



**HAL**  
open science

# Subadditive theorems in metric measure spaces and homogenization in Cheeger-Sobolev spaces

Omar Anza Hafsa, Jean-Philippe Mandallena

► **To cite this version:**

Omar Anza Hafsa, Jean-Philippe Mandallena. Subadditive theorems in metric measure spaces and homogenization in Cheeger-Sobolev spaces. 2019. hal-02004618

**HAL Id: hal-02004618**

**<https://hal.science/hal-02004618>**

Preprint submitted on 1 Feb 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# SUBADDITIVE THEOREMS IN METRIC MEASURE SPACES AND HOMOGENIZATION IN CHEEGER-SOBOLEV SPACES

OMAR ANZA HAFSA AND JEAN-PHILIPPE MANDALLENNA

ABSTRACT. We prove subadditive theorems in the setting of metric measure spaces in the deterministic and stochastic case. Applications to homogenization of nonconvex integrals in Cheeger-Sobolev spaces are given.

## CONTENTS

1. Introduction	1
2. subadditive theorems	2
2.1. The deterministic case	3
2.2. The stochastic case	6
3. Applications to homogenization	15
3.1. Periodic homogenization	17
3.2. Stochastic homogenization	18
References	20

## 1. INTRODUCTION

Let  $(X, d, \mu)$  be a metric measure space with  $\mu$  a positive Borel measure on  $X$ . Let  $\mathcal{B}(X)$  be the class of Borel subsets of  $X$  and let  $\mathcal{B}_{\mu,0}(X)$  denote the class of  $Q \in \mathcal{B}(X)$  such that  $\mu(Q) < \infty$  and  $\mu(\partial Q) = 0$  with  $\partial Q = \overline{Q} \setminus \overset{\circ}{Q}$ . Let  $\text{Homeo}(X)$  be the group of homeomorphisms on  $X$  and let  $G$  be a subgroup of  $\text{Homeo}(X)$  for which  $\mu$  is  $G$ -invariant. Let

$$\mathcal{S} : \mathcal{B}_{\mu,0}(X) \rightarrow L^1(\Sigma, \mathcal{T}, \mathbb{P}) \quad (\text{resp. } \mathcal{S} : \mathcal{B}_{\mu,0}(X) \rightarrow \mathbb{R})$$

be a subadditive and  $G$ -covariant (resp.  $G$ -invariant), where  $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in G})$  is a measurable dynamical  $G$ -system, and let  $\{Q_n\}_{n \in \mathbb{N}^*} \subset \mathcal{B}_{\mu,0}(X)$ . In this paper we are concerned with the problem of characterizing the following limit

$$\lim_{n \rightarrow \infty} \frac{\mathcal{S}(Q_n)(\omega)}{\mu(Q_n)} \text{ for } \mathbb{P}\text{-a.a. } \omega \in \Sigma \quad \left( \text{resp. } \lim_{n \rightarrow \infty} \frac{\mathcal{S}(Q_n)}{\mu(Q_n)} \right).$$

Such limit problems are of interest for the development of homogenization of integrals of the calculus of variations in the setting of Cheeger-Sobolev spaces (see Section 3 and also [AHM17]). Other motivations can be found in the study of percolation theory (see [HW65]).

---

*Key words and phrases.* Subadditive process, Metric measure space, Amenable group, Homogenization, Cheeger-Sobolev space.

Motivated by problems of statistical mechanics, additive theorems were first proved in 1931 by von Neumann (see [Neu32]) and Birkhoff (see [Bir31]) in the context of measure preserving  $\mathbb{Z}$ -actions. Later, in 1972, Tempelman generalized these results to the multidimensional case in the context of measure  $\mathbb{Z}^N$ -actions (see [Tem72] and also Nguyen and Zessin [NZ79]) but also to the setting of amenable semi-groups (see [Kre85, Theorem 4.4]). Then, in 1999, the theorems of von Neumann and Birkhoff were also extended by Lindenstrauss to the setting of amenable groups under weaker conditions than those of Templeman (see [Lin99, Lin01]). Beside this, motivated by the study of percolation theory, subadditive theorems were also proved, in the context of measure preserving  $\mathbb{Z}^N$ -actions, first in 1968 by Kingman in the unidimensional case (see [Kin68, Kin73]) and then in 1981 by Akcoglu and Krengel in the multidimensional case (see [AK81, Kre85] and also Derriennic [Der75], Smythe [Smy76], Nguyen [Ngu79] and Licht and Michaille [LM02]). In 2014, Dooley, Golodets and Zhang extended Kingman's theorem to the setting of amenable group (see [DGZ14] and also [DZ15]). The results of the present paper can be seen as extensions of the multidimensional Akcoglu-Krengel's theorem to the setting of metric measure space  $(X, d, \mu)$  where  $\mu$  is  $G$ -invariant, with  $G$  a subgroup of  $\text{Homeo}(X)$ , having in mind applications to homogenization.

Multidimensional subadditive results of Akcoglu-Krengel type were adapted first in 1986 by Dal Maso and Modica for dealing with homogenization of convex integral functionals of the calculus of variations defined on Sobolev spaces (see [DMM86a, DMM86b]) and then in 1994 by Messaoudi and Michaille for studying nonconvex homogenization problems (see [MM94, LM02]). In the same spirit, the object of this paper is to establish subadditive theorems allowing to deal with nonconvex homogenization problems in Cheeger-Sobolev spaces.

The plan of the paper is as follows. In the next section we state and prove the main results of the paper, see Theorems 2.3 and 2.11. To establish such theorems it is necessary to make some assumptions on the sequence of sets  $\{Q_n\}_{n \in \mathbb{N}^*}$ , see Definitions 2.1 and 2.7. The deterministic case and the stochastic case are developed in §2.1 and §2.2 respectively. Finally, to illustrate our results, in Section 3 we give applications to homogenization of nonconvex integrals in Cheeger-Sobolev spaces, see Theorems 3.6 and 3.8.

## 2. SUBADDITIVE THEOREMS

Let  $(X, d, \mu)$  be a metric measure space with  $\mu$  a positive Radon measure on  $X$ . Let  $\mathcal{B}(X)$  be the class of Borel subsets of  $X$  and let  $\mathcal{B}_{\mu,0}(X)$  denote the class of  $Q \in \mathcal{B}(X)$  such that  $\mu(Q) < \infty$  and  $\mu(\partial Q) = 0$  with  $\partial Q = \overline{Q} \setminus \overset{\circ}{Q}$ , where  $\overline{Q}$  (resp.  $\overset{\circ}{Q}$ ) denotes the closure (resp. the interior) of  $Q$ . Let  $\text{Homeo}(X)$  be the group of homeomorphisms on  $X$  and let  $G$  be a subgroup of  $\text{Homeo}(X)$  for which  $\mu$  is  $G$ -invariant.

From now on, we consider  $\{\mathbb{U}_k\}_{k \in \mathbb{N}^*} \subset \mathcal{B}_{\mu,0}(X)$  with  $\mu(\mathbb{U}_k) > 0$  for all  $k \in \mathbb{N}^*$  and, for each  $k \in \mathbb{N}^*$ , we consider the class  $\mathcal{U}_k(G)$  defined by

$$\mathcal{U}_k(G) := \left\{ H \subset G : \{g(\mathbb{U}_k)\}_{g \in H} \text{ is disjoint} \right\}.$$

In what follows,  $|\cdot|$  denotes the counting measure on  $G$  and, for any  $H \subset G$ ,  $\mathcal{P}_f(H)$  denotes the class of finite subsets of  $H$ .

**2.1. The deterministic case.** The following definition set a framework for establishing a subadditive theorem in the deterministic case and in the setting of metric measure spaces (see Theorem 2.3).

**Definition 2.1.** Let  $\{Q_n\}_{n \in \mathbb{N}^*} \subset \mathcal{B}_{\mu,0}(X)$ . We say that  $\{Q_n\}_{n \in \mathbb{N}^*}$  is weakly  $G$ -asymptotic with respect to  $\{\mathbb{U}_k\}_{k \in \mathbb{N}^*}$  if for all  $k \in \mathbb{N}^*$  there exists  $\mathbb{G}_k \in \mathcal{U}_k(G)$  with the property that for all  $n \in \mathbb{N}^*$  there exist  $m_{n,k} \in \mathbb{N}^*$ ,  $g_{n,k} \in G$  and  $F_{n,k}, G_{n,k}^-, G_{n,k}^+ \in \mathcal{P}_f(\mathbb{G}_k)$  such that:

$$\bigcup_{g \in G_{n,k}^-} g(\mathbb{U}_k) \subset Q_n \subset \bigcup_{g \in G_{n,k}^+} g(\mathbb{U}_k); \quad (2.1)$$

$$\lim_{n \rightarrow \infty} \frac{\mu \left( \bigcup_{g \in G_{n,k}^+} g(\mathbb{U}_k) \setminus \bigcup_{g \in G_{n,k}^-} g(\mathbb{U}_k) \right)}{\mu(Q_n)} = 0; \quad (2.2)$$

$$G_{n,k}^+ \subset F_{n,k} \text{ and } \bigcup_{g \in F_{n,k}} g(\mathbb{U}_k) = g_{n,k}(\mathbb{U}_{m_{n,k}}); \quad (2.3)$$

$$\overline{\lim}_{n \rightarrow \infty} \frac{|F_{n,k}|}{|G_{n,k}^+|} \leq 1. \quad (2.4)$$

Let us recall the definition of a subadditive and  $G$ -invariant set function.

**Definition 2.2.** Let  $\mathcal{S} : \mathcal{B}_{\mu,0}(X) \rightarrow \mathbb{R}$  be a set function.

(a) The set function  $\mathcal{S}$  is said to be subadditive if

$$\mathcal{S}(A \cup B) \leq \mathcal{S}(A) + \mathcal{S}(B).$$

for all  $A, B \in \mathcal{B}_{\mu,0}(X)$  such that  $A \cap B = \emptyset$ .

(b) The set function  $\mathcal{S}$  is said to be  $G$ -invariant if

$$\mathcal{S}(g(A)) = \mathcal{S}(A)$$

for all  $A \in \mathcal{B}_{\mu,0}(X)$  and all  $g \in G$ .

Here is the first main result of the paper.

**Theorem 2.3.** *Let  $\mathcal{S} : \mathcal{B}_{\mu,0}(X) \rightarrow \mathbb{R}$  be a subadditive and  $G$ -invariant set function with the following boundedness condition:*

$$|\mathcal{S}(Q)| \leq c\mu(Q) \quad (2.5)$$

for all  $Q \in \mathcal{B}_{\mu,0}(X)$  and some  $c > 0$ . Then, for any  $\{Q_n\}_{n \in \mathbb{N}^*} \subset \mathcal{B}_{\mu,0}(X)$  such that  $\{Q_n\}_{n \in \mathbb{N}^*}$  is weakly  $G$ -asymptotic with respect to  $\{\mathbb{U}_k\}_{k \in \mathbb{N}^*}$ , one has

$$\lim_{n \rightarrow \infty} \frac{\mathcal{S}(Q_n)}{\mu(Q_n)} = \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)}.$$

**Proof of Theorem 2.3.** First of all, let  $\{k_j\}_{j \in \mathbb{N}^*}$  be such that

$$\lim_{j \rightarrow \infty} \frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})} = \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)}. \quad (2.6)$$

We divide the proof into three steps.

**Step 1: establishing lower bound and upper bound.** Fix any  $j \in \mathbb{N}^*$  and any  $n \in \mathbb{N}^*$  and set:

$$Q_{n,j}^- := \bigcup_{g \in G_{n,k_j}^-} g(\mathbb{U}_{k_j});$$

$$Q_{n,j}^+ := \bigcup_{g \in G_{n,k_j}^+} g(\mathbb{U}_{k_j}),$$

where  $G_{n,k_j}^-, G_{n,k_j}^+ \in \mathcal{P}_f(\mathbb{G}_{k_j})$  with  $\mathbb{G}_{k_j} \in \mathcal{U}_{k_j}(G)$  given by Definition 2.1.

