONDITIONS

The value of $\sigma_1$ and that of tension $\gamma_1$ are derived from the amplitude of the surface deformation.

By introducing $f$, with a unit modulus at the origin, in Eq. (14), we obtain

$$\left(-\omega^2 + \frac{E}{\sigma_0} K^2 - i \frac{E_v}{\sigma_0} \omega K^2\right) \bar{\sigma}_1 = -\sigma_0 g K^2, \tag{31}$$

with

$$\sigma_1 = \bar{\sigma}_1 \exp[i(Kx - \omega t)].$$

Equation (13) also becomes

$$\bar{\gamma}_1 = \left(-\frac{E}{\sigma_0} + i \frac{E_v}{\sigma_0} \omega\right) \bar{\sigma}_1. \tag{32}$$

When the film is elastic, $\epsilon\bar{\sigma}_1$ and $\epsilon\bar{\gamma}_1$ are real numbers equal to the mass per unit area and the surface tension;

$$\epsilon\bar{\sigma}_1 = \frac{\rho_0 \epsilon}{C^2 - \eta^2}, \tag{33}$$

$$\epsilon\bar{\gamma}_1 = -\frac{\sigma_0 g \eta^2 \epsilon}{C^2 - \eta^2}. \tag{34}$$

If the function $C(\Omega)$ for each of the modes and the film breakage limit are known, one can determine the maximum movement amplitude and frequency ensuring nonbreakage of the film.

If $R$ is the breakage tension, the nonbreakage condition is

$$|C^2 - \eta^2| > \sigma_0 g \eta^2 \epsilon / R. \tag{35}$$

Moreover, a reduction of mass per unit area can be interpreted as a decrease in thickness for a constant-density film. In this case, we have

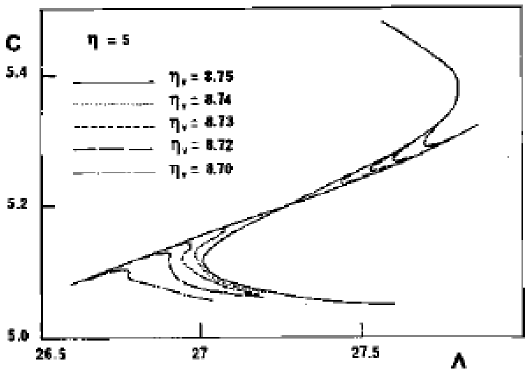


FIG. 6. Wave velocity in the case of a viscoelastic film: evolution near the singular point.

$$\frac{\Delta e}{e}=\frac{\Delta\sigma}{\sigma}=\epsilon\frac{\sigma_1}{\sigma_0}, \tag{36}$$

where $e$ is the film thickness. If $e_m$ is the minimum thickness ensuring impermeability, one then has the condition

$$\left|\epsilon\frac{\sigma_1}{\sigma_0}\right|<\frac{e_0-e_m}{e_0}, \tag{37}$$

which can also be written

$$|C^2-\eta^2|>\frac{\rho_0\epsilon e_0}{\sigma_0(e_0-e_m)}. \tag{38}$$

The two foregoing conditions must be satisfied in order to avoid gas entrainment during a disturbance of amplitude $\epsilon$. Conversely, given $R$ and $e_m$, one can calculate the maximum amplitude $\epsilon$ of the surface wave as a function of its pulsation. The limit curve is always of the form

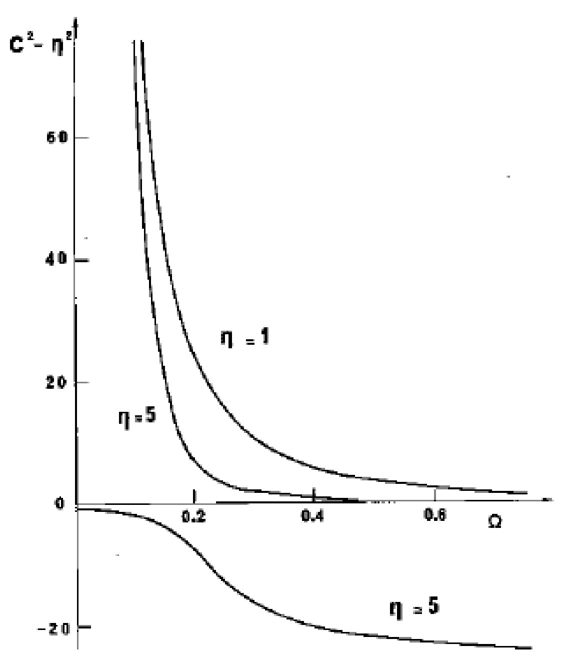


FIG. 7. Impermeability limit of an elastic film.

$$\epsilon=a(C^2-\eta^2), \tag{39}$$

where $a$ is a proportionality factor (Fig. 7).

