HAL
open science

Extremal Cylinder Configurations I: Configuration C_m

Oleg Ogievetsky, Senya Shlosman

To cite this version:

Oleg Ogievetsky, Senya Shlosman. Extremal Cylinder Configurations I: Configuration C_m. Discrete and Computational Geometry, 2021, 66 (1), pp.140-164. 10.1007/s00454-020-00244-6 . hal-02003983

HAL Id: hal-02003983

https://hal.science/hal-02003983v1

Submitted on 1 Feb 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

arXiv:1812.09543v1 [math.MG] 22 Dec 2018

Extremal Cylinder Configurations I: Configuration $C_{\mathfrak{m}}$

Oleg Ogievetsky[◇∗1] **and Senya Shlosman**[◇†‡]

[◇]Aix Marseille Université, Université de Toulon, CNRS,
CPT UMR 7332, 13288, Marseille, France
[†]Inst. of the Information Transmission Problems, RAS, Moscow, Russia
[‡] Skolkovo Institute of Science and Technology, Moscow, Russia
[∗]Kazan Federal University, Kremlevskaya 17, Kazan 420008, Russia

I do not ask for better than not to be believed.
Axel Munthe, *The story of San Michele*

Abstract

We study the path $\Gamma = \{C_{6,x} \mid x \in [0,1]\}$ in the moduli space of configurations of 6 equal cylinders touching the unit sphere. Among the configurations $C_{6,x}$ is the record configuration $C_{\mathfrak{m}}$ of [OS]. We show that $C_{\mathfrak{m}}$ is a local sharp maximum of the distance function, so in particular the configuration $C_{\mathfrak{m}}$ is not only unlockable but rigid. We show that if $\frac{(1+x)(1+3x)}{3}$ is a rational number but not a square of a rational number, the configuration $C_{6,x}$ has some hidden symmetries, part of which we explain.

[1]Also at Lebedev Institute, Moscow, Russia.

Contents

1 Introduction **3**

2 Configuration manifold **8**

3 The critical point $C_6(\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}, \varkappa_{\mathfrak{m}})$ **9**

3.1 Main maximality result 12

3.2 Toy example 12

4 Proof of Theorem 1 **13**

4.1 Variation of distances in the first order 14

4.2 Linear dependence of differentials 16

4.3 Relevant quadratic form 17

5 Local maximum **18**

5.1 First proof of Theorem 2 19

5.2 Second proof of Theorem 2 21

5.2.1 Concluding remark 26

6 Hidden symmetry **27**

6.1 Galois symmetry 27

6.2 Discussion 31

References **38**

1 Introduction

This is a continuation of our work [OS]. In that paper we were considering the configurations of six (infinite) nonintersecting cylinders of the same radius r touching the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. We were interested in the maximal value of r for which this is possible. We have constructed in [OS] the 'record' configuration $C_{\mathfrak{m}}$ of six cylinders of radius

$$r_{\mathfrak{m}} = \frac{1}{8}\left(3+\sqrt{33}\right) \approx 1.093070331, \tag{1}$$

thus we know that the maximal value of r is at least $r_{\mathfrak{m}}$. We believe that $r_{\mathfrak{m}}$ is in fact the maximal possible value for r, but we have no proof of that.

In [OS] we have constructed the deformation $C_{6,x}$ of the configuration C_6 of six vertical unit nonintersecting cylinders. The configuration C_6 corresponds to $x = 1$ while $C_{\mathfrak{m}}$ – to $x = 1/2$. These configurations are shown on Figure 1 (the green unit ball is in the center).

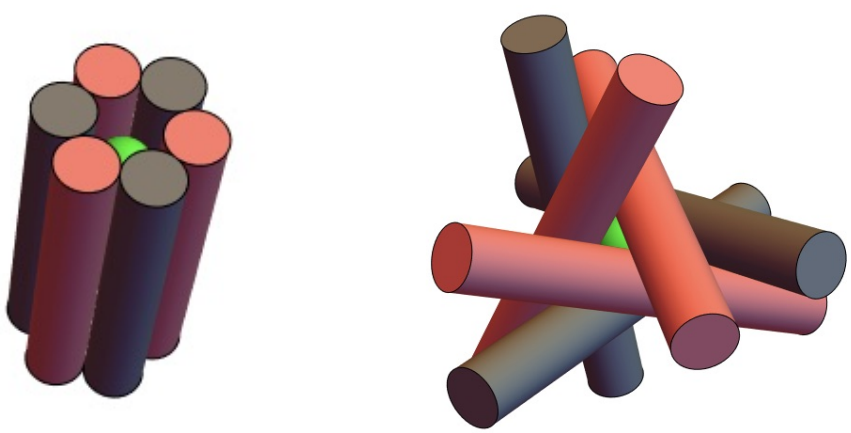

Figure 1: Two configurations of cylinders: the configuration C_6 of six parallel cylinders of radius 1 (on the left) and the configuration $C_{\mathfrak{m}}$ of six cylinders of radius ≈ 1.0931 (on the right)

To explain the results of the present paper, we introduce some notation.

A cylinder ς touching the unit sphere $\mathbb{S}^2$ has a unique generator (a line parallel to the axis of the cylinder) $\iota(\varsigma)$ touching $\mathbb{S}^2$. We will usually represent a configuration $\{\varsigma_1, \ldots, \varsigma_L\}$ of cylinders touching the unit sphere by the configuration $\{\iota(\varsigma_1), \ldots, \iota(\varsigma_L)\}$ of tangent to $\mathbb{S}^2$ lines. The manifold of all such six-tuples we denote by M^6.

For example, let $C_6 \equiv C_6\,(0,0,0)$ be the configuration of six nonintersecting cylinders of radius 1, parallel to the z direction in $\mathbb{R}^3$ and touching the unit ball centered at the origin. The configuration of tangent lines associated to the configuration C_6 is shown on Figure 2.

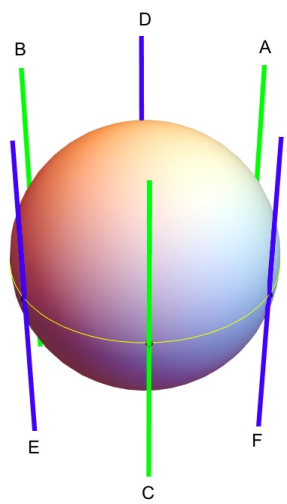

Figure 2: Configuration C_6 of tangent lines

Let ς', ς'' be two equal cylinders of radius r touching $\mathbb{S}^2$, which also touch each other, while ι', ι'' are the corresponding tangents to $\mathbb{S}^2$. If $d = d_{\iota'\iota''}$ is the distance between ι', ι'' then we have

$$r = \frac{d}{2-d},$$

so it is really the same - to study the manifold of six-tuples of cylinders of equal radii, some of which are touching, or to study the manifold M^6 and the function D on it:

$$D\,(\iota_1, \ldots, \iota_6) = \min_{1 \le i < j \le 6} d_{\iota_i \iota_j}.$$

In this paper we prove that the configuration $C_{\mathfrak{m}}$ is a sharp local maximum of the function D. In the process of the proof we also show that $(\mathrm{mod}\, SO\,(3))$ the 15-dim tangent space at $C_{\mathfrak{m}}$ contains a 4-dimensional subspace along which the function $D\,(\mathbf{m})^2$ decays quadratically, while along any other tangent direction it decays linearly.

It turns out that the question of finding sufficient conditions for the extrema of the *min* functions can be quite delicate.

For the configuration $C_{\mathfrak{m}}$ we distinguish twelve relevant distances $l_1, \ldots,$ l_{12} out of the total of fifteen pairwise distances between the tangent lines. Thus the question is about the local maximum of the non-analytic function (the minimum of the squares $F_1, \ldots, F_{12}$ of these twelve distances) in fifteen variables.

We make a general remark. Let $F_1, \ldots, F_m$ be analytic functions in n variables, $n \geq m$, and let

$$\mathsf{F}\,(x) := \min\left\{F_1\,(x)\,, \ldots, F_m\,(x)\right\}\ .$$

We assume that $F_j(0) = 0$, $j = 1, \ldots, m$. Suppose that the point $0 \in \mathbb{R}^n$ is a local maximum of the function $\mathsf{F}\,(x)$. Then the differentials $dF_1, \ldots, dF_m$ are necessarily linearly dependent at 0. Indeed, let Π_j^+, $j = 1, \ldots, m$, be the half-space in $\mathbb{R}^n$ on which the differential $dF_j(0)$ is positive. If the differentials $dF_1, \ldots, dF_m$ are independent at 0 then the intersection of the half-spaces Π_j is non-empty, so there is a direction from 0 along which all m functions F_i are increasing, thus the point 0 is not a local maximum of the function $\mathsf{F}\,(x)$. This remark is a generalization of the case $m = 1$ (just one analytic function): if the point 0 is a local maximum of an analytic function $F(x)$ then its differential vanishes at the point 0, $dF(0) = 0$.

We return for a moment to the configuration $C_{\mathfrak{m}}$. We calculate explicitly the differentials of the squares of the twelve relevant distances $l_1, \ldots, l_{12}$. Our first observation is that they are indeed not linearly independent. More precisely, there is a *single* linear combination λ of the differentials which vanish, $\lambda(dl_1^2, \ldots, dl_{12}^2) = 0$.

We continue the general remark. Suppose that the point $0 \in \mathbb{R}^n$ is a local maximum of the function $\mathsf{F}\,(x)$ and there is exactly one linear dependency between the differentials $dF_1, \ldots, dF_m$ at 0. Then this dependency *must* be convex, in the sense that it must have the form $\lambda^1 dF_1 + \ldots \lambda^m dF_m = 0$ with

$\lambda^j > 0$, $j = 1, \ldots, m$. Indeed, let us assume that the single linear combination of the differentials is not convex. Then, renumbering, if necessary, the functions $F_j(x)$, we write the linear dependency between the differentials in the following form

$$dF_1(0) = \mu^2 dF_2(0) \pm \cdots \pm \mu^m dF_m(0) \ ,$$

with $\mu^j > 0$, $j = 2, \ldots, m$. The differentials $dF_2, \ldots, dF_m$ are independent and the subset of $\mathbb{R}^n$ where $\mu^2 dF_2(0) \pm \cdots \pm \mu^m dF_m(0) > 0$ is a non-empty open convex cone in which all the differentials are positive, so again the point 0 cannot be a local maximum of the function $\mathsf{F}(x)$.

This is exactly what happens for the configuration $C_{\mathfrak{m}}$. The unique linear combination $\lambda(l_1, \ldots, l_{12}) = 0$ of the differentials of the twelve relevant distances is convex. We thus have a four-dimensional linear subspace E of the tangent space on which all twelve differentials vanish. Here $4 = 15$ (dimension of the configuration space mod $SO(3)$) - 12 (the number of relevant distances) + 1 (the number of relations between the differentials).

The presence of the linear convex dependency between the differentials is necessary, but not sufficient, and we have to continue the analysis. Let q_1, $\ldots$, q_{12} denote the second differentials of the functions F_1, $\ldots$, F_{12}. Let q be the restriction of the same convex combination λ of the second differentials to the space E, $q = \lambda(q_1, \ldots, q_{12})|_E$.

Our second observation is that the form q is negatively defined. We prove (and it is not immediate) that the local maximality is implied by these two observations.

Our results imply that the configuration $C_{\mathfrak{m}}$ is unlockable and, moreover, rigid.