**Step 1-1: lower bound.** By the right inclusion in (2.1) we have  $Q_n \subset Q_{n,j}^+$  and so  $Q_{n,j}^+ = Q_n \cup (Q_{n,j}^+ \setminus Q_n)$ . Hence

$$\mathcal{S}(Q_{n,j}^+) \leq \mathcal{S}(Q_n) + \mathcal{S}(Q_{n,j}^+ \setminus Q_n),$$

and consequently

$$\frac{\mathcal{S}(Q_{n,j}^+)}{\mu(Q_{n,j}^+)} \leq \frac{\mathcal{S}(Q_n)}{\mu(Q_n)} + \frac{\mathcal{S}(Q_{n,j}^+ \setminus Q_n)}{\mu(Q_n)}.$$

As  $Q_{n,j}^- \subset Q_n$  by the left inclusion in (2.1), we see that  $Q_{n,j}^+ \setminus Q_n \subset Q_{n,j}^+ \setminus Q_{n,j}^-$  and so

$$\mathcal{S}(Q_{n,j}^+ \setminus Q_n) \leq c\mu(Q_{n,j}^+ \setminus Q_{n,j}^-)$$

with  $c_2 > 0$  given by (2.5). It follows that

$$\frac{\mathcal{S}(Q_{n,j}^+)}{\mu(Q_{n,j}^+)} \leq \frac{\mathcal{S}(Q_n)}{\mu(Q_n)} + \frac{c\mu(Q_{n,j}^+ \setminus Q_{n,j}^-)}{\mu(Q_n)}.$$

Letting  $n \rightarrow \infty$  and using (2.2) we obtain

$$l_j := \liminf_{n \rightarrow \infty} \frac{\mathcal{S}(Q_{n,j}^+)}{\mu(Q_{n,j}^+)} \leq \liminf_{n \rightarrow \infty} \frac{\mathcal{S}(Q_n)}{\mu(Q_n)} =: l. \quad (2.7)$$

**Step 1-2: upper bound.** By the left inclusion in (2.1) we have  $Q_{n,j}^- \subset Q_n$  and so  $Q_n = Q_{n,j}^- \cup (Q_n \setminus Q_{n,j}^-)$ . Hence

$$\mathcal{S}(Q_n) \leq \mathcal{S}(Q_{n,j}^-) + \mathcal{S}(Q_n \setminus Q_{n,j}^-),$$

and consequently

$$\frac{\mathcal{S}(Q_n)}{\mu(Q_n)} \leq \frac{\mathcal{S}(Q_{n,j}^-) \mu(Q_{n,j}^-)}{\mu(Q_{n,j}^-) \mu(Q_n)} + \frac{\mathcal{S}(Q_n \setminus Q_{n,j}^-)}{\mu(Q_n)}.$$

As  $Q_n \subset Q_{n,j}^+$  by the right inclusion in (2.1), we see that  $Q_n \setminus Q_{n,j}^- \subset Q_{n,j}^+ \setminus Q_{n,j}^-$  and so

$$\mathcal{S}(Q_n \setminus Q_{n,j}^-) \leq c\mu(Q_{n,j}^+ \setminus Q_{n,j}^-)$$

with  $c > 0$  given by (2.5). It follows that

$$\begin{aligned} \frac{\mathcal{S}(Q_n)}{\mu(Q_n)} &\leq \frac{\mathcal{S}(Q_{n,j}^-) \mu(Q_{n,j}^-)}{\mu(Q_{n,j}^-) \mu(Q_n)} + \frac{c\mu(Q_{n,j}^+ \setminus Q_{n,j}^-)}{\mu(Q_n)} \\ &\leq \frac{\mathcal{S}(Q_{n,j}^-)}{\mu(Q_{n,j}^-)} + \frac{c\mu(Q_{n,j}^+ \setminus Q_{n,j}^-)}{\mu(Q_n)} \end{aligned}$$

because  $\mu(Q_{n,j}^-) \leq \mu(Q_n)$  since  $Q_{n,j}^- \subset Q_n$ . Letting  $n \rightarrow \infty$  and using (2.2) we obtain

$$\bar{l} := \overline{\lim}_{n \rightarrow \infty} \frac{\mathcal{S}(Q_n)}{\mu(Q_n)} \leq \overline{\lim}_{n \rightarrow \infty} \frac{\mathcal{S}(Q_{n,j}^-)}{\mu(Q_{n,j}^-)} =: \bar{l}_j. \quad (2.8)$$

**Step 2: we prove that  $\underline{l} = \bar{l}$ .** It is sufficient to prove that for each  $\varepsilon > 0$ , one has

$$\bar{l} - \underline{l} < \varepsilon. \quad (2.9)$$

Fix  $\varepsilon > 0$ . From (2.7) and (2.8) we see that  $\bar{l} - \underline{l} \leq \bar{l}_j - \underline{l}_j$ . So, to prove (2.9) it suffices to show that there exists  $j \in \mathbb{N}^*$  such that

$$\bar{l}_j - \underline{l}_j < \varepsilon. \quad (2.10)$$

Let  $\mathcal{S}_j : \mathcal{P}_f(\mathbb{G}_{k_j}) \rightarrow \mathbb{R}$  be defined by

$$\mathcal{S}_j(E) := \frac{1}{\mu(\mathbb{U}_{k_j})} \left[ \mathcal{S}\left(\bigcup_{g \in E} g(\mathbb{U}_{k_j})\right) - |E| \mathcal{S}(\mathbb{U}_{k_j}) \right]. \quad (2.11)$$

As  $\mathcal{S}$  is subadditive, we can assert that  $\mathcal{S}_j$  is negative, i.e.,

$$\mathcal{S}_j(E) = \frac{1}{\mu(\mathbb{U}_{k_j})} \left[ \mathcal{S}\left(\bigcup_{g \in E} g(\mathbb{U}_{k_j})\right) - |E| \mathcal{S}(\mathbb{U}_{k_j}) \right] \leq 0 \quad (2.12)$$

for all  $E \in \mathcal{P}_f(\mathbb{G}_{k_j})$ . Moreover, it is easily seen that  $\mathcal{S}_j$  is decreasing, i.e., for all  $E, F \in \mathcal{P}_f(\mathbb{G}_{k_j})$ , if  $E \subset F$  then  $\mathcal{S}_j(E) \geq \mathcal{S}_j(F)$ . Consider  $m_{n,k_j} \in \mathbb{N}^*$ ,  $g_{n,k_j} \in G$  and  $F_{n,k_j} \in \mathcal{P}_f(\mathbb{G}_{k_j})$  given by Definition 2.1. From (2.3) it follows that

$$\begin{aligned} \mathcal{S}_j(G_{n,k_j}^+) \geq \mathcal{S}_j(F_{n,k_j}) &= \frac{1}{\mu(\mathbb{U}_{k_j})} \left[ \mathcal{S}\left(\bigcup_{g \in F_{n,k_j}} g(\mathbb{U}_{k_j})\right) - |F_{n,k_j}| \mathcal{S}(\mathbb{U}_{k_j}) \right] \\ &= \frac{1}{\mu(\mathbb{U}_{k_j})} \left[ \mathcal{S}(g_{n,k_j}(\mathbb{U}_{m_{n,k_j}})) - |F_{n,k_j}| \mathcal{S}(\mathbb{U}_{k_j}) \right]. \end{aligned}$$

Hence, since  $\frac{1}{|G_{n,k_j}^+|} \geq \frac{1}{|F_{n,k_j}|}$  and  $\mathcal{S}$  and  $\mu$  are  $G$ -invariant, we get

$$\begin{aligned} \frac{\mathcal{S}_j(G_{n,k_j}^+)}{|G_{n,k_j}^+|} &\geq \frac{1}{|G_{n,k_j}^+| \mu(\mathbb{U}_{k_j})} \left[ \mathcal{S}(g_{n,k_j}(\mathbb{U}_{m_{n,k_j}})) - |F_{n,k_j}| \mathcal{S}(\mathbb{U}_{k_j}) \right] \\ &\geq \frac{\mathcal{S}(g_{n,k_j}(\mathbb{U}_{m_{n,k_j}}))}{|F_{n,k_j}| \mu(\mathbb{U}_{k_j})} - \frac{|F_{n,k_j}| \mathcal{S}(\mathbb{U}_{k_j})}{|G_{n,k_j}^+| \mu(\mathbb{U}_{k_j})} \\ &= \frac{\mathcal{S}(g_{n,k_j}(\mathbb{U}_{m_{n,k_j}}))}{\mu(g_{n,k_j}(\mathbb{U}_{m_{n,k_j}}))} - \frac{|F_{n,k_j}| \mathcal{S}(\mathbb{U}_{k_j})}{|G_{n,k_j}^+| \mu(\mathbb{U}_{k_j})} \\ &= \frac{\mathcal{S}(\mathbb{U}_{m_{n,k_j}})}{\mu(\mathbb{U}_{m_{n,k_j}})} - \frac{|F_{n,k_j}| \mathcal{S}(\mathbb{U}_{k_j})}{|G_{n,k_j}^+| \mu(\mathbb{U}_{k_j})} \\ &\geq \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)} - \frac{|F_{n,k_j}| \mathcal{S}(\mathbb{U}_{k_j})}{|G_{n,k_j}^+| \mu(\mathbb{U}_{k_j})}. \end{aligned}$$

Letting  $n \rightarrow \infty$  and taking (2.4) into account, we deduce that

$$\liminf_{n \rightarrow \infty} \frac{\mathcal{S}_j(G_{n,k_j}^+)}{|G_{n,k_j}^+|} \geq \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)} - \frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})}. \quad (2.13)$$

By (2.6) we can assert that there exists  $j_\varepsilon \in \mathbb{N}^*$  such that for all  $j \geq j_\varepsilon$ , one has

$$\frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})} - \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)} < \varepsilon. \quad (2.14)$$

Combining (2.13) with (2.14) we conclude that

$$\liminf_{n \rightarrow \infty} \frac{\mathcal{S}_j(G_{n,k_j}^+)}{|G_{n,k_j}^+|} > -\varepsilon \quad (2.15)$$

for all  $j \geq j_\varepsilon$ . On the other hand, by using (2.11) with  $E = G_{n,k_j}^+$  and (2.12) with  $E = G_{n,k_j}^-$  we get:

$$\frac{\mathcal{S}(Q_{n,j}^+)}{\mu(Q_{n,j}^+)} - \frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})} = \frac{\mathcal{S}_j(G_{n,k_j}^+)}{|G_{n,k_j}^+|}, \quad (2.16)$$

$$\frac{\mathcal{S}(Q_{n,j}^-)}{\mu(Q_{n,j}^-)} - \frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})} \leq 0. \quad (2.17)$$

Letting  $n \rightarrow \infty$  in (2.16) and (2.17) and taking (2.15) into account, we deduce that:

$$\underline{l}_j - \frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})} > -\varepsilon \text{ for all } j \geq j_\varepsilon; \quad (2.18)$$

$$\bar{l}_j - \frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})} \leq 0 \text{ for all } j \in \mathbb{N}^*, \quad (2.19)$$

and (2.10) follows with  $j = j_\varepsilon$ . We set  $l := \underline{l} = \bar{l}$  and  $\gamma := \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)}$ .

**Step 3: we prove that  $l = \gamma$ .** Combining (2.8) with (2.19) we see that  $l \leq \frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})}$  for all  $j \in \mathbb{N}^*$ , and so  $l \leq \gamma$  by letting  $j \rightarrow \infty$  and using (2.6). On the other hand, combining (2.7) with (2.18) we see that  $l > -\varepsilon + \frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})}$  for all  $j \geq j_\varepsilon$ . Letting  $j \rightarrow \infty$  and using (2.6) we deduce that  $l \geq -\varepsilon + \gamma$  for all  $\varepsilon > 0$ , and so  $l \geq \gamma$  by letting  $\varepsilon \rightarrow 0$ . ■

**2.2. The stochastic case.** We begin with the following definition.

**Definition 2.4.** The metric measure space  $(X, d, \mu)$  is said to be meshable with respect to  $\{\mathbb{U}_k\}_{k \in \mathbb{N}^*}$  if for all  $k \in \mathbb{N}^*$  there exists  $\mathbb{H}_k \in \mathcal{U}_k(G)$  with the property that for all  $n \in \mathbb{N}^*$  there

exist  $H_{n,k}^-, H_{n,k}^+ \in \mathcal{P}_f(\mathbb{H}_k)$  such that:

$$\bigcup_{g \in H_{n,k}^-} g(\mathbb{U}_k) \subset \mathbb{U}_n \subset \bigcup_{g \in H_{n,k}^+} g(\mathbb{U}_k); \quad (2.20)$$

$$\lim_{n \rightarrow \infty} \frac{\mu \left( \bigcup_{g \in H_{n,k}^+} g(\mathbb{U}_k) \setminus \bigcup_{g \in H_{n,k}^-} g(\mathbb{U}_k) \right)}{\mu(\mathbb{U}_n)} = 0. \quad (2.21)$$

The interest of Definition 2.4 comes from the following lemma (which will be used in the proof of Theorem 2.11).

