



HAL
open science

Moduli of connections on smooth varieties

Tony Pantev, Bertrand Toën

► **To cite this version:**

Tony Pantev, Bertrand Toën. Moduli of connections on smooth varieties. Algebraic Geometry, 2022, 10.14231/AG-2022-009 . hal-02003691

HAL Id: hal-02003691

<https://hal.science/hal-02003691>

Submitted on 1 Feb 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Moduli of connections on smooth varieties

Bertrand Toën* and Tony Pantev

Draft, December 2018

Abstract

This paper is a companion to [Pa-To]. We study the moduli functor of flat bundles on smooth, possibly non-proper, algebraic variety X (over a field of characteristic zero). For this we introduce the notion of *formal boundary* of X , denoted by $\hat{\partial}X$, which is a formal analogue of the boundary at ∞ of the Betti topological space associated to X studied in [Pa-To]. We explain how to construct two derived moduli functors $\mathbf{Vect}^\nabla(X)$ and $\mathbf{Vect}^\nabla(\hat{\partial}X)$, of flat bundles on X and on $\hat{\partial}X$, as well as a restriction map $R : \mathbf{Vect}^\nabla(X) \rightarrow \mathbf{Vect}^\nabla(\hat{\partial}X)$.

This work contains two main results. First of all we prove that the morphism R comes equipped with a canonical shifted lagrangian structure in the sense of [PTVV]. This first result can be understood as the de Rham analogue of the existence of Poisson structures on moduli of local systems studied in [Pa-To]. As a second statement, we prove that the geometric fibers of R are representable by *quasi-algebraic spaces*, a slight weakening of the notion of algebraic spaces.

Contents

1	Preliminaries on connections, graded mixed modules and equivariant objects	3
1.1	Connections as graded mixed modules	3
1.2	Graded mixed modules and equivariant objects	5
2	The formal boundary of smooth varieties	7
2.1	Perfect complexes on the formal boundary	7
2.2	Flat perfect complexes on the formal boundary	12
2.3	De Rham cohomology of the formal boundary and compactly supported cohomology . .	15
3	Formal properties of moduli functors	19
3.1	Infinitesimal properties of \mathbf{Perf}^∇	21
3.2	Cotangent complexes	23
4	The lagrangian restriction map	24
4.1	Reminders on forms and symplectic structures	24
4.2	Orientation on the formal boundary	26

*Partially supported by ERC-2016-ADG-741501 and ANR-11-LABX-0040-CIMI within the program ANR-11-IDEX-0002-02.

5	The relative representability theorem	31
A	Perfect relative \mathcal{D}-modules	35
B	Derived quasi-algebraic spaces and Artin's representability	38

Introduction

This work is a sequel of [Pa-To] in which we studied moduli of local systems on the underlying topological space of a smooth non-proper complex algebraic variety X . One the main result of [Pa-To] asserts that this moduli is a derived Artin stack endowed with a natural shifted Poisson structure whose symplectic leaves can be studied by fixing monodromies of local systems at infinity.

In this paper we start to study the de Rham analogue of the results of [Pa-To]. The results of this work can be somehow subsumed by stating that $\mathbf{Vect}^\nabla(X)$, the derived moduli of flat bundles on a smooth variety X (over a field k of characteristic 0), carries a canonical shifted Poisson structure. However, this can only be a moral statement as $\mathbf{Vect}^\nabla(X)$ is not representable outside of the proper case and we thus have had to overcome many technical difficulties in order to state and prove this existence statement.

The key ingredient of this work is the notion of *formal boundary* of a smooth variety X , denoted by $\widehat{\partial}X$ in the sequel. The object itself $\widehat{\partial}X$ has no formal existence in its own, but it is possible to define the notions of vector bundles and flat bundles on $\widehat{\partial}X$ has categories or stacks. The non-existing object is morally the punctured formal completion of \bar{X} along D , for \bar{X} a smooth compactification of X with D the divisor at infinity. This notion already appeared in [Be-Te, Ef, He-Po-Ve] and the novelty here is the systematic study of its de Rham theory: vector bundles with connections and their de Rham complex. We construct derived stacks $\mathbf{Vect}^\nabla(X)$ and $\mathbf{Vect}^\nabla(\widehat{\partial}X)$, of flat bundles on X and $\widehat{\partial}X$, together with a restriction map $R : \mathbf{Vect}^\nabla(X) \longrightarrow \mathbf{Vect}^\nabla(\widehat{\partial}X)$. We study infinitesimal properties of these derived stacks, and show in particular that they are formally representable at any field valued points. This formal representability can be used in order to define the notion of shifted symplectic structures, as well as shifted lagrangian structure, on these derived stacks, even though they are not representable. A first main result of this work is the following theorem.

