



**HAL**  
open science

# Analysis of transsonic flow with chemical reactions by the small perturbation method. The one and two-dimensional problem

Roger Prud'Homme

► **To cite this version:**

Roger Prud'Homme. Analysis of transsonic flow with chemical reactions by the small perturbation method. The one and two-dimensional problem. Acta Astronautica, 1970. hal-02005032

**HAL Id: hal-02005032**

**<https://hal.science/hal-02005032>**

Submitted on 25 Apr 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Analysis of Transonic Flow with Chemical Reactions by the Small Perturbation Method. The One and Two-Dimensional Problem

**Roger Prud'homme**

Office National d'Etudes et de Recherches Aéropatiales (ONERA), France

*(Presented at the International Colloquium on Gazdynamics of Explosions. Novosibirsk. August 24-29, 1969)*

**Abstract:** The present study is concerned with setting up and solving the transonic quasi-one-dimensional equations of a flow with chemical reactions. Particular solutions are examined for such a flow when two-dimensional. The equations are written by the method of singular small perturbations around a uniform flow in chemical equilibrium.

## 1. Introduction

There are few published studies of transonic flow with relaxation. And yet the problem at issue is an important one, because aerodynamics is increasingly concerned with the high-temperature range where chemical relaxation as well as that of the internal degrees of freedom in molecules occur. The same phenomena are also incident to rocket propulsion where the nozzle throat region is transonic.

As yet there seem to exist but few approaches to the problem of relaxed transonic flows. The complexity of the equations concerned precludes the hodograph method applicable to simple gases; this leaves the computer and the small disturbances method. The mathematical singularity involved makes the former difficult to use, although, as will be seen, it does stand in good stead for tackling quasi-one-dimensional flows. That is why we have turned our attention to the method of singular small perturbations, based on the asymptotic expansion theory. The relevant equations were written by L. G. Napolitano [1] who confined himself to first-order terms. In a very broad way, his approach can also be applied, as demonstrated below, to the quasi-one-dimensional hypothesis.

The present paper is primarily devoted to solving the equations arrived at in this latter case and for two-dimensional plane flows. One-dimensional flows present no major problem, for the model is quite straightforward and amenable to correct analysis. This is not the case with two-dimensional flows, as the partial-derivative equations may be non-linear third order ones.

The initial and limit conditions are a matter of all the greater difficulty as the non-uniform asymptotic expansions result in three or four equations instead of a single one to describe the transonic zone. These expansions depend on a dimensionless parameter (Damköhler's first parameter), which equals the ratio of two lengths (or two times).

The method adopted requires the chosen flow to be disturbed, simple and uniform i.e. chemically in equilibrium. The drawback limiting the inquiry to small disturbances is that this leaves room for near-equilibrium; still the results should be capable of generalisation.

Lastly, analysis is here restricted to the first term of the asymptotic expansion, carrying it on to the following terms would unquestionably yield valuable results.

## 2. Quasi-One-Dimensional-Flows

What we are largely concerned with here are processes evolving in the Laval nozzle. Allowing for the initial conditions and a particular nozzle design, it is set out to determine the flow in the geometric throat region.

With the problem posed in these terms, there is known to exist a singular point for the equations at which the flow velocity equals the speed of frozen sound  $a_f$  or  $\sqrt{[(\partial p / \partial \rho_{s,\xi})]}$  where  $p$ ,  $\rho$ ,  $s$  and  $\xi$  respectively denote the pressure, density, entropy, and the extent of the reaction (or a chemical composition).

Conditions at the singular point marked  $F$ , are given in Tables 1 to 4.

$\frac{\delta - \alpha}{2}$	$D_1$	$\bar{x}$	Equations
$\frac{1}{2}$	1	$\varepsilon^{-\frac{1}{2}} \frac{x^+}{l_c^+}$	$z\xi_1 = \frac{\Gamma + 1}{2} v_1^2 - \mathcal{A}_1 + \dot{m}_2,$ $\xi_1 = e^{\mathcal{A}_1}, \quad v_1 = v_1, \quad s_1 = 0.$
0	$\varepsilon^{\frac{1}{2}}$	$\frac{x^+}{l_c^+}$	$z\xi_1 = \frac{\Gamma + 1}{2} v_1^2 - \mathcal{A}_1 + \dot{m}_2,$ $\frac{d\xi_1}{d\bar{x}} + \xi_1 + v_1 = 0,$ $v_1 = v_1, \quad s_1 = 0.$
$-\frac{1}{2}$	$\varepsilon$	$\varepsilon^{\frac{1}{2}} \frac{x^+}{l_c^+}$	$z\xi_1 = \frac{\Gamma + 1}{2} v_1^2 - \mathcal{A}_1 + \dot{m}_2,$ $\xi_1 + v_1 = 0,$ $v_1 = v_1, \quad s_1 = 0.$
$\Gamma = k_{r,0}, \quad \frac{a_{f,0}^2}{a_{s,0}^2} - 1 = z\varepsilon^{\frac{1}{2}}, \quad v_0 = a_{f,0} = v_D = 1, \quad l_c^{\frac{1}{2}} = (z_{s,0})_0$			