**Lemma 2.5.** *Let  $\mathcal{S} : \mathcal{B}_{\mu,0}(X) \rightarrow \mathbb{R}$  be a subadditive and  $G$ -invariant set function satisfying (2.5). If  $(X, d, \mu)$  is meshable with respect to  $\{\mathbb{U}_k\}_{k \in \mathbb{N}^*}$  then*

$$\lim_{n \rightarrow \infty} \frac{\mathcal{S}(\mathbb{U}_n)}{\mu(\mathbb{U}_n)} = \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)}. \quad (2.22)$$

**Proof of Lemma 2.5.** First of all, it is clear that  $\frac{\mathcal{S}(\mathbb{U}_n)}{\mu(\mathbb{U}_n)} \geq \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)}$  for all  $n \in \mathbb{N}^*$ , and so

$$\liminf_{n \rightarrow \infty} \frac{\mathcal{S}(\mathbb{U}_n)}{\mu(\mathbb{U}_n)} \geq \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)}. \quad (2.23)$$

On the other hand, fix any  $k \in \mathbb{N}^*$  and any  $n \in \mathbb{N}^*$  and set:

$$\begin{aligned} \mathbb{U}_{n,k}^- &:= \bigcup_{g \in H_{n,k}^-} g(\mathbb{U}_k); \\ \mathbb{U}_{n,k}^+ &:= \bigcup_{g \in H_{n,k}^+} g(\mathbb{U}_k), \end{aligned}$$

where  $H_{n,k}^-$  and  $H_{n,k}^+ \in \mathcal{P}_f(\mathbb{H}_k)$  with  $\mathbb{H}_k$  given by Definition 2.4. By the left inclusion in (2.20) we have  $\mathbb{U}_{n,k}^- \subset \mathbb{U}_n$  and so  $\mathbb{U}_n = \mathbb{U}_{n,k}^- \cup (\mathbb{U}_n \setminus \mathbb{U}_{n,k}^-)$ . Hence

$$\mathcal{S}(\mathbb{U}_n) \leq \mathcal{S}(\mathbb{U}_{n,k}^-) + \mathcal{S}(\mathbb{U}_n \setminus \mathbb{U}_{n,k}^-)$$

because  $\mathcal{S}$  is subadditive, and consequently

$$\frac{\mathcal{S}(\mathbb{U}_n)}{\mu(\mathbb{U}_n)} \leq \frac{\mathcal{S}(\mathbb{U}_{n,k}^-)}{\mu(\mathbb{U}_{n,k}^-)} \frac{\mu(\mathbb{U}_{n,k}^-)}{\mu(\mathbb{U}_n)} + \frac{\mathcal{S}(\mathbb{U}_n \setminus \mathbb{U}_{n,k}^-)}{\mu(\mathbb{U}_n)}.$$

Using again the subadditivity of  $\mathcal{S}$  and its  $G$ -invariance (resp. the  $G$ -invariance of  $\mu$ ) we have

$$\begin{aligned} \mathcal{S}(\mathbb{U}_{n,k}^-) &\leq |H_{n,k}^-| \mathcal{S}(\mathbb{U}_k) \\ (\text{resp. } \mu(\mathbb{U}_{n,k}^-)) &= |H_{n,k}^-| \mu(\mathbb{U}_k). \end{aligned}$$

Moreover,  $\mathbb{U}_n \subset \mathbb{U}_{n,k}^+$  by the right inclusion in (2.20), which implies that  $\mathbb{U}_n \setminus \mathbb{U}_{n,k}^- \subset \mathbb{U}_{n,k}^+ \setminus \mathbb{U}_{n,k}^-$  and so

$$\mathcal{S}(\mathbb{U}_n \setminus \mathbb{U}_{n,k}^-) \leq c \mu(\mathbb{U}_{n,k}^+ \setminus \mathbb{U}_{n,k}^-)$$



with  $c > 0$  given by (2.5). It follows that

$$\begin{aligned} \frac{\mathcal{S}(\mathbb{U}_n)}{\mu(\mathbb{U}_n)} &\leq \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)} \frac{\mu(\mathbb{U}_{n,k}^-)}{\mu(\mathbb{U}_n)} + \frac{c\mu(\mathbb{U}_{n,k}^+ \setminus \mathbb{U}_{n,k}^-)}{\mu(\mathbb{U}_n)} \\ &\leq \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)} + \frac{c\mu(\mathbb{U}_{n,k}^+ \setminus \mathbb{U}_{n,k}^-)}{\mu(\mathbb{U}_n)} \end{aligned}$$

because  $\mu(\mathbb{U}_{n,k}^-) \leq \mu(\mathbb{U}_n)$  since  $\mathbb{U}_{n,k}^- \subset \mathbb{U}_n$ . Letting  $n \rightarrow \infty$  and using (2.21), and then passing to the infimum on  $k$ , we obtain

$$\overline{\lim}_{n \rightarrow \infty} \frac{\mathcal{S}(\mathbb{U}_n)}{\mu(\mathbb{U}_n)} \leq \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)}, \quad (2.24)$$

and (2.6) follows by combining (2.23) with (2.24). ■

In what follows,  $\Delta$  denotes the symmetric difference of sets, i.e.,  $E\Delta F := (E \setminus F) \cup (F \setminus E)$  for any  $E, F \subset G$ , and we adopt the following notation:  $EF := \{gof : (g, f) \in E \times F\}$  and  $E^{-1}F := \{g^{-1}of : (g, f) \in E \times F\}$  and, for any  $g \in G$ ,  $gF := \{gof : f \in F\}$ . From now on, for each  $k \in \mathbb{N}^*$ , we consider the class  $\mathcal{U}_k^a(G)$  defined by

$$\mathcal{U}_k^a(G) := \left\{ H \in \mathcal{U}_k(G) : H \text{ is countable, discrete and amenable group} \right\},$$

where amenability of  $H$  means that for each  $E \in \mathcal{P}_f(H)$  and each  $\delta > 0$  there exists  $F \in \mathcal{P}_f(H)$  such that

$$|F\Delta EF| \leq \delta|F|.$$

(For more details about the theory of amenability, we refer to [Gre69, OW87, Pat88, Tem92, AAB<sup>+</sup>10, DZ15] and the references therein, see also [Kre85, §6.4].)

The property of Følner-Tempelmann stated in the definition below is needed to use both Lindenstrauss's ergodic theorem (see Theorem 2.13) which is valid for general amenable groups and a maximal inequality (see Lemma 2.14) which is valid for countable discrete amenable groups. (These two results will be used in the proof of Theorem 2.11.)

**Definition 2.6.** Let  $H \in \mathcal{U}_k^a(G)$  and let  $\{G_n\}_{n \in \mathbb{N}^*} \subset \mathcal{P}_f(H)$ . We say that  $\{G_n\}_{n \in \mathbb{N}^*}$  is of Følner-Tempelmann type with respect to  $H$  if it satisfies the following two conditions:

(a) Følner's condition: for every  $g \in H$ , one has

$$\lim_{n \rightarrow \infty} \frac{|gG_n \Delta G_n|}{|G_n|} = 0;$$

(b) Tempelman's condition: there exists  $M > 0$ , which called the Templeman constant associated with  $\{G_n\}_{n \in \mathbb{N}^*}$ , such that for every  $n \in \mathbb{N}^*$ , one has

$$\left| \bigcup_{i=1}^n G_i^{-1}G_n \right| \leq M|G_n|.$$

Together with Definition 2.4, the following definition set a framework for establishing a subadditive theorem in the stochastic case and in the setting of metric measure spaces (see Theorem 2.11).

**Definition 2.7.** Let  $\{Q_n\}_{n \in \mathbb{N}^*} \subset \mathcal{B}_{\mu,0}(X)$ . We say that  $\{Q_n\}_{n \in \mathbb{N}^*}$  is strongly  $G$ -asymptotic with respect to  $\{\mathbb{U}_k\}_{k \in \mathbb{N}^*}$  if there exists  $\{\mathbb{G}_k\}_{k \in \mathbb{N}^*}$  with  $\mathbb{G}_k \in \mathcal{U}_k^a(G)$  for all  $k \in \mathbb{N}^*$  and  $G = \cup_{k \in \mathbb{N}^*} \mathbb{G}_k$  such that for all  $k \in \mathbb{N}^*$  and all  $n \in \mathbb{N}^*$  there exist  $m_{n,k} \in \mathbb{N}^*$ ,  $g_{n,k} \in G$  and  $F_{n,k}, G_{n,k}^-, G_{n,k}^+ \in \mathcal{P}_f(\mathbb{G}_k)$  such that (2.1), (2.2), (2.3) and (2.4) are satisfied with the additional assumption that  $\{G_{n,k}^-\}_{n \in \mathbb{N}^*}$  and  $\{G_{n,k}^+\}_{n \in \mathbb{N}^*}$  are of Følner-Tempelman type with respect to  $\mathbb{G}_k$ .

Let  $(\Sigma, \mathcal{T}, \mathbb{P})$  be a probability space and let  $\{\tau_g : \Sigma \rightarrow \Sigma\}_{g \in G}$  be satisfying the following three properties:

- (mesurability)  $\tau_g$  is  $\mathcal{T}$ -measurable;
- (group property)  $\tau_g \circ \tau_f = \tau_{g \circ f}$  and  $\tau_{g^{-1}} = \tau_g^{-1}$  for all  $g, f \in G$ ;
- (mass invariance)  $\mathbb{P}(\tau_g(E)) = \mathbb{P}(E)$  for all  $E \in \mathcal{T}$  and all  $g \in G$ .

**Definition 2.8.** Such a  $\{\tau_g\}_{g \in G}$  is said to be a group of  $\mathbb{P}$ -preserving transformation on  $(\Sigma, \mathcal{T}, \mathbb{P})$  and the quadruplet  $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in G})$  is called a measurable dynamical  $G$ -system. (Note that if  $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in G})$  is a measurable dynamical  $G$ -system, then  $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in H})$  is a measurable dynamical  $H$ -system for all subgroups  $H$  of  $G$ .)

Let  $\mathcal{I} := \{E \in \mathcal{T} : \mathbb{P}(\tau_g(E) \Delta E) = 0 \text{ for all } g \in G\}$  be the  $\sigma$ -algebra of invariant sets with respect to  $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in G})$ . (For any subgroup  $H$  of  $G$ , we denote the  $\sigma$ -algebra of invariant sets with respect to  $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in H})$  by  $\mathcal{I}_H$ .)

**Definition 2.9.** When  $\mathbb{P}(E) \in \{0, 1\}$  for all  $E \in \mathcal{I}$ , the measurable dynamical  $G$ -system  $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in G})$  is said to be ergodic.

In what follows, we assume that  $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in G})$  is a measurable dynamical  $G$ -system. Let us recall the definition of a subadditive process.

**Definition 2.10.** A set function  $\mathcal{S} : \mathcal{B}_{\mu,0}(X) \rightarrow L^1(\Sigma, \mathcal{T}, \mathbb{P})$  is called a subadditive process if it is subadditive in the sense of Definition 2.2(a) and  $G$ -covariant, i.e.,

$$\mathcal{S}(g(A)) = \mathcal{S}(A) \circ \tau_g$$

for all  $A \in \mathcal{B}_{\mu,0}(X)$  and all  $g \in G$ . If in addition the measurable dynamical  $G$ -system  $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in G})$  is ergodic, then  $\mathcal{S}$  is called an ergodic subadditive process.

Here is the second main result of the paper.