The precise meaning of the *unlocking* is the following. Let Π be a collection of non-intersecting open solid bodies, $\Pi = \{\Lambda_1, \ldots, \Lambda_k\}$, where each Λ_i touches the unit central ball, while some distances between bodies of Π are zero. We call a family $\Pi(t) = \{\Lambda_1(t), \ldots, \Lambda_k(t)\}$, $t \geq 0$, of collections of non-intersecting open solid bodies, touching the unit central ball, a continuous deformation of the collection Π if $\Lambda_j(t) = g_j(t)\Lambda_j$, $j = 1, \ldots, k$, where $g_j(t)$ is a continuous curve in the group of Euclidean motions of $\mathbb{R}^3$ with $g_j(0) = \mathrm{Id}$. We say that Π can be unlocked if there exists a continuous deformation $\Pi(t)$ of Π such that some of zero distances between the members of the configuration Π are positive in $\Pi(t)$ for all $t > 0$.

We say that a configuration Π is *rigid* if the only continuous deformations of Π are global rotations in the three-dimensional space.

In [K] W. Kuperberg suggested another configuration of six unit non-intersecting cylinders touching the unit sphere and asked whether it can be unlocked. It is the configuration O_6 shown on Figure 3. We are planning to address this question in the forthcoming work [OS-O6].

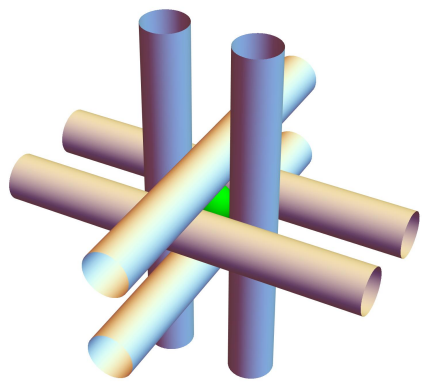

Figure 3: Configuration O_6 of cylinders

While calculating variations of distances we observed that the coefficients of the Taylor decompositions of squares of the pairwise distances around the record point $C_{\mathfrak{m}}$ belong, after a certain normalization, to the field $\mathbb{Q}[\tau]$, where τ is the golden ratio. This miraculous fact allows us to reveal a hidden symmetry of the formulas for the coefficients of the Taylor expansions of distances around the point $C_{\mathfrak{m}}$. Namely, the Galois conjugation in the field $\mathbb{Q}[\tau]$ restores the original $\mathbb{D}_6$-symmetry of the configuration C_6.

Puzzled by the hidden Galois symmetry, we performed several experiments wondering whether this symmetry is specific for the record point $C_{\mathfrak{m}}$ or it is inherent at some other points of the curve $C_{6,x}$. It turns out that for all rational x such that $\sqrt{(1+x)(1+3x)/3}$ is not rational, we have a similar

phenomenon. Namely, when we perturb the configuration $C_{6,x}$, the Taylor coefficients, after a certain normalization, belong to a (real) quadratic extension of the rational field $\mathbb{Q}$, and Galois conjugation restores the $\mathbb{D}_6$-symmetry of the formulas for variations.

The paper is organized as follows. The next section introduces further notation, concerning our manifold M^6. In Section 3 we formulate our main local maximality results. Section 4 contains the proofs of assertions from Section 3. Our general results concerning sufficient conditions for the local extrema of the *min* function are collected in Section 5. The last Section 6 is devoted to the hidden symmetry of the configuration $C_{\mathfrak{m}}$ and, more generally, to the hidden symmetry of the configurations $C_{6,x}$.

2 Configuration manifold

Here we collect the notation of [OS] used below.

Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere, centered at the origin. For every $x \in \mathbb{S}^2$ we denote by TL_x the set of all (unoriented) tangent lines to $\mathbb{S}^2$ at x. We denote by M the manifold of tangent lines to $\mathbb{S}^2$. We represent a point in M by a pair (x, ξ), where ξ is a unit tangent vector to $\mathbb{S}^2$ at x, though such a pair is not unique: the pair $(x, -\xi)$ is the same point in M.

We shall use the following coordinates on M. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be the standard coordinate axes in $\mathbb{R}^3$. Let $R^\alpha_{\mathbf{x}}$, $R^\alpha_{\mathbf{y}}$ and $R^\alpha_{\mathbf{z}}$ be the counterclockwise rotations about these axes by an angle α, viewed from the tips of axes.

We call the point $\mathsf{N} = (0, 0, 1)$ the North pole, and $\mathsf{S} = (0, 0, -1)$ – the South pole. By *meridians* we mean geodesics on $\mathbb{S}^2$ joining the North pole to the South pole. The meridian in the plane $\mathbf{xz}$ with positive $\mathbf{x}$ coordinates will be called Greenwich. The angle φ will denote the latitude on $\mathbb{S}^2$, $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and the angle $\varkappa \in [0, 2\pi)$ – the longitude, so that Greenwich corresponds to $\varkappa = 0$. Every point $x \in \mathbb{S}^2$ can be written as $x = (\varphi_x, \varkappa_x)$.

Finally, for each $x \in \mathbb{S}^2$, we denote by R^α_x the rotation by the angle α about the axis joining $(0, 0, 0)$ to x, counterclockwise if viewed from its tip, and by $(x, \uparrow)$ we denote the pair (x, ξ_x), $x \neq \mathsf{N}, \mathsf{S}$, where the vector ξ_x points to the North. We also abbreviate the notation $(x, R^\alpha_x \uparrow)$ to $(x, \uparrow_\alpha)$.

Let $u = (x', \xi')$, $v = (x'', \xi'')$ be two lines in M. We denote by d_{uv} the

distance between u and v; clearly $d_{uv}=0$ iff $u\cap v\neq\varnothing$. If the lines u,v are not parallel then the square of d_{uv} is given by the formula

$$d_{uv}^2=\frac{\det^2[\xi',\xi'',x''-x']}{1-(\xi',\xi'')^2}\ , \tag{2}$$

where $(*,*)$ is the scalar product.

We note that if $d_{uv}=d>0$ then the cylinders $\mathrm{C}_u(r)$ and $\mathrm{C}_v(r)$, touching $\mathbb{S}^2$ at x',x'', having directions ξ',ξ'', and radius r, touch each other iff

$$r=\frac{d}{2-d}. \tag{3}$$

Indeed, if the cylinders touch each other, we have the proportion:

$$\frac{d}{1}=\frac{2r}{1+r}. \tag{4}$$

We denote by M^6 the manifold of 6-tuples

$$\mathbf{m}=\{u_1,\ldots,u_6:u_i\in M,i=1,\ldots,6\}. \tag{5}$$

We are studying the critical points of the function

$$D(\mathbf{m})=\min_{1\leq i<j\leq 6}d_{u_iu_j}.$$

Note that $D(C_6)=1$.

3 The critical point $C_6(\varphi_{\mathfrak{m}},\delta_{\mathfrak{m}},\varkappa_{\mathfrak{m}})$

The configuration $C_6\equiv C_6(0,0,0)$ in our notation can be written as

$$\begin{aligned}C_6=\ &\left\{A=\left[\left(0,\tfrac{\pi}{6}\right),\uparrow\right],D=\left[\left(0,\tfrac{\pi}{2}\right),\uparrow\right],\right.\\ &B=\left[\left(0,\tfrac{5\pi}{6}\right),\uparrow\right],E=\left[\left(0,\tfrac{7\pi}{6}\right),\uparrow\right],\\ &\left.C=\left[\left(0,\tfrac{3\pi}{2}\right),\uparrow\right],F=\left[\left(0,\tfrac{11\pi}{6}\right),\uparrow\right]\right\}.\end{aligned}$$

We need also the configurations $C_6\left(\varphi,\delta,\varkappa\right)$:

$$\begin{aligned} C_6\left(\varphi,\delta,\varkappa\right) = \big\{ A = \big[\big(\varphi, \tfrac{\pi}{6} - \varkappa\big), \uparrow_\delta\big], D = \big[\big(-\varphi, \tfrac{\pi}{2} + \varkappa\big), \uparrow_\delta\big], \\ B = \big[\big(\varphi, \tfrac{5\pi}{6} - \varkappa\big), \uparrow_\delta\big], E = \big[\big(-\varphi, \tfrac{7\pi}{6} + \varkappa\big), \uparrow_\delta\big], \\ C = \big[\big(\varphi, \tfrac{3\pi}{2} - \varkappa\big), \uparrow_\delta\big], F = \big[\big(-\varphi, \tfrac{11\pi}{6} + \varkappa\big), \uparrow_\delta\big]\big\}. \end{aligned} \tag{6}$$

In [OS] we have constructed a a continuous curve

$$\gamma(\varphi) = C_6\big(\varphi, \delta\left(\varphi\right), \varkappa\left(\varphi\right)\big)\ ,\ \varphi \in \left[0; \frac{\pi}{2}\right]\ , \tag{7}$$

on which the function $D\big(\gamma(\varphi)\big)$ grows for $\varphi \in [0, \varphi_{\mathfrak{m}}]$ and decays for $\varphi > \varphi_{\mathfrak{m}}$. For the 'record' point $C_{\mathfrak{m}} = C_6\left(\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}, \varkappa_{\mathfrak{m}}\right)$ we have

$$D\big(\gamma(\varphi_{\mathfrak{m}})\big) = \sqrt{\frac{12}{11}}\ ,$$

with

$$\varphi_{\mathfrak{m}} = \arcsin\sqrt{\frac{3}{11}}\ ,\ \varkappa_{\mathfrak{m}} = -\arctan\frac{1}{\sqrt{15}}\ ,\ \delta_{\mathfrak{m}} = \arctan\sqrt{\frac{5}{11}}\ .$$

The radii of the corresponding cylinders are equal to $r_{\mathfrak{m}}$ (see formula (1)) which, we believe is the maximal possible common radius for six non-intersecting cylinders touching the unit ball.

Note that placing the cylinders of radius 1 instead of $r_{\mathfrak{m}}$ leaves a spacing $2(r_{\mathfrak{m}} - 1)$ for each cylinder. Even if we could manage to move these unit cylinders in such a way that the spacings would behave additively then the total spacing would be $6 \cdot 2(r_{\mathfrak{m}} - 1) \approx 1.116843972$ which does not make enough room for a seventh unit cylinder (the problem of whether seven infinite circular non-intersecting unit cylinders can be arranged about a central unit ball is open).

The configuration $C_{\mathfrak{m}}$ is shown on Figures 4, 5 and 6.

There is now an animation, on the page of Yoav Kallus [Ka], demonstrating the motion of the configuration of 6 cylinders along the curve $\gamma(\varphi)$.

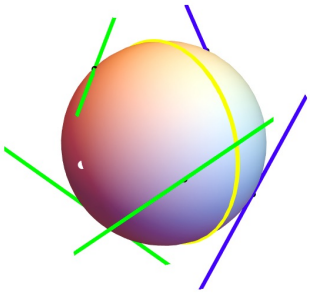

Figure 4: Record configuration, side view, the equator is yellow, the north pole is white

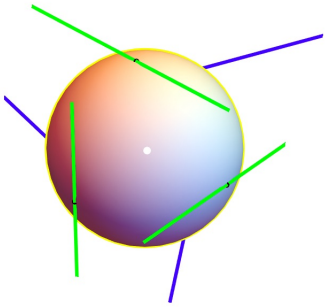

Figure 5: Record configuration again, three upper tangency points shown

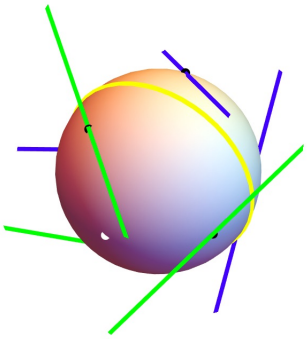

Figure 6: Record configuration once more, two upper and one lower tangency points shown

3.1 Main maximality result

In [OS] we have shown that the function $D\left(C_6\left(\varphi,\delta\left(\varphi\right),\varkappa\left(\varphi\right)\right)\right)$ has a global maximum at the point $\varphi_{\mathfrak{m}}$, corresponding to the configuration $C_{\mathfrak{m}}$. We now study the function D in the vicinity of the point $C_{\mathfrak{m}}$ in the whole space M^6.