Table 1. Transonic equations for adjacent sound velocities

$\frac{\delta - \alpha}{2}$	$D_1$	$\bar{x}$	Equations
$\frac{1}{2}$	1	$\varepsilon^{-\frac{1}{2}} \frac{x^+}{l_c^+}$	$x\xi_1 = \frac{\Gamma + 1}{2} v_1^2 - \mathcal{A}_1 + \dot{m}_1,$ $\xi_1 = c^x, \quad v_1 = v_1, \quad s_1 = 0.$
0	$\varepsilon^{\frac{1}{2}}$	$\frac{x^+}{l_c^+}$	$x\xi_1 = \frac{\Gamma + 1}{2} v_1^2 - \mathcal{A}_1 + \dot{m}_1,$ $\frac{d\xi_1}{d\bar{x}} + \xi_1 + v_1 = 0,$ $v_1 = v_1, \quad s_1 = 0.$
$-\frac{1}{2}$	$\varepsilon$	$\varepsilon^{\frac{1}{2}} \frac{x^+}{l_c^+}$	$x\xi_1 = \frac{\Gamma + 1}{2} v_1^2 - \mathcal{A}_1 + \dot{m}_1,$ $\xi_1 + v_1 = 0,$ $v_1 = v_1, \quad s_1 = 0.$

$\Gamma = k_{\gamma\gamma_0}, \quad \frac{a_{f0}^2}{a_{e0}^2} - 1 = \varepsilon\varepsilon^{\frac{1}{2}}, \quad v_0 = a_{f0} = v_0 = 1, \quad l_c^+ = (x_{e,0}^+)_0.$

Table 2. Transonic flow for non-adjacent sound velocities. Neighbourhood of point  $E$  ( $v=a_e$ ).

$\delta - \alpha$	$D_1$	$\bar{x}$	Equations
1	1	$\varepsilon^{-1} \frac{x^+}{l_c^+}$	$(M_{f0}^2 - 1)v_1 - \mathcal{A}_1 + \dot{m}_1 = \left(1 - \frac{M_{f0}^2}{M_{e0}^2}\right)\xi_1,$ $\xi_1 = c^e, \quad v_1 = v_1 + \mathcal{A}_1 - \dot{m}_1, \quad s_1 = 0.$
0	$\varepsilon$	$\frac{x^+}{l_c^+}$	$(M_{f0}^2 - 1)v_1 - \mathcal{A}_1 + \dot{m}_1 = \left(1 - \frac{M_{f0}^2}{M_{e0}^2}\right)\xi_1,$ $\frac{d\xi_1}{d\bar{x}} + \xi_1 + v_1 = 0, \quad v_1 = v_1 + \mathcal{A}_1 - \dot{m}_1, \quad s_1 = 0.$
-1	$\varepsilon^2$	$\varepsilon \frac{x^+}{l_c^+}$	$(M_{f0}^2 - 1)v_1 - \mathcal{A}_1 + \dot{m}_1 = 0,$ $v_1 = -\xi_1 = v_1 + \mathcal{A}_1 - \dot{m}_1, \quad s_1 = 0.$

$\frac{a_{f0}^2}{a_{e0}^2} - 1 = O(1), \quad a_{e0} < v_0 < a_{f0}, \quad n = 1, \quad v_0 = v_0 = 1, \quad l_c^+ = (x_{e,0}^+)_0.$

Table 3. Transonic flow for non-adjacent sound velocities. Zone laying between  $E$  and  $F$ . Excluding neighbourhood of these points.