**Theorem 2.11.** *Assume that  $(X, d, \mu)$  is meshable with respect to  $\{\mathbb{U}_k\}_{k \in \mathbb{N}^*}$  and consider  $\mathcal{S} : \mathcal{B}_{\mu,0}(X) \rightarrow L^1(\Sigma, \mathcal{T}, \mathbb{P})$  a subadditive process satisfying (2.5). Then, for any  $\{Q_n\}_{n \in \mathbb{N}^*} \subset \mathcal{B}_{\mu,0}(X)$  such that  $\{Q_n\}_{n \in \mathbb{N}^*}$  is strongly  $G$ -asymptotic with respect to  $\{\mathbb{U}_k\}_{k \in \mathbb{N}^*}$ , one has*

$$\lim_{n \rightarrow \infty} \frac{\mathcal{S}(Q_n)(\omega)}{\mu(Q_n)} = \inf_{k \in \mathbb{N}^*} \frac{\mathbb{E}^{\mathcal{I}}[\mathcal{S}(\mathbb{U}_k)](\omega)}{\mu(\mathbb{U}_k)} \text{ for } \mathbb{P}\text{-a.a. } \omega \in \Sigma,$$

where  $\mathbb{E}^{\mathcal{I}}[\mathcal{S}(\mathbb{U}_k)]$  denotes the conditional expectation of  $\mathcal{S}(\mathbb{U}_k)$  over  $\mathcal{I}$  with respect to  $\mathbb{P}$ . If in addition  $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in G})$  is ergodic, then

$$\lim_{n \rightarrow \infty} \frac{\mathcal{S}(Q_n)(\omega)}{\mu(Q_n)} = \inf_{k \in \mathbb{N}^*} \frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_k)]}{\mu(\mathbb{U}_k)} \text{ for } \mathbb{P}\text{-a.a. } \omega \in \Sigma,$$

where  $\mathbb{E}[\mathcal{S}(\mathbb{U}_k)]$  denotes the expectation of  $\mathcal{S}(\mathbb{U}_k)$  with respect to  $\mathbb{P}$ .

**Proof of Theorem 2.11.** The proof is divided into four steps.

**Step 1: establishing lower bound and upper bound.** Fix any  $k \in \mathbb{N}^*$  and any  $n \in \mathbb{N}^*$  and set:

$$Q_{n,k}^- := \bigcup_{g \in G_{n,k}^-} g(\mathbb{U}_k);$$

$$Q_{n,k}^+ := \bigcup_{g \in G_{n,k}^+} g(\mathbb{U}_k),$$

where  $G_{n,k}^-, G_{n,k}^+ \in \mathcal{P}_f(\mathbb{G}_k)$  with  $\mathbb{G}_k \in \mathcal{U}_k^a(G)$  given by Definition 2.7. Arguing as in Step 1 of the proof of Theorem 2.3, for each  $\omega \in \Sigma$ , we get:

$$\underline{l}_k(\omega) := \liminf_{n \rightarrow \infty} \frac{\mathcal{S}(Q_{n,k}^+)(\omega)}{\mu(Q_{n,k}^+)} \leq \liminf_{n \rightarrow \infty} \frac{\mathcal{S}(Q_n)(\omega)}{\mu(Q_n)} =: \underline{l}(\omega) \quad (2.25)$$

$$\bar{l}(\omega) := \overline{\lim}_{n \rightarrow \infty} \frac{\mathcal{S}(Q_n)(\omega)}{\mu(Q_n)} \leq \overline{\lim}_{n \rightarrow \infty} \frac{\mathcal{S}(Q_{n,k}^-)(\omega)}{\mu(Q_{n,k}^-)} =: \bar{l}_k(\omega). \quad (2.26)$$

**Remark 2.12.** Arguing as in Step 1-1 of the proof of Theorem 2.3, we see that we also have

$$\overline{\lim}_{n \rightarrow \infty} \frac{\mathcal{S}(Q_{n,k}^+)(\omega)}{\mu(Q_{n,k}^+)} \leq \bar{l}(\omega) \quad (2.27)$$

for all  $\omega \in \Sigma$ . (This will be used in Step 3.)

**Step 2: we prove that  $\underline{l}(\omega) = \bar{l}(\omega)$  for  $\mathbb{P}$ -a.a.  $\omega \in \Sigma$ .** It is sufficient to prove that for each  $\alpha > 0$ , one has

$$\mathbb{P}\left(\left\{\omega \in \Sigma : \bar{l}(\omega) - \underline{l}(\omega) > \alpha\right\}\right) = 0. \quad (2.28)$$

Fix  $\alpha > 0$ . From (2.25) and (2.26) we see that for each  $k \in \mathbb{N}^*$ , one has

$$\left\{\omega \in \Sigma : \bar{l}(\omega) - \underline{l}(\omega) > \alpha\right\} \subset \left\{\omega \in \Sigma : \bar{l}_k(\omega) - \underline{l}_k(\omega) > \alpha\right\} =: W_{k,\alpha}. \quad (2.29)$$

So, to prove (2.28) it suffices to show that for each  $\varepsilon > 0$  there exists  $k \in \mathbb{N}^*$  such that

$$\mathbb{P}(W_{k,\alpha}) \leq \frac{M_k}{\alpha} \varepsilon, \quad (2.30)$$

where  $M_k > 0$  is the Tempelman constant associated with  $\{G_{n,k}^+\}_{n \in \mathbb{N}^*}$ . Fix  $\varepsilon > 0$ .

**Step 2-1: constructing a decreasing negative subadditive process on  $\mathcal{P}_f(\mathbb{G}_k)$ .** Let  $\mathcal{A}_k : \mathcal{P}_f(\mathbb{G}_k) \rightarrow L^1(\Sigma, \mathcal{T}, \mathbb{P})$  be defined by

$$\mathcal{A}_k(E) := \sum_{g \in E} \mathcal{S}(\mathbb{U}_k) \circ \tau_g,$$

where  $\mathbb{G}_k \in \mathcal{U}_k^a(G)$  is (a countable discrete and amenable subgroup of  $G$ ) given by Definition 2.7, and let  $\mathcal{S}_k : \mathcal{P}_f(\mathbb{G}_k) \rightarrow L^1(\Sigma, \mathcal{T}, \mathbb{P})$  be defined by

$$\mathcal{S}_k(E) := \frac{1}{\mu(\mathbb{U}_k)} \left[ \mathcal{S}\left(\bigcup_{g \in E} g(\mathbb{U}_k)\right) - \mathcal{A}_k(E) \right]. \quad (2.31)$$

As  $\mathcal{S}$  is subadditive and  $G$ -covariant (and so  $\mathbb{G}_k$ -covariant) and  $\mathcal{A}_k$  is additive and  $\mathbb{G}_k$ -covariant, we can assert that  $\mathcal{S}_k$  is a subadditive process<sup>1</sup> on  $\mathcal{P}_f(\mathbb{G}_k)$  which is negative, i.e.,

$$\mathcal{S}_k(E)(\omega) = \frac{1}{\mu(\mathbb{U}_k)} \left[ \mathcal{S}\left(\bigcup_{g \in E} g(\mathbb{U}_k)\right)(\omega) - \mathcal{A}_k(E)(\omega) \right] \leq 0 \quad (2.32)$$

for all  $E \in \mathcal{P}_f(\mathbb{G}_k)$  and all  $\omega \in \Sigma$ . Moreover, it is easily seen that  $\mathcal{S}_k$  is decreasing, i.e., for all  $E, F \in \mathcal{P}_f(\mathbb{G}_k)$ , if  $E \subset F$  then  $\mathcal{S}_k(E) \geq \mathcal{S}_k(F)$ . Consider  $m_{n,k} \in \mathbb{N}^*$ ,  $g_{n,k} \in G$  and  $F_{n,k} \in \mathcal{P}_f(\mathbb{G}_k)$  given by Definition 2.7. From (2.3) it follows that

$$\begin{aligned} \mathcal{S}_k(G_{n,k}^+) \geq \mathcal{S}_k(F_{n,k}) &= \frac{1}{\mu(\mathbb{U}_k)} \left[ \mathcal{S}\left(\bigcup_{g \in F_{n,k}} g(\mathbb{U}_k)\right) - \mathcal{A}_k(F_{n,k}) \right] \\ &= \frac{1}{\mu(\mathbb{U}_k)} \left[ \mathcal{S}(g_{n,k}(\mathbb{U}_{m_{n,k}})) - \mathcal{A}_k(F_{n,k}) \right]. \end{aligned}$$

By using the  $G$ -covariance of  $\mathcal{S}$  we see that

$$\begin{aligned} \int_{\Sigma} \mathcal{S}_k(G_{n,k}^+)(\omega) d\mathbb{P}(\omega) &\geq \frac{1}{\mu(\mathbb{U}_k)} \left[ \int_{\Sigma} \mathcal{S}(g_{n,k}(\mathbb{U}_{m_{n,k}}))(\omega) d\mathbb{P}(\omega) - \int_{\Sigma} \mathcal{A}_k(F_{n,k})(\omega) d\mathbb{P}(\omega) \right] \\ &= \frac{1}{\mu(\mathbb{U}_k)} \left[ \int_{\Sigma} \mathcal{S}(\mathbb{U}_{m_{n,k}})(\omega) d\mathbb{P}(\omega) - |F_{n,k}| \mathbb{E}[\mathcal{S}(\mathbb{U}_k)] \right] \\ &= \frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_{m_{n,k}})]}{\mu(\mathbb{U}_k)} - |F_{n,k}| \frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_k)]}{\mu(\mathbb{U}_k)}. \end{aligned}$$

Consequently, since  $\frac{1}{|G_{n,k}^+|} \geq \frac{1}{|F_{n,k}|}$  and  $\mu$  is  $G$ -invariant, we get

$$\begin{aligned} \frac{\mathbb{E}[\mathcal{S}_k(G_{n,k}^+)]}{|G_{n,k}^+|} &\geq \frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_{m_{n,k}})]}{\mu(\mathbb{U}_{m_{n,k}})} - \frac{|F_{n,k}| \mathbb{E}[\mathcal{S}(\mathbb{U}_k)]}{|G_{n,k}^+| \mu(\mathbb{U}_k)} \\ &\geq \inf_{m \in \mathbb{N}^*} \frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_m)]}{\mu(\mathbb{U}_m)} - \frac{|F_{n,k}| \mathbb{E}[\mathcal{S}(\mathbb{U}_k)]}{|G_{n,k}^+| \mu(\mathbb{U}_k)}. \end{aligned}$$

Letting  $n \rightarrow \infty$  and taking (2.4) into account, we deduce that

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}[\mathcal{S}_k(G_{n,k}^+)]}{|G_{n,k}^+|} \geq \inf_{m \in \mathbb{N}^*} \frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_m)]}{\mu(\mathbb{U}_m)} - \frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_k)]}{\mu(\mathbb{U}_k)}. \quad (2.33)$$

As  $\mathcal{S}$  is subadditive and  $G$ -covariant, we see that the set function  $\mathbb{E}[\mathcal{S}(\cdot)]$  is subadditive and  $G$ -invariant. From Lemma 2.5 it follows that there exists  $k_{\varepsilon} \in \mathbb{N}^*$  such that for all  $k \geq k_{\varepsilon}$ , one has

$$\frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_k)]}{\mu(\mathbb{U}_k)} - \inf_{m \in \mathbb{N}^*} \frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_m)]}{\mu(\mathbb{U}_m)} < \varepsilon. \quad (2.34)$$

<sup>1</sup>The set function  $\mathcal{S}_k : \mathcal{P}_f(\mathbb{G}_k) \rightarrow L^1(\Sigma, \mathcal{T}, \mathbb{P})$  is said to be a subadditive process on  $\mathcal{P}_f(\mathbb{G}_k)$  if it is subadditive, i.e.,  $\mathcal{S}_k(E \cup F) \leq \mathcal{S}_k(E) + \mathcal{S}_k(F)$  for all  $E, F \in \mathcal{P}_f(\mathbb{G}_k)$  such that  $E \cap F = \emptyset$ , and  $\mathbb{G}_k$ -covariant, i.e.,  $\mathcal{S}_k(Eg) = \mathcal{S}_k(E) \circ \tau_g$  for all  $E \in \mathcal{P}_f(\mathbb{G}_k)$  and all  $g \in \mathbb{G}_k$ .

Combining (2.33) with (2.34) we conclude that

$$\overline{\lim}_{n \rightarrow \infty} \frac{\mathbb{E}[\mathcal{S}_k(G_{n,k}^+)]}{|G_{n,k}^+|} > -\varepsilon \quad (2.35)$$

for all  $k \geq k_\varepsilon$ .