Theorem 1 *The configuration $C_{\mathfrak{m}}$ is a point of a sharp local maximum of the function D : for any point $\mathbf{m}$ in a vicinity of $C_{\mathfrak{m}}$ we have*

$$D\left(\mathbf{m}\right)<\sqrt{\frac{12}{11}}=D\left(C_{\mathfrak{m}}\right).$$

In the process of the proof, we will see that there exists a 4-dimensional subspace L_{quadr} in the tangent space of M^6 at $C_{\mathfrak{m}}$, such that for any $l\in L_{quadr}$ we have

$$-c_u\left\|l\right\|t^2\leq D\left(C_{\mathfrak{m}}+tl\right)-D\left(C_{\mathfrak{m}}\right)\leq -c_d\left\|l\right\|t^2$$

for t small enough. Here c_d and c_u are some constants, $0<c_d\leq c_u<+\infty$ and $C_{\mathfrak{m}}+tl\in M^6$ stands for the exponential map applied to the tangent vector tl.

For the tangent vectors l outside L_{quadr} we have

$$-c'_u\left(l\right)t\leq D\left(C_{\mathfrak{m}}+tl\right)-D\left(C_{\mathfrak{m}}\right)\leq -c'_d\left(l\right)t,$$

where now $c'_d\left(l\right)$ and $c'_u\left(l\right)$ are some positive valued functions of l, $0<c'\left(l\right)\leq c''\left(l\right)<+\infty$.

Note, however, that the last two inequalities do not imply our Theorem, as the example of the Section 3.2 shows.

3.2 Toy example

Suppose that we know that along any tangent direction to the critical point the function D decays, either linearly or at least quadratically. We emphasize that this property by itself *does not* imply our main result about the local maximality of this point. An extra work is needed to prove the maximality statements. The following example explains this.

Let f be a function of two variables defined by

$$f:=\min\{u_1,u_2\}\ \text{ where }\ u_1=-y+3x^2,u_2=y-x^2\ .$$

The function f equals 0 at the origin. Consider an arbitrary ray l starting at the origin. Clearly, for some time this ray evades the 'horns' – the region between the parabolas $y = 3x^2$ and $y = x^2$. But outside the horns the function f is negative. Indeed, inside the the narrow parabola $y = 3x^2$ we have $u_1 < 0, u_2 > 0$ so f is negative there; outside the wide parabola $y = x^2$ we have $u_1 > 0, u_2 < 0$ so f is negative there as well. Therefore the origin is a local maximum of f restricted to l, for any l. Yet the origin is not a local maximum of the function f on the plane, because inside the horns the functions u_1 and u_2 are positive so f there is positive as well [2].

Note that there is a convex linear combination of the differentials of the functions u_1 and u_2 which vanishes (the sum of the differentials); both differentials vanish on the line $y = 0$. However the restriction of the sum of the second differentials to this line is positive, in line with our Theorem 2, Subsection 5.1.

This toy example captures the essential features of questions we encounter in the study of local maxima of the distance functions. Some general theorems needed to establish maximality assertions are formulated and proven in Section 5.

4 Proof of Theorem 1

For any $\varphi, \delta, \varkappa$ the configuration $C_6(\varphi, \delta, \varkappa)$, defined by the formula (6), possesses the dihedral symmetry group $\mathbb{D}_3 \equiv \mathbb{Z}_2 \ltimes \mathbb{Z}_3$. The group $\mathbb{D}_3$ is generated by the rotations $R_{\mathbf{z}}^{120^\circ}$ and $R_{\mathbf{x}}^{180^\circ}$. Because of the $\mathbb{D}_3$-symmetry, the fifteen pairwise distances between the lines in the configuration $C_6(\varphi, \delta, \varkappa)$ split into four groups: we have $d_{AB} = d_{BC} = d_{CA} = d_{DE} = d_{EF} = d_{FD}$ for the six-plet $\{AB, BC, CA, DE, EF, FD\}$, and $d_{AD} = d_{BE} = d_{CF}$, $d_{AF} = d_{BD} = d_{CE}$ and $d_{AE} = d_{BF} = d_{CD}$ for the three triplets $\{AD, BE, CF\}$, $\{AF, BD, CE\}$ and $\{AE, BF, CD\}$. For any configuration $\gamma(\varphi)$ lying on the curve γ, see (7), twelve of the fifteen distances coincide (we additionally have $d_{AB} = d_{AD} = d_{AF}$). In the proof we shall study variations of distances in a vicinity of the point $C_6(\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}, \varkappa_{\mathfrak{m}}) \equiv \gamma(\varphi_{\mathfrak{m}})$ in M^6. The distances from the $\{AE, BF, CD\}$-triplet are greater than the other twelve

[2] Compare with the 'Example of a differentiable function possessing no extremum at the origin but for which the restriction to an arbitrary line through the origin has a strict relative minimum there', Chapter 9 in [GO].

(equal) distances for the configuration $C_6\,(\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}, \varkappa_{\mathfrak{m}})$, see [OS], so these three distances are not relevant in the study of the local maximality of the configuration $C_6\,(\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}, \varkappa_{\mathfrak{m}})$.

The perturbed position of a tangent line $J \in \{A, B, C, D, E, F\}$ is

$$J = J(\varkappa_{\mathfrak{m}} + \Delta_J^{\varkappa}, \varphi_{\mathfrak{m}} + \Delta_J^{\varphi}, \delta_{\mathfrak{m}} + \Delta_J^{\delta})$$

where

$$\{\varkappa_{\mathfrak{m}}, \varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}\}$$

is the maximal point on our D_3-symmetric curve and

$$\begin{aligned} \Delta_J^{\varkappa} &= \frac{11}{32\sqrt{3}} \left(J_{\varkappa,1} t + J_{\varkappa,2} t^2 + o(t^2) \right) \ , \\ \Delta_J^{\varphi} &= \frac{11}{4\sqrt{6}} \left(J_{\varphi,1} t + J_{\varphi,2} t^2 + o(t^2) \right) \ , \\ \Delta_J^{\delta} &= \frac{11\sqrt{11}}{48} \left(J_{\delta,1} t + J_{\delta,2} t^2 + o(t^2) \right) \ . \end{aligned} \tag{8}$$

The numerical factors here are introduced for convenience, because with this normalization the coefficients of the Taylor decompositions of squares of distances around the record point $C_6\,(\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}, \varkappa_{\mathfrak{m}})$ belong (we do not understand the reason for this) to the field $\mathbb{Q}[\tau]$, where τ is the golden section,

$$\tau = \frac{1+\sqrt{5}}{2} \ , \ \bar{\tau} = \frac{1-\sqrt{5}}{2} \ .$$

To fix the rotational symmetry we keep the tangent line A at its place, that is, $A_{\varphi,j} = A_{\varkappa,j} = A_{\delta,j} = 0$, $j = 1, 2, \ldots$, working therefore in the 15-dimensional space M^6 mod $SO(3)$.

4.1 Variation of distances in the first order

We first list all differentials.

Let

$$\begin{aligned} \beta_{11} &= \frac{2}{11}(4-\sqrt{5}) \ , \ \bar{\beta}_{11} = \frac{2}{11}(4+\sqrt{5}) \ , \\ \gamma_{-19} &= -1-2\sqrt{5} \ , \ \bar{\gamma}_{-19} = -1+2\sqrt{5} \ . \end{aligned} \tag{9}$$

The seemingly strange indices of β and γ are used due to the prime decompositions

$$11 = (4+\sqrt{5})(4-\sqrt{5})\ ,\ -19 = (-1-2\sqrt{5})(-1+2\sqrt{5})$$

in the ring $\mathbb{Z}[\tau]$ of golden integers.

The differentials of the $\{AF, CE, BD\}$-triplet are

$$\left[d(AF)^2\right]_1 = \beta_{11}\left(\tau^2\beta_{11}F_{\varkappa,1} + \tau^3 F_{\varphi,1} + F_{\delta,1}\right),$$

$$\left[d(CE)^2\right]_1 = \beta_{11}\left(\tau^2\beta_{11}(C_{\varkappa,1}+E_{\varkappa,1}) + \tau^3(C_{\varphi,1}+E_{\varphi,1}) + (C_{\delta,1}+E_{\delta,1})\right),$$

$$\left[d(BD)^2\right]_1 = \beta_{11}\left(\tau^2\beta_{11}(B_{\varkappa,1}+D_{\varkappa,1}) + \tau^3(B_{\varphi,1}+D_{\varphi,1}) + (B_{\delta,1}+D_{\delta,1})\right).$$

The differentials of the $\{CF, BE, AD\}$-triplet are

$$\left[d(CF)^2\right]_1 = -\bar{\beta}_{11}\left(\bar{\tau}^2\bar{\beta}_{11}(C_{\varkappa,1}+F_{\varkappa,1}) - \bar{\tau}^3(C_{\varphi,1}+F_{\varphi,1}) + (C_{\delta,1}+F_{\delta,1})\right),$$

$$\left[d(BE)^2\right]_1 = -\bar{\beta}_{11}\left(\bar{\tau}^2\bar{\beta}_{11}(B_{\varkappa,1}+E_{\varkappa,1}) - \bar{\tau}^3(B_{\varphi,1}+E_{\varphi,1}) + (B_{\delta,1}+E_{\delta,1})\right),$$

$$\left[d(AD)^2\right]_1 = -\bar{\beta}_{11}\left(\bar{\tau}^2\bar{\beta}_{11}D_{\varkappa,1} - \bar{\tau}^3 D_{\varphi,1} + D_{\delta,1}\right).$$

For the 6-plet $\{AB, BC, CA, DE, EF, FD\}$ we have

$$\left[d(AB)^2\right]_1 = \frac{1}{5}\left(B_{\varkappa,1} - 2\tau\bar{\gamma}_{-19}B_{\varphi,1} - 2\bar{\tau}B_{\delta,1}\right),$$

$$\left[d(BC)^2\right]_1 = \frac{1}{5}\left(C_{\varkappa,1} - B_{\varkappa,1} - 2\bar{\tau}\gamma_{-19}B_{\varphi,1} - 2\tau\bar{\gamma}_{-19}C_{\varphi,1} + 2\tau B_{\delta,1} - 2\bar{\tau}C_{\delta,1}\right),$$

$$\left[d(CA)^2\right]_1 = \frac{1}{5}\left(-C_{\varkappa,1} - 2\bar{\tau}\gamma_{-19}C_{\varphi,1} + 2\tau C_{\delta,1}\right),$$

$$\left[d(DE)^2\right]_1 = \frac{1}{5}\left(D_{\varkappa,1} - E_{\varkappa,1} - 2\bar{\tau}\gamma_{-19}E_{\varphi,1} - 2\tau\bar{\gamma}_{-19}D_{\varphi,1} + 2\tau E_{\delta,1} - 2\bar{\tau}D_{\delta,1}\right),$$

$$\left[d(EF)^2\right]_1 = \frac{1}{5}\left(E_{\varkappa,1} - F_{\varkappa,1} - 2\bar{\tau}\gamma_{-19}F_{\varphi,1} - 2\tau\bar{\gamma}_{-19}E_{\varphi,1} + 2\tau F_{\delta,1} - 2\bar{\tau}E_{\delta,1}\right),$$

$$\left[d(FD)^2\right]_1 = \frac{1}{5}\left(F_{\varkappa,1} - D_{\varkappa,1} - 2\bar{\tau}\gamma_{-19}D_{\varphi,1} - 2\tau\bar{\gamma}_{-19}F_{\varphi,1} + 2\tau D_{\delta,1} - 2\bar{\tau}F_{\delta,1}\right).$$

The expressions for the differentials $[d(AF)^2]_1$, $[d(AD)^2]_1$, $[d(AB)^2]_1$ and $[d(CA)^2]_1$ look shorter but this is only because of our convention to keep the tangent line A fixed.