$n$	$(\delta - \alpha)n$	$D_1$	$\bar{x}$	Equations
$\frac{1}{2}$	1	$a^{-1}$	$\varepsilon^{-1} \frac{x^*}{l_c^*}$	$\frac{\Gamma + 1}{2} v_1^2 + [(1 - \Gamma) e_{v0} + h_{v0}] \zeta_1 v_1 - \mathcal{A}_1 + \dot{m}_2$ $+ h_{t0} v_2 + \left[ \left( \frac{\Gamma}{2} - 1 \right) e_{v0}^2 + \frac{h_{v0}}{2} - h_{v0} e_{v0} \right] \zeta_1^2 = 0,$ $\dot{m}_1 = e_{v0} \zeta_1, \quad v_1 = v_2 - \dot{m}_1, \quad \zeta_1 = c^e, \quad \zeta_2 = c^e.$
$\frac{1}{2}$	$\frac{1}{2}$	1	$\varepsilon^{-1} \frac{x^*}{l_c^*}$	$\frac{d}{d\bar{x}} \frac{\Gamma + 1}{2} v_1^2 + [(1 - \Gamma) e_{v0} + h_{v0}] \zeta_1 v_1 - \mathcal{A}_1$ $+ \frac{M_{v0}^2 - 1}{M_{v0}^2} v_1 + \frac{e_{v0}}{M_{v0}^2} \zeta_1 = 0,$ $\dot{m}_1 = e_{v0} \zeta_1, \quad v_1 = v_2 - \dot{m}_1, \quad \zeta_1 = c^e.$
$\frac{1}{2}$	0		$\frac{x^*}{l_c^*}$	Banal equations
1	1		$\varepsilon^{-1} \frac{x^*}{l_c^*}$	Banal equations
1	0	$\varepsilon$	$\frac{x^*}{l_c^*}$	$(M_{v0}^2 - 1) v_1 = M_{v0}^2 \frac{d\mathcal{A}_1}{d\bar{x}} + \mathcal{A}_1 - \dot{m}_1,$ $(M_{v0}^2 - 1) v_1 = M_{v0}^2 \left( \frac{d\mathcal{A}_1}{d\bar{x}} + \mathcal{A}_1 - \dot{m}_1 \right),$ $(M_{v0}^2 - 1) \zeta_1 = -M_{v0}^2 (\mathcal{A}_1 - \dot{m}_1).$
1	-1	$\varepsilon^2$	$\varepsilon \frac{x^*}{l_c^*}$	$(M_{v0}^2 - 1) v_1 = \mathcal{A}_1 - \dot{m}_1,$ $(M_{v0}^2 - 1) v_1 = M_{v0}^2 (\mathcal{A}_1 - \dot{m}_1),$ $(M_{v0}^2 - 1) \zeta_1 = -M_{v0}^2 (\mathcal{A}_1 - \dot{m}_1).$
$\Gamma - h_{v0}, \quad \frac{a_{f0}^2}{a_{v0}^2} - 1 = O(1), \quad v_0 = a_{f0} = v_0 = 1, \quad \tau_c^* = (\tau_{c0}^*)_0$				

Table 4. Transonic flow for non-adjacent sound velocities.  
Neighbourhood of point  $F$  ( $v=a_f$ ).

with  $s_2$ , which expresses the entropy variation of the relaxing flow.

The tables give the orders of magnitude of Damköhler's first parameter  $D_1$ ; it can be shown that [4]:

$$D_1 = O[\varepsilon^{(1-\delta+\alpha)n}] \quad (13)$$

where the number  $n$  corresponds to the variation in equilibrium of the same entropy and stagnation enthalpy.

In the case of a nozzle flow, it is generally true to say that a known  $D_1$  as derived from the results of equilibrium calculations, determine the choice of the equation to be used in a given table.

The most typical instance is the case of Table 1, where  $D_1 = O(1)$  stands for a frozen flow  $D_1 = O(\varepsilon)$  for an equilibrium flow (except for the second order) and  $D = O(\varepsilon^{2/3})$  for a relaxed flow properly so called. This last-named case alone provides a suitable model for use in the present study, which is limited the terms of the first order.

The relevant equations are:

$$Z \xi_1 = \frac{\Gamma+1}{2} v_1^2 - \mathcal{A}_1 + \dot{m}_2 \quad (14)$$

$$\frac{d\xi_1}{d\bar{x}} + \xi_1 + v_1 = 0 \quad (15)$$

$$v_1 = v_1, \quad s_1 = 0 \quad (16)$$

With  $\xi_1$  eliminated, in between these equations is found:

$$(\Gamma+1)v_1 \frac{dv_1}{d\bar{x}} + \frac{\Gamma+1}{2} v_1^2 + Z v_1 = \frac{d\mathcal{A}_1}{d\bar{x}} + \mathcal{A}_1 - \dot{m}_2 \quad (17)$$

which leads to the following conclusions:

(a) There is a singular point  $F$  viz.

$$v_{1F} = 0, \quad (v = a_f) \quad (18)$$

$$\left( \frac{d\mathcal{A}_1}{d\bar{x}} \right)_F + \mathcal{A}_{1F} - \dot{m}_2 = 0 \quad (19)$$

(b) Point  $F$  lies downstream of the geometrical throat  $C$  ( $x_F > x_C$ ), if  $d\xi/d\bar{x}$  has the same sign as  $(d\xi/d\bar{x})_{\text{equilibrium}}$ , i.e. if  $d\xi_3/d\bar{x}$  is negative.

We have

$$Z \left( \frac{d\xi_1}{d\bar{x}} \right)_F = - \left( \frac{d\mathcal{A}_1}{d\bar{x}} \right)_F < 0 \quad (20)$$

from equations (14) and (18).

