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ON QUANTUM COMPLEX FLOWS

THIERRY PAUL

ABSTRACT. We study the propagation of quantum Töplitz observables through quantized complex linear canonical transformation of one degree of freedom systems. We associate to such a propagated observable a non local “Töplitz” expression involving off diagonal terms. We study the link of this construction with the usual Weyl symbolic paradigm.

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1. INTRODUCTION AND MAIN RESULT

Complex quantum Hamiltonians have (re)gained a lot of interest these last years, see e.g. the book [L10] and all the references quoted there. It seems that, at the contrary, quantization of complex symplectic flows didn’t get the interest it deserves as it did at the period of the birth of Fourier integral operators, see [MS74] for example. Inside this category, representation of *linear* complex symplectic flows, namely the complex symplectic group and its corresponding complex metaplectic representation has also lost most of interest since the golden years of “group theoretical methods in physics”, see [G75].

Let us remark that, since complex numbers are (according to us) necessary to the formulation of quantum mechanics, considering the Schrödinger equation on a Hilbert space \mathcal{H} with complex (e.g. bounded) Hamiltonian H does not create any intrinsic a priori difficulty, a property that classical mechanics (on a real symplectic phase space \mathcal{P}) doesn’t share with its quantum counterpart. The quantum flow is still given by $e^{-itH/\hbar}$ which exists (of course it is not anymore unitary) for all time for example when H is bounded. $e^{-itH/\hbar}$ defines for any $t \in \mathbb{R}$ a bounded operator on \mathcal{H} , at the contrary of the Lie exponential \mathcal{L}^{-th} associated to the symbol h of H which doesn’t apply on \mathcal{P} because the Hamiltonian vector field associated to h is not tangent to \mathcal{P} anymore.

This complex setting is not the only one leading to a situation where quantum mechanics is well defined while its underlying classical counterpart is not. Little regular potentials (or more generally Hamiltonians) generating vector fields below Cauchy-Lipschitz regularity condition are also examples of perfectly well defined quantum dynamics having a classical counterpart ill-defined, at least in the standard sense of this word. Another situation deals with classical chaotic systems, and the semiclassical approximation of its quantum counterpart in the case where the Planck constant \hbar vanishes and the time evolution diverges, the two limits being taken at the same time and being correlated. This twofold limit correspond to the classical (in the sense of $\hbar \rightarrow 0$) limit of the quantum flow taken at time infinite and could be naively though being the classical dynamics at time $t = \infty$, which doesn't exist, especially for chaotic systems.

We have been studied these singular (with respect to the classical underlying dynamics) situations in a series of articles dealing with the semiclassical approximation of the quantum dynamics. In [AP13] was shown on examples potentials giving rise to BV (and not more) vector field how quantum initial conditions of the Schrödinger equation select between the several¹ possible classical trajectories the one obtained at the classical limit ("superposition of them are also reachable). In [AFFGP11, FLP12] is shown in the general BH case how the Di Perna-Lions [DPL88] classical flow associated to these singular Hamiltonians is recovered at the classical limit, see [P13] for a (short) review of results concerning singular potentials. Long time semiclassical "chaotic" evolution was first apprehended in [P08], where splitting and reconstruction of evolved coherent states was shown to happen around hyperbolic fixed points of the classical dynamics, and is studied more systematically in [P12] where we show that the limit $\hbar \rightarrow 0, t \rightarrow \infty$ leads to non standard classical limit : the phase space - locus where the symbol of the evolved observable is defined - becomes a noncommutative space (space of leaves of the invariant foliation of the classical flow). Finally in [P16] we studied "quantum" observables constructed in the framework of topological quantum fields theory involving even more singular (below continuous) naive symbols. A change of paradigm dealing with operator valued symbols is introduced in order to define the "right" symbol (and the right underlying phase-space).

In all these papers, the phase space (and therefore the dynamics) obtained by taking the classical limit $\hbar \rightarrow 0$ has had to be changed from the (standard symplectic manifold) expected ones: [P08, AP13] the limit dynamics becomes probabilistic (or ubiquitous), in [AFFGP11, FLP12] the limit flow is only defined almost everywhere and in [P12, P16] the phase space becomes a noncommutative space. Moreover in [P12, P16], though it is the standard quantum dynamics which is studied, new types of quantizations were needed both in a form of, say, non local Töplitz quantization. The goal of the present little paper is to show how the *standard* quantization of complex linear canonical transformations can be understood in terms of *real* flows (actually two) and a genuine change of type of quantization.

¹due to ill-definiteness of the dynamics not satisfying Cauchy-Lipschitz condition

We will study conjugation of quantum observables given by the Töplitz (anti-Wick) construction by complex metaplectic operators. In order to avoid heavy notation, we will be, in most of the paper, concern with the one degree of freedom case: a two dimensional phase space that will be, for simplicity, \mathbb{C} .