Theorem 0.0.1 *There exists a canonical shifted lagrangian structure of degree $3 - 2d$ on the restriction map*

$$R : \mathbf{Vect}^\nabla(X) \longrightarrow \mathbf{Vect}^\nabla(\widehat{\partial}X).$$

At the linear level of tangent complexes, the above theorem is an incarnation of Poincaré duality in de Rham cohomology and de Rham cohomology with compact supports. The existence of the lagrangian structure globally is itself a version of Poincaré duality relative to various derived base schemes, together with the general existence result of [To3]. Morally, theorem 0.0.1 implies the existence of a shifted Poisson structure of degree $2 - 2d$, thanks to [Me-Sa].

A second main result of this work is the following representability result. We fix a flat bundle at infinity $V_\infty \in \mathbf{Vect}^\nabla(\widehat{\partial}X)(k)$ and consider the fiber of R at V_∞ denoted by $\mathbf{Vect}_{V_\infty}^\nabla(X)$. Our original goal was to prove that $\mathbf{Vect}_{V_\infty}^\nabla(X)$ is representable by a derived Artin stack (even algebraic space if no components of X are proper) locally of finite presentation over k . Though we haven't been able to prove this last statement, we prove the following weakened version of it.

Theorem 0.0.2 *The derived stack $\mathbf{Vect}_{V_\infty}^\nabla(X)$ is a derived quasi-algebraic space locally of finite presentation in the sense of definition B.0.2.*

Derived quasi-algebraic spaces are almost algebraic spaces, they satisfy all of Artin-Lurie's criterion for representability except they might not be of locally of finite presentation as a functor. They only satisfy local presentability generically, and the result is that these derived stacks only have a smooth atlas generically (i.e. whose image is Zariski dense in some sense).

Notations and conventions: k a field of characteristic zero. We use the expression *symmetric monoidal dg-categories* to mean E_∞ -algebra object inside the symmetric monoidal ∞ -category of locally presentable dg-categories (see [To2]).

1 Preliminaries on connections, graded mixed modules and equivariant objects

In this section we have gathered some results concerning \mathcal{D}_X -modules on smooth varieties considered as modules over the de Rham algebra. The results of this part do not pretend of being original.

1.1 Connections as graded mixed modules

We remind the notion of graded mixed k -modules from [PTVV, CPTVV], whose ∞ -category is denoted by $k - \mathbf{dg}_\epsilon^{gr}$. It comes equipped with an ∞ -functor

$$|-| := \mathbb{R}\underline{Hom}(k(0), -) : k - \mathbf{dg}_\epsilon^{gr} \longrightarrow k - \mathbf{dg}$$

where $k(0)$ denotes the unit graded mixed complex pure of weight 0. Explicitly $|-|$ sends a graded mixed complex E to $\prod_i E(i)[-2i]$ endowed with the total differential, sum of the cohomological differential and the mixed structure. This ∞ -functor is lax symmetric monoidal and thus induces a corresponding ∞ -functor on algebras, modules etc.

We let $X = \mathit{Spec} A$ be an affine smooth variety over k . Let \mathcal{D}_X be the k -algebra of differential operators on X . We let $\mathbf{DR}_X = \mathit{Sym}_A(\Omega_A^1[-1])$ be the de Rham algebra of X , considered as a graded mixed cdga for its natural structure of graded algebra and for which the mixed structure is defined to be the de Rham differential (see [PTVV]). The dg-category of left \mathcal{D}_X -dgmodules where quasi-isomorphisms have been inverted will be denoted by $\mathcal{D}_{qcoh}(\mathcal{D}_X)$ (see appendix for more on dg-categories of \mathcal{D} -modules). Recall that a model for $\mathcal{D}_{qcoh}(\mathcal{D}_X)$ is the dg-category of all cofibrant \mathcal{D}_X -dgmodules. In the same way, we denote by $\mathbf{DR}_X - \mathbf{dg}_\epsilon^{gr}$ the dg-category of graded mixed \mathbf{DR}_X -dgmodules up to quasi-isomorphisms (again an explicit model is the dg-category of cofibrant graded mixed dg-modules). We have a natural dg-functor

$$\mathbf{DR} : \mathcal{D}_{qcoh}(\mathcal{D}_X) \longrightarrow \mathbf{DR}_X - \mathbf{dg}_\epsilon^{gr},$$

from dg-modules over \mathcal{D}_X to graded mixed \mathbf{DR}_X -modules. The dg-functor \mathbf{DR} is defined by sending a (cofibrant) \mathcal{D}_X -dgmodule E to its de Rham complex $\mathbf{DR}(E) := \mathbf{DR}_X \otimes_A E$. By definition, $\mathbf{DR}(E)$ is free as a graded module over \mathbf{DR}_X , and its mixed structure is induced by the connection $\nabla : E \longrightarrow \Omega_A^1 \otimes_A E$ coming from the left \mathcal{D}_X -module structure on E .