(c) Point  $E$ , where  $v=a_e$  corresponds to

$$(v_1)_{\bar{x}} = - \frac{Z}{\Gamma+1} \quad (21)$$

and is not a singular point.

(d) Choice of one of the values  $(\xi_1)_F$  or  $(d\xi_1/d\bar{x})_F$ , determines the abscissa of point  $F$  through relation (20) and the mass flow rate on the strength of relation (14).

(e) Any solution that does not bear out both equation (18) and equation (19) presents two extremes, or two points at which  $dv_1/d\bar{x}$  is infinite.

Thus, the given initial condition and the continuous nature of the solution obtained define this solution and the corresponding flow rate uniquely.

A case in point is a solution  $v_l$  common to both relaxed and the equilibrium flows. They can be shown to share the following conditions between them:

(1) the nozzle equation is  $\mathcal{A}_1 - \mathcal{A}_{1c} = (ax+b)^2$

(2)  $v_1 = (v_1)_e$ , irrespective of the value of  $\bar{x}$  is a particular solution of the relaxed flow.

(3)  $d\xi_1/d\bar{x} = (d\xi_1/d\bar{x})_e$  is a particular solution of the relaxed flow (Fig.1).

The solution common to both flows is accordingly:

$$\left. \begin{aligned} v_1 &= (v_1)_e = m\bar{x} + q \\ \xi_1 &= (\xi_1)_e + m \\ (\xi_1)_e &= -(m\bar{x} + q) \\ \mathcal{A}_1 &= \mathcal{A}_{1C} + \frac{\Gamma+1}{2} \left( m\bar{x} + q + \frac{Z}{\Gamma+1} \right)^2 \end{aligned} \right\} \quad (22)$$

where  $m$  and  $q$  are two constant numbers, the first of which is positive;

$$\left. \begin{aligned} \dot{m}_2 &= (\mathcal{A}_1)_C + Zm + \frac{Z^2}{2(\Gamma+1)^2} \\ (\dot{m}_2)_e &= (\mathcal{A}_1)_C + \frac{Z^2}{2(\Gamma+1)^2} \end{aligned} \right\} \quad (23)$$

At point  $F$  the conditions are as follows:

$$\left. \begin{aligned} \bar{x}_F &= -\frac{q}{m} = \bar{x}_C + \frac{Z}{m(\Gamma+1)^2} \\ (\mathcal{A}_1)_F &= (\mathcal{A}_1)_C + \frac{Z^2}{2(\Gamma+1)^2} \\ (v_1)_F &= 0, \quad (\xi_1)_F = m \end{aligned} \right\} \quad (24)$$

Over and above the particular solution given above, there are of course alternative ones covering different mass flow rates for the same nozzle.