More precisely we consider operators of the form

$$H = \int_{\mathbb{C}} h_{\beta}(z) |\psi_z^{\beta}\rangle \langle \psi_z^{\beta}| \frac{dz d\bar{z}}{2\pi\hbar}.$$

where the family of coherent states ψ_z^{β} is defined, for $\Im\beta > 0$, by

$$\psi_z^{\beta} = \frac{e^{-\frac{\beta(x-z^*)^2}{2\hbar}} e^{i\frac{z^*x}{\hbar}}}{(\pi\hbar/\beta)^{\frac{1}{4}}}$$

Note that the standard Töplitz quantization correspond to $\beta = 1$, but β will show to be a true dynamical variable, so we need to consider it as a degree of freedom. the link between different $h_{\beta}(\cdot)$ leading to the same operator H (in particular the standard case) is given below, in Appendix B, Lemma 17. One has

$$\int_{\mathbb{C}} h_{\beta}(z) |\psi_z^{\beta}\rangle \langle \psi_z^{\beta}| \frac{dz d\bar{z}}{2\pi\hbar} = \int_{\mathbb{C}} h_{\beta'}(z) |\psi_z^{\beta'}\rangle \langle \psi_z^{\beta'}| \frac{dz d\bar{z}}{2\pi\hbar} \iff h_{\beta} = e^{-\frac{\hbar}{4}(\beta-\beta')\Delta_x - \frac{\hbar}{4}(\frac{1}{\beta} - \frac{1}{\beta'})\Delta_{\xi}} h_{\beta'}, \quad (1.1)$$

for $\Re(\beta - \beta') < 0$. $\Re(\frac{1}{\beta} - \frac{1}{\beta'}) < 0$.

An easy computation shows that, when $h_{\beta}(z) := 1$, $H = \text{Id}$ for any value of β .

We consider the operator $U(S)^{-1}HU(S)$, conjugated of H by the operator $U(S)$ where S is a complex 2×2 matrix of determinant one and U is the metaplectic representation. More precisely $U(S)$ is the operator of integral kernel given by

$$\begin{aligned} U(x, y) &= \frac{1}{\sqrt{b2\pi\hbar}} e^{-\frac{i}{2b\hbar}(dx^2 - 2xy + ay^2)} & b \neq 0 \\ &= \frac{1}{\sqrt{a}} \delta\left(\frac{x}{a} - y\right) e^{i\frac{c}{a}\frac{x^2}{2}} & b = 0 \end{aligned} \quad (1.2)$$

In order to show of (1.2) can be derived, let us recall that one way to define the metaplectic representation is through the formula

$$S \begin{pmatrix} x \\ -i\hbar\partial_x \end{pmatrix} = \begin{pmatrix} U(S)xU(S)^{-1} \\ U(S)(-i\hbar\partial_x)U(S)^{-1} \end{pmatrix}. \quad (1.3)$$

Writing that $U(S)$ is unitary by $U(S)^{-1} = U(S)^*$, that is $U^{-1}(x, y) = \overline{U(y, x)}$ we get for $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, an equation whose solution is (1.2), of course modulo a global phase.

It is well known and easy to derive after (1.3) that, when S is real the Weyl symbol of $U(S)HU(S)^{-1}$ is the push-forward of the Weyl symbol of H by S . namely

$$\sigma_{U(S)HU(S)^{-1}}^{Weyl} \begin{pmatrix} q \\ p \end{pmatrix} = \sigma_H^{Weyl} \left(S^{-1} \begin{pmatrix} q \\ p \end{pmatrix} \right). \quad (1.4)$$

An easy computation, see Lemma 18 below, shows that this implies the following result.

$$U(S)HU(S)^{-1} = \int h_{\beta_S(\beta)}(S^{-1}z) |\psi_z^\beta\rangle\langle\psi_z^\beta| \frac{dzd\bar{z}}{2\pi\hbar} \text{ with } \beta_S(\beta) = iS \cdot (-i\beta),$$

where $S \cdot z = \frac{az+b}{cz+d}$.

Of course this formula doesn't make any sense when S is not real any more for general symbols h . But we will see that there is a general "off-diagonal" Töplitz representation of the form

$$U(S)HU(S)^{-1} = C_S \int h_{\beta_S(\beta)}(T_S^{-1}(z)) |\psi_{z_S}^{\beta_S^{-1}\bar{S}}\rangle\langle\psi_z^{\beta'_S}| \frac{dzd\bar{z}}{2\pi\hbar}.$$

The main result of the present short note is the following theorem.

Notation: through the all paper we will denote by $z_\star \in \mathbb{R}$ and $z^\star \in \mathbb{R}$ the real and imaginary parts of $z \in \mathbb{C}$, i.e. $z = z_\star + iz^\star$ and S^c the matrix complex conjugate to S .