**Step 2-2: using Lindenstrauss's ergodic theorem.** We need the following pointwise additive ergodic theorem<sup>2</sup> due to Lindenstrauss (see [Lin01, Theorem 1.2] and also [DGZ14, Theorem 2.1]).

**Theorem 2.13.** *Let  $\Theta \in L^1(\Sigma, \mathcal{T}, \mathbb{P})$  and let  $\{G_n\}_{n \in \mathbb{N}^*} \subset \mathcal{P}_f(\mathbb{G}_k)$ . If  $\{G_n\}_{n \in \mathbb{N}^*}$  is of Følner-Tempelman type with respect to  $\mathbb{G}_k$  then*

$$\lim_{n \rightarrow \infty} \frac{1}{|G_n|} \sum_{g \in G_n} \Theta(\tau_g(\omega)) = \mathbb{E}^{\mathcal{I}_{\mathbb{G}_k}}[\Theta](\omega) \text{ for } \mathbb{P}\text{-a.a } \omega \in \Sigma,$$

where  $\mathcal{I}_{\mathbb{G}_k}$  is the  $\sigma$ -algebra of invariant sets with respect to  $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in \mathbb{G}_k})$  and  $\mathbb{E}^{\mathcal{I}_{\mathbb{G}_k}}[\Theta]$  denotes the conditional expectation over  $\mathcal{I}_{\mathbb{G}_k}$  with respect to  $\mathbb{P}$ .

As  $\{G_{n,k}^-\}_{n \in \mathbb{N}^*}$  and  $\{G_{n,k}^+\}_{n \in \mathbb{N}^*}$  are of Følner-Tempelman type with respect to  $\mathbb{G}_k$ , applying Theorem 2.13 with  $\Theta = \mathcal{S}(\mathbb{U}_k)$  we deduce that there exists  $\widehat{\Sigma} \in \mathcal{T}$  with  $\mathbb{P}(\widehat{\Sigma}) = 1$  such that

$$\lim_{n \rightarrow \infty} \frac{\mathcal{A}_k(G_{n,k}^-)(\omega)}{|G_{n,k}^-|} = \lim_{n \rightarrow \infty} \frac{\mathcal{A}_k(G_{n,k}^+)(\omega)}{|G_{n,k}^+|} = \mathbb{E}^{\mathcal{I}_{\mathbb{G}_k}}[\mathcal{S}(\mathbb{U}_k)](\omega) \text{ for all } \omega \in \widehat{\Sigma}. \quad (2.36)$$

On the other hand, by using (2.31) with  $E = G_{n,k}^+$  and (2.32) with  $E = G_{n,k}^-$  we get:

$$\begin{aligned} \frac{\mathcal{S}(Q_{n,k}^+)(\omega)}{\mu(Q_{n,k}^+)} - \frac{1}{\mu(\mathbb{U}_k)} \frac{\mathcal{A}_k(G_{n,k}^+)(\omega)}{|G_{n,k}^+|} &= \frac{\mathcal{S}_k(G_{n,k}^+)(\omega)}{|G_{n,k}^+|} \geq \inf_{q \in \mathbb{N}^*} \frac{\mathcal{S}_k(G_{q,k}^+)(\omega)}{|G_{q,k}^+|}; \\ \frac{\mathcal{S}(Q_{n,k}^-)(\omega)}{\mu(Q_{n,k}^-)} - \frac{1}{\mu(\mathbb{U}_k)} \frac{\mathcal{A}_k(G_{n,k}^-)(\omega)}{|G_{n,k}^-|} &\leq 0 \end{aligned} \quad (2.37)$$

for all  $\omega \in \Sigma$ . Letting  $n \rightarrow \infty$  we deduce that:

$$l_k(\omega) - \frac{\mathbb{E}^{\mathcal{I}_{\mathbb{G}_k}}[\mathcal{S}(\mathbb{U}_k)](\omega)}{\mu(\mathbb{U}_k)} \geq \inf_{n \in \mathbb{N}^*} \frac{\mathcal{S}_k(G_{n,k}^+)(\omega)}{|G_{n,k}^+|} \text{ for all } k \in \mathbb{N}^* \text{ and all } \omega \in \widehat{\Sigma}; \quad (2.38)$$

$$\bar{l}_k(\omega) - \frac{\mathbb{E}^{\mathcal{I}_{\mathbb{G}_k}}[\mathcal{S}(\mathbb{U}_k)](\omega)}{\mu(\mathbb{U}_k)} \leq 0 \text{ for all } k \in \mathbb{N}^* \text{ and all } \omega \in \widehat{\Sigma}; \quad (2.39)$$

In what follows, without loss of generality, we assume that  $\widehat{\Sigma} = \Sigma$ .

**Step 2-3: using a maximal inequality.** We need the following lemma (see [DGZ14, Lemma 3.5] and also [AK81, Theorem 4.2]).

<sup>2</sup>Lindenstrauss's ergodic theorem is established under the weaker condition that  $\{G_n\}_{n \in \mathbb{N}^*}$  is a tempered Følner sequence (see [Lin01, Definition 1.1] and [DGZ14, §2] for more details). The tempered Følner condition implies the Følner-Tempelman condition, but the converse is not true (see [Lin01, DGZ14]).

**Lemma 2.14.** *Let  $\mathcal{K} : \mathcal{P}_f(\mathbb{G}_k) \rightarrow L^1(\Sigma, \mathcal{T}, \mathbb{P})$  be a negative subadditive process and let  $\{G_n\}_{n \in \mathbb{N}^*} \subset \mathcal{P}_f(\mathbb{G}_k)$ . Fix  $\alpha > 0$  and consider  $V_\alpha^\mathcal{K} \in \mathcal{T}$  given by*

$$V_\alpha^\mathcal{K} := \left\{ \omega \in \Sigma : \inf_{n \in \mathbb{N}^*} \frac{\mathcal{K}(G_n)(\omega)}{|G_n|} < -\alpha \right\}.$$

*If  $\{G_n\}_{n \in \mathbb{N}^*}$  is of Følner-Templeman type with respect to  $\mathbb{G}_k$  then*

$$\mathbb{P}(V_\alpha^\mathcal{K}) \leq -\frac{M}{\alpha} \overline{\lim}_{n \rightarrow \infty} \frac{\mathbb{E}[\mathcal{K}(G_n)]}{|G_n|},$$

*where  $M > 0$  is the Templeman constant associated with  $\{G_n\}_{n \in \mathbb{N}^*}$ .*

As  $\mathcal{S}_k : \mathcal{P}_f(\mathbb{G}_k) \rightarrow L^1(\Sigma, \mathcal{T}, \mathbb{P})$  defined by (2.31) is a negative subadditive process, we can apply Theorem 2.14 with  $\mathcal{K} = \mathcal{S}_k$ . Hence, since  $\{G_{n,k}^+\}_{n \in \mathbb{N}^*}$  is of Følner-Templeman type with respect to  $\mathbb{G}_k$ , one has

$$\mathbb{P}(V_\alpha^{\mathcal{S}_k}) \leq -\frac{M_k}{\alpha} \overline{\lim}_{n \rightarrow \infty} \frac{\mathbb{E}[\mathcal{S}_k(G_{n,k}^+)]}{|G_{n,k}^+|},$$

where  $M_k > 0$  is the Templeman constant associated with  $\{G_{n,k}^+\}_{n \in \mathbb{N}^*}$ . Consequently, taking (2.35) into account, we get

$$\mathbb{P}(V_\alpha^{\mathcal{S}_k}) \leq \frac{M_k}{\alpha} \varepsilon \text{ for all } k \geq k_\varepsilon. \tag{2.40}$$

**Step 2-4: end of Step 2.** From (2.38) and (2.39) it follows that

$$\bar{l}_k - \underline{l}_k \leq -\inf_{n \in \mathbb{N}^*} \frac{\mathcal{S}_k(G_{n,k}^+)}{|G_{n,k}^+|}.$$

Hence  $W_{k,\alpha} \subset V_\alpha^{\mathcal{S}_k}$ , where  $W_{k,\alpha}$  is defined in (2.29). From (2.40) we conclude that (2.30) is satisfied with  $k = k_\varepsilon$ .

In what follows we set  $l := \bar{l} = \underline{l}$  and  $\gamma := \inf_{k \in \mathbb{N}^*} \gamma_k$  with  $\gamma_k := \frac{\mathbb{E}^{\mathcal{I}_{\mathbb{G}_k}}[S(\mathbb{U}_k)]}{\mu(\mathbb{U}_k)}$  for all  $k \in \mathbb{N}^*$ .

**Step 3: we prove that  $l(\omega) = \gamma(\omega)$  for  $\mathbb{P}$ -a.a.  $\omega \in \Sigma$ .** First of all, from (2.26) and (2.39) we see that  $l(\omega) \leq \gamma_k(\omega)$  for  $\mathbb{P}$ -a.a.  $\omega \in \Sigma$  and all  $k \in \mathbb{N}^*$ , and so

$$l(\omega) \leq \gamma(\omega) \text{ for } \mathbb{P}\text{-a.a. } \omega \in \Sigma. \tag{2.41}$$

On the other hand, letting  $n \rightarrow \infty$  in (2.37) and using (2.36) we get

$$\overline{\lim}_{n \rightarrow \infty} \frac{\mathcal{S}(Q_{n,k}^+)(\omega)}{\mu(Q_{n,k}^+)} - \gamma_k(\omega) \geq \overline{\lim}_{n \rightarrow \infty} \frac{\mathcal{S}_k(G_{n,k}^+)(\omega)}{|G_{n,k}^+|} \text{ for } \mathbb{P}\text{-a.a. } \omega \in \Sigma$$

and so, taking (2.27) into account, one has

$$l(\omega) - \gamma_k \geq \overline{\lim}_{n \rightarrow \infty} \frac{\mathcal{S}_k(G_{n,k}^+)(\omega)}{|G_{n,k}^+|} \text{ for } \mathbb{P}\text{-a.a. } \omega \in \Sigma.$$

It follows that

$$\int_\Sigma [l(\omega) - \gamma_k] d\mathbb{P}(\omega) \geq \int_\Sigma \overline{\lim}_{n \rightarrow \infty} \frac{\mathcal{S}_k(G_{n,k}^+)(\omega)}{|G_{n,k}^+|} d\mathbb{P}(\omega).$$

But, by using Fatou's lemma and (2.35) we see that for any  $k \geq k_\varepsilon$ , one has

$$\int_{\Sigma} \overline{\lim}_{n \rightarrow \infty} \frac{\mathcal{S}_k(G_{n,k}^+)(\omega)}{|G_{n,k}^+|} d\mathbb{P}(\omega) > -\varepsilon, \quad (2.42)$$

and consequently

$$\begin{aligned} \int_{\Sigma} l(\omega) d\mathbb{P}(\omega) &\geq \int_{\Sigma} \gamma_k(\omega) d\mathbb{P}(\omega) - \varepsilon \\ &\geq \int_{\Sigma} \gamma(\omega) d\mathbb{P}(\omega) - \varepsilon. \end{aligned}$$

Letting  $\varepsilon \rightarrow 0$  we deduce that

$$\int_{\Sigma} [l(\omega) - \gamma(\omega)] d\mathbb{P}(\omega) \geq 0, \quad (2.43)$$

and the result follows by combining (2.41) with (2.43).

In what follows, we set  $\gamma^{\mathcal{I}} := \inf_{k \in \mathbb{N}^*} \gamma_k^{\mathcal{I}}$  with  $\gamma_k^{\mathcal{I}} := \frac{\mathbb{E}^{\mathcal{I}}[\mathcal{S}(\mathbb{U}_k)]}{\mu(\mathbb{U}_k)}$  for all  $k \in \mathbb{N}^*$ .