Proposition 1.1.1 *The above dg-functor*

$$\mathbf{DR} : \mathcal{D}_{qcoh}(\mathcal{D}_X) \longrightarrow \mathbf{DR}_X - \mathbf{dg}_\epsilon^{gr}$$

is fully faithful. Its essential image consists of all objects that are free as a graded dg-module (i.e. of the form $\mathbf{DR}_X \otimes_A E_0$ for some A -dg-module E_0).

Proof: To prove fully faithfulness we use the following method to compute the mapping complexes inside $\mathbf{DR}_X - \mathbf{dg}_\epsilon^{gr}$.

Let B be a graded mixed cdga and E and F be two graded mixed B -dgmodule. We assume that E and F are cofibrant as graded B -modules. Consider the complex

$$H(E, F) := \prod_{p \geq 0} \underline{Hom}_{B-\mathbf{dg}^{gr}}(E, F(p))[-p],$$

where $F(p)$ is the graded B -module defined by shifting the grading by p (so $\underline{Hom}_{B-\mathbf{dg}^{gr}}(E, F(p))$ consists of graded maps of degree p). The complex $H(E, F)$ is endowed with the total differential D , sending a family of elements $\{f_p\}_{p \geq 0}$ to

$$D(\{f_p\}) := \{\nabla_F f_p + f_{p-1} \nabla_E + d(f_{p+1})\}_{p \geq 0},$$

where ∇_E and ∇_F are the mixed structures on E and F , and d is the cohomological differential. Then, we have a natural quasi-isomorphism of complexes of k -modules $H(E, F) \sim \underline{Hom}_{B-\mathbf{dg}_\epsilon^{gr}}(E, F)$, as this can be seen by describing an explicit cofibrant model for E . Using this it is easy to see that the dg-functor \mathbf{DR} is fully faithful: for two \mathcal{D}_X -dg-modules E and F , it sends $\mathbb{R}\underline{Hom}_{\mathcal{D}_X}(E, F)$ to the de Rham complex of the \mathcal{D}_X -module $\mathbb{R}\underline{Hom}_A(E, F)$.

For the second part of the proposition, let E be a graded mixed \mathbf{DR}_X -module which is of the form $E_0 \otimes_A \mathbf{DR}_X$ as a graded module. We can write E_0 as a filtered colimit of perfect complexes of A -modules. As the dg-functor \mathbf{DR} is continuous and fully faithful, it is enough to restrict to the case where E_0 is perfect. By a cell decomposition induction we can furthermore reduce to the case where $E_0 = M$ is a projective A -module of finite rank. We thus have a graded mixed \mathbf{DR}_X -module E whose underlying graded module is quasi-isomorphic to $M \otimes_A \mathbf{DR}_X$. We can recover a \mathcal{D}_X -module structure on M simply by considering the map $M \longrightarrow M \otimes_A \Omega_A^1$ induced by the mixed structure on E . By the method mentioned above, we can construct a canonical morphism of graded mixed dg-modules $\mathbf{DR}(M) \longrightarrow E$, which by construction is a quasi-isomorphism. \square

The previous proposition extends by stackification to the case where X is a smooth scheme over k , or even a smooth DM-stack over k . It can be stated as fully faithful embedding of dg-categories

$$DR : \mathcal{D}_{qcoh}(\mathcal{D}_X) \hookrightarrow \mathbf{DR}_X - \mathbf{dg}_\epsilon^{gr},$$

where the dg-categories $\mathbf{DR}_{qcoh}(\mathcal{D}_X)$ and $\mathbf{DR}_X - \mathbf{dg}_\epsilon^{gr}$ are simply defined by descent

$$\mathcal{D}_{qcoh}(\mathcal{D}_X) := \lim_{U=Spec A \rightarrow X} \mathcal{D}_U - \mathbf{dg} \quad \mathbf{DR}_X - \mathbf{dg}_\epsilon^{gr} := \lim_{U=Spec A \rightarrow X} \mathbf{DR}_U - \mathbf{dg}_\epsilon^{gr},$$

where the limit are taken over the small etale site of X and inside the ∞ -category of presentable dg-categories (see [To2]). The essential image of the dg-functor DR consists of all graded mixed \mathbf{DR}_X -dgmodules which, as graded modules, are of the form $E \otimes_{\mathcal{O}_X} \mathbf{DR}_X$ for some quasi-coherent \mathcal{O}_X -module E .

Similarly, it is also possible to extend the statement to a relative situation. Let B be a connective cdga and X a smooth DM-stack. Let us consider on the one side $\mathcal{D}_X \otimes_k B$, as a sheaf of dg-algebras, and on the other side $\mathbf{DR}_X \otimes_k B$ as a sheaf of graded mixed B -linear cdga's (over the small etale site of X). The above full embedding extends to a fully faithful embedding of presentable dg-categories

$$DR : \mathcal{D}_{qcoh}(\mathcal{D}_{X,B}) \hookrightarrow (\mathbf{DR}_X \otimes_k B) - \mathbf{dg}_\epsilon^{gr},$$

whose essential image consists of graded mixed modules which, as graded modules, are of the form $E \otimes_{\mathcal{O}_X} \mathbf{DR}_X$ for E a quasi-coherent $\mathcal{O}_X \otimes_k B$ -dgmodule.