Theorem 1 (Off-diagonal Töplitz representation). *Let $\psi_z^\beta = \frac{e^{-\frac{\beta(x-z_\star)^2}{2\hbar}} e^{iz^\star x}}{(\pi\hbar/\beta)^{\frac{1}{4}}}$, $\beta_\star > 0$ and*

$$H = \int h(z) |\psi_z^\beta\rangle\langle\psi_z^\beta| \frac{dzd\bar{z}}{2\pi\hbar}.$$

Then, for $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ satisfying

$$(\bar{a}d - \bar{c}b)_\star - \beta(\bar{d}b + \bar{c}a)_\star > 0, \quad (\star)$$

$$U(S)^{-1}HU(S) = \int h_{\beta_{S^c}(\beta)}(\beta T_{S^c} z) \frac{e^{\frac{\mathbb{Q}_S(z)}{\hbar}}}{D_S^\beta D_{S^{-1}}^\beta} |\psi_{\beta T_{S^{-1}S^c}(z)}^{\beta_{S^{-1}S^c}(\beta)}\rangle\langle\psi_z^\beta| \frac{dzd\bar{z}}{2\pi\hbar\beta},$$

where, for all $S' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2, \mathbb{Z})$,

$$\beta_{S'}(\beta) = \frac{-c' + id'\beta}{b'\beta + ia'} = iS'^{-1} \cdot (-i\beta)$$

$${}_{\beta}T_{S'}(q + ip) = q^{S'} + ip^{S'} \text{ defined through}$$

$$\beta_{S'}(\beta)q^{S'} + ip^{S'} = \beta_{S'}(\beta)q_{S'} + ip_{S'}, \quad \begin{pmatrix} q_{S'} \\ p_{S'} \end{pmatrix} := S' \begin{pmatrix} q \\ p \end{pmatrix}$$

$h_{\beta'}(z)$ is given by (B.1)

$$D_{S'}^{\beta} = \frac{(\beta_{\star}/(\beta_{S'}_{\star}))^{\frac{1}{4}}}{(b'\beta + ia')^{1/2}}$$

and

$$\delta = \left[\frac{b}{b\beta - id} + \frac{\bar{b}}{\bar{b}\beta + i\bar{d}} \right] ((T_{\bar{S}}^{-1}z)_{\star} + i\beta(T_{\bar{S}}^{-1}z)_{\star})^2$$

$$2\mathbb{Q}_S(z) = (T_{S^{-1}}T_{\bar{S}}^{-1}z)_{\star}^2\beta_{S^{-1}} - 2(T_{\bar{S}}^{-1}z)_{\star}^2\beta + z_{\star}^2\beta_{\bar{S}} - \delta.$$

Note that, in particular

$$T_{\bar{S}}^{-1} = \frac{1}{(\beta_{\bar{S}})_{\star}\beta|b\beta - id|^2} \begin{pmatrix} ((\overline{d - ib\beta})\beta_{\bar{S}})_{\star} & -(\overline{d - ib\beta})_{\star} \\ -((d - ib\beta)\bar{\beta}_{\bar{S}}\beta)_{\star} & ((d - ib\beta)\beta)_{\star} \end{pmatrix}, \quad (1.5)$$

$$T_{S^{-1}}T_{\bar{S}}^{-1} = \frac{1}{(\beta_{\bar{S}})_{\star}(\beta_{S^{-1}})_{\star}\beta|b\beta - id|^4} \begin{pmatrix} ((d^2 + b^2\beta^2)\bar{\beta}_{\bar{S}}\beta)_{\star} & ((d^2 + b^2\beta^2)\beta)_{\star} \\ ((d^2 + b^2\beta^2)\beta_{S^{-1}}\bar{\beta}_{\bar{S}}\beta)_{\star} & ((d^2 + b^2\beta^2)\beta_{S^{-1}}\beta)_{\star} \end{pmatrix}. \quad (1.6)$$

Remark 2. We could have thought to try to use $U(S)^{-1}HU(S)\psi_z^{\beta}$ for H pseudo, but this creates $O(\hbar^{\infty})$ terms in competition with the large, as $\hbar \rightarrow 0$, terms of the norm of $U(S)\psi_z^{\beta}$.

The main result of this paper shows that, associated to the complex mapping (flow) S are associated two real (non flow) mappings.

An example. $a = d = 1$, $b = 0$, $c = -it$ (quantum multiplication by $e^{-tx^2/2}$) treated in Appendix 3. we have

$$\beta_S(\beta) = \beta + t, \quad \begin{pmatrix} q_S \\ p_S \end{pmatrix} = \begin{pmatrix} \frac{\beta_{\star}}{\beta_{\star} + t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \quad C = \frac{-t\beta}{\beta + t} \times \frac{q^2}{2}, \text{ for } \beta > 0$$

we get ($\beta > 0$)

$$U(S)^{-1}HU(S) = \int h \left(\frac{\beta + t}{\beta} q, p \right) e^{\frac{t^2(\beta+t)q^2}{\beta(\beta-t)\hbar}} |\psi_{(\frac{\beta+t}{\beta-t}q, p)}^{\beta-t} \rangle \langle \psi_z^{\beta+t} | \frac{dzd\bar{z}}{2\pi\hbar\beta},$$

We computed in Section 3 several simple examples.