**Step 4: we prove that  $l(\omega) = \gamma^{\mathcal{I}}(\omega)$  for  $\mathbb{P}$ -a.a.  $\omega \in \Sigma$ .** Since  $\gamma_k$  is  $\mathcal{I}_{\mathbb{G}_k}$ -measurable for all  $k \in \mathbb{N}^*$ ,  $\gamma = \inf_{k \in \mathbb{N}^*} \gamma_k$  is  $\cap_{k \in \mathbb{N}^*} \mathcal{I}_{\mathbb{G}_k}$ -measurable. But  $\cap_{k \in \mathbb{N}^*} \mathcal{I}_{\mathbb{G}_k} = \mathcal{I}$  because  $\cup_{k \in \mathbb{N}^*} \mathbb{G}_k = G$ , hence  $\gamma$  is  $\mathcal{I}$ -measurable and so  $l$  is  $\mathcal{I}$ -measurable by Step 3. we have  $\mathcal{I} = \cap_{k \in \mathbb{N}^*} \mathcal{I}_{\mathbb{G}_k}$ . It follows that

$$\mathbb{E}^{\mathcal{I}}[l] = l. \quad (2.44)$$

As  $\mathcal{I} \subset \mathcal{I}_{\mathbb{G}_k}$  for all  $k \in \mathbb{N}^*$  we also have

$$\mathbb{E}^{\mathcal{I}}[\gamma_k] = \gamma_k^{\mathcal{I}} \text{ for all } k \in \mathbb{N}^*. \quad (2.45)$$

Arguing as in Step 3, for each  $k \in \mathbb{N}^*$ , we have  $l \leq \gamma_k$  hence  $\mathbb{E}^{\mathcal{I}}[l] \leq \mathbb{E}^{\mathcal{I}}[\gamma_k]$  and so  $l \leq \gamma_k^{\mathcal{I}}$  by using (2.44) and (2.45). Consequently

$$l \leq \gamma^{\mathcal{I}}. \quad (2.46)$$

Fix any  $E \in \mathcal{I}$ . Arguing again as in Step 3 we see that for any  $k \geq k_\varepsilon$ , one has

$$\int_E l(\omega) d\mathbb{P}(\omega) \geq \int_E \gamma_k(\omega) d\mathbb{P}(\omega) - \varepsilon.$$

But  $\int_E \gamma_k(\omega) d\mathbb{P}(\omega) = \int_E \mathbb{E}^{\mathcal{I}}[\gamma_k](\omega) d\mathbb{P}(\omega)$  by definition of the conditional expectation, hence  $\int_E \gamma_k(\omega) d\mathbb{P}(\omega) = \int_E \gamma_k^{\mathcal{I}}(\omega) d\mathbb{P}(\omega)$  by (2.45), and so

$$\begin{aligned} \int_E l(\omega) d\mathbb{P}(\omega) &\geq \int_E \gamma_k^{\mathcal{I}}(\omega) d\mathbb{P}(\omega) - \varepsilon \\ &\geq \int_E \gamma^{\mathcal{I}}(\omega) d\mathbb{P}(\omega) - \varepsilon. \end{aligned}$$

Letting  $\varepsilon \rightarrow 0$  we get

$$\int_E l(\omega) d\mathbb{P}(\omega) \geq \int_E \gamma^{\mathcal{I}}(\omega) d\mathbb{P}(\omega) \text{ for all } E \in \mathcal{I}. \quad (2.47)$$

Combining (2.46) with (2.47) we deduce that

$$\int_E l(\omega) d\mathbb{P}(\omega) = \int_E \gamma^{\mathcal{I}}(\omega) d\mathbb{P}(\omega) \text{ for all } E \in \mathcal{I},$$

which implies that  $l = \mathbb{E}^{\mathcal{I}}[\gamma^{\mathcal{I}}]$  by unicity of the conditional expectation. But  $\gamma^{\mathcal{I}}$  is  $\mathcal{I}$ -measurable because  $\gamma_k^{\mathcal{I}}$  is  $\mathcal{I}$ -measurable for all  $k \in \mathbb{N}^*$ , hence  $\mathbb{E}^{\mathcal{I}}[\gamma^{\mathcal{I}}] = \gamma^{\mathcal{I}}$  and consequently  $l = \gamma^{\mathcal{I}}$ . ■

### 3. APPLICATIONS TO HOMOGENIZATION

Here  $(X, d, \mu)$  is a metric measure space, with  $(X, d)$  a length space, i.e., the distance between any two points equals infimum of lengths of curves connecting the points, which is complete, separable and locally compact, and satisfies a weak  $(1, p)$ -Poincaré inequality with  $p > 1$  and such that  $\mu$  is a doubling positive Radon measure on  $X$ . Let  $m \geq 1$  be an integer, let  $\Omega \subset X$  be a bounded open set, let  $\mathcal{O}(\Omega)$  be the class of open subsets of  $\Omega$  and let  $(\Sigma, \mathcal{F}, \mathbb{P})$  be a probability space. For each  $n \in \mathbb{N}^*$ , let  $E_n : H_{\mu}^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \times \Sigma \rightarrow [0, \infty]$  be the variational stochastic integral defined by

$$E_n(u, A, \omega) := \int_A L_n(x, \nabla_{\mu} u(x), \omega) d\mu(x), \quad (3.1)$$

where the stochastic integrand  $L_n : \Omega \times \mathbb{M} \times \Sigma \rightarrow [0, \infty]$  is Borel measurable and has  $p$ -growth, i.e., there exist  $\alpha, \beta > 0$ , which do not depend on  $n$ , such that

$$\alpha|\xi|^p \leq L_n(x, \xi, \omega) \leq \beta(1 + |\xi|^p) \quad (3.2)$$

for  $\mu$ -a.a.  $x \in \Omega$ , all  $\xi \in \mathbb{M}$  and  $\mathbb{P}$ -a.a.  $\omega \in \Sigma$ , with  $\mathbb{M}$  denoting the space of real  $m \times N$  matrices. The space  $H_{\mu}^{1,p}(\Omega; \mathbb{R}^m)$  denotes the class of  $p$ -Cheeger-Sobolev functions from  $\Omega$  to  $\mathbb{R}^m$  and  $\nabla_{\mu} u$  is the  $\mu$ -gradient of  $u$  (see [BB11, HKST15] and the references therein for more details on the theory of metric Sobolev spaces). In this section we deal with the problem of computing the almost sure (a.s.)  $\Gamma$ -convergence with respect to the strong convergence of  $L_{\mu}^p(\Omega; \mathbb{R}^m)$  (see Definitions 3.1 and 3.2) of the stochastic sequence  $\{E_n\}_{n \in \mathbb{N}^*}$  as  $n \rightarrow \infty$  toward a variational stochastic integral  $E_{\infty} : H_{\mu}^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \times \Sigma \rightarrow [0, \infty]$  of the type

$$E_{\infty}(u, A, \omega) = \int_A L_{\infty}(x, \nabla_{\mu} u(x), \omega) d\mu(x) \quad (3.3)$$

with  $L_{\infty} : \Omega \times \mathbb{M} \times \Sigma \rightarrow [0, \infty]$  which does not depend on  $n$ . When  $L_{\infty}$  is independent of the variable  $x$ , the procedure of passing from (3.1) to (3.3) is referred as stochastic homogenization. If furthermore  $L_{\infty}$  is independent of the variable  $\omega$  then  $E_{\infty}$  is said to be deterministic. When  $\{L_n\}_{n \in \mathbb{N}^*}$  is deterministic, i.e.,  $L_n$  is independent of the variable  $\omega$  for all  $n \in \mathbb{N}^*$ , the procedure of passing from (3.1) to (3.3) is referred as deterministic homogenization. (Deterministic and stochastic homogenization were studied by many authors in the euclidean case, i.e., when the metric measure space  $(X, d, \mu)$  is equal to  $\mathbb{R}^N$  endowed with the euclidean distance and the Lebesgue measure, see for instance [DG16] and the references therein.)

Let us recall the definition of  $\Gamma$ -convergence and a.s  $\Gamma$ -convergence. (For more details on the theory of  $\Gamma$ -convergence we refer to [DM93].)



**Definition 3.1.** For each  $n \in \mathbb{N}^*$ , let  $E_n : H_\mu^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$  and let  $E_\infty : H_\mu^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ . We say that  $\{E_n\}_{n \in \mathbb{N}^*}$   $\Gamma$ -converges with respect to the strong convergence of  $L_\mu^p(\Omega; \mathbb{R}^m)$ , or simply  $\Gamma(L_\mu^p)$ -converges, to  $E_\infty$  as  $n \rightarrow \infty$  if

$$\Gamma(L_\mu^p)\text{-}\underline{\lim}_{n \rightarrow \infty} E_n(u, A) \geq E_\infty(u, A) \geq \Gamma(L_\mu^p)\text{-}\overline{\lim}_{n \rightarrow \infty} E_n(u, A)$$

for any  $u \in H_\mu^{1,p}(\Omega; \mathbb{R}^m)$  and any  $A \in \mathcal{O}(\Omega)$ , with:

$$\begin{aligned} \Gamma(L_\mu^p)\text{-}\underline{\lim}_{n \rightarrow \infty} E_n(u, A) &:= \inf \left\{ \underline{\lim}_{n \rightarrow \infty} E_n(u_n, A) : u_n \xrightarrow{L_\mu^p} u \right\}; \\ \Gamma(L_\mu^p)\text{-}\overline{\lim}_{n \rightarrow \infty} E_n(u, A) &:= \inf \left\{ \overline{\lim}_{n \rightarrow \infty} E_n(u_n, A) : u_n \xrightarrow{L_\mu^p} u \right\}. \end{aligned}$$

Then we write

$$\Gamma(L_\mu^p)\text{-}\lim_{n \rightarrow \infty} E_n(u, A) = E_\infty(u, A).$$

Almost sure  $\Gamma$ -convergence is defined from Definition 3.1 as follows.

**Definition 3.2.** For each  $n \in \mathbb{N}^*$ , let  $E_n : H_\mu^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \times \Sigma \rightarrow [0, \infty]$  and let  $E_\infty : H_\mu^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \times \Sigma \rightarrow [0, \infty]$ . We say that  $\{E_n\}_{n \in \mathbb{N}^*}$  a.s.  $\Gamma$ -converges with respect to the strong convergence of  $L_\mu^p(\Omega; \mathbb{R}^m)$ , or simply a.s.  $\Gamma(L_\mu^p)$ -converges, to  $E_\infty$  as  $n \rightarrow \infty$  if for  $\mathbb{P}$ -a.e.  $\omega \in \Sigma$ , one has

$$\Gamma(L_\mu^p)\text{-}\lim_{n \rightarrow \infty} E_n(u, A, \omega) = E_\infty(u, A, \omega).$$

for any  $u \in H_\mu^{1,p}(\Omega; \mathbb{R}^m)$  and any  $A \in \mathcal{O}(\Omega)$ .

For each  $n \in \mathbb{N}^*$  and each  $\rho > 0$ , let  $\mathcal{H}_\mu^\rho L_n : \Omega \times \mathbb{M} \times \Sigma \rightarrow [0, \infty]$  be given by

$$\mathcal{H}_\mu^\rho L_n(x, \xi, \omega) := \inf \left\{ \int_{Q_\rho(x)} L_n(y, \xi + \nabla_\mu w(y), \omega) d\mu(y) : w \in H_{\mu,0}^{1,p}(Q_\rho(x); \mathbb{R}^m) \right\}$$

where the space  $H_{\mu,0}^{1,p}(Q_\rho(x); \mathbb{R}^m)$  is the closure of

$$\text{Lip}_0(Q_\rho(x); \mathbb{R}^m) := \left\{ u \in \text{Lip}(\Omega; \mathbb{R}^m) : u = 0 \text{ on } \Omega \setminus Q_\rho(x) \right\}$$

with respect to the  $H_\mu^{1,p}$ -norm, where  $\text{Lip}(\Omega; \mathbb{R}^m) := [\text{Lip}(\Omega)]^m$  with  $\text{Lip}(\Omega)$  denoting the algebra of Lipschitz functions from  $\Omega$  to  $\mathbb{R}$ . When  $\{L_n\}_{n \in \mathbb{N}^*}$  is deterministic, in [AHM17, Theorem 2.2 and Corollary 2.3] we proved the following deterministic  $\Gamma$ -convergence result.

**Theorem 3.3.** *Assume that (3.2) holds and*

$$\underline{\lim}_{n \rightarrow \infty} \mathcal{H}_\mu^\rho L_n(x, \xi) = \overline{\lim}_{n \rightarrow \infty} \mathcal{H}_\mu^\rho L_n(x, \xi)$$

for  $\mu$ -a.a.  $x \in \Omega$ , all  $\rho > 0$  and all  $\xi \in \mathbb{M}$ . Then

$$\Gamma(L_\mu^p)\text{-}\lim_{n \rightarrow \infty} E_n(u, A) = \int_A \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \mathcal{H}_\mu^\rho L_n(x, \nabla_\mu u(x)) d\mu(x)$$

for all  $u \in H_\mu^{1,p}(\Omega; \mathbb{R}^m)$  and all  $A \in \mathcal{O}(\Omega)$ .

