



HAL
open science

Local (sub) Finslerian geometry for the maximum norms in dimension 2

A.-L Ali, Grégoire Charlot

► **To cite this version:**

A.-L Ali, Grégoire Charlot. Local (sub) Finslerian geometry for the maximum norms in dimension 2. Journal of Dynamical and Control Systems, 2019, 10.1007/s10883-019-09435-8 . hal-02004271

HAL Id: hal-02004271

<https://hal.science/hal-02004271v1>

Submitted on 1 Feb 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Local (sub) Finslerian geometry for the maximum norms in dimension 2*

Entisar A.-L. Ali[♣] and G. Charlot[♣]

[♣]*Univ. Grenoble Alpes, CNRS, Institut Fourier, F-38000 Grenoble, France
Dyala University, Irak*

`entisar.ali@univ-grenoble-alpes.fr`

[♣]*Univ. Grenoble Alpes, CNRS, Institut Fourier, F-38000 Grenoble, France
gregoire.charlot@univ-grenoble-alpes.fr*

February 1, 2019

Abstract

We consider specific sub-Finslerian structures in the neighborhood of 0 in \mathbb{R}^2 , defined by fixing a family of vector fields (F_1, F_2) and considering the norm defined on the non constant rank distribution $\Delta = \text{vect}\{F_1, F_2\}$ by

$$|G| = \inf_u \{\max\{|u_1|, |u_2|\} \mid G = u_1 F_1 + u_2 F_2\}.$$

If F_1 and F_2 are not proportionnal at p then we obtain a Finslerian structure; if not, the structure is sub-Finslerian on a distribution with non constant rank. We are interested in the study of the local geometry of these Finslerian and sub-Finslerian structures: generic properties, normal form, short geodesics, cut locus, switching locus and small spheres.

MSC classes: 53B40, 53C22, 49J15, 49K15.

Keywords: sub-Finslerian geometry, maximum norm, geodesics, small spheres, cut locus.

1 Introduction

From the 80's, the interest for the sub-Riemannian geometry increases with a lot of contributions in several domains as PDEs, analysis, probability, geometry and control. One of the question was to understand the local geometry of sub-riemannian metrics, as the singularities of small spheres, local cut locus, local conjugate locus and so on, motivated in particular by new results on the heat kernel in the sub-Riemannian context, see [10, 22, 23, 9]. The contact and the Martinet cases were deeply studied (see [1, 12, 11, 19, 2]). The quasi-contact case in dimension 4 also (see [15]). These results allowed to give new results on the asymptotics of the heat kernel at cut and conjugate loci in the 3D contact and 4D quasi-contact cases ([7, 6]).

In this article, we start the same work for Finslerian and sub-Finslerian metrics associated with a maximum norm: let consider a manifold M , a vector bundle $\pi : E \rightarrow M$ with fibers of same dimension as M endowed with a maximum norm, and a morphism of vector bundles $f : E \rightarrow TM$

*This research has been supported by ANR-15-CE40-0018.

such that the map from $\Gamma(E) \rightarrow \text{Vec}(M)$ defined by $\sigma \mapsto f \circ \sigma$ is injective. An admissible curve is a curve γ in M such that exists a lift σ in E with $\dot{\gamma}(t) = f(\sigma(t))$ a.e. The length of such a curve is the infimum of the $\int_0^T |\sigma(t)| dt$ for all possible such σ and the distance between two points q_0 and q_1 is the infimum of the lengths of the curves joining q_0 and q_1 . Remark that the map f itself is not assumed to be injective everywhere: at points where f is injective the structure is Finslerian when at points where it is not it is sub-Finslerian.

Here we concentrate our attention on the local study of such structures in dimension 2, that is when M and the fibers of E have dimension 2.

Equivalently, with a control point of view and since we are interested in local properties, we consider control systems in a neighborhood of 0 in \mathbb{R}^2 of the type

$$\dot{q} = u_1 F_1(q) + u_2 F_2(q) \tag{1}$$

where F_1 and F_2 are smooth vector fields and u_1 and u_2 are control functions satisfying

$$|u_1| \leq 1 \text{ and } |u_2| \leq 1. \tag{2}$$

Up to reparameterization, minimizing the distance in the geometric context is equivalent to minimizing the time of transfer in the control context.

We are interested in the study of the time optimal synthesis of such systems. Of course, the general situation cannot be completely described since singular cases may have very special behaviour. For example, in the case $F_1 = \partial_x$ and $F_2 = \partial_y$, any admissible trajectory with $u_1 \equiv 1$ and $\int_0^1 u_2(t) dt = 0$ joins optimally $(0, 0)$ to $(1, 0)$. Hence in the following, we will consider only "generic" situations as defined in section 2.1.

Few works exist concerning sub-Finsler geometry since it is a new subject. Let mention the works [17, 18] for dimension 3, considering norms which are assumed to be smooth outside the zero section. In [14], the sphere of a left invariant sub-Finsler structure associated to a maximum norm in the Heisenberg group is described. In the preprint [5], the authors describe the extremals (and discuss in particular their number of switches before the loss of optimality) for the Heisenberg, Grushin and Martinet distributions. In the preprint [4], we describe, in the 3D generic contact case, the small sphere and the local cut locus.

The paper is organised as follows.

In section 2 we recall Thom's transversality theorem and some of its corollaries, define what we mean by generic, give generic properties of the couples of vector fields on 2 dimensional manifolds and give a normal form for the generic couples.

In section 3, we give first general results about the optimal synthesis; recalling classical results as Chow-Rashevski, Filippov and Pontryagin theorems; analysing the possibilities for extremals to switch or to be singular depending on their initial condition; giving details on the weights of coordinates in the normal form and on the associated nilpotent approximation.

In section 4, we present the local synthesis in all the generic cases presented in the normal form of section 2.

2 Normal form

In this section, the goal is to give a list of properties of generic couples (F_1, F_2) and to construct a normal form for the couple (G_1, G_2) defined by $G_1 = F_1 + F_2$ and $G_2 = F_1 - F_2$. As we will see,

$\pm G_1$ and $\pm G_2$ are the velocities of a large class of the minimizers of the optimal control system defined by (1) and (2).

In order to do that we use the Thom's transversality theorem and some of its corollaries.

2.1 Generic properties of couples of smooth vector fields on 2d-manifolds

2.1.1 Thom's transversality theorem

Denote $J^k(M, N)$ the set of k -jets of maps from M to N .

Theorem 1 (Thom Transversality Theorem, [21], Page 82). *Let M, N be smooth manifolds and $k \geq 1$ an integer. If S_1, \dots, S_r are smooth submanifolds of $J^k(M, N)$ then the set*

$$\{f \in C^\infty(M, N) : J^k f \pitchfork S_i \text{ for } i = 1, 2, \dots, r\},$$

is residual in the C^∞ -Whitney topology.

Corollary 2. *Assume that $\text{codim } S_i > \dim M$ for $i = 1, \dots, r$ and $k \geq 1$. Then the set*

$$\{f \in C^\infty(M, N) : J^k f(M) \cap S_i = \emptyset \text{ for } i = 1, \dots, r\},$$

is residual in the C^∞ -Whitney topology.

Corollary 3. *For every f in the residual set defined in Theorem 1, the inverse images $\tilde{S}_i := (J^k f)^{-1}(S_i)$ is a smooth submanifold of M and $\text{codim } S_i = \text{codim } \tilde{S}_i$ for $i = 1, \dots, r$.*

Remark 4. Let φ be a diffeomorphism of M and ϕ be a diffeomorphism of N . The map

$$\sigma_{\varphi, \phi} : \begin{cases} C^\infty(M, N) & \longrightarrow & C^\infty(M, N) \\ f & \longmapsto & \varphi \circ f \circ \phi \end{cases}$$

induces a diffeomorphism $\sigma_{\varphi, \phi}^*$ of $J^k(M, N)$ which sends submanifolds of $J^k(M, N)$ on submanifolds of $J^k(M, N)$. Moreover, f is in the residual set defined in theorem 1, if and only if $\sigma_{\varphi, \phi}(f)$ is in the residual set

$$\{g \in C^\infty(M, N) : J^k g \pitchfork \sigma_{\varphi, \phi}^*(S_i) \text{ for } i = 1, \dots, r\}.$$

This remark is important to facilitate the presentation of the proofs of the generic properties given in the next section.

Definition 5. In the following, we will say that a property of maps is generic if it is true on a residual set defined as in Thom's theorem.

2.1.2 First generic properties

We give a list of generic properties for couples of vector fields on 2d-manifolds. In order to use Thom transversality theorem, we work locally in coordinates. Locally one can consider a couple of vector fields as the data of a map

$$g : \begin{cases} U \subset \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \times \mathbb{R}^2 \\ (x, y) & \mapsto & ((g_1(x, y), g_2(x, y)), (g_3(x, y), g_4(x, y))) \end{cases}$$

and the k -jet at $q = (0, 0) \in U$ of g as the data of the map

$$J^k g : \begin{cases} \mathbb{R}^2 & \rightarrow \mathbb{R}_k[x, y]^4 \\ (x, y) & \mapsto (P_1(x, y), \dots, P_4(x, y)) \end{cases}$$

where P_i ($1 \leq i \leq 4$) is the Taylor series of order k of g_i at q .

In order to describe submanifolds of $\mathbb{R}_k[x, y]^4$ in coordinates, we write:

$$P_1(x, y) = \sum_{i=0}^k \sum_{j=0}^{k-i} p_{1,i,j} x^i y^j, \dots, P_4(x, y) = \sum_{i=0}^k \sum_{j=0}^{k-i} p_{4,i,j} x^i y^j.$$

In the following (g_1, g_2) are the coordinates of G_1 and (g_3, g_4) the coordinates of G_2 in a local coordinate system.

Generic property 1 (GP1): for generic couples of vector fields (F_1, F_2) on M , the set of points where $G_1 = G_2 = 0$ is empty.

Indeed in coordinates such points correspond to jets with $p_{1,0,0} = p_{2,0,0} = p_{3,0,0} = p_{4,0,0} = 0$ which form a submanifold of $\mathbb{R}_k[x, y]^4$ of codimension 4. Hence, thanks to corollary 2, the property is proven.

Let call J_N^k the set of k -jets such that $P_1 \equiv 1$ and $P_2 \equiv 0$. Once assumed that we choose a coordinate system such that $G_1 = (1, 0)$ then $J^k g$ is in J_N^k .

Assume that a set S of $J^k(\mathbb{R}^2, \mathbb{R}^4)$ is defined has the zero level of a finite number of functions h_i , $i = 1 \dots k$, which differentials form a free family when restricted to TJ_N^k . Then locally the differentials of the functions h_i form a free family and hence, close to $J_N^k \cap S$, the set S is locally a submanifold. In this context, the codimension of S in $J^k(\mathbb{R}^2, \mathbb{R}^4)$ is equal to the codimension of $S' = S \cap J_N^k$ in J_N^k .

Thanks to remark 4, up to a permutation between $\pm F_1$ and $\pm F_2$ and a good choice of coordinates, we will assume in all the following that $G_1 \equiv (1, 0)$ locally that is $g_1 \equiv 1$ and $g_2 \equiv 0$. It corresponds to jets in J_N^k . As a consequence, if a set S is defined by a finite number of functions h_i , $i = 1 \dots k$, which differentials form a free family when restricted to TJ_N^k , then to apply Thom's theorem and its corollaries we are reduced to apply them to the map

$$g : \begin{cases} U \subset \mathbb{R}^2 & \rightarrow \mathbb{R}^2 \\ (x, y) & \mapsto (g_3(x, y), g_4(x, y)) \end{cases}$$

and the set $S' = S \cap J_N^k$ seen as a submanifold of $J^k(\mathbb{R}^2, \mathbb{R}^2)$.

Generic property 2 (GP2): for generic couples of vector fields (F_1, F_2) on M , the set of points where $G_2 = 0$ is a discrete set. The same holds for the set where $F_1 = 0$ or the set where $F_2 = 0$.

Indeed such points correspond to jets with $p_{3,0,0} = p_{4,0,0} = 0$ which is a submanifold of $\mathbb{R}_k[x, y]^2$ of codimension 2. Hence, thanks to corollary 3, the set where $G_2 = 0$ is generically a submanifold of M of codimension 2 that is a discrete set. For $F_2 = 0$ the equations are $p_{3,0,0} = 1$ and $p_{4,0,0} = 0$ and for $F_1 = 0$ the equations are $p_{3,0,0} = -1$ and $p_{4,0,0} = 0$.

Generic property 3 (GP3): for generic couples of vector fields (F_1, F_2) on M , the set Δ_A of points where G_2 is parallel to G_1 is an imbedded submanifold of codimension 1.

Indeed Δ_A is exactly the set of points where $g_4 = 0$, corresponding to jets with $p_{4,0,0} = 0$. This last set is an imbedded submanifold of $\mathbb{R}_k[x, y]^2$ of codimension 1. Thanks to (GP1) and to corollary 3, we can conclude that generically Δ_A is an imbedded submanifold of codimension 1.

Generic property 4 (GP4): for generic couples of vector fields (F_1, F_2) on M , the set Δ_1 of points where F_1 is parallel to $[F_1, F_2]$ is an imbedded submanifold of codimension 1. The same holds for Δ_2 where F_2 is parallel to $[F_1, F_2]$.

In order to prove (GP4), compute $[F_1, F_2]$ and describe Δ_1 in coordinates. $[F_1, F_2] = -\frac{1}{2}[G_1, G_2]$ hence has coordinates $-\frac{1}{2}p_{3,1,0}$ and $-\frac{1}{2}p_{4,1,0}$ and F_1 has coordinates $\frac{1}{2}(1 + p_{3,0,0})$ and $\frac{1}{2}p_{4,0,0}$. Hence Δ_1 corresponds to jets satisfying

$$\begin{vmatrix} -\frac{1}{2}p_{3,1,0} & \frac{1}{2}(1 + p_{3,0,0}) \\ -\frac{1}{2}p_{4,1,0} & \frac{1}{2}p_{4,0,0} \end{vmatrix} = 0.$$

The differential of this determinant is not degenerate hence the set of $\mathbb{R}_k[x, y]^2$ satisfying this equality is an imbedded submanifold of codimension 1. Hence, generically, Δ_1 is the preimage of an immersed submanifold of codimension 1 which, thanks to corollary 3, permits to conclude that Δ_1 is an immersed submanifold of codimension 1.

Generic property 5 (GP5): for generic couples of vector fields (F_1, F_2) on M , the sets $(\Delta_A \cap \Delta_1)$, $(\Delta_A \cap \Delta_2)$ and $(\Delta_1 \cap \Delta_2)$ are discrete.

Since $G_1 = (1, 0)$, the set $(\Delta_1 \cap \Delta_2) \setminus \Delta_A$ is the set of points where (F_1, F_2) is free and $[F_1, F_2] = 0$ that is

$$\begin{aligned} p_{4,0,0} &\neq 0, \\ p_{3,1,0} &= 0 \\ p_{4,1,0} &= 0. \end{aligned}$$

This set is an immersed submanifold of codimension 2 of $\mathbb{R}_k[x, y]^2$ hence, thanks to corollary 3, the set $(\Delta_1 \cap \Delta_2) \setminus \Delta_A$ is generically a discrete set.

The set $(\Delta_A \cap \Delta_2) \setminus \Delta_1$ is a set of points where $F_2 = 0$. By (GP2) it is a discrete set. The same holds for $(\Delta_A \cap \Delta_1) \setminus \Delta_2$ which is a set of points where $F_1 = 0$.

The set $\Delta_A \cap \Delta_1 \cap \Delta_2$ is the union of the subset where $F_1 \neq 0$ and $F_1 \parallel F_2 \parallel [F_1, F_2]$ with a subset where $F_1 = 0$. The second is discrete. The first set is also defined by $G_1 \parallel G_2 \parallel [G_1, G_2]$ that is $p_{4,0,0} = 0$ and $p_{4,1,0} = 0$. Hence, thanks to corollary 3, the set where $F_1 \neq 0$ and $F_1 \parallel F_2 \parallel [F_1, F_2]$ is a submanifold of codimension 2 that is a discrete set.