2. PROOF OF THEOREM 1

Lemma 3. Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) = Spn(2, \mathbb{R})$. We define $U = U(S)$ the operator on $L^2(\mathbb{R})$ defined through its integral kernel

$$\begin{aligned} U(x, y) &= \frac{1}{\sqrt{b2\pi\hbar}} e^{-\frac{i}{2b\hbar}(dx^2 - 2xy + ay^2)} & b \neq 0 \\ &= \frac{1}{\sqrt{a}} \delta\left(\frac{x}{a} - y\right) e^{i\frac{c}{a}\frac{x^2}{2}} & b = 0 \end{aligned}$$

(remark that U is continuous as an operator as $b \rightarrow 0$. Then U is unitary and

$$U(S)^{-1} \begin{pmatrix} Q \\ P \end{pmatrix} U(S) = S \begin{pmatrix} Q \\ P \end{pmatrix}$$

Lemma 4. Let, for $\beta \in \mathbb{C}$, $\Re\beta > 0$ and $(q, p) \in \mathbb{R}^2$ the coherent state $\psi_{q,p}^\beta$ defined by

$$\psi_{q,p}^\beta(x) := \left(\frac{\Re\beta}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{\beta}{2\hbar}(x-q)^2} e^{i\frac{px}{\hbar}}.$$

Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) = Spn(2, \mathbb{C})$ and let $U = U(S)$ be defined through its integral kernel

$$\begin{aligned} U(x, y) &= \frac{1}{\sqrt{b2\pi\hbar}} e^{-\frac{i}{2b\hbar}(dx^2 - 2xy + ay^2)} & b \neq 0 \\ &= \frac{1}{\sqrt{a}} \delta\left(\frac{x}{a} - y\right) e^{-i\frac{c}{a}\frac{x^2}{2}} & b = 0 \end{aligned}$$

Then (when it exists, that is when $\Re(i\frac{a}{b} + \beta) > 0$ or $b = 0$)

$$U(S)\psi_{q,p}^\beta = D e^{\frac{c}{\hbar}} \psi_{qS, pS}^{\beta_S}$$

where, denoting $z_\star = \Re z$, $z^\star = \Im z$,

$$\begin{aligned} \beta_S &= \frac{1 + id[b\beta + ia]}{b^2\beta + iab} = \frac{-c + id\beta}{b\beta + ia} \\ \begin{pmatrix} -\beta^\star & -1 \\ \beta_\star & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} &= \begin{pmatrix} ((b\beta + ia)\beta_S)_\star & -(b\beta + ia)^\star \\ ((b\beta + ia)\beta_S)^\star & (b\beta + ia)_\star \end{pmatrix} \begin{pmatrix} q_S \\ p_S \end{pmatrix} \\ &\Downarrow \text{ for } (\beta_S)_\star \neq 0 \text{ since } |b\beta + ia| \neq 0 \\ \begin{pmatrix} q_S \\ p_S \end{pmatrix} &= \frac{1}{(\beta_S)_\star |b\beta + ia|^2} \begin{pmatrix} (b\beta + ia)_\star & (b\beta + ia)^\star \\ -(\beta_S(b\beta + ia))^\star & (\beta_S(b\beta + ia))_\star \end{pmatrix} \begin{pmatrix} -\beta^\star & -1 \\ \beta_\star & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \\ &= \frac{1}{(\beta_S)_\star |b\beta + ia|^2} \times \begin{pmatrix} ((\overline{a - ib\beta})\beta)_\star & (a - ib\beta)^\star \\ ((\overline{a - ib\beta})\beta_S\beta)^\star & ((a - ib\beta)\beta_S)_\star \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \\ &= \frac{1}{(\beta_S)_\star} \begin{pmatrix} (\frac{\beta}{a - ib\beta})_\star & (\frac{1}{a - ib\beta})^\star \\ (\frac{\beta_S\beta}{a - ib\beta})^\star & (\frac{\beta_S}{a - ib\beta})_\star \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \\ C &= \frac{1}{2}(\beta_S q_S^2 - \beta q^2 + \frac{b}{b\beta + ia}(\beta q + ip)^2) \\ D = D_S^\beta &= \frac{(\beta_\star/(\beta_S)_\star)^{\frac{1}{4}}}{(b\beta + ia)^{\frac{1}{2}}} \end{aligned}$$