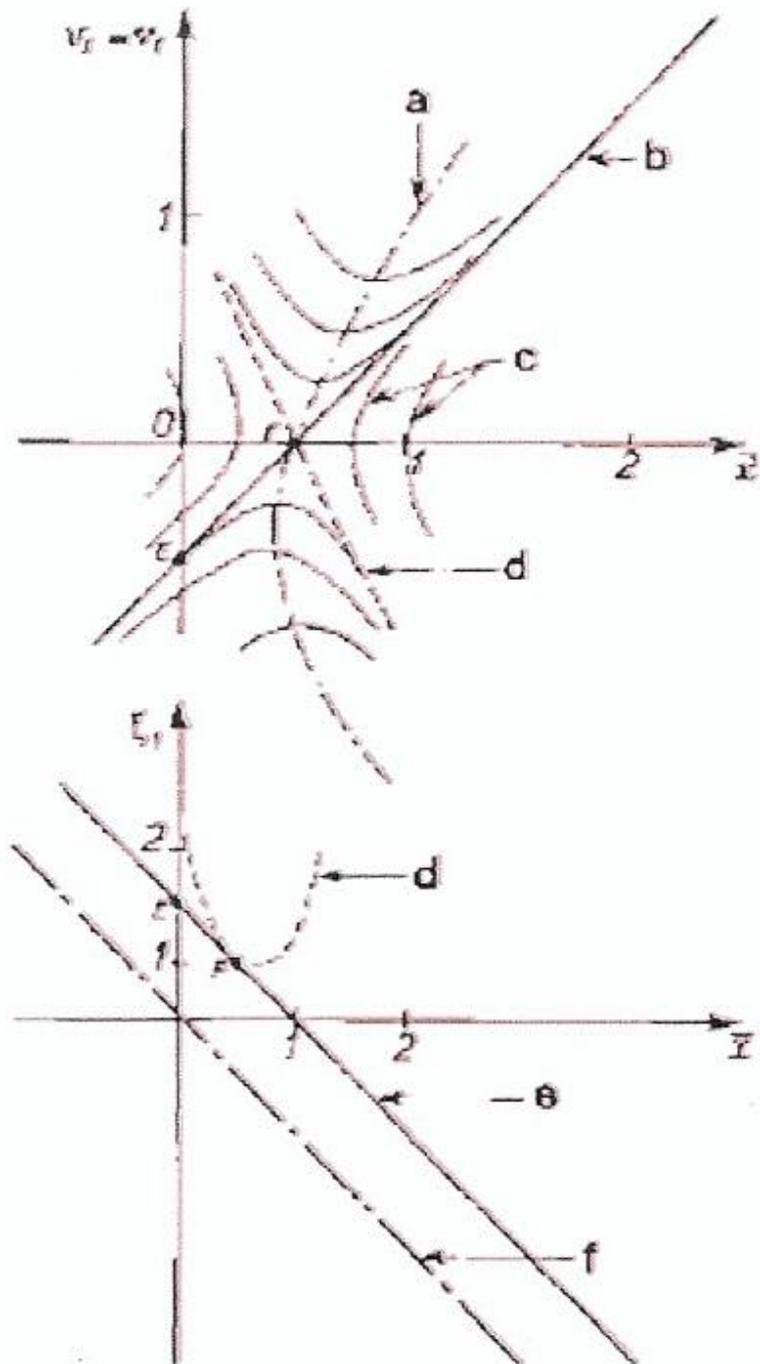


Fig.1. Solution with a fixed mass flow rate perturbation term  $\dot{m}_2$ :  $\dot{m}_2(\mathcal{A}_1)_c + 1.25$ ;  $(\dot{m}_2)_e = (\mathcal{A}_1)_c + 0.25$ . Solutions  $v_1(\bar{x})$ . Continuous solutions  $\xi_1(\bar{x})$ . ( $Z = \Gamma = 1$ )

- a: Locus of horizontal tangent points
- b: Relaxed (increasing  $v_I$ ) Equilibrium
- c: Relaxed (alternative solution)
- d: Relaxed (decreasing  $v_I$ )
- e: Relaxed (increasing  $v_I$ )
- f: Equilibrium

Amongst the equations in Table 2, there is only one, relating to  $D_I$  of order  $e$  and of the Ricatti type, whose solution presents certain difficulties. The rest of the equations contained in the tables have equally been analysed [4]; in order, however, to avoid spinning out this study to undue length, the results are not shown here.

### 3. Plane Two-Dimensional Flows

The foregoing method is also applicable to plane two-dimensional flows.

For the same basic flow and reference values as before, the velocity components  $u$  and  $w$  are now written as

$$\left. \begin{aligned} u &= u_0 + \varepsilon^n u_1 + \varepsilon^{2n} u_2 + \dots \\ w &= \varepsilon(w_1 + \varepsilon^n w_2 + \dots) \end{aligned} \right\} \quad (25)$$

and the reduced co-ordinates  $x$  and  $y$  as

$$\left. \begin{aligned} \bar{x} &= \varepsilon^{\alpha n} \frac{x^+}{l_r^+} = \varepsilon^{(\alpha-\delta)n} \frac{x^+}{l_c^+} \\ \bar{y} &= \varepsilon^{(\alpha-1)n+1} \frac{y^+}{l_r^+} = \varepsilon^{(\alpha-\delta-1)n+1} \frac{y^+}{l_c^+} \end{aligned} \right\} \quad (26)$$

Analysis of the first order admits the use of the velocity potential  $\Phi_1$  notably

$$u_1 = \Phi_{1\bar{x}}, \quad w_1 = \Phi_{1\bar{y}} \quad (27)$$

This yields Tables 5-8.

The above equations, which link up the partial derivatives of the velocity potential  $\Phi_1$  fall into three types. viz.:

1. Second or third-order linear equations
2. Non-linear equations capable of linearization on passing into the hodograph plane
3. Non-linear equations that are not simplified by transformation in the hodograph plane.

The second-order linear equations correspond to the subsonic or supersonic flow of a simple gas.

The third-order linear equations have been, studied partly by Vicenti [5], partly by Clarke [6] for particular limit conditions.

The equations of type 2 correspond to the transonic flow of a simple gas.

We have worked out some particular solutions for the equations of type 3. The method employed should also enable approximate solutions to be found for them.

Writing:

$$\left. \begin{aligned} \Phi_{1\bar{x}} &= z_1, & \Phi_{1\bar{y}} &= z_2, \\ \Phi_{1\bar{x}\bar{x}} &= z_3, & \Phi_{1\bar{x}\bar{y}} &= z_4, & \Phi_{1\bar{y}\bar{y}} &= z_5, \end{aligned} \right\} \quad (28)$$

etc.

a partial-derivative equation appears in the form

$$F(z_1, z_2, \dots) = 0 \quad (29)$$

where  $F$  is a polynomial of the second degree.