We finish this part by an analyzing inverse image for \mathcal{D} -modules in terms of graded mixed modules over de Rham algebras. Let us give a morphism of smooth varieties $f : X = \text{Spec } A' \rightarrow Y = \text{Spec } A$, given by a morphism of smooth k -algebras $A \rightarrow A'$. We have the usual pull-back functor of \mathcal{D} -modules

$$f^* : \mathcal{D}_{qcoh}(\mathcal{D}_Y) \xrightarrow{\sim} \mathcal{D}_{qcoh}(\mathcal{D}_X).$$

By the proposition 1.1.1 this can be seen as a dg-functor on dg-categories of graded mixed modules which are free as garded modules that we are now going to describe explicitly. This functor simply is the natural one given by base change. The morphism f induces a morphism of graded mixed cdga $\mathbf{DR}_Y \rightarrow \mathbf{DR}_X$ which, in turn, defines a base change functor on graded mixed modules. This base change is canonically equivalent to f^* when restricted to graded mixed modules which are free as in proposition 1.1.1. As a final note, the above discussion also makes sense without the affiness conditions on X and Y , as well as in the relative setting by tensoring with a connective cdga B .

1.2 Graded mixed modules and equivariant objects

We now turn to a more conceptual but equivalent description of the dg-category of \mathcal{D} -modules, as equivariant objects inside the dg-category of quasi-coherent modules on the shifted cotangent stacks. This will be useful later to reduce some statements for \mathcal{D} -modules to the case of quasi-coherent modules.

We let $\mathcal{H} := \text{aut}(B\mathbb{G}_a)$ be the group stack of autoequivalences of $B\mathbb{G}_a$. It can be explicitly described as a semi-direct product $B\mathbb{G}_a \rtimes \mathbb{G}_m$, of \mathbb{G}_m acting on $B\mathbb{G}_a$ by its natural action of weight 1 on \mathbb{G}_a . In this description, \mathbb{G}_m acts on $B\mathbb{G}_a$ by its standard action, and $B\mathbb{G}_a$ acts on itself by translations (using the fact that $B\mathbb{G}_a$ is itself a group stack).

We remind from [To2] that there is a derived stack $\mathbb{D}g^{lp} \in \mathbf{dSt}_k$ of locally presentable dg-categories with descent. We then set the following definition.

Definition 1.2.1 *An \mathcal{H} -equivariant (locally presentable) dg-category T consists of a morphism of derived stacks $T : B\mathcal{H} \rightarrow \mathbb{D}g^{lp}$. Locally presentable \mathcal{H} -equivariant dg-categories form an ∞ -category*

$$\mathbb{D}g^{lp}(B\mathcal{H}) := \mathbf{Map}(B\mathcal{H}, \mathbb{D}g^{lp}).$$

We also remind that $\mathbb{D}g^{lp}$ has a canonical extension to a derived stack of symmetric monoidal ∞ -categories, for the tensor product of locally presentable dg-categories of [To2]. This makes it possible to consider symmetric monoidal dg-categories with a compatible \mathcal{H} -action, as being a morphism $B\mathcal{H} \rightarrow E_\infty - \text{Alg}(\mathbb{D}g^{lp})$, from $B\mathcal{H}$ to the derived stack of E_∞ -algebra objects in $\mathbb{D}g^{lp}$. We leave the

details of these monoidal extensions to the reader.

Given an \mathcal{H} -equivariant dg-category T in the sense of the previous definition, we can form its direct image (see [To2]) by the natural projection $p : B\mathcal{H} \rightarrow \text{Spec } k$, which by definition is the dg-category of \mathcal{H} -equivariant objects in T

$$T^{\mathcal{H}} := p_*(T).$$

Assume now that X is a smooth DM-stack as in our previous paragraph, and B a connective cdga. We let $\mathbf{DR}_X \otimes_k B$, considered as a sheaf of graded cdga's on X , and its dg-category of (non-graded, non-mixed) dg-modules $(\mathbf{DR}_X \otimes_k B) - \mathbf{dg}$. The group \mathcal{H} acts on the commutative dg-algebra $(\mathbf{DR}_X \otimes_k B)$ in an obvious manner: the \mathbb{G}_m -action is the grading and the $B\mathbb{G}_a$ -action is the mixed structure. This is formalized by the following proposition.