Generic property 6 (GP6): for generic couples of vector fields (F_1, F_2) on M , the set of points where $G_1 \parallel G_2 \parallel [G_1, G_2] \parallel [G_1, [G_1, G_2]]$ is empty.

The set where $G_1 \parallel G_2 \parallel [G_1, G_2] \parallel [G_1, [G_1, G_2]]$ is such that $p_{4,0,0} = p_{4,1,0} = p_{4,2,0} = 0$. Hence, thanks to corollary 3, it is a submanifold of codimension 3 that is an empty set.

Generic property 7 (GP7): for generic couples of vector fields (F_1, F_2) on M , at the points q where $G_1(q) \parallel G_2(q) \parallel [G_1, G_2](q)$ one gets $G_1(q) \in T_q \Delta_A$.

The property $G_1(q) \parallel G_2(q) \parallel [G_1, G_2](q)$ implies that $p_{4,0,0} = p_{4,1,0} = 0$. If $p_{4,0,1} \neq 0$ then Δ_A can be written $p_{4,0,1}y = o(x)$ that is Δ_A is tangent to the x axis and $G_1 \in T_q \Delta_A$. Hence the set of points where $G_1(q) \parallel G_2(q) \parallel [G_1, G_2](q)$ and $G_1(q) \notin T_q \Delta_A$ corresponds to jets with $p_{4,0,0} = p_{4,1,0} = p_{4,0,1} = 0$ which is a submanifold of codimension 3. Hence generically, at the points q where $G_1(q) \parallel G_2(q) \parallel [G_1, G_2](q)$, one has $G_1(q) \in T_q \Delta_A$.

One can even detail more the generic properties: using Thom transversality theorem and its corollaries, we can prove that generically

Generic property 8 (GP8): along $\Delta_1 \setminus (\Delta_2 \cup \Delta_A)$, the points where G_1 or G_2 is tangent to Δ_1 are isolated. The same holds true for $\Delta_2 \setminus (\Delta_1 \cup \Delta_A)$.

Generic property 9 (GP9): at points of $(\Delta_1 \cap \Delta_2) \setminus \Delta_A$, neither G_1 nor G_2 are tangent to Δ_1 or Δ_2 .

Generic property 10 (GP10): along $\Delta_A \setminus (\Delta_1 \cup \Delta_2)$, the set of points where $G_2 = 0$ or $G_2 = \pm G_1$ is discrete.

2.2 Normal form

Thanks to the generic properties established in the previous section, we can prove :

Theorem 6 (Normal form). *For generic couples of vector fields (F_1, F_2) on a 2d manifold M , up to an exchange between $\pm F_1$ and $\pm F_2$, at each point q of the manifold $G_1 \neq 0$ and it exists a unique coordinate system (x, y) centred at q such that one of the following normal form holds:*

$$(NF_1) \quad \begin{aligned} G_1(x, y) &= \partial_x, \\ G_2(x, y) &= \partial_y + x(a_{10} + a_{20}x + a_{11}y + o(x, y))\partial_x + x(b_{10} + b_{20}x + b_{11}y + o(x, y))\partial_y, \\ &\text{and } q \notin \Delta_A. \end{aligned}$$

$$(NF_2) \quad \begin{aligned} G_1(x, y) &= \partial_x, \\ G_2(x, y) &= (a_0 + a_{10}x + a_{01}y + o(x, y))\partial_x + x(1 + x(b_{20} + O(x, y)))\partial_y, \\ &\text{with } 0 \leq a_0 \leq 1, \text{ and } q \in \Delta_A \setminus \Delta_1. \end{aligned}$$

$$(NF_3) \quad \begin{aligned} G_1(x, y) &= \partial_x, \\ G_2(x, y) &= (a_0 + o(1))\partial_x + (b_{01}y + \frac{1}{2}x^2 + b_{11}xy + b_{02}y^2 + o(x^2, y^2))\partial_y, \\ &\text{with } b_{01} > 0 \text{ and } 0 < a_0 < 1, q \in \Delta_A \cap \Delta_1 \cap \Delta_2 \text{ and } G_1(q) \in T_q \Delta_A. \end{aligned}$$

For $(NF1)$ and $(NF2)$ one of the following subcases holds:

(NF_{1a}) (NF_1) holds with $a_{10} - b_{10} \neq 0$ and $a_{10} + b_{10} \neq 0$. It corresponds to $q \notin \Delta_A \cup \Delta_1 \cup \Delta_2$.

(NF_{1b}) (NF_1) holds with $a_{10} - b_{10} = 0$ and $a_{10} + b_{10} \neq 0$. It corresponds to $q \in \Delta_1 \setminus (\Delta_A \cup \Delta_2)$.

(NF_{1c}) (NF_1) holds with $a_{10} - b_{10} \neq 0$ and $a_{10} + b_{10} = 0$. It corresponds to $q \in \Delta_2 \setminus (\Delta_A \cup \Delta_1)$.

(NF_{1d}) (NF_1) holds with $a_{10} = b_{10} = 0$. It corresponds to $q \in (\Delta_1 \cap \Delta_2) \setminus \Delta_A$.

(NF_{2a}) (NF_2) holds with $0 \leq a_0 < 1$. It corresponds to $q \in \Delta_A \setminus (\Delta_1 \cup \Delta_2)$.

(NF_{2b}) (NF_2) holds with $a_0 = 1$. It corresponds to $q \in (\Delta_A \cap \Delta_2) \setminus \Delta_1$ that is to $q \in \Delta_A \setminus \Delta_1$ such that $F_2(q) = 0$.

Such coordinate system is called the normal coordinate system associated with F_1 and F_2 .

Proof. We assume that all the generic properties given before are satisfied. Thanks to $(GP1)$, and thanks to the fact that we are working locally, we can assume that G_1 is not zero.

Thanks to $(GP3)$, we know that Δ_A is a submanifold of dimension 1. Let start by considering a point q outside Δ_A . Let define the map φ which to (x, y) in a neighborhood U of $(0, 0)$ in \mathbb{R}^2

associates the point reached by starting at q and following G_2 during time y and then G_1 during time x that is

$$\varphi : \begin{cases} U & \rightarrow M \\ (x, y) & \mapsto e^{xG_1}e^{yG_2}q \end{cases}$$

Since $\partial_x\varphi(0,0) = G_1(q)$ and $\partial_y\varphi(0,0) = G_2(q)$, φ is a local diffeomorphism hence defines a local coordinate system. One proves easily that at each point of coordinates (x, y) the vector $G_1(x, y) = (1, 0)$. Moreover, along the y axis, since $\varphi(0, y) = e^{yG_2}q$ then $G_2(0, y) = (0, 1)$. This implies the normal form (NF_1) . With the normal form (NF_1) , one gets that

$$\begin{aligned} [F_1, F_2](0) &= -\frac{1}{2}[G_1, G_2](0) = -\frac{1}{2}(a_{10}, b_{10}), \\ F_1(0) &= \frac{1}{2}(G_1(0) + G_2(0)) = \left(\frac{1}{2}, \frac{1}{2}\right), \\ F_2(0) &= \frac{1}{2}(G_1(0) - G_2(0)) = \left(\frac{1}{2}, -\frac{1}{2}\right) \end{aligned}$$

which implies that

$$[F_1, F_2](0) = -\frac{a_{10} + b_{10}}{2}F_1(0) - \frac{a_{10} - b_{10}}{2}F_2(0).$$

The subcases follow immediately.

Assume now that $q \in \Delta_A \setminus \Delta_1$. Hence $G_1(q)$ and $G_2(q)$ are parallel and since we assume that $G_1(q)$ is not 0, we can assume up to a change of role that $G_2(q) = \alpha G_1(q)$ with $\alpha \in [0, 1]$. Since $q \notin \Delta_1$, $G_1(q)$ and $[G_1, G_2](q)$ are not parallel. This implies that G_1 is not tangent to Δ_A . As a consequence, one can choose a local parameterization $\gamma(t)$ of Δ_A such that $\gamma(0) = q$ and $\dot{\gamma}(t)$ has second coordinate 1 in the basis $(G_1(\gamma(t)), [G_1, G_2](\gamma(t)))$. We can now define the map φ which to (x, y) in a neighborhood U of $(0, 0)$ in \mathbb{R}^2 associates the point reached by starting at $\gamma(y)$ and following G_1 during time x that is

$$\varphi : \begin{cases} U & \rightarrow M \\ (x, y) & \mapsto e^{xG_1}\gamma(y) \end{cases}$$

In this coordinate system, Δ_A is the y axis, $G_1(x, y) = (1, 0)$ and the second coordinate of G_2 is null at $x = 0$ hence it is the product of the function $(x \mapsto x)$ with a smooth function g . Moreover, thanks to the property of γ , $g(0, y) = 1$ which implies that $g(x, y) = 1 + xh(x, y)$ with h a smooth function. This is exactly (NF_2) . If $0 \leq a_0 < 1$ then $F_1(q)$ and $F_2(q)$ are not null and since they are parallel but not parallel to $[F_1, F_2](q)$ then $q \in \Delta_A \setminus (\Delta_1 \cup \Delta_2)$. If $a_0 = 1$ then $F_2(q) = 0$ and $q \in (\Delta_A \cap \Delta_2) \setminus \Delta_1$.

The case where $q \in (\Delta_A \cap \Delta_1) \setminus \Delta_2$ can be treated by exchanging the roles of G_1 and G_2 since in this case $G_2(q) \neq 0$.

Let assume finally that $q \in \Delta_A \cap \Delta_1 \cap \Delta_2$. Thanks to $(GP6)$ and $(GP7)$ at such a point G_1 and $[G_1, [G_1, G_2]]$ are not parallel. Hence we can define the map φ which to (x, y) in a neighborhood U of $(0, 0)$ in \mathbb{R}^2 associates the point reached by starting at q and following $[G_1, [G_1, G_2]]$ during time y and then G_1 during time x that is

$$\varphi : \begin{cases} U & \rightarrow M \\ (x, y) & \mapsto e^{xG_1}e^{y[G_1, [G_1, G_2]]}q \end{cases}$$

The fact that G_2 and $[G_1, G_2]$ are parallel to G_1 implies $b_0 = 0$ and $b_{10} = 0$. The fact that, along the y axis, $[G_1, [G_1, G_2]] = (0, 1)$ implies in particular that $b_{20} = \frac{1}{2}$ which finishes the proof.

3 General facts about the computation of the optimal synthesis

3.1 Local controllability and existence of minimizers

In the three cases of the normal form (NF_1) , (NF_2) and (NF_3) one checks that

$$\text{span}(F_1, F_2, [F_1, F_2], [F_1, [F_1, F_2]], [F_2, [F_1, F_2]]) = \mathbb{R}^2.$$

Hence, as a consequence of Chow-Rashevski theorem (see [3, 25, 16]), generically such a control system is locally controllable that is locally, for any two points, always exists an admissible curve joining the two points.

Moreover, since at each point the set of admissible velocities is convex and compact, thanks to Filippov theorem (see [3, 20]), locally for any two points, always exists at least a minimizer.

3.2 Pontryagin Maximum Principle (PMP)

The Pontryagin Maximum Principle (PMP for short, see [3, 24]) gives necessary conditions for a curve to be a minimizer of a control problem. For our problem it takes the following form.

Theorem 7 (PMP). *Let define the Hamiltonian*

$$H(q, \lambda, u, \lambda_0) = u_1 \lambda \cdot F_1(q) + u_2 \lambda \cdot F_2(q) + \lambda_0$$

where $q \in \mathbb{R}^2$, $\lambda \in T^*\mathbb{R}^2$, $u \in \mathbb{R}^2$ and $\lambda_0 \in \mathbb{R}$. For any minimizer $(q(t), u(t))$, there exist a never vanishing Lipschitz covector $\lambda : t \mapsto \lambda(t) \in T_{q(t)}^*\mathbb{R}^2$ and a constant $\lambda_0 \leq 0$ such that

- $\dot{q}(t) = \frac{\partial H}{\partial \lambda}(q(t), \lambda(t), u(t), \lambda_0)$,
- $\dot{\lambda}(t) = -\frac{\partial H}{\partial q}(q(t), \lambda(t), u(t), \lambda_0)$,
- $0 = H(q(t), \lambda(t), u(t), \lambda_0) = \max_v \{H(q, \lambda, v, \lambda_0) \mid |v_i| \leq 1 \text{ for } i = 1, 2\}$.

If $\lambda_0 = 0$, q is said abnormal, if not q is said normal. It may be both. A solution of the PMP is called an extremal.

Proposition 8. *For a generic SF metric on a 2D manifold defined with a maximum norm, there is no abnormal extremal. Hence we can fix $\lambda_0 = -1$. This is our choice in the following.*

Proof. It is a classical fact that an abnormal extremal should correspond to a covector $\lambda \neq 0$ orthogonal to F_1 , F_2 and $[F_1, F_2]$. This implies that along the trajectory the three vectors are parallel. But generically this happens only on a discrete set, which forbids to get a non trivial curve. ■

3.3 Switchings

In this section, we follow the ideas of [13].

Definition 9. For an extremal triplet $(q(\cdot), \lambda(\cdot), u(\cdot))$, define the *switching functions*

$$\phi_i(t) = \langle \lambda(t), F_i(q(t)) \rangle, i = 1, 2,$$

and the function $\phi_3(t) = \langle \lambda(t), [F_1, F_2](q(t)) \rangle$.

Thanks to $\lambda_0 = -1$, the ϕ_i functions satisfy

$$u_1(t)\phi_1(t) + u_2(t)\phi_2(t) = 1, \quad \text{for a.e. } t.$$

A direct consequence of the maximality condition is

Proposition 10. *If $\phi_i(t) > 0$ (resp. $\phi_i(t) < 0$) then $u_i(t) = 1$ (resp. $u_i(t) = -1$).*

If $\phi_i(t) = 0$ and $\dot{\phi}_i(t) > 0$ (resp. $\dot{\phi}_i(t) < 0$) then ϕ_i changes sign at time t and the control u_i switches from -1 to $+1$ (resp. from $+1$ to -1).

Definition 11. We call *bang* an extremal trajectory corresponding to constant controls with value 1 or -1 and *bang-bang* an extremal which is a finite concatenation of bangs. We call u_i -*singular* an extremal corresponding to a null switching function ϕ_i . A time t is said to be a *switching time* if u is not bang in any neighborhood of t .

Definition 12. Outside Δ_A , let define the functions f_1 and f_2 by

$$[F_1, F_2](q) = f_2(q)F_1(q) - f_1(q)F_2(q).$$

It is clear that

$$\Delta_1 \setminus \Delta_A = f_1^{-1}(0), \quad \Delta_2 \setminus \Delta_A = f_2^{-1}(0).$$

Proposition 13 (Switching rules). *Outside $\Delta_A \cup \Delta_1 \cup \Delta_2$ the possible switches of the controls are*

- *if $f_1 > 0$ then u_1 can only switch from -1 to $+1$ when ϕ_1 goes to 0 ,*
- *if $f_1 < 0$ then u_1 can only switch from $+1$ to -1 when ϕ_1 goes to 0 ,*
- *if $f_2 > 0$ then u_2 can only switch from -1 to $+1$ when ϕ_2 goes to 0 ,*
- *if $f_2 < 0$ then u_2 can only switch from $+1$ to -1 when ϕ_2 goes to 0 .*

Proof. The fact that $\dot{\phi}_1(t) = -u_2 \cdot \lambda \cdot [F_1, F_2]$ and $\dot{\phi}_2(t) = u_1 \cdot \lambda \cdot [F_1, F_2]$ implies that, outside $\Delta_A \cup \Delta_1 \cup \Delta_2$,

$$\dot{\phi}_1(t) = u_2(t) (f_1(q(t))\phi_2(t) - f_2(q(t))\phi_1(t)) = -u_2(t)\phi_3(t), \quad (3)$$

$$\dot{\phi}_2(t) = u_1(t) (f_2(q(t))\phi_1(t) - f_1(q(t))\phi_2(t)) = u_1(t)\phi_3(t). \quad (4)$$

Now, if $\phi_1(t) = 0$ then $|\phi_2(t)| = 1$ which implies $u_2(t)\phi_2(t) = 1$ and hence $\dot{\phi}_1(t)$ and $f_1(q(t))$ have same sign and the sign of $f_1(q(t))$ determines the switch.