4.2 Linear dependence of differentials

Our 12 linear functionals are not linearly independent. To see this consider the functionals

$$\mathcal{S}_1 := \left[d(AB)^2\right]_1 + \left[d(BC)^2\right]_1 + \left[d(CA)^2\right]_1 + \left[d(DE)^2\right]_1 + \left[d(EF)^2\right]_1 + \left[d(FD)^2\right]_1 ,$$

$$\mathcal{S}_2 := \left[d(AF)^2\right]_1 + \left[d(CE)^2\right]_1 + \left[d(BD)^2\right]_1$$

and

$$\mathcal{S}_3 := \left[d(CF)^2\right]_1 + \left[d(BE)^2\right]_1 + \left[d(AD)^2\right]_1 .$$

Let also (keeping in mind that $A_{\varphi,1} = A_{\varkappa,1} = A_{\delta,1} = 0$)

$$\mathcal{F} := (A_{\varphi,1} + B_{\varphi,1} + C_{\varphi,1} + D_{\varphi,1} + E_{\varphi,1} + F_{\varphi,1})$$

and

$$\mathcal{D} := (A_{\delta,1} + B_{\delta,1} + C_{\delta,1} + D_{\delta,1} + E_{\delta,1} + F_{\delta,1}) .$$

A direct computation shows that

$$\mathcal{S}_1 = -\frac{18}{5}\mathcal{F} + \frac{2}{\sqrt{5}}\mathcal{D} ,$$

and

$$(23 + 3\sqrt{5})\mathcal{S}_2 + (23 - 3\sqrt{5})\mathcal{S}_3 = 36\mathcal{F} - 4\sqrt{5}\mathcal{D} ,$$

so the following strictly convex combination (that is, the linear combination with positive coefficients) of the differentials vanish:

$$10\mathcal{S}_1 + (23 + 3\sqrt{5})\mathcal{S}_2 + (23 - 3\sqrt{5})\mathcal{S}_3 = 0 . \tag{10}$$

This is the only relation between the differentials: a direct computation shows that the linear space, spanned by the differentials, is 11-dimensional.

Let E denote the null-space of our differentials, i.e. the linear subspace of the tangent space on which all the differentials vanish. It has the dimension $4 = 15 - 11$.

4.3 Relevant quadratic form

We have now to study our functions to the next order, t^2. We have performed the calculation of the 12 quadratic forms using Mathematica [W]. The formulas are quite lengthy and we do not reproduce the full details since it is just an intermediate result.

The only quadratic form important to us is, as we will explain in Section 5, the same combination (10) but calculated for $[d(**)^2]_2$ instead of $[d(**)^2]_1$.

This is a quadratic form in 15 variables. However, due to the results of Section 5, it is enough to calculate the restriction of this form to the null-space E of the differentials. If this restriction to E would be negatively defined, that will prove our result.

The subspace E has dimension 4. As independent variables on E we choose

$$w_1 := E_{\varkappa,1} \ , \ w_2 := E_{\varphi,1} \ , \ w_3 := B_{\delta,1} \ \text{ and } \ w_4 := C_{\delta,1} \ .$$

The resulting quadratic form on E is $\sum \Phi_{ij} w_i w_j$ where

$$\Phi = \frac{11}{9} \begin{pmatrix} -\frac{919}{24} & \frac{5663}{12\sqrt{5}} & -\frac{\bar{\mu}_1}{30} & -\frac{\mu_1}{30} \\ \frac{5663}{12\sqrt{5}} & -\frac{18663}{6} & -\frac{7\bar{\mu}_2}{15} & \frac{7\mu_2}{15} \\ -\frac{\bar{\mu}_1}{30} & -\frac{7\bar{\mu}_2}{15} & -\frac{4\bar{\mu}_3}{15} & 700 \\ -\frac{\mu_1}{30} & \frac{7\mu_2}{15} & 700 & -\frac{4\mu_3}{15} \end{pmatrix} . \tag{11}$$

Here $\mu_1 = 2865 + 1438\sqrt{5}$, $\mu_2 = 3530 + 939\sqrt{5}$, $\mu_3 = 5335 + 1878\sqrt{5}$, and bar stands for the Galois conjugation (the replacement of $\sqrt{5}$ by $-\sqrt{5}$) in the field $\mathbb{Q}[\tau]$.

A direct calculation shows that Φ is indeed negatively defined and the result follows, by applying the theorem 2, Section 5.

5 Local maximum

This section provides a sufficient condition which ensures that the point $\mathbf{0} \in \mathbb{R}^n$ is a sharp local maximum of the function

$$\mathsf{F}(x) := \min\{F_1(x), \ldots, F_m(x)\} \ , \tag{12}$$

where the functions $F_1(x), \ldots, F_m(x)$ are analytic in a neighborhood of $0 \in \mathbb{R}^n$ and $F_u(0) = 0$, $u = 1, \ldots, m$ (for the configurations of tangent lines in Theorem 1 the functions F_u are the differences between the squares of distances in the perturbed and initial configurations). This sufficient condition is needed to complete the proofs of Theorem 1.

We denote by l_{uj} and q_{ujk} the coefficients of the linear and quadratic parts of the function $F_u(x)$, $u = 1, \ldots, m$,

$$F_u(x) = l_{uj}x^j + q_{ujk}x^j x^k + o(2) \ , \tag{13}$$

where $o(2)$ stand for higher order terms. Here and till the end of the Section the summation over repeated coordinate indices is assumed.

Let ξ^j, $j = 1, \ldots, n$, be the coordinates, corresponding to the coordinate system $x^1, \ldots, x^n$, in the tangent space to $\mathbb{R}^n$ at the origin. We define the linear and quadratic forms $l_u(\xi) \equiv l_{uj}\xi^j$ and $q_u(\xi) \equiv q_{ujk}\xi^j\xi^k$ on the tangent space $T_0\mathbb{R}^n$. Let E be the subspace in $T_0\mathbb{R}^n$ defined as the intersection of kernels of the linear forms $l_u(\xi)$,

$$E = \bigcap_{u=1}^{m} \ker l_u(\xi) \ .$$

The configuration $C_6(\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}, \varkappa_{\mathfrak{m}})$ provides a particular example, see Section 4, of a family $\{F_1(x), \ldots, F_m(x)\}$ of m analytic functions in n variables, $m \leq n$, possessing the following two properties:

(A) The linear space, generated by the linear forms $l_1(\xi), \ldots, l_m(\xi)$ is $(m-1)$-dimensional; moreover, the linear relation between $l_1(\xi), \ldots, l_m(\xi)$ is strictly convex,

$$\lambda^1 l_1(\xi) + \ldots + \lambda^m l_m(\xi) = 0 \ , \tag{14}$$

with $\lambda^i > 0$, $1 \leq i \leq m-1$. Therefore,

$$\text{If } l_u(\xi) \geq 0 \text{ for all } u = 1, \ldots, m \text{ then } \xi \in E \ . \tag{15}$$

(B) The inequality

$$\left(\lambda^1 q_1(\xi) + \ldots + \lambda^m q_m(\xi)\right)\big|_{\xi\in E} \geq 0 \tag{16}$$

admits only the trivial solution $\xi = 0$.

In the setting of Section 4, $n = 15$ and $m = 12$.

For the configuration $C_6\left(\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}, \varkappa_{\mathfrak{m}}\right)$ the properties (A) and (B) hold. The only relation between the differentials is the relation (10); this is the property (A). The property (B) follows since the form (11) is negatively defined.

We note that if the linear space, spanned by the linear forms $l_u(\xi)$, $u = 1, \ldots, m$, is $(m-1)$-dimensional then the implication (15) is satisfied if and only if the unique linear dependency between the differentials is strictly convex.

Theorem 2 *Under the conditions* (A) *and* (B)*, the origin is the strict local maximum of the function* $\mathsf{F}(x)$*.*

Below we give two proofs of Theorem 2. These proofs are different in nature. We present both of them because of the importance of Theorem 2 for our conclusion about the local maximality of the configuration $C_6\left(\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}, \varkappa_{\mathfrak{m}}\right)$.

Although Theorem 2 is formulated for analytic functions, any of the two proofs show that Theorem 2 holds in fact for functions F_j of the class $\mathcal{C}^3$.

5.1 First proof of Theorem 2

Lemma 3 *The conditions* (A) *and* (B) *are invariant under an arbitrary analytic change of variables, preserving the origin,*

$$x^j = A^j_k \tilde{x}^k + A^j_{kl} \tilde{x}^k \tilde{x}^l + o(2) \ . \tag{17}$$

Here the matrix A^j_k *is non-degenerate.*

Proof. Substituting (17) into the decompositions (13) we find

$$\begin{aligned} F_u(x) &= l_{uj}\left(A^j_k \tilde{x}^k + A^j_{kl}\tilde{x}^k\tilde{x}^l\right) + q_{uij} A^i_k A^j_l \tilde{x}^k \tilde{x}^l + o(2) \\ &= \tilde{l}_{uk}\tilde{x}^k + \tilde{q}_{ukl}\tilde{x}^k\tilde{x}^l + o(2) \ , \end{aligned}$$

where

$$\tilde{l}_{uj} = l_{uj} A^j_k \quad \text{and} \quad \tilde{q}_{ukl} = l_{uj} A^j_{kl} + q_{uij} A^i_k A^j_l \ .$$

The assertion is immediate for the condition (A). As for the condition (B), it is enough to note that $\lambda^1 l_{1j} + \cdots + \lambda^m l_{mj} = 0$, $j = 1, \ldots, n$. ■

First **Proof** of Theorem 2. It follows from the condition (A) that the differentials of the functions $F_1(x), \ldots, F_{m-1}(x)$ are independent; by using the implicit function theorem we change the variables to have

$$F_1(x) = x_1 \ , \ \ldots \ , \ F_{m-1}(x) = x_{m-1} \ .$$

To make the notation lighter we use the same letters x_j instead of $\tilde{x}_j$.

We identify the tangent subspace $E \subset T_0\mathbb{R}^n$ with the plane $x_1 = \ldots = x_{m-1} = 0$.

The remaining function $F_m(x)$ is

$$F_m(x) = -\frac{1}{\lambda^m} \sum_{i=1}^{m-1} \lambda^i x_i + q(x) + o(2) \ ,$$

where $q(x)$ is a quadratic form. Let

$$\mathsf{q}(x_m, \ldots, x_n) := q(x)\big|_{x_1 = \ldots = x_{m-1} = 0}$$

be the restriction of the quadratic form $q(x)$ to E.

We consider separately two cases: (i) $x \in E$ and (ii) $x \notin E$.

(i) The property (B) refers in our situation to the quadratic form q. Due to Lemma 3, the form q is negatively defined on E. Thus there exists a small neighborhood V of the origin in E such that if $x \in V \setminus \{0\}$ then $F_m(x) < 0$, so $\mathsf{F}(x) = \min\{0, F_m(x)\} < 0$.

(ii) If U is a small enough neighborhood of the origin in $\mathbb{R}^n$ (in the chosen coordinate system the smallness depends only on the coefficients of the function $F_m(x)$) and $x \notin U \cap E$ then the property (15) implies that there exists j, $1 \le j \le m$, such that $l_j(x) < 0$. We have two subcases.

(ii)$_1$ If there is a j, $1 \le j < m$, such that $l_j(x) < 0$ then $F_j(x) = l_j(x) < 0$ hence $\mathsf{F}(x) < 0$.