Proposition 1.2.2 *Let \mathcal{H} acts trivially on the dg-category $k - \mathbf{dg}$ of complexes of k -modules. Then, there is a natural equivalence of symmetric monoidal dg-categories*

$$(k - \mathbf{dg})^{\mathcal{H}} \simeq \mathcal{D}_{qcoh}(B\mathcal{H}) \simeq k - \mathbf{dg}_\epsilon^{gr}.$$

Proof: The first equivalence is true by definition, the content of the proposition is the existence of the second equivalence. For this, we let $\pi : B\mathcal{H} \rightarrow B\mathbb{G}_m$ be the natural projection. This morphism exhibits $B\mathcal{H}$ as an affine stack over $B\mathbb{G}_m$ whose fiber is $K(\mathbb{G}_a, 2)$. In other words we have $B\mathcal{H} \simeq \text{Spec}_{B\mathbb{G}_m} A$, where $A = \pi_*(\mathcal{O})$ considered as an E_∞ -algebra in $B\mathbb{G}_m$. This algebra simply is $A = k[u]$ where u is in cohomological degree 2 and weight 1. For any affine stack $F = \text{Spec } A$, there is a symmetric monoidal ∞ -functor

$$A - Mod \rightarrow \mathcal{D}_{qcoh}(F)$$

which makes $\mathcal{D}_{qcoh}(F)$ into the left completion of the $A - Mod$ for the natural t-structure (see [Lu1]). This remains true in the relative setting over $B\mathbb{G}_m$: there is a natural symmetric monoidal ∞ -functor

$$A - Mod(\mathcal{D}_{qcoh}(B\mathbb{G}_m)) \rightarrow \mathcal{D}_{qcoh}(B\mathcal{H}),$$

which is an equivalence when restricted to objects bounded on the left for the natural t-structures on both sides. As $A = k[u]$, we have that $\mathcal{D}_{qcoh}(B\mathcal{H})$ can be identified with the left completion of the natural t-structure on the dg-category of graded $k[u]$ -dg-modules. This completion can be identified with the dg-category of graded mixed complexes via the dg-functor

$$k - \mathbf{dg}_\epsilon^{gr} \rightarrow k[u] - \mathbf{dg}^{gr},$$

sending E to the graded $k[u]$ -module whose piece of weight p is $\mathbb{R}\underline{Hom}(k(p), E)$. This dg-functor is indeed a symmetric monoidal equivalence when restricted to graded mixed complexes which are cohomologically bounded on the left. \square

Let X be an affine smooth variety over k and B a connective cdga. $\mathbf{DR}_X \otimes_k B$ is a graded mixed cdga, and thus the previous proposition can be used to exhibit $\mathbf{DR}_X \otimes_k B$ as a quasi-coherent sheaf of cdga's on the stack $B\mathcal{H}$. The dg-category $(\mathbf{DR}_X \otimes_k B) - \mathbf{dg}$ can then be seen as a natural E_∞ -algebra object in $\mathbb{D}g^{lp}(B\mathcal{H})$, or in other words as an \mathcal{H} -equivariant symmetric monoidal dg-category.

Corollary 1.2.3 *With the above notations, we have a natural equivalence of symmetric monoidal dg-categories*

$$(\mathcal{D}_X \otimes_k B) - \mathbf{dg} \simeq ((\mathbf{DR}_X \otimes_k B) - \mathbf{dg})^{\mathcal{H}}.$$

Proof: This is a consequence of the proposition. Indeed, the equivalence of the proposition is symmetric monoidal, so preserves algebras and modules over algebras. \square

2 The formal boundary of smooth varieties

In this section we present the notion of the formal boundary of a smooth algebraic variety X over a base field k of characteristic 0. As opposed to the Betti setting presented in the separate paper [Pa-To], the formal boundary does not itself exist as an algebraic variety or stacks in any form and will only be defined through its categories of perfect complexes, possibly with integrable connections. The case of perfect complexes has been studied recently by several authors (see [Be-Te, Ef, He-Po-Ve]). We follow a similar approach for the case of perfect complexes endowed with integrable connections, for which many statements can be deduced to the case without connections. However, the ∞ -category of perfect complexes with flat connections we introduce below can not be recovered from the ∞ -categories of perfect complexes on the formal boundary and therefore the results of this section are not formal consequences of the results of [Be-Te, Ef, He-Po-Ve] and are somehow new.

In this section all varieties, schemes and stacks are defined over a base field k of characteristic 0.

2.1 Perfect complexes on the formal boundary

In this section we remind the notion of the formal boundary $\widehat{\partial}X$ of a smooth variety X studied in [Be-Te, Ef, He-Po-Ve]. As we prefer to avoid any analytical aspects during the construction, we mainly follow the approach of [Ef] and [He-Po-Ve], and will use both approaches simultaneously.