If  $\mathbf{P}$  denotes the vector of co-ordinates  $z_1, z_2, \dots$  and  $\mathbf{P}_0$  a particular vector that provides a test of the preceding equation  $F(\mathbf{P}_0) = 0$ , then we can write

$$F(\mathbf{P}) = F_1(\mathbf{P} - \mathbf{P}_0) + \frac{1}{2} F_2(\mathbf{P} - \mathbf{P}_0), \quad (30)$$

where  $F_1$  and  $F_2$  are respectively first- and second-degree homogeneous polynomials in  $(z_j - z_{j0})$ .

Replacing equation  $F(\mathbf{P}) = 0$  by the linear partial differential equation  $F_1(\mathbf{P} - \mathbf{P}_0) = 0$ , the tangent hyperplane is substituted a point  $\mathbf{P}_0$  for the hyper surface  $F(\mathbf{P}) = 0$ .

The exact solutions obtained at the same time bear out the equations

$$\left. \begin{array}{l} F_1 = 0 \\ F_2 = 0 \end{array} \right\} \quad (31)$$

For example, the equation in  $\Phi_1$  (table 5) which corresponds to  $D_1 = \mathcal{O}\left(\varepsilon^{\frac{3}{2}}\right)$  is written in the form

$$F(z_1, \dots) \equiv (\Gamma + 1)z_1 z_6 + (\Gamma + 1)z_3^2 - Z z_6 - z_8 + (\Gamma + 1)z_1 z_3 - z_5 = 0 \quad (32)$$

Functions  $F_1$  and  $F_2$  are as follows:

$$\left. \begin{array}{l} F_1 \equiv (\Gamma + 1)(z_{3,0} + z_{6,0})(z_1 - z_{1,0})(\Gamma + 1)(2z_{3,0} + z_{1,0})(z_3 - z_{3,0}) \\ - (z_5 - z_{5,0}) + [(\Gamma + 1)z_{1,0} - Z](z_6 - z_{6,0}) - (z_8 - z_{8,0}), \\ F_2 \equiv 2(\Gamma + 1) \left[ (z_1 - z_{1,0})(z_3 - z_{3,0}) + (z_1 - z_{1,0})(z_6 - z_{6,0}) + (z_3 - z_{3,0})^2 \right] \end{array} \right\} \quad (33)$$

The partial-differential equation  $F_2 = 0$  is written:

$$(\Phi_{1\bar{x}} - z_{1,0})(\Phi_{1\bar{x}\bar{x}} - z_{3,0}) + (\Phi_{1\bar{x}} - z_{1,0})(\Phi_{1\bar{x}\bar{x}\bar{x}} - z_{6,0}) + (\Phi_{1\bar{x}\bar{x}} - z_{3,0})^2 = 0 \quad (34)$$

$$z_3 = z_{3,0} = m \quad (35)$$

$$z_6 + \frac{1}{2} z_3 = z_{6,0} + \frac{1}{2} z_{3,0} \quad (36)$$

This leads to the following solutions common to  $F_1 = 0$  and  $F_2 = 0$ :

$$\Gamma_1 = \frac{m}{2} \bar{x}^2 + (\alpha_2 \bar{y}^2 + \alpha_1 \bar{y} + \alpha_0) \bar{x} - \beta_4 \bar{y}^4 + \beta_3 \bar{y}^3 + \beta_2 \bar{y}^2 + \beta_1 \bar{y} + \beta_0 \quad (3?)$$

where

$$\alpha_2 = \frac{\Gamma+1}{2}m^2, \quad \beta_4 = \frac{\alpha_2^2}{6m},$$

$$\beta_3 = \frac{\alpha_1\alpha_2}{3m}, \quad \beta_2 = \frac{\alpha_0\alpha_2}{m}$$
(38)

This solution depends on four arbitrary parameters, with the constant  $m$  fixed. The other solution is

$$\Phi_1 = \alpha(\bar{y})e^{-\bar{x}/2} + \beta(\bar{y})\bar{x} + \gamma(\bar{y})$$
(39)

where

$$\left. \begin{aligned} \gamma &= \gamma_2\bar{y}^2 + \gamma_1\bar{y} + \gamma_0 \\ \beta &= \beta_2\bar{y}^2 + \beta_1\bar{y} + \beta_0 \\ \frac{d_2\alpha}{d\bar{y}^2} - \frac{1}{4}[Z + (\Gamma+1)\beta] &= 0 \end{aligned} \right\}$$
(40)

This form a class of solutions dependent of eight arbitrary parameters. The equation in  $\alpha$  can be reduced to a Riccati equation by writing

$$z = \frac{1}{\alpha} \frac{d\alpha}{d\bar{y}}$$
(41)

so that

$$\frac{dz}{d\bar{y}} + z^2 - \frac{1}{2}[Z + (\Gamma+1)\beta] = 0$$
(42)

These solutions afford the means of determining the particular lines of the flow.