The same holds true for f_2 , ϕ_2 and u_2 . ■

As a consequence, on each connected component of the complement of $\Delta_A \cup \Delta_1 \cup \Delta_2$, each control u_i can take only values -1 and $+1$ and can switch only once from -1 to $+1$ if $f_i > 0$ or from $+1$ to -1 if $f_i < 0$.

Proposition 14. *At any point q outside Δ_A , exists a $\tau > 0$ such that for any extremal issued from q and of length less than τ , only one of the two controls may switch.*

Proof. If $\phi_1(t) = 0$ then $|\phi_2(t)| = 1$. Hence, if $\phi_1(t) = 0$ and $\phi_2(t') = 0$ then ϕ_1 passes from value 0 to ± 1 in time $t' - t$ which implies that $|\dot{\phi}_1|$ takes values larger than $\frac{1}{|t'-t|}$. But, since $\dot{\phi}_1(t) = -u_2(f_2(q(t))\phi_1(q(t)) - f_1(q(t))\phi_2(q(t)))$, we have $|\dot{\phi}_1(t)| \leq |f_1(q(t))| + |f_2(q(t))|$. As a consequence, if locally $|f_1 + f_2| < M$ then $|t' - t|$ cannot be smaller than $1/M$.

Proposition 15. *At any point q outside Δ_A , consider the normal coordinate system centered at q . Any local extremal stays in one of the following domains : $\mathbb{R}_+ \times \mathbb{R}_+$, $\mathbb{R}_+ \times \mathbb{R}_-$, $\mathbb{R}_- \times \mathbb{R}_+$ or $\mathbb{R}_- \times \mathbb{R}_-$.*

Proof. Thanks to previous proposition, only one control may switch in short time. Assume that $u_1 \equiv 1$. Then at each time $u_1 F_1 + u_2 F_2 = F_1 + u_2 F_2$ hence the dynamics takes the form $\alpha G_1 + (1 - \alpha)G_2$ with $\alpha \in [0, 1]$. This dynamics leaves invariant the set $\mathbb{R}_+ \times \mathbb{R}_+$, hence the extremal does not leave this set. By the same argument one proves that if $u_1 \equiv -1$ then the extremal stays in $\mathbb{R}_- \times \mathbb{R}_-$, if $u_2 \equiv 1$ then the extremal stays in $\mathbb{R}_+ \times \mathbb{R}_-$ and that if $u_2 \equiv -1$ then the extremal stays in $\mathbb{R}_- \times \mathbb{R}_+$.

3.4 Initial conditions and their parameterization

On proves easily that in the (NF_1) case, $\max(|\lambda_x(0)|, |\lambda_y(0)|) = 1$. Hence the set of initial conditions λ is compact and extremals switching in short time or singular extremals should have a ϕ_i null or close to zero. Moreover only one control can switch in short time (see Proposition 14).

In the (NF_2) and (NF_3) cases $|\lambda_x(0)| = 1$ and there is no condition on λ_y . Hence the set of initial condition is not compact. This allows to consider initial conditions with $|\lambda_y| \gg 1$ and hence will appear optimal extremals along which the two controls switch. It is not in contradiction with the Proposition 14 since in this case the base point belongs to Δ_A .

In the (NF_{2a}) and (NF_3) cases, $\phi_1(0) = \pm \frac{1+a_0}{2}$ and $\phi_2(0) = \pm \frac{1-a_0}{2}$. Hence, if one consider a compact set of initial conditions, the corresponding extremals do not switch in short time. And are not singular. As a consequence, to consider the extremal switching at least once, one should consider initial conditions with $|\lambda_y(0)| \gg 1$.

Let us give an idea of how to estimate the $|\lambda_y(0)|$ corresponding to a u_1 -switch at small time t and the consequence in terms of choice of change of coordinates.

In the (NF_2) case, $\phi_1(0) = \frac{1+a_0}{2} \geq \frac{1}{2}$. Hence, if along an extremal the control u_1 switches for t small hence on gets, since $x(t) = O(t)$ and $y(t) = O(t^2)$,

$$0 = \lambda(t).F_1(x(t), y(t)) = \frac{1+a_0}{2} + \lambda_y(0) \frac{x(t)}{2} + O(t)$$

and it implies that if an extremal sees its control u_1 switching at t then $\lambda_y(0)$ should be like $\frac{1}{t}$. Hence, in order to make estimations of the corresponding extremals, it is natural to choose as small parameter $r_0 = \frac{1}{\lambda_y(0)}$, to make the change of coordinate $r = \frac{1}{\lambda_y}$, the change of time $s = \frac{t}{r}$ and the change of coordinate $p_x = r\lambda_x$. This is what we do in the subsections 4.2 and 4.3.

In the (NF_3) case, $\phi_1(0) = \frac{1+a_0}{2} \geq \frac{1}{2}$. Hence, if along an extremal the control u_1 switches for t small hence on gets, since $x(t) = O(t)$ and $y(t) = O(t^3)$,

$$0 = \lambda(t).F_1(x(t), y(t)) = \frac{1+a_0}{2} + \lambda_y(0) \frac{x^2(t)}{4} + O(t)$$

and it implies that if an extremal sees its control u_1 switching at t then $\lambda_y(0)$ should be like $\frac{1}{t^2}$. Hence, in order to make estimations of the corresponding extremals, it is natural to choose as small parameter r_0 such that $\lambda_y(0) = \pm \frac{1}{r_0^2}$, to make the change of coordinate $r = \frac{\pm 1}{\sqrt{|\lambda_y|}}$ and the change of time $s = \frac{t}{r}$. This is what we do in the subsection 4.4.

3.5 Weights, orders and nilpotent approximation

The definitions of privileged coordinates and nilpotent approximation are too long to be given here. We refer to [8]. The coordinates we constructed in the normal form are privileged coordinates.

In the (NF_1) case, x and y have weight 1 and ∂_x and ∂_y have weight -1 as operators of derivation. In the (NF_2) case x has weight 1 and y has weight 2, ∂_x has weight -1 and ∂_y have weight -2 . In the (NF_3) case, x has weight 1 and y has weight 3, ∂_x has weight -1 and ∂_y have weight -3 .

In privileged coordinates, one way to understand the weights of the variables naturally is to estimate how they vary with time in small time along an admissible curve. As seen before, in the (NF_1) case x and y are $O(t)$ (and may be not $o(t)$), in the (NF_2) case $x = O(t)$ and $y = O(t^2)$ and in the (NF_3) case $x = O(t)$ and $y = O(t^3)$.

In the following, $o_k(x, y)$ will denote a function whose valuation at 0 has order larger than k respectively to the weights of x and y . For example x^7 has always weight 7 and y^3 has weight 3 in the (NF_1) case but 9 in the (NF_3) case.

With this notion of weights, we define the nilpotent approximation of our normal forms in the three cases

$$\begin{aligned}
(NF_1) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= \partial_y, \\
(NF_2) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= a_0 \partial_x + x \partial_y, \\
(NF_3) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= a_0 \partial_x + \frac{1}{2} x^2 \partial_y,
\end{aligned}$$

which corresponds to an approximation to order -1 . In the following, when we will compute developments with respect to the parameter r_0 , that is for $|\lambda_y(0)| \gg 1$, we will need the approximation to order 0 for (NF_{2a}) and (NF_3) , and the approximation to order 1 for (NF_{2b})

$$\begin{aligned}
(NF_{2a}) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= (a_0 + a_{10}x) \partial_x + x(1 + b_{20}x) \partial_y, \\
(NF_{2b}) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= (1 + a_{10}x + a_{01}y + a_{20}x^2) \partial_x + x(1 + b_{20}x + b_{30}x^2) \partial_y, \\
(NF_3) \quad G_1(x, y) &= \partial_x, \\
G_2(x, y) &= (a_0 + a_{10}x) \partial_x + \left(\frac{x^2}{2} + b_{01}y + b_{30}x^3 \right) \partial_y,
\end{aligned}$$

In the (NF_1) case, we will need the approximation to order 2 in order to compute the cut locus,

when present:

$$(NF_1) \quad \begin{aligned} G_1(x, y) &= \partial_x, \\ G_2(x, y) &= x(a_{10} + a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2)\partial_x + \\ &\quad + (1 + x(b_{10} + b_{20}x + b_{11}y + b_{30}x^2 + b_{21}xy + b_{12}y^2))\partial_y, \end{aligned}$$

3.6 Symbols of extremals

As we will see in the following, the local extremals will be finite concatenations of bang arcs and u_i -singular arcs. In order to facilitate the presentation, a bang arc following $\pm G_i$ will be symbolized by $[[\pm G_i]]$, a u_1 -singular arc with control $u_2 \equiv 1$ will be symbolized by $[[S_1^+]]$, a u_1 -singular arc with control $u_2 \equiv -1$ will be symbolized by $[[S_1^-]]$, and we will combined these symbols in such a way that $[[-G_1, G_2, S_2^+]]$ symbolizes the concatenation of a bang arc following $-G_1$ with a bang arc following G_2 and a u_2 -singular arc with control $u_1 \equiv 1$.

3.7 Symmetries

One can change the roles of the vectors F_1 and F_2 and look at the effect on the functions f_i or on the invariants appearing in the normal form. For this last part, one should be careful that changing the role of F_1 and F_2 implies changing G_1 and G_2 and hence changing the coordinates x and y .

Let first look at the effect on the functions f_i on an example : $\bar{F}_1 = -F_1$ and $\bar{F}_2 = F_2$. If we define the control system with (\bar{F}_1, \bar{F}_2) , it defines the same SF structure. We compute easily that

$$[\bar{F}_1, \bar{F}_2] = [-F_1, F_2] = -[F_1, F_2] = -(f_2F_1 - f_1F_2) = f_2\bar{F}_1 - (-f_1)\bar{F}_2$$

hence $\bar{f}_1 = -f_1$ and $\bar{f}_2 = f_2$. With this choice $\bar{G}_1 = -G_2$ and $\bar{G}_2 = -G_1$. Of course, with such a change on the vectors G_1 and G_2 the change on the invariants is not so trivial to compute.

In the following we consider changes that send G_1 to $\pm G_1$ and G_2 to $\pm G_2$. These changes are interesting from a calculus point of view. Effectively, once computed the jet of a bang-bang extremals with symbol $[[G_1, G_2]]$ and of its switching times, we are able to get the expressions for the bang-bang extremals with symbols $[[\pm G_1, \pm G_2]]$. No use to repeat the computations. For example, if one gets the expression of an extremal with symbol $[[G_1, G_2]]$ as function of the initial conditions, one gets the expression of an extremal with symbol $[[-G_1, G_2]]$ by respecting the effect on the coordinates and the invariants a_0, a_{10} , etc. of the corresponding change of role of F_1 and F_2 .

3.7.1 $\bar{G}_1 = -G_1$ and $\bar{G}_2 = G_2$

Let consider the change $\bar{F}_1 = -F_2$ and $\bar{F}_2 = -F_1$. Then $\bar{G}_1 = -G_1$ and $\bar{G}_2 = G_2$. With this choice,

$$[\bar{F}_1, \bar{F}_2] = [-F_2, -F_1] = -[F_1, F_2] = -(f_2F_1 - f_1F_2) = (-f_1)\bar{F}_1 - (-f_2)\bar{F}_2$$

hence $\bar{f}_1 = -f_2$ and $\bar{f}_2 = -f_1$. Moreover, $[\bar{G}_1, \bar{G}_2] = -[G_1, G_2]$ and $[\bar{G}_1, [\bar{G}_1, \bar{G}_2]] = [G_1, [G_1, G_2]]$.

We can now consider the effect of this change of role on the coordinates and on the invariants in the three cases of the normal form

(NF₁) In this case, $\bar{x} = -x$ and $\bar{y} = y$, hence $\partial_{\bar{x}} = -\partial_x$ and $\partial_{\bar{y}} = \partial_y$ and

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (a_{10}\bar{x} - a_{20}\bar{x}^2 + a_{11}\bar{x}\bar{y} + o_2(\bar{x}, \bar{y}))\partial_{\bar{x}} + (1 - b_{10}\bar{x} + b_{20}\bar{x}^2 - b_{11}\bar{x}\bar{y} + o_2(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

(NF₂) In this case, $\bar{x} = -x$ and $\bar{y} = -y$, hence $\partial_{\bar{x}} = -\partial_x$ and $\partial_{\bar{y}} = -\partial_y$ and

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (-a_0 + a_{10}\bar{x} - a_{01}\bar{y} - a_{20}\bar{x}^2 + o_2(\bar{x}, \bar{y}))\partial_{\bar{x}} + (\bar{x} - b_{20}\bar{x}^2 + b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

(NF₃) In this case, $\bar{x} = -x$ and $\bar{y} = y$, hence $\partial_{\bar{x}} = -\partial_x$ and $\partial_{\bar{y}} = \partial_y$ and

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (-a_0 + a_{10}\bar{x} + o_1(\bar{x}, \bar{y}))\partial_{\bar{x}} + (\bar{x}^2/2 + b_{01}\bar{y} - b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

3.7.2 $\bar{G}_1 = G_1$ and $\bar{G}_2 = -G_2$

Let consider the change $\bar{F}_1 = F_2$ and $\bar{F}_2 = F_1$. Then $\bar{G}_1 = G_1$ and $\bar{G}_2 = -G_2$. With this choice,

$$[\bar{F}_1, \bar{F}_2] = [F_2, F_1] = -[F_1, F_2] = -(f_2F_1 - f_1F_2) = (f_1)\bar{F}_1 - (f_2)\bar{F}_2$$

hence $\bar{f}_1 = f_2$ and $\bar{f}_2 = f_1$. Moreover, $[\bar{G}_1, \bar{G}_2] = -[G_1, G_2]$ and $[\bar{G}_1, [\bar{G}_1, \bar{G}_2]] = -[G_1, [G_1, G_2]]$.

We can know consider the effect of this change of role on the coordinates and on the invariants in the three cases of the normal form

(NF₁) In this case, $\bar{x} = x$ and $\bar{y} = -y$, hence $\partial_{\bar{x}} = \partial_x$ and $\partial_{\bar{y}} = -\partial_y$ and

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (-a_{10}\bar{x} - a_{20}\bar{x}^2 + a_{11}\bar{x}\bar{y} + \bar{x}o(\bar{x}, \bar{y}))\partial_{\bar{x}} + (1 + b_{10}\bar{x} + b_{20}\bar{x}^2 - b_{11}\bar{x}\bar{y} + \bar{x}o(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

(NF₂) In this case, $\bar{x} = x$ and $\bar{y} = -y$, hence $\partial_{\bar{x}} = \partial_x$ and $\partial_{\bar{y}} = -\partial_y$ and

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (-a_0 - a_{10}\bar{x} + a_{01}\bar{y} - a_{20}\bar{x}^2 + o_2(\bar{x}, \bar{y}))\partial_{\bar{x}} + (\bar{x} + b_{20}\bar{x}^2 + b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

(NF₃) In this case, $\bar{x} = x$ and $\bar{y} = -y$, hence $\partial_{\bar{x}} = \partial_x$ and $\partial_{\bar{y}} = -\partial_y$ and

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (-a_0 - a_{10}\bar{x} + o_1(\bar{x}, \bar{y}))\partial_{\bar{x}} + (\bar{x}^2/2 - b_{01}\bar{y} + b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

3.7.3 $\bar{G}_1 = -G_1$ and $\bar{G}_2 = -G_2$

Let consider the change $\bar{F}_1 = -F_1$ and $\bar{F}_2 = -F_2$. Then $\bar{G}_1 = -G_1$ and $\bar{G}_2 = -G_2$. With this choice,

$$[\bar{F}_1, \bar{F}_2] = [-F_1, -F_2] = [F_1, F_2] = (f_2F_1 - f_1F_2) = (-f_2)\bar{F}_1 - (-f_1)\bar{F}_2$$

hence $\bar{f}_1 = -f_1$ and $\bar{f}_2 = -f_2$. Moreover, $[\bar{G}_1, \bar{G}_2] = [G_1, G_2]$ and $[\bar{G}_1, [\bar{G}_1, \bar{G}_2]] = -[G_1, [G_1, G_2]]$.