In the presence of viscoelasticity, one has

$$[1-\eta^2C^{-2}+\eta^2A^2-2\eta_v^2\Omega AC^{-1}+i(-2\eta^2AC^{-1}+\eta_v^2\Omega C^{-2}-\eta_v^2\Omega A^2)]\bar{\sigma}_1=\rho_0(C^{-2}-A^2+2iAC^{-1}) \tag{40}$$

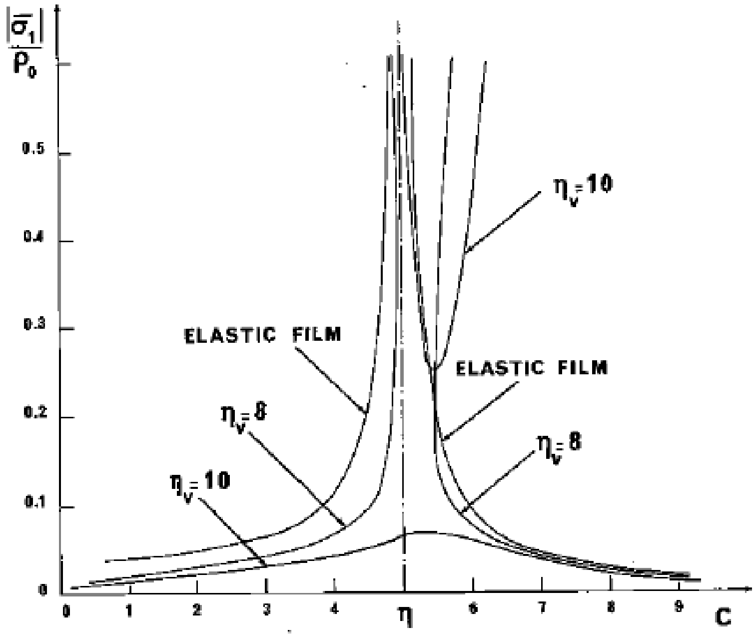


FIG. 8. Modulus of the surface density perturbation as a function of the wave velocity: elastic and viscoelastic film.

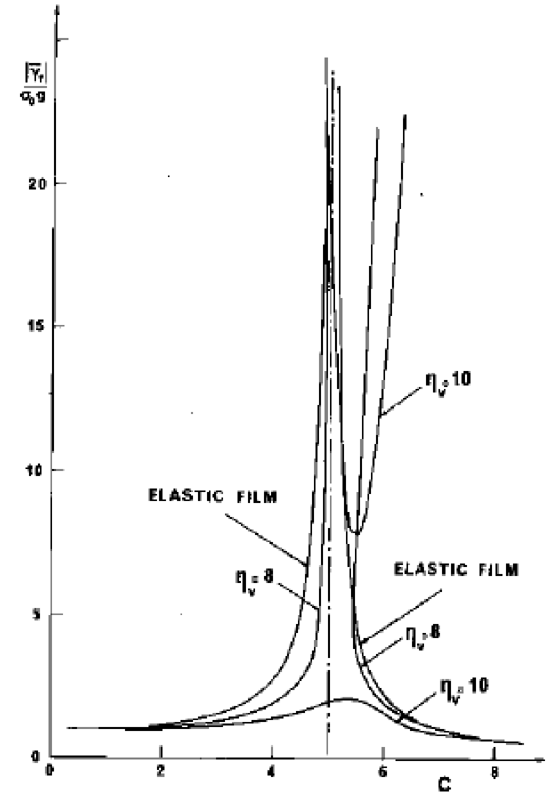


FIG. 9. Modulus of the surface tension perturbation as a function of the velocity wave: elastic and viscoelastic film.