(ii)$_2$ Otherwise, we have that all $x_i \geq 0$, $i = 1, \ldots, m-1$. We set $x_i = z_i^2$, $i = 1, \ldots, m-1$. In terms of the variables $z_1, \ldots, z_{m-1}, x_m, \ldots$, the function $F_m(x)$ has the form

$$F_m(z_1^2, \ldots, z_{m-1}^2, x_m, \ldots) = -\frac{1}{\lambda^m} \sum_{i=1}^{m-1} \lambda^i z_i^2 + \mathsf{q}(x_m, \ldots, x_n) + \text{higher order terms} .$$

The quadratic form $-\frac{1}{\lambda^m} \sum_{i=1}^{m-1} \lambda^i z_i^2 + \mathsf{q}(x_m, \ldots, x_n)$ is negatively defined, so the function $F_m(z_1^2, \ldots, z_{m-1}^2, x_m, \ldots)$ is strictly less than zero in a punctured neighborhood of the origin. This implies that the function $F_m(x)$ is strictly negative whenever all $x_i \geq 0$, $i = 1, \ldots, m-1$, and at least one of them is positive. Thus again $\mathsf{F}(x) < 0$. ■

Remark. It follows from the proof that the function $\mathsf{F}(x)$ decays quadratically at zero along any direction in E and decays linearly along any direction outside E.

5.2 Second proof of Theorem 2

The cornerstone of the second proof is the set

$$\mathcal{E} = \{x \in \mathbb{R}^n : F_1(x) = \ldots = F_m(x)\} \ . \tag{18}$$

We assume that all occuring real vector spaces are equipped with a Euclidean structure. For a vector v we denote by $\hat{v}$ the unit vector in the direction of the vector v.

Our proof will use the following observation.

Lemma 4 *Let $\underline{\lambda} = \{\lambda^1, \ldots, \lambda^m\}$ be a collection of m positive real numbers, $\lambda^j > 0$, $j = 1, \ldots, m$. Let $\mathcal{W}_{\underline{\lambda}}$ be the space of m-tuples $\{v_1, \ldots, v_m\}$ of vectors in $\mathbb{R}^{m-1}$, generating the space $\mathbb{R}^{m-1}$ and such that*

$$\lambda^1 v_1 + \ldots + \lambda^m v_m = 0 \ . \tag{19}$$

Then there exists a continuous positive-valued function $\delta \colon \mathcal{W}_{\underline{\lambda}} \to \mathbb{R}_{>0}$ such that for any unit vector $\mathsf{s} \in \mathbb{R}^{m-1}$ we have

$$\min_i \langle \mathsf{s}, \hat{v}_i \rangle < -\delta(v_1, \ldots, v_m) . \tag{20}$$

Proof. For an angle α, $0 \leq \alpha < \pi$, let $D_j(\alpha)$, $j = 1, \ldots, m$, denote the open spherical cap, centered at $(-\hat{v}_j)$, on the unit sphere $\mathbb{S}^{m-2}$, consisting of all the points $\mathsf{s} \in \mathbb{S}^{m-2}$ such that the angle $\measuredangle(\mathsf{s}, \hat{v}_j) > \alpha$.

For any unit vector s there exists an index i such that $\langle \mathsf{s}, v_i \rangle < 0$. Indeed, since the vectors $v_1, \ldots, v_m$ span the whole space $\mathbb{R}^{m-1}$, some of the scalar products $\langle \mathsf{s}, v_j \rangle$, $j = 1, \ldots, m$, are nonzero. Taking the scalar product of the relation (19) with the vector s we see that at least one of the scalar products $\langle \mathsf{s}, v_i \rangle$ has to be negative. Therefore

$$\bigcup_{i=1}^{m} D_i\left(\frac{\pi}{2}\right) = \mathbb{S}^{m-2} \ .$$

Thus,

$$\alpha_0(v_1, \ldots, v_m) > \frac{\pi}{2} \ ,$$

where the function $\alpha_0(v_1, \ldots, v_m)$ is defined by

$$\alpha_0(v_1, \ldots, v_m) = \sup\left\{\alpha : \bigcup_{i=1}^{m} D_i(\alpha) = \mathbb{S}^{m-2}\right\} \ .$$

Let

$$\bar{\alpha}(v_1, \ldots, v_m) := \frac{1}{2}\left[\alpha_0(v_1, \ldots, v_m) + \frac{\pi}{2}\right] \ .$$

Clearly, $\bigcup_{i=1}^{m} D_i(\bar{\alpha}) = \mathbb{S}^{m-2}$. Define the function δ by

$$\delta(v_1, \ldots, v_m) = -\cos\bar{\alpha}(v_1, \ldots, v_m) \, .$$

With this choice of the function δ the relation (20) clearly holds. The positivity and the continuity of the function δ are straightforward. ■

We return to the consideration of our analytic functions.

Lemma 5 *If the point $y \in \mathbb{R}^n$ happens to be away from the set $\mathcal{E}$, see (18), and the norm $\|y\|$ is small enough then one can find a point x on $\mathcal{E}$ such that $\mathsf{F}(y) < \mathsf{F}(x)$.*

Moreover, there exists a constant c such that for $y \notin \mathcal{E}$, and $x = x(y) \in \mathcal{E}$ being the point in $\mathcal{E}$ closest to y we have

$$\mathsf{F}(y) < \mathsf{F}(x) - c\,\|x - y\| \, , \tag{21}$$

provided, again, that the norm $\|y\|$ is small enough.

Proof. Since there is only one linear dependency between the differentials $l_1, \ldots, l_m$ of the functions $F_1(x), \ldots, F_m(x)$, the set $\mathcal{E}$ is a smooth manifold in a vicinity of the origin, of dimension $n - m + 1$.

We introduce the tubular neighborhood $U_r(\mathcal{E})$ of the manifold $\mathcal{E}$, which is comprised by all points y of $\mathbb{R}^n$ which can be represented as (x, s_x), where $x \in \mathcal{E}$ and s_x is a vector normal to $\mathcal{E}$ at x, with norm less than r. Let $\mathcal{E}_{r'} \subset \mathcal{E}$ be the neighborhood of the origin in $\mathcal{E}$, comprised by all $x \in \mathcal{E}$ with norm $\|x\| < r'$, and $U_r(\mathcal{E}_{r'})$ be the part of $U_r(\mathcal{E})$ formed by points hanging over $\mathcal{E}_{r'}$. If both r and r' are small enough then every $y \in U_r(\mathcal{E}_{r'})$ can be written as (x, s_x) with $x \in \mathcal{E}_{r'}$ in a unique way. Note that x is the point on $\mathcal{E}$ closest to y. Also, for any $r, r' > 0$ the set $U_r(\mathcal{E}_{r'})$ evidently contains an open neighborhood of the origin.

Now we are going to show that if $y = (x, \mathsf{s}_x) \in U_r(\mathcal{E}_{r'})$, $\mathsf{s}_x \neq 0$, and both r and r' are small enough then $\mathsf{F}(y) < \mathsf{F}(x)$. To this end, let N_x be the plane normal to $\mathcal{E}$ at x (so that $\mathsf{s}_x \in N_x$). We identify N_x with the linear space $\mathbb{R}^{m-1}$, so that x corresponds to $0 \in \mathbb{R}^{m-1}$.

Now we will use Lemma 4, applied not to a single space, but to the whole collection of the $(m-1)$-dimensional spaces N_x, $x \in \mathcal{E}_{r'}$. To do this, we equip each N_x with m vectors $v_1^x, \ldots, v_m^x \in N_x$, which generate N_x and which satisfy the same convex linear relation. All this data is readily supplied by the linear functionals $l_1, \ldots, l_m$, restricted to N_x. Indeed, each restricted functional $l_j^x \equiv l_j|_{N_x}$ can be uniquely written as $l_j^x(*) = \langle *, v_j^x \rangle$, with $v_j^x \in N_x$. Here the scalar product on N_x is the one restricted from $\mathbb{R}^n$. Clearly, for every x we have

$$\lambda^1 v_1^x + \ldots + \lambda^m v_m^x = 0 \ ,$$

since for every vector $\mathsf{s} \in N_x$ we have $\lambda^1 l_1(\mathsf{s}) + \ldots + \lambda^m l_m(\mathsf{s}) = 0$ (as for any other vector). Moreover, $l_j(\mathsf{s}) < 0$ for some $j = j(\mathsf{s})$, $1 \leq j \leq m$, see formula (15) or the proof of Lemma 4.

Since the space $N_{x=0}$ is orthogonal to the null-space E, the m vectors $v_1^0, \ldots, v_m^0$ do generate N_0. Because the spaces N_x depend on x continuously, all of them are transversal to E, provided r' is small. Thus, the vectors $v_1^x, \ldots, v_m^x$ do generate the spaces N_x for all $x \in \mathcal{E}_{r'}$, provided again that r' is small enough. Lemma 4 provides us now with a collection of functions δ^x on the spaces $\mathcal{W}_{\underline{\lambda}}^x$ of m-tuples of vectors from N_x. It follows from the continuity, in x, of the spaces N_x and the m-tuples $\{v_1^x, \ldots, v_m^x\}$, and from the Lemma 4 that the functions δ^x can be chosen in such a way that the

resulting positive function $\Delta(x) := \delta^x(v_1^x, \ldots, v_m^x)$ on $\mathcal{E}_{r'}$ is continuous in x and also is uniformly positive, that is,

$$\Delta(x) > 2c \text{ for all } x \in \mathcal{E}_{r'} \ ,$$

for some $c > 0$, provided r' is small enough.

In virtue of Lemma 4, for every $x \in \mathcal{E}_{r'}$ and each vector $\mathsf{s} \in N_x$ there exists an index $j(\mathsf{s})$ for which the value $l_{j(\mathsf{s})}(\mathsf{s})$ of the functional $l_{j(\mathsf{s})}$ is not only negative but moreover satisfies

$$l_{j(\mathsf{s})}(\mathsf{s}) < -2c\|\mathsf{s}\| \ . \tag{22}$$

Hence for $y = (x, \mathsf{s}_x) \in U_r(\mathcal{E}_{r'})$ we have

$$F_{j(\mathsf{s}_x)}(y) < F_{j(\mathsf{s}_x)}(x) - c\|\mathsf{s}_x\| \tag{23}$$

provided both r and r' are small. Therefore

$$\min_j \{F_j(y)\} \leq F_{j(\mathsf{s}_x)}(y) < F_{j(\mathsf{s}_x)}(x) - c\|\mathsf{s}_x\| = \min_j \{F_j(x)\} - c\|\mathsf{s}_x\| \ ,$$

where the last equality holds since $F_1(x) = \ldots = F_m(x)$, so we are done. ■

Theorem 2 is a straightforward consequence of the next Proposition.

Proposition 6 *The point $x = 0$ is a sharp local maximum of the function* F if *the form*

$$\sum_{u=1}^{m} \lambda^u q_u \tag{24}$$

is negative definite on E.

In the special case when all the functions $F_u(x)$, $u = 1, \ldots, m$, are linear-quadratic, F_u are sums of linear and quadratic forms,

$$F_u(x) = l_{uj}x^j + q_{ujk}x^j x^k \ , \tag{25}$$

the if *statement becomes the* iff *statement.*

Proof. In view of Lemma 5 we can restrict our search of the maximum of the function F to the submanifold $\mathcal{E}$.

Note that the plane E is the tangent plane to $\mathcal{E}$ at the point $0 \in \mathcal{E}$, so the coordinate projection of $\mathcal{E}$ to E introduces the local coordinates on $\mathcal{E}$ in a vicinity of 0. As a result, $\mathcal{E}$ can be viewed as a graph of a function Z on E, $Z(\mathbf{x}) \in \mathbb{R}^{m-1}$:

$$\mathcal{E} = \{\mathbf{x}, \mathbf{z} : \mathbf{x} \in E, \mathbf{z} = (z_1(\mathbf{x}), \ldots, z_{m-1}(\mathbf{x}))\}.$$

This is an instance of the implicit function theorem. The point $\mathbf{x} = 0$ is a critical point of all the functions $z_l(\mathbf{x})$.