We can know consider the effect of this change of role on the coordinates and on the invariants in the three cases of the normal form

(NF₁) In this case, $\bar{x} = -x$ and $\bar{y} = -y$, hence $\partial_{\bar{x}} = -\partial_x$ and $\partial_{\bar{y}} = -\partial_y$. Moreover

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (-a_{10}\bar{x} + a_{20}\bar{x}^2 + a_{11}\bar{x}\bar{y} + \bar{x}o(\bar{x}, \bar{y}))\partial_{\bar{x}} + (1 - b_{10}\bar{x} + b_{20}\bar{x}^2 + b_{11}\bar{x}\bar{y} + \bar{x}o(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

(NF₂) In this case, $\bar{x} = -x$ and $\bar{y} = y$, hence $\partial_{\bar{x}} = -\partial_x$ and $\partial_{\bar{y}} = \partial_y$. Moreover

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (a_0 - a_{10}\bar{x} + a_{01}\bar{y} + a_{20}\bar{x}^2 + o_2(\bar{x}, \bar{y}))\partial_{\bar{x}} + (\bar{x} - b_{20}\bar{x}^2 + b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

(NF₃) In this case, $\bar{x} = -x$ and $\bar{y} = -y$, hence $\partial_{\bar{x}} = -\partial_x$ and $\partial_{\bar{y}} = -\partial_y$. Moreover

$$\bar{G}_1 = \partial_{\bar{x}}, \quad \bar{G}_2 = (a_0 - a_{10}\bar{x} + o_1(\bar{x}, \bar{y}))\partial_{\bar{x}} + (\bar{x}^2/2 - b_{01}\bar{y} - b_{30}\bar{x}^3 + o_3(\bar{x}, \bar{y}))\partial_{\bar{y}}.$$

4 The generic local optimal synthesis

We present for generic couples (F_1, F_2) the local synthesis issued from a point q . The coordinates (x, y) , centred at q , are those which have been constructed in the corresponding normal form in section 2.

4.1 (NF_1) case

At points q where (NF_1) holds, one can compute that

$$\begin{aligned}
 f_1(x, y) &= \frac{1}{2}(a_{10} - b_{10}) \\
 &\quad + (2(a_{20} - b_{20}) - b_{10}(a_{10} - b_{10}))\frac{x}{2} + (a_{11} - b_{11})\frac{y}{2} \\
 &\quad + (3(a_{30} - b_{30}) - b_{10}(a_{20} - b_{20}) - (2b_{20} - b_{10}^2)(a_{10} - b_{10}))\frac{x^2}{2} \\
 &\quad + (2(a_{21} - b_{21}) - b_{11}(a_{10} - b_{10}) - b_{10}(a_{11} - b_{11}))\frac{xy}{2} + (a_{12} - b_{12})\frac{y^2}{2} + o_2(x, y), \\
 f_2(x, y) &= -\frac{1}{2}(a_{10} + b_{10}) \\
 &\quad - (2(a_{20} + b_{20}) - b_{10}(a_{10} + b_{10}))\frac{x}{2} - (a_{11} + b_{11})\frac{y}{2} \\
 &\quad - (3(a_{30} + b_{30}) - b_{10}(a_{20} + b_{20}) - (2b_{20} - b_{10}^2)(a_{10} + b_{10}))\frac{x^2}{2} \\
 &\quad - (2(a_{21} + b_{21}) - b_{11}(a_{10} + b_{10}) - b_{10}(a_{11} + b_{11}))\frac{xy}{2} - (a_{12} + b_{12})\frac{y^2}{2} + o_2(x, y).
 \end{aligned}$$

Hence, thanks to Proposition 13, if $a_{10} - b_{10} > 0$ (resp. < 0) then u_1 is bang-bang and the only possible switch is $-1 \rightarrow +1$ (resp. $+1 \rightarrow -1$) and if $a_{10} + b_{10} < 0$ (resp. > 0) then u_2 is bang-bang and the only possible switch is $-1 \rightarrow +1$ (resp. $+1 \rightarrow -1$).

Remark 16 (Generic invariants). Let remark that generically, in the (NF_1) case, one of the following situation occurs

- $|a_{10}| \neq |b_{10}|$ (NF_{1a}),
- $a_{10} = b_{10} \neq 0$ and $a_{20} - b_{20} \neq 0$ and $a_{11} - b_{11} \neq 0$,
- $a_{10} = b_{10} \neq 0$ and $a_{20} - b_{20} = 0$ and $a_{30} - b_{30} \neq 0$ and $a_{11} - b_{11} \neq 0$,
- $a_{10} = b_{10} \neq 0$ and $a_{20} - b_{20} \neq 0$ and $a_{11} - b_{11} = 0$ and $a_{12} - b_{12} \neq 0$,
- $a_{10} = -b_{10} \neq 0$ and $a_{20} + b_{20} \neq 0$ and $a_{11} + b_{11} \neq 0$,
- $a_{10} = -b_{10} \neq 0$ and $a_{20} + b_{20} = 0$ and $a_{30} + b_{30} \neq 0$ and $a_{11} + b_{11} \neq 0$,
- $a_{10} = -b_{10} \neq 0$ and $a_{20} + b_{20} \neq 0$ and $a_{11} + b_{11} = 0$ and $a_{12} + b_{12} \neq 0$.
- $a_{10} = b_{10} = 0$ and $a_{20} + b_{20} \neq 0$ and $a_{11} + b_{11} \neq 0$.

4.1.1 Singular extremals

We consider now the properties of singular extremals and their support.

Proposition 17. *Under the generic assumption that Δ_A , Δ_1 and Δ_2 are submanifolds transversal by pair then*

1. *The support of a u_i -singular is included in Δ_i .*
2. *A u_1 -singular extremal can follow Δ_1 being optimal only if, at each point $q(t)$ of the singular, $G_1(q(t))$ and $G_2(q(t))$ are pointing on the same side of Δ_1 (or one is tangent to Δ_1) where $f_1 > 0$.*
3. *A u_2 -singular extremal can follow Δ_2 being optimal only if, at each point $q(t)$ of the singular, $G_1(q(t))$ and $-G_2(q(t))$ are pointing on the same side of Δ_2 (or one is tangent to Δ_2) where $f_2 > 0$.*
4. *Let consider a u_i -singular $q(\cdot)$ satisfying 2 or 3. If it does not intersect Δ_A and if at each time $G_1(q(t))$ and $G_2(q(t))$ are not tangent to Δ_i then $q(\cdot)$ is a local minimizer that is at each time t exists ϵ such that $q(\cdot)$ realizes the SF-distance between $q(t_1)$ and $q(t_2)$ for any t_1 and t_2 in $]t - \epsilon, t + \epsilon[$.*

Proof.

1. Outside $\Delta_A \cup \Delta_i$, ϕ_i has isolated zero hence any u_i -singular should live in $\Delta_A \cup \Delta_i$. Moreover, since generically the set of points of Δ_A where the dynamics is tangent to Δ_A is isolated, a u_i -singular crosses Δ_A only at isolated times, which are consequently also in Δ_i .
2. Same proof as point 3.
3. If a u_2 -singular $q(\cdot)$ has $u_1 = 1$ then its speed is $F_1(q(t)) + u_2(t)F_2(q(t))$ which is tangent to Δ_2 . But $u_2 \in [-1, 1]$ hence either $|u_2(t)| = 1$ and G_1 or G_2 are tangent to Δ_2 or $|u_2(t)| < 1$ and $G_2(q(t)) = F_1(q(t)) - F_2(q(t))$ and $G_1(q(t)) = F_1(q(t)) + F_2(q(t))$ point on opposite side. Now, assume that $f_2 < 0$ in the domain where points $G_1(q(t))$. With the expression given before, this corresponds to $(a_{10} + b_{10} = 0$ and $a_{20} + b_{20} > 0)$ or $(a_{10} + b_{10} = 0$ and $a_{20} + b_{20} = 0$ and $a_{30} + b_{30} > 0)$. Thanks to the previous results, if the extremal leave Δ_2 at time t_0 it starts by a bang. Assume for example that this bang follows G_1 . Then during this bang

$$\dot{\phi}_2 = u_1 \phi_3 = u_1(f_2 \phi_1 - f_1 \phi_2)$$

and hence

$$\ddot{\phi}_2 = u_1(\partial_x f_2 \phi_1 + f_2 \dot{\phi}_1 - \partial_x f_1 \phi_2 - f_1 \dot{\phi}_2)$$

which implies $\ddot{\phi}_2(t_0) = u_1(t_0) \partial_x f_2(q(t_0)) \phi_1(t_0) = \partial_x f_2(q(t_0))$ since $u_1(t_0) = 1$ and $\phi_1(t_0) = 1$. But if $a_{20} + b_{20} > 0$ then $\partial_x f_2(q(t_0)) = -(a_{20} + b_{20}) + O(t_0) < 0$. Then, since $\phi_2(t_0) = \dot{\phi}_2(t_0) = 0$ and $\ddot{\phi}_2(t_0) < 0$, for t just after t_0 $\phi_2(t) < 0$ which is in contradiction with the fact that the trajectory follows G_1 just after t_0 .

The same proof can be done for the other cases: a u_2 -singular with $u_1 = 1$ and switching to G_2 , or u_2 -singulars with $u_1 = -1$. Hence in this case, no extremal following Δ_2 can leave Δ_2 .

Now, assume that Δ_2 is such that G_1 and $-G_2$ point in the same side where $f_2 < 0$ at q and that the u_2 -singular is optimal. Consider the normal coordinate system centered at q and the domain $\mathbb{R}_+ \times \mathbb{R}_+$. One can show, with the previous analysis, that the only possible extremals issued from q and entering the domain are the singular arc following Δ_2 and the bang-bang extremals starting with symbol $[[G_1, G_2]]$ or $[[G_2, G_1]]$.

Let us prove that these last ones do not switch again before crossing Δ_2 . If an extremal starts with $[[G_2, G_1]]$, switching for the first time at $t = \epsilon$ and hence at $y = \epsilon$ then along the second bang $x = t - \epsilon$, $y = \epsilon$, $\lambda \equiv (1, 1)$ and one computes easily that for $t > \epsilon$

$$\phi_2(t) = -\frac{1}{2}((a_{20} + b_{20})(t - \epsilon)^2 + (a_{11} + b_{11})(t - \epsilon)\epsilon + o_2(\epsilon, (t - \epsilon))).$$

If $(a_{20} + b_{20})(a_{11} + b_{11}) < 0$ then the second time of switch satisfies $t - \epsilon = -\frac{a_{11} + b_{11}}{a_{20} + b_{20}}\epsilon + o(\epsilon)$ and hence the second switching locus has the form $(-\frac{a_{11} + b_{11}}{a_{20} + b_{20}}\epsilon, \epsilon)$. But Δ_2 satisfies that $x = -\frac{1}{2}\frac{a_{11} + b_{11}}{a_{20} + b_{20}}y + o(y)$ and hence the second bang crosses Δ_2 before ending. In the case $a_{20} + b_{20} = 0$ hence $(a_{11} - b_{11})(a_{30} + b_{30}) < 0$ and one shows that the second switching locus has the form $(\sqrt{-\frac{a_{11} + b_{11}}{a_{30} + b_{30}}}\epsilon, \epsilon)$ and Δ_2 satisfies that $x = \sqrt{-\frac{a_{11} + b_{11}}{3(a_{30} + b_{30})}}y + o(y)$ hence again the second bang crosses Δ_2 before ending. The same kind of computations show the same result when $a_{11} + b_{11} = 0$ and $(a_{20} + b_{20})(a_{12} + b_{12}) < 0$. The same holds for extremal starting by $[[G_1, G_2]]$.

Finally, the different extremals with symbol $[[G_1, G_2]]$ do not intersect each other after their first switch hence they cannot lose optimality by crossing each other. Idem for those with symbol $[[G_2, G_1]]$. Hence they can lose optimality by crossing the singular extremal or extremals with the other symbol. Anyway, this implies that optimal extremals are coming back to Δ_2 . But this is not possible since in this case an extremal with symbol $[[G_1, G_2, S_2^+]]$ would exist which is not the case since the switching is coming strictly after the crossing with Δ_2 .

Hence, the u_2 -singular is not optimal.

4. It is a consequence of the analysis coming further but we can give a quick idea: in this case, if q is a point on Δ_1 and if we construct normal coordinates centered at q , then the only local extremals entering the domains $\{xy < 0\}$ are the one starting by a u_1 -singular and switching or not locally only once to $u_1 = \pm 1$. Hence the u_1 -singular is locally optimal.

Remark 18. For what concerns the point 4, assume that q is a point where G_1 or G_2 is tangent to Δ_1 and $\Delta_1 \cap \{xy < 0\}$ is such that at each point G_1 and G_2 are transverse to Δ_1 and point in the domain $\{f_1 > 0\}$. Then, starting from q , a u_1 -singular can run on $\Delta_1 \cap \{xy < 0\}$ and is locally optimal. The same arguments than those exposed at point 4 work.

Definition 19. If a connected part of Δ_1 (resp. Δ_2) is such that at each point G_1 and G_2 (resp. G_1 and $-G_2$) point on the same side where $f_1 > 0$ (resp. $f_2 > 0$), it is called a turnpike. If it does not at each point, it is called an anti-turnpike (see [13]).

Remark 20. Along a u_i -singular extremal the control u_i is completely determined by the fact that the dynamics should be tangent to Δ_i .

4.1.2 Optimal synthesis in the domain $\mathbb{R}_+ \times \mathbb{R}_+$

Let consider a point q and the normal coordinate system (x, y) centered at q . The dynamics entering $\mathbb{R}_+^* \times \mathbb{R}_+^*$ is with $u_1 \equiv 1$ since u_2 switches (Propositions 14 and 15). Three different cases can be identified.

1st. case. $\Delta_2 \cap (\mathbb{R}_+ \times \mathbb{R}_+ \setminus \{0\})$ is empty locally. No u_2 -singular enters the domain. It corresponds to the case (NF_{1a}) where $|a_{10}| \neq |b_{10}|$ and to the cases (NF_{1c}) and (NF_{1d}) where $a_{10} + b_{10} = 0$ and

- $(a_{20} + b_{20})(a_{11} + b_{11}) > 0$,
- or $a_{20} + b_{20} = 0$ and $(a_{30} + b_{30})(a_{11} + b_{11}) > 0$,
- or $a_{11} + b_{11} = 0$ and $(a_{20} + b_{20})(a_{12} + b_{12}) > 0$.

Only one u_2 -switch can occur along the extremal. One has $f_2 > 0$ in the domain if

- $a_{10} + b_{10} < 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} < 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} < 0$,

and in this case the possible extremals of the domain have symbol $[[G_1]]$ or $[[G_2]]$ or $[[G_2, G_1]]$. One has $f_2 < 0$ in the domain if

- $a_{10} + b_{10} > 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} > 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} > 0$.

and in this case the possible extremals of the domain have symbol $[[G_1]]$ or $[[G_2]]$ or $[[G_1, G_2]]$.

In this case 1, the picture of the synthesis is given in the Figure 1.

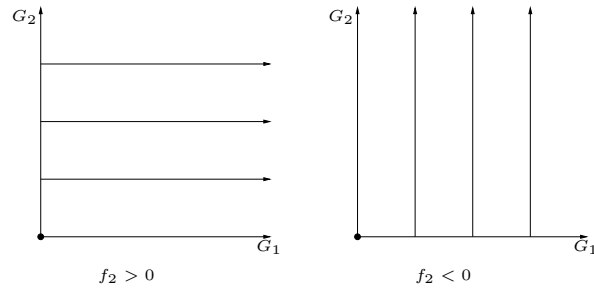


Figure 1: The syntheses when $f_2 \neq 0$ in $(\mathbb{R}_+ \times \mathbb{R}_+) \setminus \{0\}$

2nd. case. $\Delta_2 \cap (\mathbb{R}_+^* \times \mathbb{R}_+^*)$ is not empty locally and is a turnpike. It corresponds to the cases where $a_{10} + b_{10} = 0$ and

- $a_{20} + b_{20} < 0$ and $a_{11} + b_{11} > 0$,
- or $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} > 0$ and $a_{30} + b_{30} < 0$,

- or $a_{11} + b_{11} = 0$ and $a_{20} + b_{20} < 0$ and $a_{12} + b_{12} > 0$.