Where there is a symmetry with respect to axis  $O\bar{x}$  equation (37) yields a solution for the symmetrical nozzle:

$$\left. \begin{aligned} \Phi_1 &= \frac{m}{2}\bar{x}^2 + \left(\frac{\Gamma+1}{2}m^2\bar{y}^2 + \alpha_0\right)\bar{x} + \frac{(\Gamma+1)^2}{24}m^3\bar{y}^4 + \alpha_0\frac{\Gamma+1}{2}m\bar{y}^2 + \beta_0 \\ u_1 &= m\bar{x} + \frac{\Gamma+1}{2}m^2\bar{y}^2 + \alpha_0 \\ w_1 &= (\Gamma+1)m^2\bar{x}\bar{y} + \frac{(\Gamma+1)^2}{6}m^3\bar{y}^3 + \alpha_0(\Gamma+1)m\bar{y} \\ \xi_1 &= -m\bar{x} + m - \frac{\Gamma+1}{2}m^2\bar{y}^2 - \alpha_0 \end{aligned} \right\}$$
(43)

It will be noted that this solution also verifies the equilibrium equation for  $\Phi_1$ , though not for  $\xi_1$ ; thus

$$\Phi_1 = (\Phi_1)_e, \quad \xi_1 = (\xi_1)_e + m$$
(44)

is a particular solution of the relaxed flow equation.

Since E and F are two symmetry axis points of the same definition as in the one-dimensional case, we have

$$\left. \begin{aligned} (u_1)_E &= 0, \\ (u_1)_F &= \frac{Z}{\Gamma + 1} \end{aligned} \right\}$$

The corresponding flow is shown in Fig. 2.

Equation (39) also provides symmetric solutions with respect to  $O\bar{x}$ . With  $\gamma = Z = 1$ , one of these solutions is

$$\Phi_1 = \cos \frac{\bar{y}}{2} e^{-\bar{x}/2} - \bar{x}$$

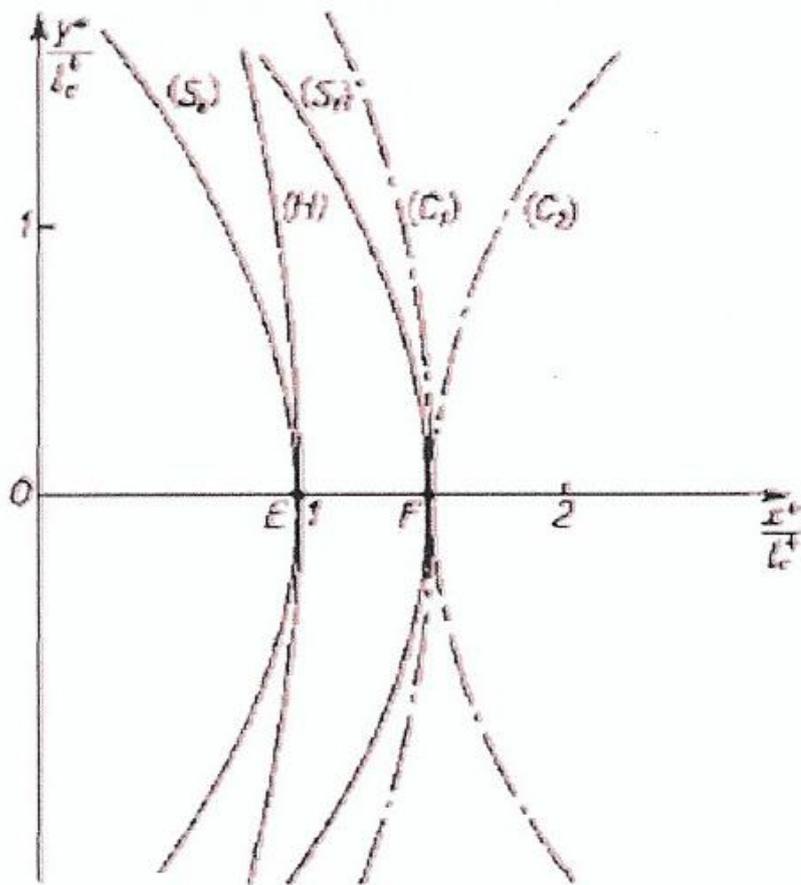


Fig. 2. Particular lines for a “nozzle solution”.

- ( $S_e$ ) Equilibrium sonic line
- ( $S_f$ ) Frozen sonic line
- (H) Throat line
- ( $C_1$ ), ( $C_2$ ) Transonic characteristic at F

A schematic view of the relevant flow can be seen in Fig. 3.

Particular solutions may be obtained for the equations in the other tables; notably, there are invariably polynomial solutions for  $\Phi_1$  according to  $\bar{x}$  and  $\bar{y}$ .