Note that, as mentioned already, this is uniform as $b \rightarrow 0$ as, as $b \rightarrow 0$, $d \rightarrow \frac{1}{a} \neq 0$ and therefore $\beta_S \rightarrow i\frac{c}{a} + \frac{\beta}{a^2}$. One get

$$\begin{pmatrix} q_S \\ p_S \end{pmatrix} = \frac{1}{(\beta_S)_\star |a|^2} \begin{pmatrix} (\overline{a}\beta)_\star & a^\star \\ (\overline{a}\beta_S\beta)^\star & (\beta_S a)_\star \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

We will also denote

$$\begin{pmatrix} q^S \\ p^S \end{pmatrix} = S \begin{pmatrix} q \\ p \end{pmatrix}$$

Note that one has the following

Lemma 5.

$$\beta_S q_S + ip_S = \frac{i}{b\beta + ia}(\beta q + ip) = \beta_S q^S + ip^S.$$

3. EXAMPLES

- $a = d = 1$, $b = 0$, $c = -it$ (quantum multiplication by $e^{-tx^2/2}$)

$$\beta_S = \beta + t, \quad \begin{pmatrix} q_S \\ p_S \end{pmatrix} = \begin{pmatrix} \frac{\beta_\star}{\beta_\star + t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \quad C = \frac{-t\beta}{\beta + t} \times \frac{q^2}{2}, \text{ for } \beta > 0$$

For $\beta = 1$ we have

$$\beta_S = 1 + t, \quad \begin{pmatrix} q_S \\ p_S \end{pmatrix} = \begin{pmatrix} \frac{1}{1+t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \quad C = -\frac{t}{1+t} \times \frac{q^2}{2}$$

- $a = d = 1, b = it$ (complex free evolution)

$$\beta_S = \frac{\beta}{t\beta + 1}, \quad \begin{pmatrix} q_S \\ p_S \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1+t\beta} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \quad C = \frac{-t\beta^2}{t\beta + 1} \times \frac{q^2}{2}, \text{ for } \beta > 0$$

- $a = d^{-1} = e^{it}, b = c = 0$ (complex dilation) (write $\beta = |\beta|e^{-2is}$)

$$\beta_S = \beta e^{-2it}, \quad \begin{pmatrix} q_S \\ p_S \end{pmatrix} = \frac{1}{\cos 2(t+s)} \begin{pmatrix} \cos(t+2s) & \frac{\sin t}{|\beta|} \\ |\beta| \sin t & \cos(t+2s) \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \quad C =$$

When $\beta = 1$ we get

$$\begin{pmatrix} q_S \\ p_S \end{pmatrix} = \frac{1}{\cos 2t} \begin{pmatrix} \cos t & \sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \quad C =$$

- $S = \begin{pmatrix} \cosh t & i \sinh t \\ -i \sinh t & \cosh t \end{pmatrix}$ (complex time harmonic oscillator)
(we take $\beta > 0$)

$$\beta_S = \frac{\sinh t + \beta \cosh t}{\cosh t + \beta \sinh t}, \quad \begin{pmatrix} q_S \\ p_S \end{pmatrix} = \begin{pmatrix} \frac{\beta}{\sinh t + \beta \cosh t} & 0 \\ 0 & \frac{\beta}{\cosh t + \beta \sinh t} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix},$$

$$C = \beta^2 \left(\frac{1}{(\sinh t + \beta \cosh t)(\cosh t + \beta \sinh t)} - 1 \right) \frac{q^2}{2}$$

When $\beta = 1$ we get

$$\beta_S = 1, \quad \begin{pmatrix} q_S \\ p_S \end{pmatrix} = e^{-t} \begin{pmatrix} q \\ p \end{pmatrix}, \quad C = (e^{-2t} - 1) \frac{q^2}{2}$$

4. MORE FEATURES

4.1. Link with Weyl.

Theorem 6. *Let H with Töplitz symbol h . Let us denote by σ_H^{\hbar} the Weyl symbol of H (note that σ_H^{\hbar} is an entire function). Then*

$$\sigma_{U(S)^{-1}HU(S)}^{\hbar} = \sigma^{\hbar} \circ S$$

and

$$\sigma_{U(S)^{-1}HU(S)}^0(q, p) = \int \widehat{h}(\xi, x) e^{-i(q^S \xi + p^S x)} dx d\xi.$$

4.2. Flows on extended phase-space. Consider on the extended phase space $\mathbb{C} \times \mathbb{C} = \{(z, \beta)\}$ the mapping

$$\Phi_S : (\beta, z) \mapsto (\beta_S(\beta), T_S^\beta z).$$

.