Denote by M the restriction of any of the functions F_i to $\mathcal{E}$. Clearly, it is a smooth function, and the differential dM vanishes at $0 \in \mathcal{E}$. So our proposition would follow once we check that the second quadratic form of M at 0 is twice the form (24). To see that, let us compute the derivative $\frac{d^2M}{dx_1^2}$ at the origin; the computation of other second derivatives repeats this computation. We have

$$\frac{d}{dx_1} M(\mathbf{x}, \mathbf{z}(\mathbf{x})) = \left(\frac{\partial}{\partial x_1} M\right)(\mathbf{x}, \mathbf{z}(\mathbf{x}))$$
$$+ \left(\frac{\partial}{\partial x_{n-m+2}} M\right)(\mathbf{x}, \mathbf{z}(\mathbf{x})) \frac{\partial}{\partial x_1} z_1(\mathbf{x}) + \ldots + \left(\frac{\partial}{\partial x_n} M\right)(\mathbf{x}, \mathbf{z}(\mathbf{x})) \frac{\partial}{\partial x_1} z_{m-1}(\mathbf{x}),$$

and then

$$\frac{d^2}{dx_1^2} M(\mathbf{x}, \mathbf{z}(\mathbf{x}))|_{\mathbf{x}=0} = 2[q_1]_{1,1} + [l_1]_1 \cdot 0 \text{ (since all } \frac{\partial}{\partial x_1} z_l(\mathbf{0}) = 0)$$
$$+ [l_1]_{n-m+2} \cdot \frac{\partial^2}{\partial x_1^2} z_1(\mathbf{0}) + \ldots + [l_1]_n \cdot \frac{\partial^2}{\partial x_1^2} z_{m-1}(\mathbf{0}).$$

Let us introduce the vector

$$\Delta = \left(0, \ldots, \frac{\partial^2}{\partial x_1^2} z_1(\mathbf{0}), \ldots, \frac{\partial^2}{\partial x_1^2} z_{m-1}(\mathbf{0})\right).$$

Then we have

$$\frac{d^2}{dx_1^2} M_1(\mathbf{x}, \mathbf{z}(\mathbf{x}))|_{\mathbf{x}=0} = 2[q_1]_{1,1} + l_1(\Delta).$$

Since we have $m-1$ identities

$$M_1(\mathbf{x}, \mathbf{z}(\mathbf{x})) = M_2(\mathbf{x}, \mathbf{z}(\mathbf{x})) = M_m(\mathbf{x}, \mathbf{z}(\mathbf{x})),$$

we can write also

$$\frac{d^2}{dx_1^2} M\left(\mathbf{x}, \mathbf{z}\left(\mathbf{x}\right)\right)|_{\mathbf{x}=0} = 2\left[q_l\right]_{1,1} + l_l\left(\Delta\right), \ l = 2, \ldots, m.$$

By (14) we then have

$$\frac{d^2}{dx_1^2} M\left(\mathbf{x}, \mathbf{z}\left(\mathbf{x}\right)\right)|_{\mathbf{x}=0} = 2\left(\sum_l \lambda^l \left[q_l\right]_{1,1}\right),$$

so our claim follows. ■

5.2.1 Concluding remark

We stress that the space $\mathcal{E}$, on which all the functions $F_u\left(x\right)$, $u = 1, \ldots, m$, are equal, is a very natural object in the study of a local maximum of the function $\mathsf{F}(x)$, see (12). As a supporting evidence we provide a simple proof of a weakened form of Lemma 5.

Lemma 7 *Assume that any $m-1$ differentials among $dF_j\left(0\right)$, $u = 1, \ldots, m$, are linearly independent. If the equalities $F_1\left(y\right) = \ldots = F_m\left(y\right)$ are not satisfied at a point $y \in \mathbb{R}^n$ with small enough $\|y\|$ then y cannot be a local maximum of the function F.*

Proof. Suppose that for some r, $1 \leq r < m$, we have

$$F_1\left(y\right) = \ldots = F_r\left(y\right) < F_{r+1}\left(y\right) \leq \ldots \leq F_m\left(y\right) \ .$$

Since the linear functionals $dF_1, \ldots, dF_r$ are independent, there exists a vector v such that all the values $dF_i\left(v\right)$, $i = 1, \ldots, r$, are positive. Therefore all the functions $N_i\left(t\right) := F_i\left(y + tv\right)$, $i = 1, \ldots, r$, are increasing in t at $t = 0$, provided both $\|y\|$ and t are small enough. ■

In [OS], we were guided by this kind of logic in our search of the curve $\gamma(\varphi)$: for each φ the point $\gamma(\varphi) \in M^6$ is defined by the condition that the distances under consideration coincide.

However, Lemma 7 does not suffice for the proof of Proposition 6, as the following example shows. Let $n = m = 2$ and

$$F_1(x_1, x_2) = x_2 - x_1^2 \ , \ F_2(x_1, x_2) = 2x_2 - x_1^2 \ .$$

The set $\mathcal{E}$ is the x_1-axis. The differential $dF_1(0)$ and $dF_2(0)$ are linearly dependent but the dependency is not convex. The restriction of any of functions F_u on $\mathcal{E}$ is the function $-x_1^2$ having the maximum at the origin. However this is not a local maximum $\mathsf{F}(x_1, x_2)$: for instance, the restriction of the function F on the x_2-axis is the monotone function

$$\begin{cases} x_2 & \text{for } x_2 > 0 \ , \\ 2x_2 & \text{for } x_2 \leq 0 \ . \end{cases}$$

For the proof of Theorem 2, we need, in the relation (23), a quantitative statement established in the claim (21) of Lemma 5.

6 Hidden symmetry

In this Section we discuss and generalize the observations from Section 4 about the algebraic nature of the Taylor coefficients for the deformations around the configuration $C_6\,(\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}, \varkappa_{\mathfrak{m}})$.

6.1 Galois symmetry

As we mentioned in the proof from Section 4, the coefficients of the differentials of squares of distances around the record point $C_6\,(\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}, \varkappa_{\mathfrak{m}})$ belong - after the strange normalization (8) - to the field $\mathbb{Q}[\tau]$. The same holds for the coefficients of all 12 quadratic forms in 15 variables. This looks miraculous. We believe that all coefficients of the Taylor decompositions of all 15 distances (not only 12 relevant distances) around the record point $C_6\,(\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}, \varkappa_{\mathfrak{m}})$ belong to the field $\mathbb{Q}[\tau]$. We have checked that it is so to some orders of t (for some distances up to t^8).

The evidence that the Galois symmetry is global – that is, that the Galois symmetry holds on the level of functions, not only their Taylor decompositions – is given in the Subsection 6.2, see Proposition 8 and especially formula (37).

We reveal now a hidden symmetry, based on the above observation, of the formulas for the coefficients of the Taylor expansions of distances around the point $C_6\,(\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}, \varkappa_{\mathfrak{m}})$.

Let $\mathcal{W}$ be a $\mathbb{Q}$-vector space with the basis

$$\{J_\upsilon\} \ \text{ where } \ J \in \{A, B, C, D, E, F\} \ \text{ and } \ \upsilon \in \{\varkappa, \varphi, \delta\} \ .$$

Let $\widetilde{\mathcal{W}} := \mathbb{Q}[\tau] \otimes_{\mathbb{Q}} \mathcal{W}$. We consider $\widetilde{\mathcal{W}}$ as a vector space over $\mathbb{Q}$. The differentials of distances (see formulas in Subsection 4.1) are naturally interpreted as elements of $\widetilde{\mathcal{W}}$. The Galois conjugation $\tau \mapsto \bar{\tau}$ turns into an involutive automorphism of the space $\widetilde{\mathcal{W}}$ which we denote by ι.

Let Π_ϖ and Π_ϱ be the operators in $\widetilde{\mathcal{W}}$ realizing the following permutations:

$$\varpi := (A, B, C)(D, E, F)$$

and

$$\varrho := (A, D)(B, F)(C, E) \ .$$

It is straightforward to see that the operators Π_ϖ and Π_ϱ preserve the differentials of distances from Subsection 4.1. The operators Π_ϖ and Π_ϱ (as well as the permutations ϖ and ϱ) generate the group $\mathbb{D}_3$.

Let Π°_ς be the operator in $\widetilde{\mathcal{W}}$ defined by

$$J_\varkappa \mapsto -\varsigma(J)_\varkappa \ , \ J_\varphi \mapsto \varsigma(J)_\varphi \ , \ J_\delta \mapsto -\varsigma(J)_\delta \ , \tag{26}$$

for the following permutation ς:

$$\varsigma := (B, C)(D, F) \ ,$$

the elements A and E are fixed by ς. Finally, let Π_ς be the composition of the Galois involution and Π°_ς,

$$\Pi_\varsigma := \iota \circ \Pi^\circ_\varsigma \ . \tag{27}$$

The direct inspection shows that the operator Π_ς preserves the differentials of distances from Subsection 4.1; it sends the differentials of the $\{AF, CE, BD\}$-triplet to the differentials of the $\{CF, BE, AD\}$-triplet and permutes the differentials of the 6-plet $\{AB, BC, CA, DE, EF, FD\}$.

Let G be the group generated by the operators Π_ϖ, Π_ϱ and Π_ς. We have

$$\Pi_\varpi = \Pi_\varrho \Pi_\varsigma \Pi_\varrho \Pi_\varsigma \ ,$$

so the group G is generated by the operators Π_ϱ and Π_ς alone. The underlying permutation of $\Pi_\varrho\Pi_\varsigma$ is

$$(A, F, C, E, B, D) \ ,$$

so

$$(\Pi_\varrho\Pi_\varsigma)^6 = \mathrm{id} \ .$$

Thus, the group G is the dihedral group $\mathbb{D}_6$. Note that the underlying permutations of tangent lines form exactly the the symmetry group $\mathbb{D}_6$ of the initial configuration C_6.

We have checked that the operator Π_ς preserves the second differentials as well and we believe that it is so for all orders of the Taylor expansion. In the same way as for the first differentials, the operator Π_ς sends the second differentials of the $\{AF, CE, BD\}$-triplet to the second differentials of the $\{CF, BE, AD\}$-triplet and permutes the second differentials of the 6-plet $\{AB, BC, CA, DE, EF, FD\}$. To convince the reader we present three formulas for the second differentials, one differential for each of the triplets $\{AF, CE, BD\}$ and $\{CF, BE, AD\}$ and one differential for the 6-plet $\{AB, BC, CA, DE, EF, FD\}$:

$$\begin{aligned}
\left[d(BD)^2\right]_2 &= \frac{2597-1017\sqrt{5}}{127776}(B_{\varkappa,1}+D_{\varkappa,1})^2 - \frac{651+236\sqrt{5}}{792}(B_{\delta,1}^2+D_{\delta,1}^2) \\
&+\frac{-97+60\sqrt{5}}{528}(B_{\varphi,1}^2+D_{\varphi,1}^2) + \frac{265-3\sqrt{5}}{132}B_{\varphi,1}D_{\varphi,1} + \frac{219+124\sqrt{5}}{198}B_{\delta,1}D_{\delta,1} \\
&+\frac{29-109\sqrt{5}}{2904}(B_{\varkappa,1}+D_{\varkappa,1})(B_{\delta,1}+D_{\delta,1}) + \frac{5-48\sqrt{5}}{132}(B_{\varphi,1}B_{\delta,1}+D_{\varphi,1}D_{\delta,1}) \\
&+\frac{181+29\sqrt{5}}{132}(B_{\delta,1}D_{\varphi,1}+B_{\varphi,1}D_{\delta,1}) + \frac{90-17\sqrt{5}}{726}(B_{\varkappa,1}+D_{\varkappa,1})(B_{\varphi,1}+D_{\varphi,1})
\end{aligned}$$