The setting. We let X be a smooth algebraic variety. We assume that we have fixed an open dense embedding $X \hookrightarrow \bar{X}$ where \bar{X} is a smooth and proper scheme over k . We moreover assume that the reduced closed complement $D \subset \bar{X}$ of X inside \bar{X} is a simple normal crossing divisor on \bar{X} . We will at some point need a situation where \bar{X} is only a smooth and proper DM-stack, for which the arguments are similar. We call such an embedding $X \hookrightarrow \bar{X}$ a *good compactification*.

For any affine scheme $\text{Spec } A$ with an étale map $u : \text{Spec } A \rightarrow \bar{X}$, we consider $I \subset A$ the ideal of the pull-back $u^*(D) \subset \text{Spec } A$ as well as $\hat{A} = \varprojlim_n A/I^n$ the formal completion of A along I . When u varies in the small étale site of \bar{X} we obtain this way a presheaf of commutative rings on $\bar{X}_{\text{ét}}$, sending $u : \text{Spec } A \rightarrow \bar{X}$ to \hat{A} . This presheaf of commutative rings comes equipped with a presheaf of ideals, which simply is the ideal generated by I inside \hat{A} .

Definition 2.1.1 *The ∞ -category of perfect complexes on $\widehat{\partial}X$ is defined by*

$$\text{Parf}(\widehat{\partial}X) := \lim_{\text{Spec } A \rightarrow \bar{X}} \text{Parf}(\text{Spec } \hat{A} - V(I)).$$

The above definition has a version with coefficients in any derived affine scheme $S = \text{Spec } B$, defined as follows. For each $u : \text{Spec } A \rightarrow \bar{X}$ in $\bar{X}_{\text{ét}}$ we can form the cgda $\widehat{A \otimes B} := \lim_n (A/I^n \otimes_k B)$. The ideal I defines an open subset in the derived scheme $\text{Spec } \widehat{A \otimes B}$ which simply is the pull-back of $\text{Spec } \hat{A} - V(I)$ by the natural projection $\text{Spec } \widehat{A \otimes B} \rightarrow \text{Spec } \hat{A}$. We continue to denote by $\text{Spec } \widehat{A \otimes B} - V(I)$ this open derived sub-scheme. We then set

$$\text{Parf}(\hat{\partial}X)(S) := \lim_{\text{Spec } A \rightarrow \bar{X}} \text{Parf}(\text{Spec } \widehat{A \otimes B} - V(I)) \in \mathbf{dgCat},$$

and call it the ∞ -category of families of perfect complexes on $\hat{\partial}X$ parametrized by S . When S varies in the ∞ -category of derived affine schemes \mathbf{dAff} , $S \mapsto \text{Parf}(\hat{\partial}X)(S)$ defines an ∞ -functor

$$\mathbf{Perf}(\hat{\partial}X) : \mathbf{dAff}^{op} \rightarrow \mathbf{dgCat}.$$

By [He-Po-Ve, Prop. 3.23] this ∞ -functor is a derived stack for the étale topology on \mathbf{dAff} . In the same manner, we have the derived stack $\mathbf{Perf}(\hat{X})$, of perfect complexes on the formal completion of \bar{X} along D . Its S -points can be defined as before (with $S = \text{Spec } B$)

$$\mathbf{Perf}(\hat{X})(S) = \lim_{\text{Spec } A \rightarrow \bar{X}} \text{Parf}(\text{Spec } \widehat{A \otimes B}) \in \mathbf{dgCat}.$$

Another equivalent description is a derived mapping stack

$$\mathbf{Perf}(\hat{X}) \simeq \mathbf{Map}_{\mathbf{dSt}_k}(\hat{X}, \mathbf{Perf}),$$

where the formal scheme \hat{X} is here defined as $\text{colim}_n D_n$, where the colimit is taken in \mathbf{dSt}_k and $D_n = \text{Spec } \mathcal{O}_X/I^n \subset X$ is the n -th infinitesimal neighborhood of D inside \bar{X} .

Definition 2.1.2 1. The derived stack of perfect complexes on \hat{X} is the derived stack $\mathbf{Perf}(\hat{X}) \in \mathbf{dSt}_k$ defined above.

2. The derived stack of perfect complexes on $\hat{\partial}X$ is the derived stack $\mathbf{Perf}(\hat{\partial}X) \in \mathbf{dSt}_k$ defined above.

The derived stacks defined above also possess sheaf theoretical interpretations as follows. The structure sheaf $\hat{\mathcal{O}}_D$ of \hat{X} can be considered as a sheaf of commutative \mathcal{O}_X -algebras, sending an étale map $\text{Spec } A \rightarrow \bar{X}$ to the A -algebra \hat{A} . We also have $\hat{\mathcal{O}}_D \simeq \lim_n \mathcal{O}_{D_n}$, where the limit is taken in the category of all sheaves of \mathcal{O}_X -algebras. Note that $\hat{\mathcal{O}}_D$ is in general not a quasi-coherent sheaf on X . In the same manner, if $S = \text{Spec } B$ is a derived affine scheme, we have a sheaf of commutative \mathcal{O}_X -dg-algebras $\hat{\mathcal{O}}_{D,B}$, sending an étale map $\text{Spec } A \rightarrow \bar{X}$ to $\widehat{A \otimes_k B} = \lim_n (A/I^n \otimes_k B)$. Again, this is, in general, not a quasi-coherent sheaf on X .