Then $f_2 > 0$ locally along $\{x > 0, y = 0\}$ and $f_2 < 0$ along $\{x = 0, y > 0\}$. Hence no bang-bang extremal with symbol $[[G_1, G_2]]$ or $[[G_2, G_1]]$ exists and any extremal entering the domain starts with a u_2 -singular arc. If it switches to G_1 then it enters the domain $(\mathbb{R}_+^* \times \mathbb{R}_+^*) \cap \{f_2 > 0\}$ which is invariant by G_1 hence it does not switch anymore. If it switches to G_2 it enters the domain $(\mathbb{R}_+^* \times \mathbb{R}_+^*) \cap \{f_2 < 0\}$ which is invariant by G_2 hence it does not switch anymore.

As a consequence, the only possible symbols for extremals are $[[G_1]]$, $[[G_2]]$, $[[S_2^+, G_1]]$ and $[[S_2^+, G_2]]$.

In this case 2, the picture of the synthesis is given in the Figure 2.

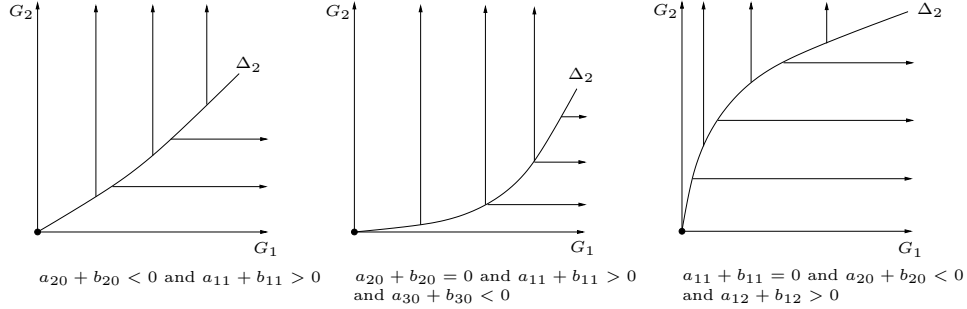


Figure 2: The syntheses when $a_{10} + b_{10} = 0$ and Δ_2 is a turnpike

3rd. case. $\Delta_2 \cap (\mathbb{R}_+^* \times \mathbb{R}_+^*)$ is not empty locally and is a anti-turnpike. It corresponds to the cases where $a_{10} + b_{10} = 0$ and

- $a_{20} + b_{20} > 0$ and $a_{11} + b_{11} < 0$,
- or $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} < 0$ and $a_{30} + b_{30} > 0$,
- or $a_{11} + b_{11} = 0$ and $a_{20} + b_{20} > 0$ and $a_{12} + b_{12} < 0$.

Then, as seen in Proposition 17, no u_2 -singular is extremal. hence the possible beginning of symbols entering the domain are $[[G_1, G_2]]$ and $[[G_2, G_1]]$. In order to complete the synthesis in this case, we have to compute the cut time and cut locus. In fact the two kind of extremals intersect before their second switching time. Let prove it.

Let fix an $\epsilon_2 > 0$ and consider at time $t > \epsilon_2$ the extremal with symbol $[[G_2, G_1]]$ switching at time ϵ_2 . One computes easily that $x(t) = t - \epsilon_2$ and $y(t) = \epsilon_2$. For an $\epsilon_1 > 0$ and the extremal

with symbol $[[G_1, G_2]]$ switching at time ϵ_1 , one gets by integrating the equations that

$$\begin{aligned}
x(t) &= \epsilon_1 + a_{10}\epsilon_1(t - \epsilon_1) + a_{20}\epsilon_1^2(t - \epsilon_1) + \frac{1}{2}(a_{10}^2 + a_{11})\epsilon_1(t - \epsilon_1)^2 \\
&\quad + a_{30}\epsilon_1^3(t - \epsilon_1) + \frac{1}{2}(3a_{10}a_{20} + a_{21} + a_{11}b_{10})\epsilon_1^2(t - \epsilon_1)^2 \\
&\quad + \frac{1}{3}\left(\frac{1}{2}a_{10}^3 + \frac{3}{2}a_{10}a_{11} + a_{12}\right)\epsilon_1(t - \epsilon_1)^3 \\
y(t) &= (t - \epsilon_1) + b_{10}\epsilon_1(t - \epsilon_1) + b_{20}\epsilon_1^2(t - \epsilon_1) + \frac{1}{2}(a_{10}b_{10} + b_{11})\epsilon_1(t - \epsilon_1)^2 \\
&\quad + b_{30}\epsilon_1^3(t - \epsilon_1) + \frac{1}{2}(a_{20}b_{10} + b_{10}b_{11} + 2a_{10}b_{20} + b_{21})\epsilon_1^2(t - \epsilon_1)^2 \\
&\quad + \frac{1}{3}\left(\frac{1}{2}(a_{10}^2 + a_{11})b_{10} + a_{10}b_{11} + b_{12}\right)\epsilon_1(t - \epsilon_1)^3
\end{aligned}$$

Let assume first that $a_{20} + b_{20} > 0$ and $a_{11} + b_{11} < 0$. Along the first front (depending on ϵ_2) $x + y = t$ when along the second $x + y = t + \epsilon_1(t - \epsilon_1)((a_{20} + b_{20})\epsilon_1 + \frac{1}{2}(a_{11} + b_{11}))$ hence they are transverse at

$$\epsilon_1 = \frac{t}{1 - \frac{2(a_{20} + b_{20})}{a_{11} + b_{11}}}$$

and they intersect at a point such that $y = -2\frac{a_{20} - b_{20}}{a_{11} - b_{11}}x + o(x)$. As seen previously, the switching locus for extremals with symbol $[[G_2, G_1]]$ satisfies $y = -\frac{a_{20} - b_{20}}{a_{11} - b_{11}}x + o(x)$ hence it stops to be optimal before switching. The same holds true for the extremals with symbol $[[G_1, G_2]]$. Finally the cut locus satisfies

$$y_{cut} = -2\frac{a_{20} - b_{20}}{a_{11} - b_{11}}x_{cut} + o(x_{cut})$$

and is tangent to Δ_2 .

The same computations can be done when G_1 or G_2 is tangent to Δ_2 . Then one computes that the extremals lose optimality by crossing the cut before the second switch and that

- if $a_{20} + b_{20} = 0$ then

$$y_{cut} = -3\frac{a_{30} + b_{30}}{a_{11} + b_{11}}x_{cut}^2 + o(x_{cut}^2),$$

- if $a_{11} + b_{11} = 0$ then

$$x_{cut} = -\frac{1}{2}\frac{a_{12} + b_{12}}{a_{20} + b_{20}}y_{cut}^2 + o(y_{cut}^2).$$

In all cases the cut is tangent to Δ_2 and the contact is of order 2 when $(a_{20} + b_{20})(a_{11} + b_{11}) = 0$.

In this case 3, the picture of the synthesis is given in the Figure 3.

Remark 21. Using the symmetries presented in section 3.7, one can obtain from the optimal synthesis in the domain $\mathbb{R}_+ \times \mathbb{R}_+$ the optimal synthesis in the three other domains.

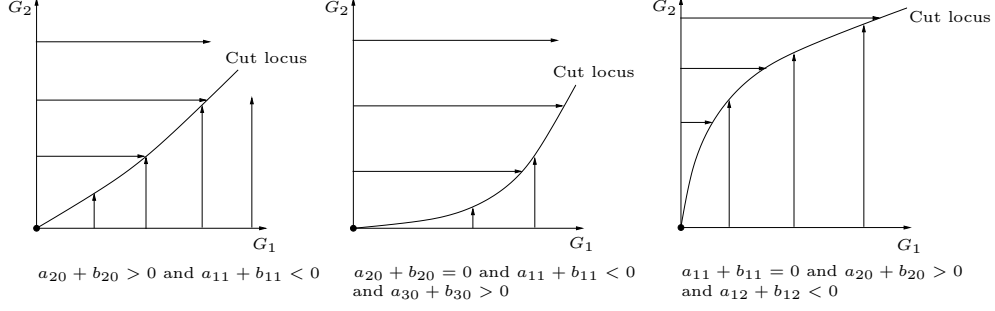


Figure 3: The syntheses when $a_{10} + b_{10} = 0$ and Δ_2 is not a turnpike

4.1.3 Optimal synthesis in the domain $\mathbb{R}_- \times \mathbb{R}_-$

The dynamics entering $\mathbb{R}_*^* \times \mathbb{R}_*^*$ is with $u_1 \equiv -1$ since u_2 switches (Propositions 14 and 15). Three different cases can be identified.

1st. case. $\Delta_2 \cap (\mathbb{R}_- \times \mathbb{R}_- \setminus \{0\})$ is empty locally. No u_2 -singular enters the domain. It corresponds to the case (NF_{1a}) where $|a_{10}| \neq |b_{10}|$ and to the cases (NF_{1c}) and (NF_{1d}) where $a_{10} + b_{10} = 0$ and

- $(a_{20} + b_{20})(a_{11} + b_{11}) > 0$,
- or $a_{20} + b_{20} = 0$ and $(a_{30} + b_{30})(a_{11} + b_{11}) < 0$,
- or $a_{11} + b_{11} = 0$ and $(a_{20} + b_{20})(a_{12} + b_{12}) < 0$.

Only one u_2 -switch can occur along the extremal. One has $f_2 > 0$ in the domain if

- $a_{10} + b_{10} < 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} > 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} > 0$,

and in this case the possible extremals of the domain have symbol $[-G_1]$ or $[-G_2]$ or $[-G_1, -G_2]$. One has $f_2 < 0$ in the domain if

- $a_{10} + b_{10} > 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} < 0$,
- or $a_{10} + b_{10} = 0$ and $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} < 0$.

and in this case the possible extremals of the domain have symbol $[-G_1]$ or $[-G_2]$ or $[-G_2, -G_1]$.

2nd. case. $\Delta_2 \cap (\mathbb{R}_*^* \times \mathbb{R}_*^*)$ is not empty locally and is a turnpike. It corresponds to the cases where $a_{10} + b_{10} = 0$ and

- $a_{20} + b_{20} < 0$ and $a_{11} + b_{11} > 0$,
- or $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} > 0$ and $a_{30} + b_{30} > 0$,
- or $a_{11} + b_{11} = 0$ and $a_{20} + b_{20} < 0$ and $a_{12} + b_{12} < 0$.

In this case, the possible symbols for extremals are $[-G_1]$, $[-G_2]$, $[[S_2^-, -G_1]]$ and $[[S_2^-, -G_2]]$.

3rd. case. $\Delta_2 \cap (\mathbb{R}_+^* \times \mathbb{R}_-^*)$ is not empty locally and is a anti-turnpike. It corresponds to the cases where $a_{10} + b_{10} = 0$ and

- $a_{20} + b_{20} > 0$ and $a_{11} + b_{11} < 0$,
- or $a_{20} + b_{20} = 0$ and $a_{11} + b_{11} < 0$ and $a_{30} + b_{30} < 0$,
- or $a_{11} + b_{11} = 0$ and $a_{20} + b_{20} > 0$ and $a_{12} + b_{12} > 0$.

Then the only optimal symbols are $[-G_1]$, $[-G_2]$, $[-G_1, -G_2]$ and $[-G_2, -G_1]$. Moreover

- if $a_{20} + b_{20} > 0$ and $a_{11} + b_{11} < 0$, the cut locus satisfies

$$y_{cut} = -2 \frac{a_{20} + b_{20}}{a_{11} + b_{11}} x_{cut} + o(x_{cut}),$$

- if $a_{20} + b_{20} = 0$ then

$$y_{cut} = -3 \frac{a_{30} + b_{30}}{a_{11} + b_{11}} x_{cut}^2 + o(x_{cut}^2),$$

- if $a_{11} + b_{11} = 0$ then

$$x_{cut} = -\frac{1}{2} \frac{a_{12} + b_{12}}{a_{20} + b_{20}} y_{cut}^2 + o(y_{cut}^2).$$

In all cases the cut is tangent to Δ_2 and the contact is of order 2 when $(a_{20} + b_{20})(a_{11} + b_{11}) = 0$.

4.1.4 Optimal synthesis in the domain $\mathbb{R}_+ \times \mathbb{R}_-$

The dynamics entering $\mathbb{R}_+^* \times \mathbb{R}_-^*$ is with $u_2 \equiv 1$ since u_1 switches (Propositions 14 and 15). Three different cases can be identified.

1st. case. $\Delta_1 \cap (\mathbb{R}_+ \times \mathbb{R}_- \setminus \{0\})$ is empty locally. No u_1 -singular enters the domain. It corresponds to the case (NF_{1a}) where $|a_{10}| \neq |b_{10}|$ and to the cases (NF_{1b}) and (NF_{1d}) where $a_{10} - b_{10} = 0$ and

- $(a_{20} - b_{20})(a_{11} - b_{11}) < 0$,
- or $a_{20} - b_{20} = 0$ and $(a_{30} - b_{30})(a_{11} - b_{11}) < 0$,
- or $a_{11} - b_{11} = 0$ and $(a_{20} - b_{20})(a_{12} - b_{12}) > 0$.

Only one u_1 -switch can occur along the extremal. One has $f_1 > 0$ in the domain if

- $a_{10} - b_{10} > 0$,
- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} > 0$,
- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} = 0$ and $a_{11} + b_{11} < 0$,

and in this case the possible extremals of the domain have symbol $[[G_1]]$ or $[[-G_2]]$ or $[[-G_2, G_1]]$. One has $f_1 < 0$ in the domain if

- $a_{10} - b_{10} < 0$,

- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} < 0$,
- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} = 0$ and $a_{11} - b_{11} > 0$.

and in this case the possible extremals of the domain have symbol $[[G_1]]$ or $[[-G_2]]$ or $[[G_1, -G_2]]$.

2nd. case. $\Delta_1 \cap (\mathbb{R}_+^* \times \mathbb{R}_-^*)$ is not empty locally and is a turnpike. It corresponds to the cases where $a_{10} - b_{10} = 0$ and

- $a_{20} - b_{20} > 0$ and $a_{11} - b_{11} > 0$,
- or $a_{20} - b_{20} = 0$ and $a_{11} - b_{11} > 0$ and $a_{30} - b_{30} > 0$,
- or $a_{11} - b_{11} = 0$ and $a_{20} - b_{20} > 0$ and $a_{12} - b_{12} < 0$.

In this case, the possible symbols for extremals are $[[G_1]]$, $[[-G_2]]$, $[[S_1^+, G_1]]$ and $[[S_1^+, -G_2]]$.

3rd. case. $\Delta_1 \cap (\mathbb{R}_+^* \times \mathbb{R}_-^*)$ is not empty locally and is a anti-turnpike. It corresponds to the cases where $a_{10} - b_{10} = 0$ and

- $a_{20} - b_{20} < 0$ and $a_{11} - b_{11} < 0$,
- or $a_{20} - b_{20} = 0$ and $a_{11} - b_{11} < 0$ and $a_{30} - b_{30} < 0$,
- or $a_{11} - b_{11} = 0$ and $a_{20} - b_{20} < 0$ and $a_{12} - b_{12} > 0$.

Then the only optimal symbols are $[[G_1]]$, $[[-G_2]]$, $[[G_1, -G_2]]$ and $[[-G_2, G_1]]$. Moreover

- if $a_{20} - b_{20} < 0$ and $a_{11} + b_{11} < 0$, the cut locus satisfies

$$y_{cut} = -2 \frac{a_{20} - b_{20}}{a_{11} - b_{11}} x_{cut} + o(x_{cut}),$$

- if $a_{20} - b_{20} = 0$ then

$$y_{cut} = -3 \frac{a_{30} - b_{30}}{a_{11} - b_{11}} x_{cut}^2 + o(x_{cut}^2),$$

- if $a_{11} - b_{11} = 0$ then

$$x_{cut} = -\frac{1}{2} \frac{a_{12} - b_{12}}{a_{20} - b_{20}} y_{cut}^2 + o(y_{cut}^2).$$

In all cases the cut is tangent to Δ_1 and the contact is of order 2 when $(a_{20} - b_{20})(a_{11} - b_{11}) = 0$.