for the $\{AF, CE, BD\}$-triplet,

$$\begin{aligned}
\left[d(CF)^2\right]_2 &= \frac{2597+1017\sqrt{5}}{127776}(C_{\varkappa,1}+F_{\varkappa,1})^2 - \frac{651-236\sqrt{5}}{792}(C_{\delta,1}^2+F_{\delta,1}^2) \\
&-\frac{97+60\sqrt{5}}{528}(C_{\varphi,1}^2+F_{\varphi,1}^2) + \frac{265+3\sqrt{5}}{132}C_{\varphi,1}F_{\varphi,1} + \frac{219-124\sqrt{5}}{198}C_{\delta,1}F_{\delta,1} \\
&+\frac{29+109\sqrt{5}}{2904}(C_{\varkappa,1}+F_{\varkappa,1})(C_{\delta,1}+F_{\delta,1}) + \frac{5+48\sqrt{5}}{132}(C_{\varphi,1}B_{\delta,1}+F_{\varphi,1}F_{\delta,1})
\end{aligned}$$

$$+\frac{181-29\sqrt{5}}{132}(C_{\delta,1}F_{\varphi,1}+C_{\varphi,1}F_{\delta,1})+\frac{90+17\sqrt{5}}{726}(C_{\varkappa,1}+F_{\varkappa,1})(C_{\varphi,1}+F_{\varphi,1})$$

for the $\{CF, BE, AD\}$-triplet and

$$\frac{150}{11}\left[d(BC)^2\right]_2=\frac{1}{32}(B_{\varkappa,1}-C_{\varkappa,1})^2+\frac{133+9\sqrt{5}}{16}B_{\varphi,1}^2+\frac{133-9\sqrt{5}}{16}C_{\varphi,1}^2$$

$$+\frac{27-2\sqrt{5}}{16}B_{\varphi,1}(B_{\varkappa,1}-C_{\varkappa,1})+\frac{27+2\sqrt{5}}{16}C_{\varphi,1}(B_{\varkappa,1}-C_{\varkappa,1})+\frac{109}{4}B_{\varphi,1}C_{\varphi,1}$$

$$+\frac{19+5\sqrt{5}}{4}B_{\delta,1}C_{\varphi,1}-\frac{19-5\sqrt{5}}{4}B_{\varphi,1}C_{\delta,1}-\frac{53}{3}B_{\delta,1}C_{\delta,1}$$

$$-\frac{103+39\sqrt{5}}{24}B_{\delta,1}^2-\frac{103-39\sqrt{5}}{24}C_{\delta,1}^2-\frac{43+4\sqrt{5}}{2}B_{\varphi,1}B_{\delta,1}+\frac{43-4\sqrt{5}}{2}C_{\varphi,1}C_{\delta,1}$$

$$+\frac{7\bar{\tau}}{4}B_{\delta,1}(B_{\varkappa,1}-C_{\varkappa,1})-\frac{7\tau}{4}C_{\delta,1}(B_{\varkappa,1}-C_{\varkappa,1})$$

for the 6-plet.

We believe that this action of the group $\mathbb{D}_6$ extends to all orders of the Taylor decompositions of the distances.

Remark. We were not discussing the remaining $\{AE, BF, CD\}$-triplet because it was not relevant for the proof. Still, the same phenomenon holds for the Taylor coefficients of the squares of the distances in this triplet. As an illustration we present the first and second differentials for the squares of the distances between the lines B and F:

$$\frac{169}{6}\left[d(BF)^2\right]_1=2\sqrt{5}(B_{\varkappa,1}+F_{\varkappa,1})+5(B_{\varphi,1}+F_{\varphi,1})-\sqrt{5}(B_{\delta,1}+F_{\delta,1})\ ,$$

and

$$-\frac{105456}{11}\left[d(BF)^2\right]_2=209(B_{\varkappa,1}+F_{\varkappa,1})^2+560\sqrt{5}(B_{\varkappa,1}+F_{\varkappa,1})(B_{\varphi,1}+F_{\varphi,1})$$

$$+57445(B_{\varphi,1}+F_{\varphi,1})^2-115600B_{\varphi,1}F_{\varphi,1}-404(B_{\varkappa,1}+F_{\varkappa,1})(B_{\delta,1}+F_{\delta,1})$$

$$+5492\sqrt{5}(B_{\varphi,1}B_{\delta,1}+F_{\varphi,1}F_{\delta,1})-6208\sqrt{5}(B_{\varphi,1}F_{\delta,1}+B_{\delta,1}F_{\varphi,1})$$

$$+1466(B_{\delta,1}^2+F_{\delta,1}^2)-968B_{\delta,1}F_{\delta,1}\ .$$

6.2 Discussion

Puzzled by the hidden Galois symmetry, described in Subsection 6.1, we performed several experiments wondering whether this hidden symmetry is specific for the record point $C_6\,(\varphi_{\mathfrak{m}}, \delta_{\mathfrak{m}}, \varkappa_{\mathfrak{m}})$ or it is inherent at some other points of the curve $\gamma(\varphi)$, see formula (7), Section 3. We computationally clarify this question in the present section. Namely we explain the geometric origin of the symmetry and show that it becomes the Galois symmetry for 'rational' points of the curve $\gamma(\varphi)$.

We recall some information about the curve $\gamma(\varphi)$ [OS].

The curve $\gamma(\varphi)$ is related to a part Γ of the plane algebraic curve $\Psi = 0$ where

$$\Psi = 4S^2 - 8T^2 - 3S^4 + 29S^2T^2 - 4T^4 - 22S^4T^2 + 14S^2T^4 + 4S^6T^2 - 7S^4T^4 + S^2T^6\ ,$$

see Fig. 7.

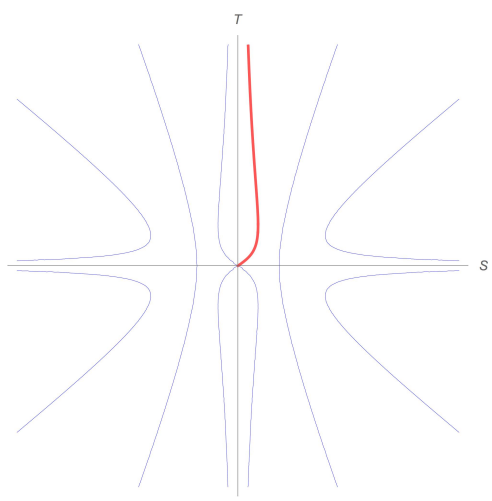

Figure 7: Curve $\Psi = 0$, the part Γ is depicted in red

The curve Γ admits the following parameterization:

$$S(x) = 2\sqrt{\frac{(1-x)x(1+x)}{1+7x+4x^2}}\ , \tag{28}$$

$$T(x) = \sqrt{\frac{(1-x)(1+3x)}{x(1+7x+4x^2)}}\ , \tag{29}$$

where x ranges from 1 to 0.

The parameterization of the curve $\gamma(\varphi)$ is given by

$$S(x) \equiv \sin\bigl(\varphi(x)\bigr) \ , \ T(x) \equiv \tan\bigl(\delta(x)\bigr) \ , \tag{30}$$

and

$$\tan\bigl(\varkappa(x)\bigr) = \frac{x-1}{\sqrt{(1+x)(1+3x)}} \ . \tag{31}$$

For brevity, we denote by $C_{6,x}$ the configuration $C_6\bigl(\varphi(x), \delta(x), \varkappa(x)\bigr)$ of six tangent lines. The squares of the relevant twelve distances between the lines of the configuration $C_{6,x}$ are all equal to

$$\frac{12x}{1+7x+4x^2} \ .$$

In our experiments we were fixing various rational values of the parameter x, then making a general (involving all 15 parameters) perturbation of the configuration $C_{6,x}$ and studying the nature and the structure of the Taylor coefficients of the squares of distances in a vicinity of $C_{6,x}$.

For example, at $x = 1/3$, the Taylor coefficients, after an appropriate normalization, belong to $\mathbb{Q}[\sqrt{2}]$ and the Galois automorphism of $\mathbb{Q}[\sqrt{2}]$ restores the $\mathbb{D}_6$ symmetry, as in Section 6.1. However, at $x = 1/5$, the Taylor coefficients, after an appropriate normalization, belong to $\mathbb{Q}$.

To summarize the results of our study, let

$$\mathfrak{p}_x = \sqrt{\frac{(1+x)(1+3x)}{3}} \ . \tag{32}$$

As in Section 4 the perturbed position of a line $J \in \{A, B, C, D, E, F\}$ in the configuration $C_{6,x}$ is

$$J = J(\varkappa(x) + \Delta_J^{\varkappa}, \varphi(x) + \Delta_J^{\varphi}, \delta(x) + \Delta_J^{\delta})$$

and

$$\begin{aligned} \Delta_J^{\varkappa} &= \vartheta_{\varkappa} \cdot \bigl(J_{\varkappa,1}t + J_{\varkappa,2}t^2 + o(t^2)\bigr) \ , \\ \Delta_J^{\varphi} &= \vartheta_{\varphi} \cdot \bigl(J_{\varphi,1}t + J_{\varphi,2}t^2 + o(t^2)\bigr) \ , \\ \Delta_J^{\delta} &= \vartheta_{\delta} \cdot \bigl(J_{\delta,1}t + J_{\delta,2}t^2 + o(t^2)\bigr) \ . \end{aligned} \tag{33}$$

Here $\vartheta_\varkappa, \vartheta_\varphi$ and ϑ_δ are normalization constants.

As before, to fix the rotational symmetry we keep the tangent line A at its place, that is, $A_{\varphi,j} = A_{\varkappa,j} = A_{\delta,j} = 0$, $j = 1, 2, \ldots$.

Proposition 8 *Let x be a rational number between 0 and 1 such that $\mathfrak{p}_x$ is not rational.*

(i) *There exists a choice of the normalization constants $\vartheta_\varkappa, \vartheta_\varphi$ and ϑ_δ such that the Taylor coefficients of the squares of distances belong to $\mathbb{Q}[\mathfrak{p}_x]$.*

(ii) *The operator Π_ς, defined by the formula (27), where ι is the Galois conjugation $\mathfrak{p}_x \to -\mathfrak{p}_x$ of $\mathbb{Q}[\mathfrak{p}_x]$, restores the $\mathbb{D}_6$ symmetry.*

Note. For $x = 1/5$ we find that $\mathfrak{p}_x$ is rational, $\mathfrak{p}_x = 4/5$. The equation $\mathfrak{p}_x = \mathfrak{r}$, $x \in \mathbb{Q}$ and $\mathfrak{r} \in \mathbb{Q}$, can be easily solved: substituting $x = (y-2)/3$ we find $y^2 = 9\mathfrak{r}^2 + 1$, so the question reduces to Pythagorean triples.

The irrationality $\mathfrak{p}_x$ can be found among the geometric objects in the configuration $C_{6,x}$. This irrationality is related to the cosine γ_x of the angle between the tangent lines A and D. We calculate:

$$\gamma_x = -\frac{3(1-x)}{2(1+2x)}\,\mathfrak{p}_x + \frac{x(1+5x)}{2(1+2x)}\ .$$

In particular, under the conditions of Proposition 8, $\mathbb{Q}[\mathfrak{p}_x] = \mathbb{Q}[\gamma_x]$ and we can reformulate the part (i) of Proposition 8: the coefficients of the Taylor expansions belong to the field $\mathbb{Q}[\gamma_x]$.

The cosine of the angle between the tangent lines A and F is equal to $\overline{\gamma}_x$, the Galois conjugate of γ_x in the field $\mathbb{Q}[\mathfrak{p}_x]$,

$$\overline{\gamma}_x = \frac{3(1-x)}{2(1+2x)}\,\mathfrak{p}_x + \frac{x(1+5x)}{2(1+2x)}\ .$$

Sketch of the **Proof** of Proposition 8. We do not furnish the complete details in order not to overload the presentation. Namely, we will work with only two distances, $d(AD)$ and $d(AF)$. Since we keep the line A fixed (which makes the formulas more tangible), the distance $d(AD)$ depends, for a given x, on the three parameters, characterizing the position of the perturbed line D. We consider only a simplified situation, namely we perturb only the angle[3]

[3] Due to the character of the formulas, the angle δ is the most manageable of the three angles. We will write the formula for the distance between the line A and the perturbed line D 'im großen', without decomposing in the Taylor series.