Theorem 7.

$$\Phi_{SS'} = \Phi_S \Phi_{S'}.$$

4.3. Passing shocks.

Proposition 8. *Let us suppose that $b\beta + ia \neq 0$ (note that it is enough to change the value of β for this condition to be satisfied). Then*

$$\forall z \in \mathbb{C}, \quad U(S)\psi_z^\beta \in \mathcal{D}'(\mathbb{R}).$$

Theorem 9. *Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $b\beta - id \neq 0 \neq d\beta + ic$, then $U(S^{-1})HU(S)$ is a bounded operator $L^2(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R})$ and is given by the formula in Theorem 1.*

Coming back to the example $a = d = 1$, $b = 0$, $c = -it$ (quantum multiplication by $e^{-itx^2/2}$), we get, at $t = \beta$,

$$U(S)^{-1}HU(S) = \int h(2q, p) e^{\frac{c_z}{\hbar}} |\psi_{(q,0)}^0\rangle \langle \psi_z^{2\beta}| \frac{dzd\bar{z}}{2\pi\hbar\beta},$$

where we denote $\forall q, \psi_{(q,0)}^0(x) = 1$.

APPENDIX A. MORE LEMMAS

Lemma 10. *Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that $S^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.*

$$U(S)^\star = U(\bar{S}) \text{ with } \bar{S} := \begin{pmatrix} \bar{d} & -\bar{b} \\ -\bar{c} & \bar{a} \end{pmatrix}$$

$$U(S)^{-1} = U(S^{-1})$$

Note that

$$\overline{SS'} = \overline{S'S}.$$

Moreover

Lemma 11.

$$(\beta_S)^\star = \beta \frac{(\bar{a}d - \bar{b}c)^\star + \beta(\bar{a}\bar{b} + \bar{c}\bar{d})^\star}{|b\beta + ia|^2}$$

$$(\beta_{\bar{S}})^\star = \beta \frac{(\bar{a}d - \bar{c}b)^\star - \beta(\bar{d}\bar{b} + \bar{c}a)^\star}{|b\beta - id|^2}$$

$$(\beta_{S^{-1}})^\star = \beta \frac{(a\bar{d} - \bar{b}c)^\star - \beta(d\bar{b} + \bar{c}a)^\star}{|b\beta - id|^2}$$

$$\begin{aligned}\beta_{S^{-1}} + \overline{\beta_{\overline{S}}} &= \frac{2\beta}{b^2\beta^2 + d^2} \\ \beta_{S^{-1}} - \overline{\beta_{\overline{S}}} &= \frac{-2i\beta}{b^2\beta^2 + d^2}\end{aligned}$$

Let us define

$$\begin{pmatrix} q_S \\ p_S \end{pmatrix} := {}_{\beta}T_S \begin{pmatrix} q \\ p \end{pmatrix}$$

Lemma 12.

$$\text{Det}_{\beta}T_S = \beta_{\star}$$

$${}_{\beta}T_{\overline{S}}^{-1} = \frac{1}{(\beta_{\overline{S}})_{\star}\beta|b\beta - id|^2} \begin{pmatrix} ((\overline{d - ib\beta})\beta_{\overline{S}})_{\star} & -(\overline{d - ib\beta})_{\star} \\ -((d - ib\beta)\beta_{\overline{S}}\beta)_{\star} & ((d - ib\beta)\beta)_{\star} \end{pmatrix}$$

and

$${}_{\beta}T_{(S)^{-1}} = \frac{1}{(\beta_{S^{-1}})_{\star}|b\beta - id|^2} \begin{pmatrix} ((\overline{d + ib\beta})\beta)_{\star} & (d + ib\beta)_{\star} \\ ((d + ib\beta)\beta_{S^{-1}}\beta)_{\star} & ((d + ib\beta)\beta_{S^{-1}})_{\star} \end{pmatrix}$$

Lemma 13.

$$\begin{aligned}q^{S^{-1}} &= \frac{\beta_{S^{-1}}(\beta)q_{S^{-1}} + \overline{\beta_{\overline{S}}(\beta)}q_{\overline{S}} + i(p_{S^{-1}} - p_{\overline{S}})}{\beta_{S^{-1}}(\beta) + \overline{\beta_{\overline{S}}(\beta)}} = \frac{z_{S^{-1}} + \overline{z_{\overline{S}}}}{\beta_{S^{-1}}(\beta) + \overline{\beta_{\overline{S}}(\beta)}} \\ p^{S^{-1}} &= \frac{\overline{\beta_{\overline{S}}(\beta)}p_{S^{-1}} - \beta_{S^{-1}}(\beta)p_{\overline{S}} - i\overline{\beta_{\overline{S}}(\beta)}\beta_{S^{-1}}(\beta)(q_{S^{-1}} - q_{\overline{S}})}{\overline{\beta_{\overline{S}}(\beta)} - \beta_{S^{-1}}(\beta)} = \frac{-i(\overline{\beta_{\overline{S}}}z_{S^{-1}} - \beta_{S^{-1}}(\beta)\overline{z_{\overline{S}}})}{\overline{\beta_{\overline{S}}(\beta)} - \beta_{S^{-1}}(\beta)}\end{aligned}$$

Proposition 14. ${}_{\beta}T_{\overline{S}}^{-1}$ and ${}_{\beta}T_{S^{-1}}$ determine S and β .