4.1.5 Optimal synthesis in the domain $\mathbb{R}_- \times \mathbb{R}_+$

The dynamics entering $\mathbb{R}_-^* \times \mathbb{R}_+^*$ is with $u_2 \equiv -1$ since u_1 switches (Propositions 14 and 15). Three different cases can be identified.

1st. case. $\Delta_1 \cap (\mathbb{R}_- \times \mathbb{R}_+ \setminus \{0\})$ is empty locally. No u_1 -singular enters the domain. It corresponds to the case (NF_{1a}) where $|a_{10}| \neq |b_{10}|$ and to the cases (NF_{1b}) and (NF_{1d}) where $a_{10} - b_{10} = 0$ and

- $(a_{20} - b_{20})(a_{11} - b_{11}) < 0$,
- or $a_{20} - b_{20} = 0$ and $(a_{30} - b_{30})(a_{11} - b_{11}) > 0$,

- or $a_{11} - b_{11} = 0$ and $(a_{20} - b_{20})(a_{12} - b_{12}) < 0$.

Only one u_1 -switch can occur along the extremal. One has $f_1 > 0$ in the domain if

- $a_{10} - b_{10} > 0$,
- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} < 0$,
- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} = 0$ and $a_{11} + b_{11} > 0$,

and in this case the possible extremals of the domain have symbol $[[-G_1]]$ or $[[G_2]]$ or $[[-G_1, G_2]]$. One has $f_1 < 0$ in the domain if

- $a_{10} - b_{10} < 0$,
- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} > 0$,
- or $a_{10} - b_{10} = 0$ and $a_{20} - b_{20} = 0$ and $a_{11} - b_{11} < 0$,

and in this case the possible extremals of the domain have symbol $[[-G_1]]$ or $[[G_2]]$ or $[[G_2, -G_1]]$.

2nd. case. $\Delta_1 \cap (\mathbb{R}_-^* \times \mathbb{R}_+^*)$ is not empty locally and is a turnpike. It corresponds to the cases where $a_{10} - b_{10} = 0$ and

- $a_{20} - b_{20} > 0$ and $a_{11} - b_{11} > 0$,
- or $a_{20} - b_{20} = 0$ and $a_{11} - b_{11} > 0$ and $a_{30} - b_{30} < 0$,
- or $a_{11} - b_{11} = 0$ and $a_{20} - b_{20} > 0$ and $a_{12} - b_{12} > 0$.

In this case, the possible symbols for extremals are $[[-G_1]]$, $[[G_2]]$, $[[S_1^-, -G_1]]$ and $[[S_1^-, G_2]]$.

3rd. case. $\Delta_1 \cap (\mathbb{R}_-^* \times \mathbb{R}_+^*)$ is not empty locally and is an anti-turnpike. It corresponds to the cases where $a_{10} - b_{10} = 0$ and

- $a_{20} - b_{20} < 0$ and $a_{11} - b_{11} < 0$,
- or $a_{20} - b_{20} = 0$ and $a_{11} - b_{11} < 0$ and $a_{30} - b_{30} > 0$,
- or $a_{11} - b_{11} = 0$ and $a_{20} - b_{20} < 0$ and $a_{12} - b_{12} < 0$.

Then the only optimal symbols are $[[-G_1]]$, $[[G_2]]$, $[[-G_1, G_2]]$ and $[[G_2, -G_1]]$. Moreover

- if $a_{20} - b_{20} < 0$ and $a_{11} + b_{11} < 0$, the cut locus satisfies

$$y_{cut} = -2 \frac{a_{20} - b_{20}}{a_{11} - b_{11}} x_{cut} + o(x_{cut}),$$

- if $a_{20} - b_{20} = 0$ then

$$y_{cut} = -3 \frac{a_{30} - b_{30}}{a_{11} - b_{11}} x_{cut}^2 + o(x_{cut}^2),$$

- if $a_{11} - b_{11} = 0$ then

$$x_{cut} = -\frac{1}{2} \frac{a_{12} - b_{12}}{a_{20} - b_{20}} y_{cut}^2 + o(y_{cut}^2).$$

In all cases the cut is tangent to Δ_1 and the contact is of order 2 when $(a_{20} - b_{20})(a_{11} - b_{11}) = 0$.

4.2 (NF_{2a}) case

Recall that the normal form (NF_{2a}) gives

$$G_1(x, y) = \partial_x, \quad G_2(x, y) = (a_0 + a_{10}x + o_1(x, y))\partial_x + (x + b_{20}x^2 + o(x, y))\partial_y,$$

with $0 \leq a_0 < 1$.

Such a point is neither in Δ_1 nor Δ_2 . Hence no singular extremal will appear in the study of the local synthesis.

One can compute easily that, for any extremal starting at 0, $\phi_1(0) = \frac{1}{2}\lambda_x(0)(1 + a_0)$ and $\phi_2(0) = \frac{1}{2}\lambda_x(0)(1 - a_0)$. With $H = 0$ it gives $|\lambda_x(0)| = 1$. Hence, since $\dot{\phi}_1 = -u_2\phi_3$ and $\dot{\phi}_2 = u_1\phi_3$, if we want to study extremals that switch in short time, we need to consider ϕ_3 large that is $|\lambda_y|$ large.

Moreover, since along an extremal issued from 0 $|\dot{x}(t)| \leq 1$ for t small, one gets easily that $|x(t)| \leq t$ and $|y(t)| \leq t^2$ for t small enough. Hence $\phi_1(t) = \frac{1+a_0}{2}\lambda_x(0) + x(t)\lambda_y(0) + o(t, x(t)\lambda_y(0))$ and $\phi_2(t) = \frac{1-a_0}{2}\lambda_x(0) + x(t)\lambda_y(t) + o(t, x(t)\lambda_y(t))$. This implies that if one wants to consider an extremal switching at time τ small, he should consider initial conditions $\lambda_y(0) \sim \frac{1}{\tau}$. Inversing the point of view, if we consider an initial condition $\lambda_y(0) = \frac{1}{r_0}$ with r_0 small, the switching time should be of order 1 in r_0 . This motivates the following change of coordinates on the fibers of the cotangent: $r = \frac{1}{\lambda_y}$, $p = r\lambda_x$ and the change of time $s = t/r$.

4.2.1 equations of the dynamics

With the new variables (x, y, p, r) and the new time s , the Hamiltonian equations become

$$\begin{aligned} x' &= r \frac{\partial H}{\partial \lambda_x}(x, y, p, -1), \\ y' &= r \frac{\partial H}{\partial \lambda_y}(x, y, p, -1), \\ p' &= -r \frac{\partial H}{\partial x}(x, y, p, -1) + rp \frac{\partial H}{\partial y}(x, y, p, -1), \\ r' &= r^2 \frac{\partial H}{\partial y}(x, y, p, -1). \end{aligned}$$

Now, looking for the solutions as taylor series in r_0 , that is under the form

$$\begin{aligned} x(r_0, s) &= x_1(s)r_0 + x_2(s)r_0^2 + o(r_0^2), & p(r_0, s) &= p_1(s)r_0 + p_2(s)r_0^2 + o(r_0^2), \\ y(r_0, s) &= y_2(s)r_0^2 + y_3(s)r_0^3 + o(r_0^3), & r(r_0, s) &= r_0 + r_2(s)r_0^2 + o(r_0^2), \end{aligned}$$

one finds the equations

$$\begin{aligned} x'_1 &= \frac{u_1+u_2}{2} + \frac{u_1-u_2}{2}a_0, & x'_2 &= \frac{u_1-u_2}{2}a_{10}x_1, \\ y'_2 &= \frac{u_1-u_2}{4}x_1, & y'_3 &= \frac{u_1-u_2}{2}(b_{20}x_1^2 + x_2), \\ p'_1 &= -\frac{u_1-u_2}{2}x_1, & p'_2 &= -\frac{u_1-u_2}{2}(a_{10}p_1 + 2b_{20}x_1), \\ r'_2 &= 0, \end{aligned}$$

4.2.2 Computation of the jets

Using these equations, we are able to compute the jets with respect to r_0 of four types of extremals: depending on the sign of $p(0) = \pm 1$ and of r_0 . For each of these types we can compute the functions $x_1, x_2, y_2, y_3, p_1, p_2$ and $r_2 \equiv 0$ of the variable s for the first bang. We can then compute the jets of ϕ_1 and ϕ_2 for the first bang and look for the first switching time under the form $s_1 = s_{10} + s_{11}r_0$ and then repeat the procedure for the second bang and so on. Finally, if we denote $\delta_p = \text{sign}(p(0))$ and $\delta_r = \text{sign}(r_0)$ then the controls during the first bang are $u_1 = u_2 = \delta_p$. The first time of switch is

$$s_1 = \delta_r(1 - \delta_r a_0) - \delta_p(1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20})r_0 + o(r_0)$$

and corresponds to $\phi_2(s_1) = 0$ if $\delta_r = 1$ or $\phi_1(s_1) = 0$ if $\delta_r = -1$. The second bang corresponds to $u_1 = \delta_p \delta_r$ and $u_2 = -\delta_p \delta_r$ and the second switch is at

$$s_2 = \delta_r(3 - \delta_r a_0) - \delta_p((1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20}) + 4b_{20})r_0 + o(r_0)$$

where $\phi_1(s_2) = 0$ if $\delta_r = 1$ and $\phi_2(s_2) = 0$ if $\delta_r = -1$. At this time

$$\begin{aligned} x(s_2) &= \delta_p(\delta_r + a_0)r_0 - \delta_r(\delta_r + a_0)(-\delta_r a_{10} + b_{20} + \delta_r a_0 b_{20})r_0^2 + o(r_0^2), \\ y(s_2) &= 2\delta_r r_0^2 - \delta_p \frac{4}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})r_0^3 + o(r_0^3). \end{aligned}$$

The third bang corresponds to $u_1 = u_2 = -1$ if $\delta_p = 1$ and to $u_1 = u_2 = 1$ if $\delta_p = -1$. The third switching time satisfies $s_3 = \delta_r(5 - \delta_r a_0) + O(r_0)$ and the corresponding time t_3 is larger than the cut time as we will see later.

Let us analyze a little the situation in terms of cut locus for these extremals: if we consider the extremals with $\delta_p = \delta_r = 1$, they all start following G_1 , without losing optimality. Then they switch to G_2 at $t = r_0(1 - a_0) + o(r_0)$. During this second bang, they do not intersect one each other since they are all following G_2 with a different initial condition on $\{x > 0, y = 0\}$. Then they switch to $-G_1$ but at a different y hence again they cannot intersect. The loss of optimality cannot come from an intersection with extremals with $\delta_r = -1$ since these last one live in $\{y \leq 0\}$. As we will see in the following, the loss of optimality will come from the intersection with an extremal with $-\delta_p = \delta_r = 1$ during the third bang. Of course, the same occurs for extremals with $\delta_r = -1$.

Let fix a small parameter $\rho > 0$. Since the dynamics during the third bang of all the extremals is given by $\pm G_1 = \pm \partial_x$, y is constant during these third bangs. Hence, for the extremals with $\delta_r = 1$, we can look for the r_0 , as a jet in ρ , such that $y = 2\rho^2$ during the third bang, and for the extremals with $\delta_r = -1$, we can look for the r_0 , as a jet in ρ , such that $y = -2\rho^2$ during the third bang. The result is

$$r_0 = \delta_r \rho + \delta_r \delta_p \frac{1}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})\rho^2 + o(\rho^2)$$

which allows to compute

$$t_2 = (3 - \delta_r a_0)\rho - \delta_r \delta_p \frac{3a_{10} - a_0^2 a_{10} + \delta_r 6b_{20} - 3a_0 b_{20} + a_0^3 b_{20}}{3}\rho^2 + o(\rho^2).$$

Hence we can compute $x(t) = x(t_2) + (t - t_2)$ for this r_0 that is

$$x(t) = -\delta_p t + \delta_p 4\rho - \frac{2}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})\rho^2 + o(\rho^2).$$

We are now in situation to complete the computation of the jet of the cut locus: an extremal intersects an extremal of same length at the time $t_{cut} = 4\rho + o(\rho^2)$ which is less than $t_3 = (5 - \delta_r a_0)\rho$ hence t_{cut} is the cut time. When $\delta_r = 1$ the cut point satisfies

$$x_{cut} = -\frac{2}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})\rho^2 + o(\rho^2), \quad y_{cut} = 2\rho^2,$$

and when $\delta_r = -1$ the cut point satisfies

$$x_{cut} = -\frac{2}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})\rho^2 + o(\rho^2), \quad y_{cut} = -2\rho^2.$$

Finally, if one wants to describe the sphere at time t small, one have that the first switching time is

$$t_1 = \delta_r(1 - \delta_r a_0)r_0 - \delta_p(1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20})r_0^2 + o(r_0^2)$$

and hence, at t small, the r_0 corresponding to a first switching point is

$$r_1 = \frac{t}{\delta_r(1 - \delta_r a_0)} + \delta_r \delta_p \frac{\delta_r a_{10} + b_{20}(1 - \delta_r a_0)}{(1 - a_0)^2} t^2 + o(t^2).$$

The second switching time is

$$t_2 = \delta_r(3 - \delta_r a_0)r_0 - \delta_p((1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20}) + 4b_{20})r_0^2 + o(r_0^2)$$

which implies that, at t small, the r_0 corresponding to a second switching point is

$$r_2 = \frac{t}{\delta_r(3 - \delta_r a_0)} + \delta_p \frac{(1 - \delta_r a_0)(\delta_r a_{10} + b_{20} - \delta_r a_0 b_{20}) + 4b_{20}}{\delta_r(3 - \delta_r a_0)^3} t^2 + o(t^2).$$

And the cut time is

$$t_{cut} = 4\delta_r(r_0 - \delta_r \delta_p \frac{1}{3}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})r_0^2) + o(r_0^2)$$

which implies that at t small the r_0 corresponding to a cut point is

$$r_{cut} = \frac{\delta_r}{4}(t + \frac{\delta_p}{12}(-a_0 a_{10} + 3b_{20} + a_0^2 b_{20})t^2) + o(t^2).$$

4.3 (NF_{2b}) case

Recall that the normal form (NF_{2b}) gives $G_1(x, y) = \partial_x$, and

$$G_2(x, y) = (1 + a_{10}x + a_{01}y + a_{20}x^2 + o_2(x, y))\partial_x + (x + b_{20}x^2 + b_{30}x^3 + o_3(x, y))\partial_y.$$

In this case, the extremals with initial condition $|\lambda_y(0)| \gg 1$ are the limit when a_0 goes to 1 of the extremal presented in the case (NF_{2a}). If $\lambda_y(0) \gg 1$ then the symbol starts with $[[G_2, -G_1]]$ or with $[[-G_2, G_1]]$ and if $-\lambda_y(0) \gg 1$ then the symbol starts with $[[G_1, -G_2]]$ or with $[[-G_1, G_2]]$.

But $F_2(0) = 0$ then for all extremals $\phi_2(0) = 0$. Hence, an extremal may also, depending on the invariants, have symbol starting by $[[G_2, G_1]]$, $[[G_1, G_2]]$, $[[S_2^+, G_1]]$ or $[[S_2^+, G_2]]$ if $\lambda_x(0) = 1$, and, $[[-G_2, -G_1]]$, $[[-G_1, -G_2]]$, $[[S_2^-, -G_1]]$ or $[[S_2^-, -G_2]]$ if $\lambda_x(0) = -1$.