δ of the line D and follow the dependence of the distance on this variation. Thus, for a given x, we consider the function $d\big(A, D(\xi)\big)$ where

$$A = A(\varkappa(x), \varphi(x), \delta(x)) \;\text{ and }\; D(\xi) = D(\varkappa(x), \varphi(x), \delta(x) + \xi) \ .$$

Since the tangent function has rational Taylor coefficients, we are allowed to pass to the variable

$$\Xi = \tan(\xi)$$

instead of ξ.

In the notation of Section 2, let $u = \big((\varphi_1, \varkappa_1), \uparrow_{\delta_1}\big)$ and $v = \big((\varphi_2, \varkappa_2), \uparrow_{\delta_2}\big)$ be two lines in M. The formula (2) for the square of the distance between lines u and v has the following explicit form in the coordinates $(\varphi, \varkappa, \delta)$

$$d_{uv}^2 = \frac{\mathcal{N}^2}{\big(1 + \tan^2(\delta_1)\big)\big(1 + \tan^2(\delta_2)\big) - \mathcal{D}^2} \ , \tag{34}$$

where

$$\begin{aligned}\mathcal{N} := &\big(\tan(\delta_1) + \tan(\delta_2)\big)\Big(\cos(\varphi_1)\cos(\varphi_2) - \cos(\Delta\varkappa)\big(1 - \sin(\varphi_1)\sin(\varphi_2)\big)\Big) \\ &- \big(1 - \tan(\delta_1)\tan(\delta_2)\big)\sin(\Delta\varkappa)\big(\sin(\varphi_1) - \sin(\varphi_2)\big) \ ,\end{aligned}$$

$$\begin{aligned}\mathcal{D} := &\cos(\varphi_1)\cos(\varphi_2) + \cos(\Delta\varkappa)\big(\sin(\varphi_1)\sin(\varphi_2) + \tan(\delta_1)\tan(\delta_2)\big) \\ &+ \sin(\Delta\varkappa)\big(\tan(\delta_2)\sin(\varphi_1) - \tan(\delta_1)\sin(\varphi_2)\big)\end{aligned}$$

and

$$\Delta\varkappa = \varkappa_1 - \varkappa_2 \ .$$

Assertion (i)**.** Let

$$\Xi_1 = \frac{\mathfrak{q}_x}{\mathfrak{p}_x^2}\,\Xi \ , \tag{35}$$

where

$$\mathfrak{q}_x = \sqrt{\frac{1 + x}{3x(1 - x)(1 + 7x + 4x^2)}} \ . \tag{36}$$

The formula (34) for the lines A and $D(\xi)$ gives, after numerous simplifications:

$$d\big(A, D(\xi)\big)^2 = \frac{x(1-x)(1+3x)}{1+7x+4x^2} \cdot \frac{\mathfrak{n}_{AD}^2}{\mathfrak{s} - \mathfrak{t}_{AD}^2} , \tag{37}$$

where

$$\mathfrak{s} = 1 + \Xi^2 = x(1-x)(1+3x)(1+7x+4x^2)\,\Xi_1^2 ,$$

and

$$\mathfrak{n}_{AD} = \frac{3}{x}\left(\frac{2\mathfrak{p}_x\gamma_x}{1+3x} + \frac{1}{1+2x}\right) + \big((1+2x)^2(\gamma_x - 1) + 6x\gamma_x\big)\,\mathfrak{p}_x\Xi_1 ,$$

$$\mathfrak{t}_{AD} = \gamma_x - 3(1-x)\left(\mathfrak{p}_x\gamma_x + \frac{1+3x}{2(1+2x)}\right)\mathfrak{p}_x\Xi_1 .$$

The formula (37) establishes the part (i). Indeed, the formula (35) gives the needed renormalization such that the final expression (37) is a rational function in Ξ_1 with coefficients in $\mathbb{Q}[\mathfrak{p}_x]$.

Assertion (ii)**.** A parallel computation, now for the tangent lines A and $F(\xi) = F(\varkappa(x), \varphi(x), \delta(x) + \xi)$, yields

$$d\big(A, F(\xi)\big)^2 = \frac{x(1-x)(1+3x)}{1+7x+4x^2} \cdot \frac{\mathfrak{n}_{AF}^2}{\mathfrak{s} - 3(1-x^2)\,\mathfrak{t}_{AF}^2} ,$$

where

$$\mathfrak{n}_{AF} = \frac{3}{x}\left(-\frac{2\mathfrak{p}_x\overline{\gamma}_x}{1+3x} + \frac{1}{1+2x}\right) - \big((1+2x)^2(\overline{\gamma}_x - 1) + 6x\overline{\gamma}_x\big)\,\mathfrak{p}_x\Xi_1 ,$$

$$\mathfrak{t}_{AF} = \overline{\gamma}_x + 3(1-x)\left(-\mathfrak{p}_x\overline{\gamma}_x + \frac{1+3x}{2(1+2x)}\right)\mathfrak{p}_x\Xi_1 .$$

A direct comparison shows that the expressions for $d\big(A, F(\xi)\big)^2$ are obtained from the expressions for $d\big(A, D(\xi)\big)^2$ by the simultaneous change of sign of Ξ and $\mathfrak{p}_x$, in the full accordance with the formula (26), so the Galois action of the operator Π_ς, see (27), restores the $\mathbb{D}_6$ symmetry.

In the general situation, when all three parameters, $\varkappa, \varphi$ and δ of the tangent lines D and F are perturbed, the closed formulas for the squares of distances are considerably lengthier (and as non-illustrative as the formula (37)) and we do not present them. ■

Two phenomena exhibited in Proposition 8 – (i) all irrationalities, except one, are absorbed in the normalization factors; (ii) the Galois conjugation of the remaining irrationality restores the $\mathbb{D}_6$ symmetry – shows a certain consonance between the curve Γ and the ingredients of the formula for the distance between skew lines. In the process of calculations it was important that for a rational x all angles $\varphi(x)$, $\varkappa(x)$ and $\delta(x)$ are purely geodetic in the sense of [CRS], that is, squares of trigonometric functions of these angles are rational.

We conclude by two remarks.

Remark 1. The Galois symmetry can be extended to all values of x if one works with the functional fields. We demonstrate it in the same simplified situation, as in the proof of Proposition 8, where we perturb only the angle δ of the lines D and F. Let $\mathbb{F} := \mathbb{Q}(x)$ be the field of rational functions in one variable. We consider its biquadratic extension $\mathbb{F}[\mathfrak{q}_x, \mathfrak{p}_x]$, where $\mathfrak{p}_x$ is defined by the formula (32) and $\mathfrak{q}_x$ – by the formula (36). The Galois group of the extension $\mathbb{F}[\mathfrak{q}_x, \mathfrak{p}_x]$ of the field $\mathbb{F}$ is the Klein four-group $C_2 \times C_2$ generated by the sign changes of $\mathfrak{q}_x$ and $\mathfrak{p}_x$. The Galois involution $\iota\colon \mathfrak{p}_x \to -\mathfrak{p}_x$ is continuous in the natural sense; it allows to define the involution Π_ς which restores the $\mathbb{D}_6$ symmetry. Thus, performing the involution Π_ς on the level of the functional fields and only then specializing the value of x allows to see a shadow of the continuous extension of the involution Π_ς to all points of the curve Γ.

Remark 2. We have added this comment because of some questions raised during our talks on the subject. The parameterization (28)-(29) serve only the part Γ of the curve $\Psi = 0$. What can be said about other components?

It is easier to work with the curve $\psi = 0$, where

$$\psi = 4s - 8t - 3s^2 + 29st - 4t^2 - 22s^2t + 14st^2 + 4s^3t - 7s^2t^2 + st^3 \ ,$$

since the polynomial Ψ depends only on $s = S^2$ and $t = T^2$. The real components of the curve $\psi = 0$ are shown on Fig. 8.

The parameterization (28)-(29) becomes

$$s(x) = \frac{4(1-x)x(1+x)}{1+7x+4x^2} \quad , \quad t(x) = \frac{(1-x)(1+3x)}{x(1+7x+4x^2)} \ .$$

The denominators of the rational functions $s(x)$ and $t(x)$ are singular at $x = 0$ and the roots $(-7-\sqrt{33})/8 \approx -1.5931$ and $(-7+\sqrt{33})/8 \approx -0.1569$

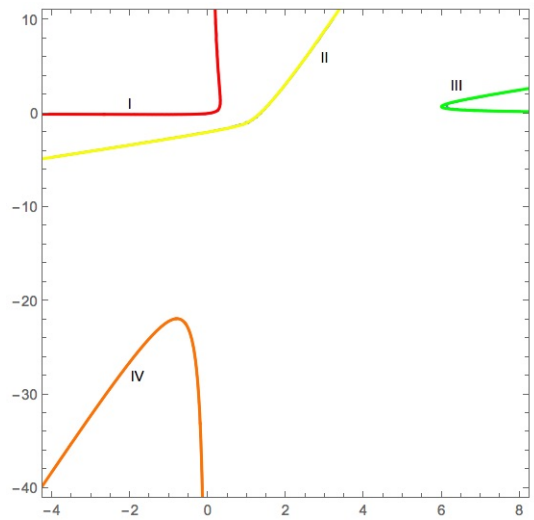

Figure 8: Real components of the curve $\psi = 0$

of the polynomial $1 + 7x + 4x^2$. For x ranging from $-\infty$ to $(-7 - \sqrt{33})/8$ we obtain the part III (in green) on Fig. 8; for x ranging from $(-7 - \sqrt{33})/8$ to $(-7 + \sqrt{33})/8$ – the part II (in yellow) on Fig. 8; for x ranging from $(-7 + \sqrt{33})/8$ to 0 – the part IV (in orange) on Fig. 8; finally, for x ranging from 0 to $+\infty$ we obtain the part I (in red) on Fig. 8.

The only singular point of the (homogenized) curve $\psi = 0$ in the complex domain is the triple point $(s = 1, t = -1)$; it is the image of three points on the complex x-plane, namely, the point $x = -1/2$ and two other points $x = (-1 \pm i\sqrt{-7})/4$.

Acknowledgements. Part of the work of S. S. has been carried out in the framework of the Labex Archimède (ANR-11-LABX-0033) and of the A*MIDEX project (ANR-11-IDEX-0001-02), funded by the Investissements d'Avenir French Government program managed by the French National Research Agency (ANR). Part of the work of S. S. has been carried out at IITP RAS. The support of Russian Foundation for Sciences (project No. 14-50-00150) is gratefully acknowledged by S. S. The work of O. O. was supported by the Program of Competitive Growth of Kazan Federal University and by the grant RFBR 17-01-00585.

References

[CRS] J. H. Conway, C. Radin and L. Sadun, *On angles whose squared trigonometric functions are rational*, Discrete & Computational Geometry (1999) 22(3), 321-332.

[GO] B. R. Gelbaum and J. M. H. Olmsted, *Counterexamples in analysis*, Dover Publications Inc., 2003.

[Ka] Y. Kallus page, https://ykallus.github.io/demo/cyl2.html

[K] W. Kuperberg, *How many unit cylinders can touch a unit ball?* Problem 3.3, in: DIMACS Workshop on Polytopes and Convex Sets, Rutgers University, January 10, 1990.

[K2] W. Kuperberg mathoverflow page, https://mathoverflow.net/questions/156008/how-many-unit-cylinders-can-touch-a-unit-ball

[OS] O. Ogievetsky and S. Shlosman, *The six cylinders problem: $\mathbb{D}_3$-symmetry approach*; arXiv:1805.09833.

[OS-O6] O. Ogievetsky and S. Shlosman, *Extremal Cylinder Configurations II: Configuration O_6*, to appear.

[W] Wolfram Research, Inc., Mathematica, Version 11.3, Champaign, IL (2018).