Proof. $T_{\overline{S}}^{-1}$ determine $b\beta + id$ and etc, and $T_{S^{-1}}$ determine $b\beta - id$ and etc. Moreover one has $ad - bc = 1$. Thos gives five equalities for five unknowns. \square

Proposition 15.

$${}_{\beta}T_{S^{-1}}{}_{\beta}T_{\overline{S}}^{-1} = \frac{1}{(\beta_{\overline{S}})_{\star}(\beta_{S^{-1}})_{\star}\beta|b - id|^4} \begin{pmatrix} ((d^2 + b^2\beta^2)\overline{\beta_{\overline{S}}}\beta)_{\star} & (d^2 + b^2\beta^2)_{\star} \\ ((d^2 + b^2\beta^2)\beta_{S^{-1}}\beta_{\overline{S}}\beta)_{\star} & ((d^2 + b^2\beta^2)\beta_{S^{-1}}\beta)_{\star} \end{pmatrix}$$

Moreover denoting ${}_{\beta}T_{\overline{S}}^{-1} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q' \\ p' \end{pmatrix}$ and ${}_{\beta}T_{S^{-1}}{}_{\beta}T_{\overline{S}}^{-1} \begin{pmatrix} q \\ p \end{pmatrix} = T_{S^{-1}} \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} q'' \\ p'' \end{pmatrix}$, we have

$$\beta q' + ip' = (\overline{d} + i\overline{b}\beta)(\beta_{\overline{S}}q + ip)$$

\Updownarrow

$$\beta q' + ip' = (i\overline{c} - \overline{d}\beta)q + i(\overline{d} + i\overline{b}\beta)p$$

$$\beta_{S^{-1}}q'' + ip'' = \frac{1}{d + ib\beta}(\beta q' + ip')$$

$$= \frac{\overline{d} + i\overline{b}\beta}{d + ib\beta}(\beta_{\overline{S}}q + ip)$$

\Updownarrow

$$(c + ia\beta)q'' + i(-b\beta + id)p'' = (\overline{c} + i\overline{a}\beta)q + i(-\overline{b} + i\overline{d})p.$$

We deduce that

$$\begin{aligned}
\beta' T_{S'} \beta T_S &= \beta T_{S'S}, & \beta' &= \beta_S(\beta) \\
\beta' T_{S'}^{-1} \beta T_S^{-1} &= \beta' T_{S'S'}^{-1}, & \beta &= \beta_{S'}(\beta') \\
\beta' T_{S'}^{-1} \beta T_S &= \beta T_{(S')^{-1}S}, & \beta_{S'}(\beta') &= \beta_S(\beta) \\
\beta_S(\beta_{S'}) &= \beta_{SS'} \\
\beta &= \beta_{S^{-1}}(\beta_S) \\
\beta T_S^{-1} &= \beta_S(\beta) T_{S^{-1}}
\end{aligned}$$

Note that, when $\beta > 0$, $q' = \frac{(\beta_S)^*}{\beta} q$ and $p' = p + (\beta_S)^* q$.

APPENDIX B. WEYL

Lemma 16.

$$\begin{aligned}
&\sigma_{Weyl}^{|\psi_z^\beta\rangle\langle\psi_{z'}^{\beta'}|}(x, \xi) = \\
&\sqrt{\frac{2\sqrt{\beta_\star\beta'_\star}}{\beta + \beta'}} \frac{1}{\hbar} e^{-\frac{\beta\beta'}{2(\beta+\beta')\hbar}(q+q'-2x)^2} e^{-\frac{1}{2(\beta+\beta')\hbar}(p+p'-2\xi)^2} e^{i[(p-p')x - (p+p'-2\xi)(\beta(x-q) - \beta'(x-q'))]/(\beta+\beta')\hbar]}
\end{aligned}$$

In particular when $z = z'$, $\beta = \beta' > 0$,

$$\sigma_{Weyl}^{|\psi_z^\beta\rangle\langle\psi_z^\beta|}(x, \xi) = \frac{1}{\pi\hbar} e^{-\frac{\beta(q-x)^2}{\hbar}} e^{-\frac{(p-\xi)^2}{\beta\hbar}}$$

Corollary 17.

$$\int h_\beta(z) |\psi_z^\beta\rangle\langle\psi_z^\beta| dz = \int h_{\beta'}(z) |\psi_z^{\beta'}\rangle\langle\psi_z^{\beta'}| dz.$$

if and only if

$$h_{\beta'}(x, \xi) = e^{\frac{\hbar\beta' - \beta}{\beta\beta'} \frac{\Delta x}{4} + \hbar(\beta - \beta') \frac{\Delta \xi}{4}} h(x, \xi), \quad \beta'_\star < \beta_\star, \quad \frac{\beta'_\star}{|\beta'|^2} < \frac{\beta_\star}{|\beta|^2}. \quad (\text{B.1})$$

Proof. From Lemma 16 we get that

$$e^{\frac{\hbar}{\beta} \frac{\Delta x}{4} + \hbar\beta \frac{\Delta \xi}{4}} h(x, \xi) = e^{\frac{\hbar}{\beta'} \frac{\Delta x}{4} + \hbar\beta' \frac{\Delta \xi}{4}} h_{\beta'}(x, \xi).$$

□

Lemma 18.