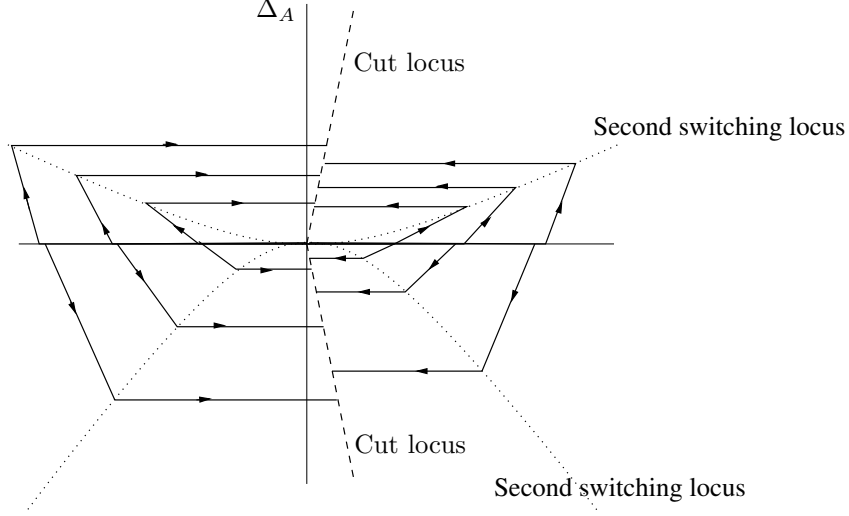


Figure 4: The optimal synthesis in the (NF_{2a}) case

If $\lambda_x(0) = 1$ then at least for small time $u_1(t) = 1$ and $x(t) = t + o(t)$ and $y(t) = o(t)$. Then, computing ϕ_2 one finds $\phi_2(t) = -\lambda_x(t)\frac{a_{10}}{2} - \lambda_y(t)\frac{x(t)}{2} + o(t) = -(\frac{a_{10}+\lambda_y(0)}{2})t + o(t)$. Hence if $\lambda_y(0) > -a_{10}$ then, since $\phi_2(0) < 0$ for small time, the extremal starts by a bang following G_2 . If $\lambda_y(0) < -a_{10}$ then $\phi_2(0) > 0$ for small time and the extremal starts by a bang following G_1 .

If $\lambda_x(0) = -1$ then at least for small time $u_1(t) = -1$ and $x(t) = -t + o(t)$ and $y(t) = o(t)$. Then $\phi_2(t) = (\frac{a_{10}-\lambda_y(0)}{2})t + o(t)$. Hence if $\lambda_y(0) > a_{10}$ then, since $\phi_2(0) < 0$ for small time, the extremal starts by a bang following $-G_1$. If $\lambda_y(0) < a_{10}$ then $\phi_2(0) > 0$ for small time and the extremal starts by a bang following $-G_2$.

In coordinates, one can compute that

$$\det(F_2, [F_1, F_2])(x, y) = \frac{1}{4}((a_{10}b_{20} - a_{20})x^2 + a_{01}y) + o_2(x, y)$$

where x has weight 1 and y has weight 2. Since generically at such points (which are isolated points) $a_{01} \neq 0$ then an equation for Δ_2 is given by

$$y = \frac{a_{20} - a_{10}b_{20}}{a_{01}}x^2 + o(x^2).$$

Remark that generically $\frac{a_{20}-a_{10}b_{20}}{a_{01}}$ is neither 0 nor $\frac{1}{2}$. Moreover

$$f_2(x, y) = \frac{\det(F_2, [F_1, F_2])(x, y)}{\det(F_2, F_1)(x, y)} = \frac{((a_{10}b_{20} - a_{20})x^2 + a_{01}y) + o_2(x, y)}{2x}.$$

Recall that an equation of the support of the integral curve of G_1 passing by 0 is $y = 0$ and that an equation for the support of the integral curve of G_2 passing by 0 is $y = \frac{x^2}{2} + o(x^2)$.

If $\frac{a_{20}-a_{10}b_{20}}{a_{01}} < 0$ or if $\frac{a_{20}-a_{10}b_{20}}{a_{01}} > \frac{1}{2}$ then Δ_2 does not enter the domain $\mathcal{D} = \{x > 0, 0 < y < \frac{x^2}{2}\}$ and along it G_1 and G_2 point on the same side of Δ_2 hence Δ_2 is not a turnpike. In these cases:

- if $a_{10}b_{20} - a_{20} > 0$ then $f_2 > 0$ in \mathcal{D} and the new extremals, that are not described as limit of the case NF_{2a} , have symbol $[[G_2, G_1]]$.
- if $a_{10}b_{20} - a_{20} < 0$ then $f_2 < 0$ in \mathcal{D} and the new extremals, that are not described as limit of the case NF_{2a} , have symbol $[[G_1, G_2]]$.

If $\frac{a_{20}-a_{10}b_{20}}{a_{01}} > 0$ and $\frac{a_{20}-a_{10}b_{20}}{a_{01}} < \frac{1}{2}$ then Δ_2 enters \mathcal{D} and along it G_1 and G_2 point on opposite sides of Δ_2 . In this case:

- if $a_{10}b_{20} - a_{20} > 0$ then, along $\Delta_2 \cap \mathcal{D}$, G_1 points in direction of $f_2 > 0$ and Δ_2 is a turnpike. Then, the only extremals entering the domain \mathcal{D} start with a singular arc and have symbols $[[S_2^+]]$, $[[S_2^+, G_1]]$ or $[[S_2^+, G_2]]$.
- if $a_{10}b_{20} - a_{20} < 0$ then, along $\Delta_2 \cap \mathcal{D}$, G_1 points in direction of $f_2 < 0$ and Δ_2 is not a turnpike. In this case the symbols start with $[[G_1, G_2]]$ and $[[G_2, G_1]]$. One can compute, with the same techniques that in section 4.2.2, the switching times and the second switching locus for extremals that enter the domain \mathcal{D} , that is for extremal with initial condition $\lambda_y(0) = -a_{10} + \delta\epsilon$ with $\epsilon > 0$ small and $\delta = \pm 1$. If $\delta < 0$ then the symbol is $[[G_1, G_2, G_1]]$ and the switching times are $t_1 = \frac{\epsilon}{a_{20}-a_{10}b_{20}}$ and $t_2 = t_1 + \frac{2\epsilon}{a_{01}-2a_{20}+2a_{10}b_{20}}$, the second switching locus being

$$x(\epsilon) = \frac{a_{01}\epsilon}{(a_{20} - a_{10}b_{20})(a_{01} - 2a_{20} + 2a_{10}b_{20})}, \quad y(\epsilon) = \frac{2(a_{01} - a_{20} + a_{10}b_{20})\epsilon^2}{(a_{20} - a_{10}b_{20})(a_{01} - 2a_{20} + 2a_{10}b_{20})^2}.$$

If $\delta > 0$ then the symbol is $[[G_2, G_1, G_2]]$ and the switching times are $t_1 = \frac{2\epsilon}{a_{01}-2a_{20}+2a_{10}b_{20}}$ and $t_2 = t_1 + \frac{\epsilon}{a_{20}-a_{10}b_{20}}$, the second switching locus being

$$x(\epsilon) = \frac{a_{01}\epsilon}{(a_{20} - a_{10}b_{20})(a_{01} - 2a_{20} + 2a_{10}b_{20})}, \quad y(\epsilon) = \frac{2\epsilon^2}{(a_{01} - 2a_{20} + 2a_{10}b_{20})^2}.$$

One prove easily that all these extremals cut Δ_2 before the second switching. Moreover they cannot be optimal after the second switching (by considerations on the jacobian). Hence the only optimal symbols entering the domain \mathcal{D} are $[[G_1, G_2]]$ and $[[G_2, G_1]]$.

Pictures for the (NF_{2b}) case are in Figures 5 and 6.

4.4 (NF_3) case

Recall that in the (NF_3) case, x has weight 1 and y has weight 3. Hence we can write

$$G_1(x, y) = \partial_x \quad G_2(x, y) = (a_0 + a_{10}x + o_k(x, y))\partial_x + \left(\frac{x^2}{2} + b_{01}y + b_{30}x^3 + o_3(x, y) \right) \partial_y$$

with $b_{0,1} \neq 0$ and $0 < a_0 < 1$, where $o_k(x, y)$ has the meaning given in subsection 3.5. As in the (NF_{2b}) case, for any extremal starting at 0,

$$\phi_1(0) = \frac{1}{2}\lambda_x(0)(1 + a_0) \quad \text{and} \quad \phi_2(0) = \frac{1}{2}\lambda_x(0)(1 - a_0).$$

And for the same reasons, if we want to study extremals that switch in short time, we need to consider $|\lambda_y|$ large.

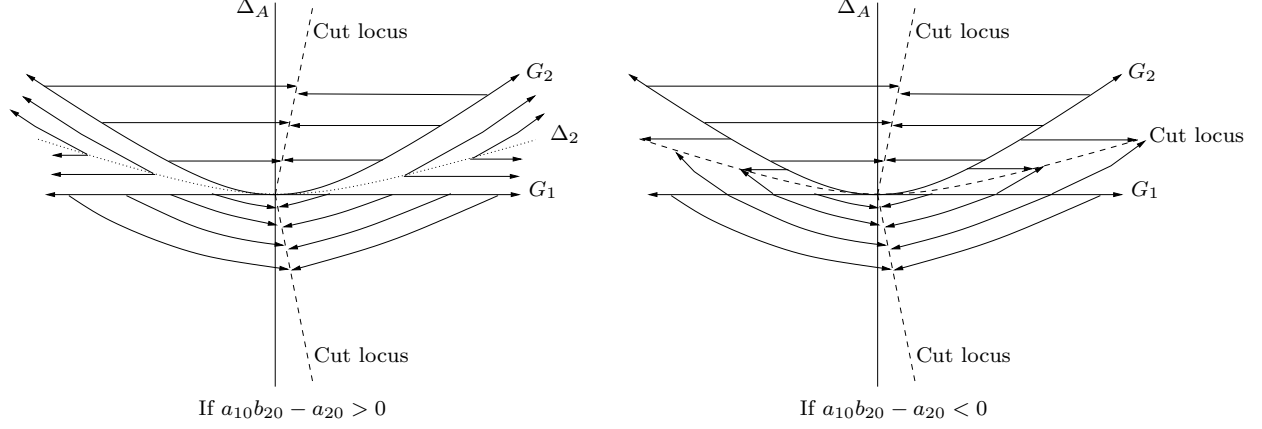


Figure 5: (NF_{2b}) case: when $\frac{a_{20}-a_{10}b_{20}}{a_{01}} > 0$ and $\frac{a_{20}-a_{10}b_{20}}{a_{01}} < \frac{1}{2}$

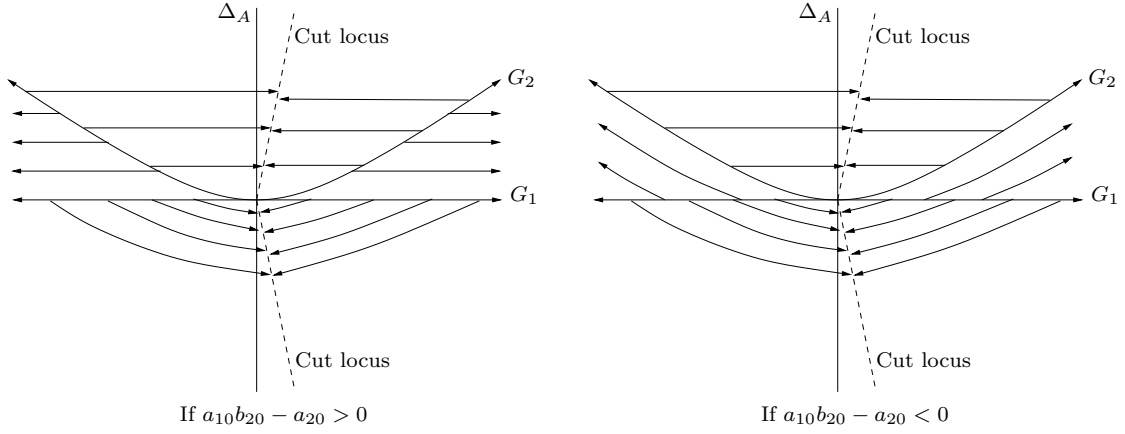


Figure 6: (NF_{2b}) case: when $\frac{a_{20}-a_{10}b_{20}}{a_{01}} < 0$ or $\frac{a_{20}-a_{10}b_{20}}{a_{01}} > \frac{1}{2}$

The set of initial condition is $\{(\lambda_x(0), \lambda_y(0)) \mid \lambda_x(0) = \pm 1\}$. We parameterize the upper part of this set by setting $\lambda_y(0) = \frac{1}{r_0^2}$ and the lower part by $\lambda_y(0) = -\frac{1}{r_0^2}$.

As explained in subsection 3.4, in order to compute extremals with $\lambda_y(0) \gg 1$ we make the change of coordinates $r = \frac{1}{\sqrt{\lambda_y}}$, $X = \frac{x}{r}$, $Y = \frac{y}{r^3}$ and the change of time $s = \frac{t}{r}$.

Now, looking for the solutions as taylor series in r_0 , that is under the form

$$\begin{aligned} X(r_0, s) &= X_0(s) + r_0 X_1(s) + o(r_0), & \lambda_x(r_0, s) &= \lambda_{x0}(s) + r_0 \lambda_{x1}(s) + o(r_0), \\ Y(r_0, s) &= Y_0(s) + r_0 Y_1(s) + o(r_0), & r(r_0, s) &= r_0 + r_0^2 r_2(s) + o_2(r_0) \end{aligned}$$

one finds the equations

$$\begin{aligned} X'_0(s) &= \frac{1}{2}(u_1 + u_2) + \frac{a_0}{2}(u_1 - u_2), & X'_1(s) &= \frac{(u_1 - u_2)}{4}(2a_{10} - b_{01})X_0(s), \\ Y'_0(s) &= \frac{1}{4}(u_1 - u_2)X_0^2(s), & Y'_1(s) &= \frac{(u_1 - u_2)}{4}(2b_{30}X_0^3(s) + 2X_0(s)X_1(s) - b_{01}Y_0(s)), \\ \lambda'_{x0}(s) &= -\frac{1}{2}(u_1 - u_2)X_0(s), & \lambda'_{x1}(s) &= -\frac{(u_1 - u_2)}{2}(a_{10}\lambda_{x0}(s) + 3b_{30}X_0^2(s) + X_1(s)), \\ r'_2(s) &= \frac{b_{01}}{4}(u_1 - u_2), \end{aligned}$$

For an initial condition $\lambda_x(0) = 1$, one find $\phi_1(0) > 0$ and $\phi_2(0) > 0$, hence $u_1(0) = u_2(0) = 1$. One can integrate the equations and look for the first switching time as a Taylor series $s^1 = s_0^1 + r_0 s_1^1 + o(r_0)$ and compute $\phi_2(r_0, s_0^1 + r_0 s_1^1 + o(r_0))$ in order to compute

$$s_0^1 = \sqrt{2}\sqrt{1-a_0} \quad \text{and} \quad s_1^1 = -a_{10} - 2b_{30}(1-a_0).$$

At the switching time

$$\begin{aligned} X(s^1) &= \sqrt{2}\sqrt{1-a_0} - (a_{10} + 2b_{30})(1-a_0)r_0, & \lambda_x(s^1) &= 1, \\ Y(s^1) &= 0, & r(s^1) &= r_0. \end{aligned}$$

After this first switch $\phi_1(0) > 0$ and $\phi_2(0) < 0$, hence $u_1(0) = 1$ and $u_2(0) = -1$. We can compute and look for the next switching time and one finds that ϕ_1 goes to 0 at $s^2 = s_0^2 + r_0 s_1^2 + o(r_0)$ with

$$s_0^2 = s_0^1 + \sqrt{2} \frac{\sqrt{1+a_0} - \sqrt{1-a_0}}{a_0}, \quad s_1^2 = s_1^1 + \frac{b_{01}((1-a_0)^{\frac{3}{2}} - \sqrt{1+a_0}(1-2a_0)) - 12b_{30}a_0^2\sqrt{1+a_0}}{3a_0^2\sqrt{1+a_0}}.$$

At the second switching time

$$\begin{aligned} X(s^2) &= (\sqrt{2}\sqrt{1+a_0}) + \left(\frac{3a_{10}a_0\sqrt{1+a_0} + b_{01}((1-a_0)^{\frac{3}{2}} - (1+a_0)^{\frac{3}{2}}) - 6b_{30}a_0(1+a_0)^{\frac{3}{2}}}{3a_0\sqrt{1+a_0}} \right) r_0, \\ Y(s^2) &= \left(\frac{\sqrt{2}((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})}{3a_0} \right) - \left(\frac{2b_{01}(1-a_0+a_0^2 - (1-a_0)^{\frac{3}{2}}\sqrt{1+a_0}) + 12a_0^2b_{30}}{3a_0^2} \right) r_0, \\ \lambda_x(s^2) &= -1, \\ r(s^2) &= r_0 + \left(\frac{(\sqrt{1+a_0} - \sqrt{1-a_0})b_{01}}{\sqrt{2}a_0} \right) r_0^2. \end{aligned}$$

After this second switch, $\phi_1(0) < 0$ and $\phi_2(0) < 0$, hence $u_1(0) = u_2(0) = -1$. One can compute the third switch as being $s^3 = s_0^3 + r_0 s_1^3 + o(r_0)$ with

$$s_0^3 = s_0^2 + 2\sqrt{2}\sqrt{1+a_0}, \quad s_1^3 = s_1^2 - \frac{2((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})b_{01}}{3a_0\sqrt{1+a_0}}.$$

At this time $X(s^3) = -\sqrt{2}\sqrt{1+a_0} + O(r_0)$ and we will see that this third switching time comes after the cut time.