Similarly, we can define a sheaf of commutative \mathcal{O}_X -algebras $\hat{\mathcal{O}}_D^\circ$ by locally inverting the equation of D in $\hat{\mathcal{O}}_D$. More precisely, for $S = \text{Spec } B$ a derived affine scheme we send the étale map $\text{Spec } A \rightarrow \hat{X}$ to $\Gamma(\text{Spec } (\widehat{A \otimes_k B}) - V(I), \mathcal{O})$. When $\text{Spec } A \rightarrow \hat{X}$ is small enough so that D becomes principal over $\text{Spec } A$ (which we can always assume to define the sheaf $\hat{\mathcal{O}}_D^\circ$), the value of $\hat{\mathcal{O}}_{D,B}^\circ$ is the cgda $(\widehat{A \otimes_k B})[t^{-1}]$, where $t \in A$ is a generator of the ideal $I \subset A$.

Both sheaves $\hat{\mathcal{O}}_{D,B}$ and $\hat{\mathcal{O}}_{D,B}^\circ$ of cdga's on \bar{X}_{et} are set theoretically supported on D , and can therefore be considered as sheaves of cdga's on the site D_{et} . It makes then sense to consider the ∞ -categories of sheaves of perfect modules on D_{et} over the sheaves of cdgas $\hat{\mathcal{O}}_{D,B}$ and $\hat{\mathcal{O}}_{D,B}^\circ$. Let us denote this ∞ -categories by $Parf(\hat{\mathcal{O}}_{D,B})$ and $Parf(\hat{\mathcal{O}}_{D,B}^\circ)$. The descent result proved in [He-Po-Ve, Prop. 3.23] precisely implies that we have natural equivalences of ∞ -categories, functorial in $S = Spec B$

$$\mathbf{Perf}(\hat{X})(S) \simeq Parf(\hat{\mathcal{O}}_{D,B}) \quad \mathbf{Perf}(\hat{\partial}X)(S) \simeq Parf(\hat{\mathcal{O}}_{D,B}^\circ).$$

One aspect of the definition 2.1.3 is it depends a priori on a choice of \bar{X} . the case of perfect complexes over \hat{X} is certainly moral, but the idea behind the definition 2.1.3 (2) is that the derived stack $\mathbf{Perf}(\hat{\partial}X)$ only depends on the variety X . Unfortunately, we do not know if this is the case and we could not deduce this from the combined results of [Be-Te, Ef, He-Po-Ve]. It is shown in [He-Po-Ve, A.4] (together with [Be-Te]) that the ∞ -category $\mathbf{Perf}(\hat{\partial}X)(k)$ of global k -points only depends on X . However, as noted in [He-Po-Ve, App. A] the setting [Be-Te] is only for smooth varieties and it is therefore unclear that $\mathbf{Perf}(\hat{\partial}X)(B)$ remains independent of the choice of \bar{X} for a general cdga B (already for a non-smooth commutative k -algebra B of finite type). To overcome this issue we will introduce a full substack $\mathbf{Perf}^{ex}(\hat{\partial}X) \subset \mathbf{Perf}(\hat{\partial}X)$ of *extendable perfect complexes* and show that $\mathbf{Perf}^{ex}(\hat{\partial}X)$ only depends on X alone by using the categorical approach of [Ef].

We consider the map of stacks in ∞ -categories

$$\mathbf{Perf}(\hat{X}_D) \longrightarrow \mathbf{Perf}(\hat{\partial}X)$$

from perfect complexes on the formal completion of \bar{X} along D to perfect complexes on $\hat{\partial}X$. This is a morphism of stacks in stable ∞ -categories and it therefore makes sense to define its Karoubian image. This is the substack of objects that are locally (for the étale topology) direct factors of objects in the essential image of the above map. More precisely, for any affine derived scheme $S \in \mathbf{dAff}$ we have a stable ∞ -functor

$$\mathbf{Perf}(\hat{X}_D)(S) \longrightarrow \mathbf{Perf}(\hat{\partial}X)(S),$$

and we denote by $\mathbf{Perf}^{ex,pr}(\hat{\partial}X)(S) \subset \mathbf{Perf}(\hat{\partial}X)(S)$ the full sub- ∞ -category of objects that are retracts of objects in the essential image of $\mathbf{Perf}(\hat{X}_D)(S) \longrightarrow \mathbf{Perf}(\hat{\partial}X)(S)$. When S varies, this defines a full sub-prestack $\mathbf{Perf}^{ex,pr}(\hat{\partial}X) \subset \mathbf{Perf}^{ex}(\hat{\partial}X)$.