By the same method as in [AHM17, Theorem 2.2 and Corollary 2.3] we can establish the following stochastic version of Theorem 3.3.

**Theorem 3.4.** *Assume that (3.2) holds and*

$$\underline{\lim}_{n \rightarrow \infty} \mathcal{H}_\mu^\rho L_n(x, \xi, \omega) = \overline{\lim}_{n \rightarrow \infty} \mathcal{H}_\mu^\rho L_n(x, \xi, \omega)$$

for  $\mu$ -a.a.  $x \in \Omega$ , all  $\rho > 0$ , all  $\xi \in \mathbb{M}$  and  $\mathbb{P}$ -a.a.  $\omega \in \Sigma$ . Then, for  $\mathbb{P}$ -a.e.  $\omega \in \Sigma$ , one has

$$\Gamma(L_\mu^p)\text{-}\lim_{n \rightarrow \infty} E_n(u, A, \omega) = \int_A \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \mathcal{H}_\mu^\rho L_n(x, \nabla_\mu u(x), \omega) d\mu(x)$$

for all  $u \in H_\mu^{1,p}(\Omega; \mathbb{R}^m)$  and all  $A \in \mathcal{O}(\Omega)$ .

In §3.1 (resp. §3.2), by using Theorems 2.3 and 3.3 (resp. Theorems 2.11 and 3.4) we establish a periodic (resp. stochastic) homogenization theorem in the setting of Cheeger-Sobolev spaces, see Theorem 3.6 (resp. Theorem 3.8). In what follows, we adopt notation of Section 2 and, from now on,  $\text{Ba}(X)$  denotes the class of open balls  $Q$  of  $X$ . As  $(X, d)$  is a Length space we have  $\mu(\partial Q) = 0$  for all  $Q \in \text{Ba}(X)$ <sup>3</sup>. Hence  $\text{Ba}(X) \subset \mathcal{B}_{\mu,0}(X)$ .

**3.1. Periodic homogenization.** Let  $L : X \times \mathbb{M} \rightarrow [0, \infty]$  be a Borel measurable integrand having  $p$ -growth, i.e., there exist  $\alpha, \beta > 0$  such that

$$\alpha|\xi|^p \leq L(x, \xi) \leq \beta(1 + |\xi|^p) \quad (3.4)$$

for  $\mu$ -a.a.  $x \in \Omega$  and all  $\xi \in \mathbb{M}$ , and assumed to be  $G$ -invariant, i.e.,

$$L(g(x), \xi) = L(x, \xi) \quad (3.5)$$

for  $\mu$ -a.a.  $x \in X$ , all  $\xi \in \mathbb{M}$  and all  $g \in G$ . Let  $\{h_n\}_{n \in \mathbb{N}^*} \subset \text{Homeo}(X)$  and, for each  $n \in \mathbb{N}^*$ , let  $L_n : X \times \mathbb{M} \rightarrow [0, \infty]$  be given by

$$L_n(x, \xi) = L(h_n(x), \xi). \quad (3.6)$$

(Then (3.4) implies (3.2) with  $L_n$  independent of  $\omega$ , and we have  $L_n((h_n^{-1} \circ g \circ h_n)(x), \xi) = L_n(x, \xi)$  for  $\mu$ -a.a.  $x \in X$ , all  $\xi \in \mathbb{M}$ , all  $n \in \mathbb{N}^*$  and all  $g \in G$ .)

**Definition 3.5.** Such a  $\{L_n\}_{n \in \mathbb{N}^*}$ , defined by (3.5)-(3.6), is called a  $(G, \{h_n\}_{n \in \mathbb{N}^*})$ -periodic sequence (of integrands) modelled on  $L$ .

Let us consider the following condition on the triplet  $((X, d, \mu), G, \{h_n\}_{n \in \mathbb{N}^*})$ :

(P) there exists  $\mathbb{U} \in \mathcal{B}_{\mu,0}(X)$  such that for all  $Q \in \text{Ba}(X)$ , the sequence  $\{h_n(Q)\}_{n \in \mathbb{N}^*}$  is weakly  $G$ -asymptotic with respect to  $\{h_k(\mathbb{U})\}_{k \in \mathbb{N}^*}$  (see Definition 2.1).

The following theorem was established in [AHM17, Theorems 2.20] under a slightly different framework. (In what follows, the symbol  $\int$  stands for the mean value integral.)

<sup>3</sup>Indeed, by Colding-Minicozzi II's inequality (see [CM98], [Che99, Proposition 6.12] and [HKST15, Proposition 11.5.3]), there exists  $\delta > 0$  such that  $\mu(Q_{\tau\rho}(x) \setminus Q_\rho(x)) \leq 2^\delta(1 - \frac{1}{\tau})\mu(Q_{\tau\rho}(x))$  for all  $x \in X$ , all  $\rho > 0$  and all  $\tau \in ]1, \infty[$ . Then, given  $x \in X$  and  $\rho > 0$ , we have  $1 \geq \mu(Q_\rho(x))/\mu(\overline{Q_\rho(x)}) \geq \mu(Q_\rho(x))/\mu(Q_{\tau\rho}(x)) \geq 1 - 2^\delta(1 - \frac{1}{\tau})$  for all  $\tau \in ]1, \infty[$ . Hence, by letting  $\tau \rightarrow 1$ ,  $\mu(Q_\rho(x))/\mu(\overline{Q_\rho(x)}) = 1$ , i.e.,  $\mu(Q_\rho(x)) = \mu(\overline{Q_\rho(x)})$ .

**Theorem 3.6.** *Assume that  $(X, d, \mu)$  satisfies (P) and consider  $\{L_n\}_{n \in \mathbb{N}^*}$  a  $(G, \{h_n\}_{n \in \mathbb{N}^*})$ -periodic sequence modelled on  $L$ . If (3.4) holds then*

$$\Gamma(L_\mu^p)\text{-}\lim_{n \rightarrow \infty} E_n(u, A) = \int_A L_{\text{hom}}(\nabla_\mu u(x)) d\mu(x)$$

for all  $u \in H_\mu^{1,p}(\Omega; \mathbb{R}^m)$  and all  $A \in \mathcal{O}(\Omega)$  with  $L^{\text{hom}} : \mathbb{M} \rightarrow [0, \infty]$  given by

$$L_{\text{hom}}(\xi) := \inf_{k \in \mathbb{N}^*} \inf \left\{ \int_{h_k(\mathring{U})} L(y, \xi + \nabla_\mu w(y)) d\mu(y) : w \in H_{\mu,0}^{1,p} \left( h_k(\mathring{U}); \mathbb{R}^m \right) \right\}.$$

The proof of Theorem 3.6 follows the same line as in the proof of Theorem 3.8, in using Theorems 3.3 and 2.3 instead of Theorems 3.4 and 2.11. So, we omit its proof and we refer to §3.2.

**3.2. Stochastic homogenization.** In what follows, we assume that  $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in G})$  is a measurable dynamical  $G$ -system (see Definition 2.8). Let  $L : X \times \mathbb{M} \times \Sigma \rightarrow [0, \infty]$  be a Borel measurable integrand having  $p$ -growth, i.e., there exist  $\alpha, \beta > 0$  such that

$$\alpha|\xi|^p \leq L(x, \xi, \omega) \leq \beta(1 + |\xi|^p) \quad (3.7)$$

for  $\mu$ -a.a.  $x \in \Omega$ , all  $\xi \in \mathbb{M}$  and  $\mathbb{P}$ -a.a.  $\omega \in \Sigma$ , and assumed to be  $G$ -covariant, i.e.,

$$L(g(x), \xi, \omega) = L(x, \xi, \tau_g(\omega)) \quad (3.8)$$

for  $\mu$ -a.a.  $x \in X$ , all  $\xi \in \mathbb{M}$ , all  $g \in G$  and  $\mathbb{P}$ -a.a.  $\omega \in \Sigma$ . Let  $\{h_n\}_{n \in \mathbb{N}^*} \subset \text{Homeo}(X)$  and, for each  $n \in \mathbb{N}^*$ , let  $L_n : X \times \mathbb{M} \times \Sigma \rightarrow [0, \infty]$  be given by

$$L_n(x, \xi, \omega) = L(h_n(x), \xi, \omega). \quad (3.9)$$

(Then (3.7) implies (3.2), and we have  $L_n((h_n^{-1} \circ g \circ h_n)(x), \xi, \omega) = L_n(x, \xi, \tau_g(\omega))$  for  $\mu$ -a.a.  $x \in X$ , all  $\xi \in \mathbb{M}$ , all  $n \in \mathbb{N}^*$ , all  $g \in G$  and  $\mathbb{P}$ -a.a.  $\omega \in \Sigma$ .)

**Definition 3.7.** Such a  $\{L_n\}_{n \in \mathbb{N}^*}$ , defined by (3.8)-(3.9), is called a  $(G, \{h_n\}_{n \in \mathbb{N}^*})$ -stochastic sequence (of integrands) modelled on  $L$ .

Let us consider the following condition on the triplet  $((X, d, \mu), G, \{h_n\}_{n \in \mathbb{N}^*})$ :

(S) there exists  $\mathbb{U} \in \mathcal{B}_{\mu,0}(X)$  such that:

- $(X, d, \mu)$  is meshable with respect to  $\{h_k(\mathbb{U})\}_{k \in \mathbb{N}^*}$  (see Definition 2.4);
- for all  $Q \in \text{Ba}(X)$ , the sequence  $\{h_n(Q)\}_{n \in \mathbb{N}^*}$  is strongly  $G$ -asymptotic with respect to  $\{h_k(\mathbb{U})\}_{k \in \mathbb{N}^*}$  (see Definition 2.7).

The following theorem is the stochastic version of Theorem 3.8.

**Theorem 3.8.** *Assume that  $(X, d, \mu)$  satisfies (S) and consider  $\{L_n\}_{n \in \mathbb{N}^*}$  a  $(G, \{h_n\}_{n \in \mathbb{N}^*})$ -stochastic sequence modelled on  $L$ . If (3.7) holds then, for  $\mathbb{P}$ -a.e.  $\omega \in \Sigma$ , one has*

$$\Gamma(L_\mu^p)\text{-}\lim_{n \rightarrow \infty} E_n(u, A, \omega) = \int_A L_{\text{hom}}(\nabla_\mu u(x), \omega) d\mu(x)$$

for all  $u \in H_\mu^{1,p}(\Omega; \mathbb{R}^m)$  and all  $A \in \mathcal{O}(\Omega)$  with  $L^{\text{hom}} : \mathbb{M} \times \Sigma \rightarrow [0, \infty]$  given by

$$L_{\text{hom}}(\xi, \omega) := \inf_{k \in \mathbb{N}^*} \mathbb{E}^{\mathcal{I}} \left[ \inf \left\{ \int_{h_k(\mathring{U})} L(y, \xi + \nabla_\mu w(y), \cdot) d\mu(y) : w \in H_{\mu,0}^{1,p} \left( h_k(\mathring{U}); \mathbb{R}^m \right) \right\} \right] (\omega),$$

where  $\mathbb{E}^{\mathcal{I}}$  denotes the conditional expectation over  $\mathcal{I}$  with respect to  $\mathbb{P}$ , with  $\mathcal{I}$  being the  $\sigma$ -algebra of invariant sets with respect to  $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in G})$ . If in addition  $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in G})$  is ergodic, then  $L_{\text{hom}}$  is deterministic and is given by

$$L_{\text{hom}}(\xi) := \inf_{k \in \mathbb{N}^*} \mathbb{E} \left[ \inf \left\{ \int_{h_k(\mathring{\mathbb{U}})} L(y, \xi + \nabla_{\mu} w(y), \cdot) d\mu(y) : w \in H_{\mu,0}^{1,p}(h_k(\mathring{\mathbb{U}}); \mathbb{R}^m) \right\} \right],$$

where  $\mathbb{E}$  denotes the expectation with respect to  $\mathbb{P}$ .