The same computations can be done for the extremals starting with $\lambda_x(0) = -1$. We use the notation \bar{z} for variables z corresponding to these extremals. During the first bang the controls are $\bar{u}_1 = \bar{u}_2 = -1$, during the second $\bar{u}_1 = 1$ and $\bar{u}_2 = -1$ and during the third one $\bar{u}_1 = \bar{u}_2 = 1$. The switching times are \bar{s}^1 and \bar{s}^2 satisfying

$$\begin{aligned} \bar{s}_0^1 &= \sqrt{2}\sqrt{1+a_0}, & \bar{s}_1^1 &= -a_{10} + 2b_{30}(1+a_0), \\ \bar{s}_0^2 &= \bar{s}_0^1 + \sqrt{2} \frac{\sqrt{1+a_0} - \sqrt{1-a_0}}{a_0}, & \bar{s}_1^2 &= \bar{s}_1^1 + \frac{b_{01}((1+a_0)^{\frac{3}{2}} - \sqrt{1-a_0}(1+2a_0)) + 12b_{30}a_0^2\sqrt{1-a_0}}{3a_0^2\sqrt{1-a_0}}. \end{aligned}$$

And at the second switching time

$$\begin{aligned} \bar{X}(\bar{s}^2) &= -(\sqrt{2}\sqrt{1-a_0}) + \left(\frac{-3a_{10}a_0\sqrt{1-a_0} + b_{01}((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}}) - 6b_{30}a_0(1-a_0)^{\frac{3}{2}}}{3a_0\sqrt{1-a_0}} \right) \bar{r}_0 + o(\bar{r}_0), \\ \bar{Y}(\bar{s}^2) &= \left(\frac{\sqrt{2}((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})}{3a_0} \right) - \left(\frac{2b_{01}(1+a_0+a_0^2 - \sqrt{1-a_0}(1+a_0)^{\frac{3}{2}}) - 12a_0^2b_{30}}{3a_0^2} \right) \bar{r}_0 + o(\bar{r}_0), \\ \bar{\lambda}_x(\bar{s}^2) &= -1, \\ \bar{r}(\bar{s}^2) &= \bar{r}_0 + \left(\frac{(\sqrt{1+a_0} - \sqrt{1-a_0})b_{01}}{\sqrt{2}a_0} \right) \bar{r}_0^2 + o(\bar{r}_0^2). \end{aligned}$$

One can compute that at the third switching time $\bar{X}(\bar{s}^3) = \sqrt{2}\sqrt{1-a_0} + O(r_0)$.

We are now ready to compute the cut locus. As one can estimate easily, an extremal starting with $\lambda_x(0) > 0$ intersects an extremal starting with $\lambda_x(0) < 0$, both during their third bang. Moreover, since $Y(s^2) = \bar{Y}(\bar{s}^2) + o(r_0)$ one have that $\bar{r}_0 = r_0 + o(r_0)$.

Let fix a ρ and look for the extremals that intersect at $y = \frac{\sqrt{2}((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})}{3a_0}\rho^3$. We write $r_0 = \rho + R_{cut}\rho^2 + o(\rho^2)$ and look for R_{cut} such that $r_0Y(s^2) = \frac{\sqrt{2}((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})}{3a_0}\rho^3 + o(\rho^4)$. We find

$$R_{cut} = \frac{\sqrt{2}((-2a_0^2 + (2+a_0)(-1 + \sqrt{1-a_0^2}))b_{01} + 6a_0^2b_{30})}{3a_0((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})}.$$

For $\bar{r}_0 = \rho + \bar{R}_{cut}\rho^2 + o(\rho^2)$ one finds $\bar{R}_{cut} = \frac{\sqrt{2}((-2a_0^2 + (2-a_0)(-1 + \sqrt{1-a_0^2}))b_{01} - 6a_0^2b_{30})}{3a_0((1+a_0)^{\frac{3}{2}} - (1-a_0)^{\frac{3}{2}})}$. With these values, we can compute the second switching times $t^2 = rs^2 = t_1^2\rho + t_2^2\rho^2 + o(\rho^3)$ and $\bar{t}_2 = \bar{r}\bar{s}^2 = \bar{t}_1^2\rho + \bar{t}_2^2\rho^2 + o(\rho^3)$ with

$$\begin{aligned} t_1^2 &= \sqrt{2} \left(\sqrt{1-a_0} + \frac{\sqrt{1+a_0} - \sqrt{1-a_0}}{a_0} \right) \\ t_2^2 &= -a_{10} + \frac{(-5 + 2a_0 - 6a_0^2 + a_0^3)\sqrt{1+a_0} - (-5 - 3a_0 + a_0^2 + 3a_0^3)\sqrt{1-a_0}}{3a_0^2\sqrt{1+a_0}(2 + \sqrt{1-a_0^2})}b_{01} \\ &\quad + \frac{2(-4 + a_0 - 2a_0^2 + a_0^3 + (-1 + a_0)\sqrt{1-a_0^2})}{3 + a_0^2}b_{30} \\ \bar{t}_1^2 &= \sqrt{2} \left(\sqrt{1+a_0} + \frac{\sqrt{1+a_0} - \sqrt{1-a_0}}{a_0} \right) \\ \bar{t}_2^2 &= -a_{10} - \frac{(-5 + 3a_0 + a_0^2 - 3a_0^3)\sqrt{1+a_0} + (5 + 2a_0 + 6a_0^2 + a_0^3)\sqrt{1-a_0}}{3a_0^2\sqrt{1-a_0}(2 + \sqrt{1-a_0^2})}b_{01} \\ &\quad + \frac{2(4 + a_0 + 2a_0^2 + a_0^3) + (1 + a_0)\sqrt{1-a_0^2}}{3 + a_0^2}b_{30} \end{aligned}$$

and the x coordinates of the point of second switching under the form $x = x_1\rho + x_2\rho^2 + o(\rho^3)$ and $\bar{x} = \bar{x}_1\rho + \bar{x}_2\rho^2 + o(\rho^3)$ with

$$\begin{aligned} x_1 &= \frac{2\sqrt{2}(1 + 3a_0^2 - (1 - a_0^2)^{\frac{3}{2}})}{a_0((1 + a_0)^{\frac{3}{2}} - (1 - a_0)^{\frac{3}{2}})}, \\ x_2 &= -\frac{5 + a_0 + 5a_0^2 - (5 + a_0)\sqrt{1 - a_0^2}}{3a_0^2}b_{01} - 4b_{30}, \\ \bar{x}_1 &= -\frac{2\sqrt{2}(1 + 3a_0^2 - (1 - a_0^2)^{\frac{3}{2}})}{a_0((1 + a_0)^{\frac{3}{2}} - (1 - a_0)^{\frac{3}{2}})}, \\ \bar{x}_2 &= \frac{5 - a_0 + 5a_0^2 + (-5 + a_0)\sqrt{1 - a_0^2}}{3a_0^2}b_{01} - 4b_{30}. \end{aligned}$$

One find easily that the cut locus is at $x_c = \frac{x_1 + \bar{x}_1}{2} \rho + \frac{x_2 + \bar{x}_2}{2} \rho^2 + o(\rho^2)$ that is

$$\begin{aligned} x_{cut}^+ &= - \left(\frac{a_0}{3(1 + \sqrt{1 - a_0^2})} b_{01} + 4b_{30} \right) \rho^2 + o(\rho^2), \\ y_{cut}^+ &= \frac{\sqrt{2}((1 + a_0)^{\frac{3}{2}} - (1 - a_0)^{\frac{3}{2}})}{3a_0} \rho^3. \end{aligned}$$

When $-\lambda_y(0) \gg 1$, then we set $r = \frac{1}{\sqrt{-\lambda_y}}$. Equations are changed but the final result is very similar

$$\begin{aligned} x_{cut}^- &= - \left(\frac{a_0}{3(1 + \sqrt{1 - a_0^2})} b_{01} + 4b_{30} \right) \rho^2 + o(\rho^2), \\ y_{cut}^- &= - \frac{\sqrt{2}((1 + a_0)^{\frac{3}{2}} - (1 - a_0)^{\frac{3}{2}})}{3a_0} \rho^3. \end{aligned}$$

Finally, the cut locus appears to be a cusp whose tangent at the singular point is the tangent to Δ_A , see Figure 7.

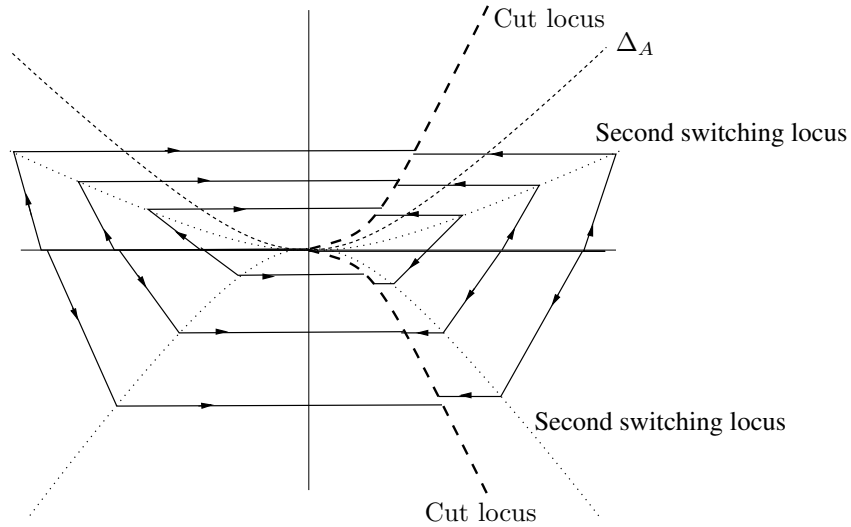


Figure 7: The synthesis in the (NF_3) case

References

- [1] A. Agrachev, B. Bonnard, M. Chyba, and I. Kupka. Sub-Riemannian sphere in Martinet flat case. *ESAIM Control Optim. Calc. Var.*, 2:377–448, 1997.
- [2] Andrei Agrachev and Jean-Paul Gauthier. On the subanalyticity of Carnot-Caratheodory distances. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 18(3):359–382, 2001.

- [3] Andrei A. Agrachev and Yuri L. Sachkov. *Control theory from the geometric viewpoint*, volume 87 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. Control Theory and Optimization, II.
- [4] E. Ali and G. Charlot. Local contact sub-finslerian geometry for maximum norms in dimension 3. *preprint*.
- [5] D. Barilari, U. Boscain, E. Le Donne, and M. Sigalotti. Sub-finsler structures from the time-optimal control viewpoint for some nilpotent distributions. *arXiv:1506.04339*, 2016.
- [6] Davide Barilari, Ugo Boscain, Grégoire Charlot, and Robert W. Neel. On the heat diffusion for generic riemannian and sub-riemannian structures. *IMRN*, 2016.
- [7] Davide Barilari, Ugo Boscain, and Robert W. Neel. Small-time heat kernel asymptotics at the sub-Riemannian cut locus. *J. Differential Geom.*, 92(3):373–416, 2012.
- [8] André Bellaïche. The tangent space in sub-Riemannian geometry. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 1–78. Birkhäuser, Basel, 1996.
- [9] G. Ben Arous. Développement asymptotique du noyau de la chaleur hypoelliptique hors du cut-locus. *Ann. Sci. École Norm. Sup. (4)*, 21(3):307–331, 1988.
- [10] G. Ben Arous and R. Léandre. Décroissance exponentielle du noyau de la chaleur sur la diagonale. II. *Probab. Theory Related Fields*, 90(3):377–402, 1991.
- [11] B. Bonnard, M. Chyba, and E. Trelat. Sub-riemannian geometry, one-parameter deformation of the martinet flat case. *J. Dyn. Control Syst.* 4, No.1, 59-76 (1998)., 4:59–76, 1998.
- [12] Bernard Bonnard and Monique Chyba. Méthodes géométriques et analytiques pour étudier l’application exponentielle, la sphère et le front d’onde en géométrie sous-riemannienne dans le cas Martinet. *ESAIM Control Optim. Calc. Var.*, 4:245–334 (electronic), 1999.
- [13] Ugo Boscain, Thomas Chambrion, and Grégoire Charlot. Nonisotropic 3-level quantum systems: complete solutions for minimum time and minimum energy. *Discrete Contin. Dyn. Syst. Ser. B*, 5(4):957–990, 2005.
- [14] E. Breuillard and E. Le Donne. On the rate of convergence to the asymptotic cone for nilpotent groups and subfinsler geometry. *Proc. Natl. Acad. Sci. USA*, 110(48):19220–19226, 2013.
- [15] Grégoire Charlot. Quasi-contact s-r metrics : normal form in \mathbb{R}^{2n} , wave front and caustic in \mathbb{R}^4 . *Acta App. Math.*, 74:217–263, 2002.
- [16] W. L. Chow. ber systeme von linearen partiellen differentialgleichungen erster ordnung. *Math. Ann.*, 117:98–105, 1939.
- [17] Jeanne Clelland and Christopher Moseley. Sub-finsler geometry in dimension three. *Differ. Geom. Appl.*, 24(6):628–651, 2006.
- [18] Jeanne Clelland, Christopher Moseley, and George Wilkens. Geometry of sub-finsler engel manifolds. *Asian J. Math.*, 11(4):699–726, 2007.

- [19] El-Houcine Chakir El Alaoui, J.-P. Gauthier, and I. Kupka. Small sub-riemannian balls on \mathbb{R}^3 . *J. Dyn. Control Syst.*, 2(3):359–421, 1996.
- [20] A.F. Filippov. On some questions in the theory of optimal regulation: existence of a solution of the problem of optimal regulation in the class of bounded measurable functions. *Vestnik Moskov. Univ. Ser. Mat. Meh. Astr.Fiz.Him.*, 2:2532, 1959.
- [21] Morris W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.
- [22] Rémi Léandre. Majoration en temps petit de la densité d’une diffusion dégénérée. *Probab. Theory Related Fields*, 74(2):289–294, 1987.
- [23] Rémi Léandre. Minoration en temps petit de la densité d’une diffusion dégénérée. *J. Funct. Anal.*, 74(2):399–414, 1987.
- [24] L. S. Pontryagin, V. G. Boltyanskiĭ, R. V. Gamkrelidze, and E. F. Mishchenko. *The Mathematical Theory of Optimal Processes*. “Nauka”, Moscow, fourth edition, 1983.
- [25] P.K. Rashevsky. About connecting two points of complete nonholonomic space by admissible curve. *Uch. Zap. Ped. Inst. Libknehta*, 2:83–94, 1938.