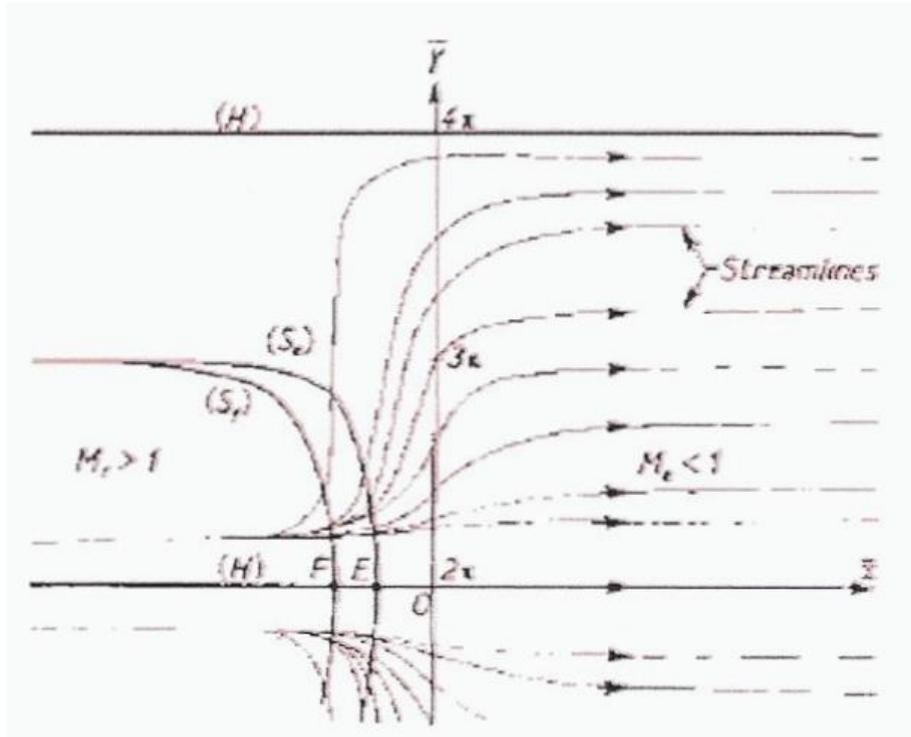


Fig. 3. Particular lines for

$$\Phi_1 = \cos \frac{\bar{y}}{2} e^{-\bar{x}/2} - \bar{x}$$

#### 4. Conclusion

This paper reviews the problems involved in solving the equations of transonic flow with chemical relaxation.

The quasi-one-dimensional equations are easy to solve, in one instance, we were able to point up, and provide the answer to the problem of the flow rate.

We have found particular solutions for the two-dimensional equations.

Lastly, there is the matter of choosing from among the equations derived from the asymptotic expansion method. While, as has been shown, Damköhler's first parameter has provided the means of making such a choice the prevailing non-uniformity may well call for the use of several equations to describe the transonic zone. If so, the solutions of these different equations will have to be tied up, a task that lay the scope of our study.

Nor have we dealt with the problem of flows fully out of chemical equilibrium. Whereas in powers of  $\epsilon$  are generally adequate for nozzle flows, this is not necessarily so in all cases, as can be demonstrated in respect of simple gases [7].

All the same, even these results will help to grapple with a certain number of particular problems.

## References

- [1] Napolitano L G., Transonic approximation for reacting mixtures, *Israel Jl Technol.* **4**, No. 1 (1966).
- [2] Gurney L.P., Napier, R. A. and Montchiloff L. N., Integration of near-equilibrium flows in propulsive nozzles. Meeting of the American Institute of Chemical Engineers, San Francisco, 16-19 May (1965).
- [3] Van Dyke M., Perturbation methods in fluid mechanics. *Appl. Math. Mech.*, N. Y. 8.
- [4] Prud'homme, R, Ecoulements relaxés dans les tuyères, ONERA Note Technique No **146** (1969).
- [5] Vincenti W. G., Non equilibrium flow over a wavy wall, *J. Fluid Mech.* **6**, part 4(1959).
- [6] Clarke J. F. The linearized flow of a dissociating gas, *J. Fluid Mech.* **7**, part 4(1960).
- [7] Cole J. O. and Messiter A. F., Expansion procedure and similarity laws for transonic flow. *Z. angew. Math Phys.* **8**, (1957).

**Résumé** - Analyse d'écoulement transsonique avec réactions chimiques par la méthode des perturbations. Le problème uni- et bi-dimensionnel. La présente étude concerne la mise en équation et la solution d'équations transsoniques quasi-unidimensionnelles d'un écoulement avec réactions chimiques. Des solutions particulières sont examinées pour un tel écoulement lorsqu'il est bi-dimensionnel.

Les équations sont écrites selon la méthode des petites perturbations uniques autour d'un écoulement uniforme en équilibre chimique.