Proof.

□

APPENDIX C. MISCELLANEOUS

$$\begin{aligned}\bar{u}_* v_* - u^* \bar{v}^* &= (u\bar{v})_* \\ u^* v_* - \bar{u}_* \bar{v}^* &= (uv)^* \\ u_* v_* + \bar{u}^* \bar{v}^* &= (u\bar{v})_*\end{aligned}$$

$$e^{-\frac{\beta}{2\hbar}(x-z)^2} e^{i\frac{z'x}{\hbar}} = e^{-\frac{\beta}{2\hbar}(x-q)^2} e^{i\frac{px}{\hbar}} e^{-\frac{z^2-q^2}{2\hbar}}, \quad \beta q + ip = \beta z + iz'$$

$$\int e^{-\beta\frac{x^2}{2}} e^{ipx} dx = (2\pi/\beta)^{\frac{1}{2}} e^{-\frac{p^2}{2\beta}}$$

$$\langle \psi_{z'}^{\beta'} | \psi_z^\beta \rangle = \frac{(\beta\bar{\beta}')^{1/4}}{(\beta + \bar{\beta}')^{1/2}} e^{-\frac{i}{\beta+\bar{\beta}'}(p-p')(\beta q + \bar{\beta}' q')} e^{-\frac{1}{2(\beta+\bar{\beta}')} (p-p')^2} e^{-\frac{\beta\bar{\beta}'}{2(\beta+\bar{\beta}')} (q-q')^2}$$

$$\begin{aligned}& \int e^{-a\frac{q^2}{2} - b\frac{p^2}{2} + icqp + idq + iep} h(q, p) dq dp \\ & e^{-\frac{\hbar}{2(c^2+ab)} (aD_q^2 + bD_p^2 + 2icD_q D_p + 2(ea+icd)D_q/\hbar + 2(db+ice)D_p/\hbar + (e^2a+d^2b+2icde)/\hbar^2)} h(0, 0) \\ & = e^{-\frac{e^2a+d^2b+2icde}{2(c^2+ab)\hbar}} \left(e^{-\frac{\hbar}{2(c^2+ab)} (aD_q^2 + bD_p^2 + 2icD_q D_p)} h \right) \left(-\frac{ea+icd}{2(c^2+ab)}, -\frac{db+ice}{2(c^2+ab)} \right)\end{aligned}$$

REFERENCES

- [AFFGP11] L. Ambrosio, A. Figalli, G. Friesecke, J. Giannoulis, T. Paul: Semiclassical limit of quantum dynamics with rough potentials and well posedness of transport equations with measure initial data, *Comm. Pure Appl. Math.*, 64 (2011),1199-1242.
- [AP13] On the selection of the classical limit for potentials with BV derivatives , A. Athanassoulis, T. Paul "Journal of Dynamics and Differential Equations", 25, p. 33-47 (2013).
- [DPL88] R.J. DiPerna, P.L. Lions: Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98 (1989), 511-547.
- [G75] A. Grosmann, real and complex canonical transforms, *Lecture Notes in Physics*, Springer 1975
- [FLP12] A. Figalli, M. Ligabo, T. Paul: ‘ Semiclassical limit for mixed states with singular and rough potentials, *Indiana University Mathematics Journal*.
- [L10] N. Lerner, “Metrics on the phase space and non-selfadjoint pseudo-differential operators”, Birkhauser Verlag, Basel, 2010.
- [MS74] A. Mellin, J. Sjostrand, Fourier integral operators with complex phase, *CPDE* 1974.
- [P08] T. Paul, chelles de temps pour l’volution quantique petite constante de Planck , "Sminaire X-EDP 2007-2008", Publications de l’cole Polytechnique, 2008.
- [P12] T. Paul, Semiclassical approximation and noncommutative geometry, *C. R. Acad. Sci. Paris, Ser. I* 349 (2011) 1177-1182.
- [P13] T. Paul, Recent results in semiclassical approximation with rough potentials, actes de la confrence "Microlocal Methods in Mathematical Physics and Global Analysis (Microlocal 2011)", Tbingen, 14-18 juin 2011, 'Trends in Mathematics 2013', Springer (2013), 49-52.
- [P16] T. Paul, Symbolic calculus for singular curve operators, hal-01361277.