**Proof of Theorem 3.8.** The proof consists of applying Theorem 3.4. For this, it suffices to prove that for every  $\xi \in \mathbb{M}$ , one has

$$\varliminf_{n \rightarrow \infty} \mathcal{H}_{\mu}^{\rho} L_n(x, \xi, \omega) = \overline{\varliminf}_{n \rightarrow \infty} \mathcal{H}_{\mu}^{\rho} L_n(x, \xi, \omega) = L_{\text{hom}}(\xi, \omega) \quad (3.10)$$

for  $\mu$ -a.e.  $x \in \Omega$ , all  $\rho > 0$  and  $\mathbb{P}$ -a.a.  $\omega \in \Sigma$ . Fix  $\xi \in \mathbb{M}$  and let  $\mathcal{S}^{\xi} : \mathcal{B}_{\mu,0}(X) \rightarrow L^1(\Sigma, \mathcal{T}, \mathbb{P})$  be defined by

$$\mathcal{S}^{\xi}(A)(\omega) := \inf \left\{ \int_{\mathring{A}} L(y, \xi + \nabla_{\mu} w(y), \omega) d\mu(y) : w \in H_{\mu,0}^{1,p}(\mathring{A}; \mathbb{R}^m) \right\},$$

where by (3.7) we have  $0 \leq \mathcal{S}^{\xi}(A)(\omega) \leq c\mu(\mathring{A}) \leq c\mu(A)$  for all  $A \in \mathcal{B}_0(X)$  and all  $\omega \in \Sigma$  with  $c := \beta(1 + |\xi|^p)$ . In particular  $\mathcal{S}^{\xi}$  satisfies the boundedness condition in (2.5). On the other hand, by using (3.9), we see that

$$\begin{aligned} \mathcal{S}^{\xi}(h_n(Q))(\omega) &= \inf \left\{ \int_{h_n(Q)} L(y, \xi + \nabla_{\mu} w(y), \omega) d\mu(y) : w \in H_{\mu,0}^{1,p}(h_n(Q); \mathbb{R}^m) \right\} \\ &= \inf \left\{ \int_Q L(h_n(y), \xi + \nabla_{\mu} w(h_n(y)), \omega) d(h_n^{\#}\mu)(y) : w \in H_{\mu,0}^{1,p}(h_n(Q); \mathbb{R}^m) \right\} \\ &= \mu(h_n(Q)) \inf \left\{ \int_Q L_n(y, \xi + \nabla_{\mu} w(y), \omega) d\mu(y) : w \in H_{\mu,0}^{1,p}(Q; \mathbb{R}^m) \right\} \end{aligned}$$

for all  $Q \in \text{Ba}(X)$ , all  $n \in \mathbb{N}^*$  and all  $\omega \in \Sigma$  (where  $h_n^{\#}\mu$  denotes the image measure of  $\mu$  by  $h_n$ ). Consequently, we have:

$$\varliminf_{n \rightarrow \infty} \mathcal{H}_{\mu}^{\rho} L_n(x, \xi, \omega) = \varliminf_{n \rightarrow \infty} \frac{\mathcal{S}^{\xi}(h_n(Q_{\rho}(x)))(\omega)}{\mu(h_n(Q_{\rho}(x)))}; \quad (3.11)$$

$$\overline{\varliminf}_{n \rightarrow \infty} \mathcal{H}_{\mu}^{\rho} L_n(x, \xi, \omega) = \overline{\varliminf}_{n \rightarrow \infty} \frac{\mathcal{S}^{\xi}(h_n(Q_{\rho}(x)))(\omega)}{\mu(h_n(Q_{\rho}(x)))} \quad (3.12)$$

for  $\mu$ -a.e.  $x \in \Omega$ , all  $\rho > 0$  and  $\mathbb{P}$ -a.a.  $\omega \in \Sigma$ . Moreover, from (3.8) it easily seen that the set function  $\mathcal{S}^{\xi}$  is  $G$ -covariant, and  $\mathcal{S}^{\xi}$  is also subadditive because, for each  $A, B \in \mathcal{B}_0(X)$ ,  $\mu(\widehat{A \cup B} \setminus (\mathring{A} \cup \mathring{B})) = 0$  since  $\widehat{A \cup B} \setminus (\mathring{A} \cup \mathring{B}) \subset \partial A \cup \partial B$  and  $\mu(\partial A) = \mu(\partial B) = 0$ . Thus, since (S) is satisfied, for  $\mu$ -a.e.  $x \in \Omega$  and all  $\rho > 0$ , we can apply Theorem 2.11 with  $\{\mathbb{U}_k\}_{k \in \mathbb{N}^*} = \{h_k(\mathbb{U})\}_{k \in \mathbb{N}^*}$  and  $\{Q_n\}_{n \in \mathbb{N}^*} = \{h_n(Q_{\rho}(x))\}_{n \in \mathbb{N}^*}$ , and, noticing that  $\mu(h_k(\mathbb{U})) =$

$\mu(\widehat{h_k(\mathbb{U})}) = \mu(h_k(\mathring{\mathbb{U}}))$  for all  $k \in \mathbb{N}^*$ , we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mathcal{S}^\xi(h_n(Q_\rho(x)))(\omega)}{\mu(h_n(Q_\rho(x)))} &= \inf_{k \in \mathbb{N}^*} \frac{\mathbb{E}^\mathcal{I}[\mathcal{S}^\xi(h_k(\mathbb{U}))](\omega)}{\mu(h_k(\mathbb{U}))} \\ &= \inf_{k \in \mathbb{N}^*} \mathbb{E}^\mathcal{I} \left[ \frac{\mathcal{S}^\xi(h_k(\mathbb{U}))}{\mu(h_k(\mathring{\mathbb{U}}))} \right](\omega) \\ &= L_{\text{hom}}(\xi, \omega), \end{aligned}$$

for  $\mathbb{P}$ -a.a.  $\omega \in \Sigma$ , and (3.10) follows from (3.11) and (3.12). ■

## REFERENCES

- [AAB<sup>+</sup>10] Claire Anantharaman, Jean-Philippe Anker, Martine Babillot, Aline Bonami, Bruno Demange, Sandrine Grellier, François Havard, Philippe Jaming, Emmanuel Lesigne, Patrick Maheux, Jean-Pierre Otal, Barbara Schapira, and Jean-Pierre Schreiber. *Théorèmes ergodiques pour les actions de groupes*, volume 41 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*. L'Enseignement Mathématique, Geneva, 2010. With a foreword in English by Amos Nevo.
- [AHM17] Omar Anza Hafsa and Jean-Philippe Mandallena.  $\Gamma$ -convergence of nonconvex integrals in Cheeger-Sobolev spaces and homogenization. *Adv. Calc. Var.*, 10(4):381–405, 2017.
- [AK81] M. A. Akcoglu and U. Krengel. Ergodic theorems for superadditive processes. *J. Reine Angew. Math.*, 323:53–67, 1981.
- [BB11] Anders Björn and Jana Björn. *Nonlinear potential theory on metric spaces*, volume 17 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2011.
- [Bir31] George D. Birkhoff. Proof of the ergodic theorem. *Proceedings of the National Academy of Sciences*, 17(12):656–660, 1931.
- [Che99] J. Cheeger. Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.*, 9(3):428–517, 1999.
- [CM98] Tobias H. Colding and William P. Minicozzi, II. Liouville theorems for harmonic sections and applications. *Comm. Pure Appl. Math.*, 51(2):113–138, 1998.
- [Der75] Yves Derriennic. Sur le théorème ergodique sous-additif. *C. R. Acad. Sci. Paris Sér. A-B*, 281(22):Aii, A985–A988, 1975.
- [DG16] Mitia Duerinckx and Antoine Gloria. Stochastic homogenization of nonconvex unbounded integral functionals with convex growth. *Arch. Ration. Mech. Anal.*, 221(3):1511–1584, 2016.
- [DGZ14] Anthony H. Dooley, Valentyn Ya. Golodets, and Guohua Zhang. Sub-additive ergodic theorems for countable amenable groups. *J. Funct. Anal.*, 267(5):1291–1320, 2014.
- [DM93] Gianni Dal Maso. *An introduction to  $\Gamma$ -convergence*. Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston Inc., Boston, MA, 1993.
- [DMM86a] Gianni Dal Maso and Luciano Modica. Nonlinear stochastic homogenization. *Ann. Mat. Pura Appl. (4)*, 144:347–389, 1986.
- [DMM86b] Gianni Dal Maso and Luciano Modica. Nonlinear stochastic homogenization and ergodic theory. *J. Reine Angew. Math.*, 368:28–42, 1986.
- [DZ15] Anthony H. Dooley and Guohua Zhang. Local entropy theory of a random dynamical system. *Mem. Amer. Math. Soc.*, 233(1099):vi+106, 2015.
- [Gre69] Frederick P. Greenleaf. *Invariant means on topological groups and their applications*. Van Nostrand Mathematical Studies, No. 16. Van Nostrand Reinhold Co., New York-Toronto, Ont.-London, 1969.
- [HKST15] Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam, and Jeremy T. Tyson. *Sobolev spaces on metric measure spaces*, volume 27 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2015. An approach based on upper gradients.

- [HW65] J. M. Hammersley and D. J. A. Welsh. First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. In *Proc. Internat. Res. Semin., Statist. Lab., Univ. California, Berkeley, Calif*, pages 61–110. Springer-Verlag, New York, 1965.
- [Kin68] J. F. C. Kingman. The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. Ser. B*, 30:499–510, 1968.
- [Kin73] J. F. C. Kingman. Subadditive ergodic theory. *Ann. Probability*, 1:883–909, 1973. With discussion by D. L. Burkholder, Daryl Daley, H. Kesten, P. Ney, Frank Spitzer and J. M. Hammersley, and a reply by the author.
- [Kre85] Ulrich Krengel. *Ergodic theorems*, volume 6 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1985. With a supplement by Antoine Brunel.
- [Lin99] Elon Lindenstrauss. Pointwise theorems for amenable groups. *Electron. Res. Announc. Amer. Math. Soc.*, 5:82–90, 1999.
- [Lin01] Elon Lindenstrauss. Pointwise theorems for amenable groups. *Invent. Math.*, 146(2):259–295, 2001.
- [LM02] Christian Licht and Gérard Michaille. Global-local subadditive ergodic theorems and application to homogenization in elasticity. *Ann. Math. Blaise Pascal*, 9(1):21–62, 2002.
- [MM94] K. Messaoudi and G. Michaille. Stochastic homogenization of nonconvex integral functionals. *RAIRO Modél. Math. Anal. Numér.*, 28(3):329–356, 1994.
- [Neu32] J. v. Neumann. Proof of the quasi-ergodic hypothesis. *Proceedings of the National Academy of Sciences*, 18(1):70–82, 1932.
- [Ngu79] Xuan-Xanh Nguyen. Ergodic theorems for subadditive spatial processes. *Z. Wahrsch. Verw. Gebiete*, 48(2):159–176, 1979.
- [NZ79] Xuan-Xanh Nguyen and Hans Zessin. Ergodic theorems for spatial processes. *Z. Wahrsch. Verw. Gebiete*, 48(2):133–158, 1979.
- [OW87] Donald S. Ornstein and Benjamin Weiss. Entropy and isomorphism theorems for actions of amenable groups. *J. Analyse Math.*, 48:1–141, 1987.
- [Pat88] Alan L. T. Paterson. *Amenability*, volume 29 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.
- [Smy76] R. T. Smythe. Multiparameter subadditive processes. *Ann. Probability*, 4(5):772–782, 1976.
- [Tem72] A. A. Tempelman. Ergodic theorems for general dynamical systems. *Trudy Moskov. Mat. Obšč.*, 26:95–132, 1972.
- [Tem92] Arkady Tempelman. *Ergodic theorems for group actions*, volume 78 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1992. Informational and thermodynamical aspects, Translated and revised from the 1986 Russian original.

(Omar Anza Hafsa) UNIVERSITE DE NIMES, LABORATOIRE MIPA, SITE DES CARMES, PLACE GABRIEL PÉRI, 30021 NÎMES, FRANCE.

*E-mail address:* `omar.anza-hafsa@unimes.fr`

(Jean-Philippe Mandallena) UNIVERSITE DE NIMES, LABORATOIRE MIPA, SITE DES CARMES, PLACE GABRIEL PÉRI, 30021 NÎMES, FRANCE.

*E-mail address:* `jean-philippe.mandallena@unimes.fr`