Finally, the tension modulus

$$\frac{|\bar{\gamma}_1|^2}{\sigma_0^2 g^2} = \frac{(\eta^4+\eta_v^4\Omega^2)(A^2+C^{-2})^2}{1+(A^2+C^{-2})^2(\eta^4+\eta_v^4\Omega^2)+2\eta^2(A^2-C^{-2})-4\eta_v^2\Omega A C^{-1}}, \tag{41}$$

and that of the variation in mass per unit area

$$\frac{|\sigma_1|^2}{\rho_0^2} = \frac{(A^2+C^{-2})^2}{1+(A^2+C^{-2})^2(\eta^4+\eta_v^4\Omega^2)+2\eta^2(A^2-C^{-2})-4\eta_v^2\Omega A C^{-1}}. \tag{42}$$

With the damping of the viscoelastic wave, it is necessary to consider the maximum amplitude and not that of the already attenuated wave (Figs. 8 and 9). One can then proceed as in the case of the elastic film and determine the limit curve $\epsilon(\Omega)$ for impermeability conditions to be guaranteed.

## IV. CONCLUSIONS

We succeeded in determining the impermeability conditions of the surface film subjected to monochromatic disturbances of low amplitude causing the propagation of a progressive wave. The amplitude of these disturbances is a function of their frequency, which we have determined. On the basic level, our results call for certain observations.

We obtain two clearly distinct branches for the dispersion curve $C(\Omega)$ when the reduced elasticity modulus $\eta$ is greater than unity. This phenomenon can be interpreted as the result of a double wave propagation, one in the fluid medium, the second in the elastic or viscoelastic film. If these phenomena were independent, one would simply obtain the propagation velocities $c = c_E$ and $c = k/\omega$ of the elastic film and the liquid with the free surface. The difference results from the interaction.

By comparing this result with a well-known case, it can be seen that this phenomenon does not occur with a free surface (with null mass per unit area) with surface tension as shown in the curve in Fig. 10. This curve clearly has two branches, but curve $C(A)$ is continuous. The interpretation of these two branches is given by Bouasse[6]; this is a question of two states, one corresponding to surface ripples, the second to the wave proper.

Our study also revealed dimensionless parameters and characteristic frequencies and velocities of which the values must be known for each specific case. These coefficients depend on the nature of the liquid, the nature of the film and its thickness.

Note that we have ignored the effect of the gas located above the surface, and that we have dealt exclusively with the impermeable character of the film. According to calculations which we made, no significant in-

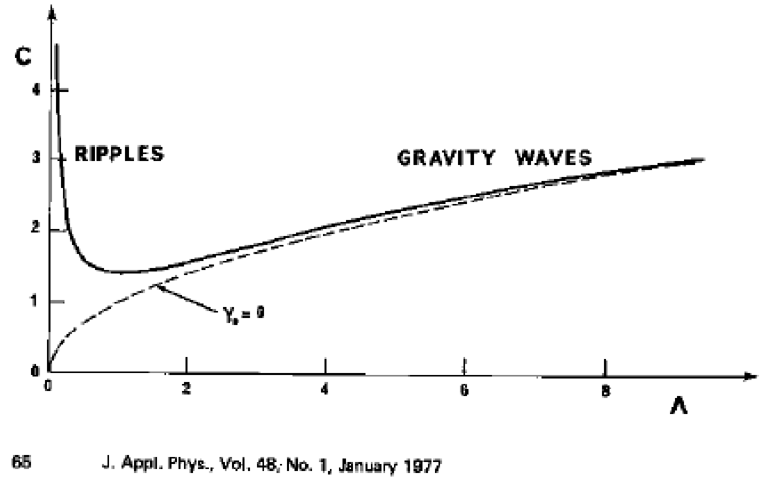


FIG. 10. Dispersion curve for free surface with constant surface tension $\gamma_0$.

teraction occurs between the gas and the liquid covered with a film for moderate-energy waves. However, a high-energy wave propagating in the gas will not fail to act both on the film and on the liquid. This is a case of strong disturbances, of which the propagation study is incompatible with the linearized theory to which we restricted our investigation. Our study nevertheless remains applicable to a wide range of disturbances which occur in practice for relatively uniform flows.


[1]K.K. Puri, J. Appl. Phys. **42**, 995 (1971).
[2]R.S. Hansen and J.A. Mann, Jr., J. Appl. Phys. **35**, 19 (1964).
[3]M. Barrère and R. Prud'Homme, *Equations fondamentales de l'Aérothermochimie* (Masson, Paris, 1973), Chap. III, p. 110.
[4]P. Germain, *Mécanique des Milieux Continus* (Masson, Paris, 1962), Chap. XIV, p. 248.
[5]I.I. Stoker, *Water Waves*, edited by R. Courant, L. Bers, and J.J. Stoker (Interscience, New York, 1957), Chap. 1, p. 9.
[6]H. Bouasse, *Houles, rides, seiches et marées* (Delagrave, Paris, 1924), p. 186.