Definition 2.1.3 *The derived stack of extendable perfect complexes on $\hat{\partial}X$ is the stack associated to the prestack $\mathbf{Perf}^{ex,pr}(\hat{\partial}X)$ defined above. It is denoted by $\mathbf{Perf}^{ex}(\hat{\partial}X)$*

Note that by definition $\mathbf{Perf}^{ex}(\hat{\partial}X)$ is a full sub-stack in $\mathbf{Perf}(\hat{\partial}X)$. An important property of the stack $\mathbf{Perf}^{ex}(\hat{\partial}X)$ is that it only depends on X alone and not on the choice of \bar{X} .

Proposition 2.1.4 *For a given $S = Spec B \in \mathbf{dAff}$, the ∞ -category $\mathbf{Perf}^{ex}(\hat{\partial}X)(S)$ can be reconstructed from the k -linear dg-category $Parf(X)$ of perfect complexes over the variety X . Moreover, this reconstruction is functorial in B .*

Proof: This is essentially the main result of [Ef, Thm. 3.2]. To start with, as $\mathbf{Perf}^{ex}(\hat{\partial}X)$ is the stack associated to the prestack $\mathbf{Perf}^{ex,pr}(\hat{\partial}X)$ it is enough to show that $\mathbf{Perf}^{ex}(\hat{\partial}X)(S)$ can be recovered from $Parf(X)$. We start by the following lemma.

Lemma 2.1.5 *Let $\mathcal{K}(S)$ be the kernel of the ∞ -functor $\mathbf{Perf}(\hat{X})(S) \rightarrow \mathbf{Perf}^{ex,pr}(\hat{\partial}X)(S)$ then the sequence of stable ∞ -categories*

$$\mathcal{K}(S) \hookrightarrow \mathbf{Perf}(\hat{X})(S) \longrightarrow \mathbf{Perf}^{ex,pr}(\hat{\partial}X)(S)$$

exhibits $\mathbf{Perf}^{ex,pr}(\hat{\partial}X)(S)$ as the triangulated quotient of $\mathbf{Perf}(\hat{X})(S)$ by $\mathcal{K}(S)$.

Proof of the lemma: By descent, the ∞ -functor of the lemma can be written as a finite limit over an affine cover \mathcal{U} of \bar{X}

$$\lim_{Spec A \in \mathcal{U}} \mathbf{Parf}(\widehat{A \otimes_k B}) \longrightarrow \lim_{Spec A \in \mathcal{U}} \mathbf{Parf}(\widehat{A \otimes_k B[t^{-1}]})$$

where the affine cover \mathcal{U} has been chosen so that D becomes principal on each $Spec A$ (and we have denoted by t an equation of D). For a given $Spec A \in \mathcal{U}$ we have an exact sequence of stable ∞ -categories

$$\mathcal{K}(S) \hookrightarrow \mathbf{Parf}(\widehat{A \otimes_k B}) \longrightarrow \mathbf{Parf}(\widehat{A \otimes_k B[t^{-1}]}).$$

It is easy to see that for finite diagram of full faithful stable ∞ -functors $T_\alpha \hookrightarrow T'_\alpha$ the induced ∞ -functor on triangulated quotients

$$(\lim_{\alpha} T'_\alpha) / (\lim_{\alpha} T_\alpha) \longrightarrow \lim_{\alpha} (T'_\alpha / T_\alpha)$$

is fully faithful. Therefore, the ∞ -functor $\mathbf{Perf}(\hat{X})(S) / \mathcal{K}(S) \rightarrow \mathbf{Perf}^{ex,pr}(\hat{\partial}X)(S)$ is always fully faithful. Finally, by definition of extendable objects it is also essentially surjective up to direct factors, which implies that it is an equivalence. \square

We come back to the proof of the proposition. We will need a more precise description of the kernel $\mathcal{K}(S)$. For this, we chose $K \in \mathbf{Parf}(\bar{X})$ a compact generator for $\mathbf{Parf}_D(\bar{X}) \subset \mathbf{Parf}(\bar{X})$, the sub dg-category of perfect complexes with supports on D . The corresponding object $K \otimes_k B \in \mathbf{Parf}(\bar{X}) \otimes_k B$ is a compact generator for $\mathbf{Parf}_D(\bar{X}) \otimes_k B$, and this remains true after Zariski localization on \bar{X} : for any Zariski open $U = Spec A \subset \bar{X}$, the object $K|_U \otimes_k B \in \mathbf{Parf}(U)$ is compact generator for $\mathbf{Parf}_D(U)$. Formal gluing for the affine U (see [He-Po-Ve]), tells us that we have a fibered square of dg-categories

$$\begin{array}{ccc} \mathbf{Parf}(A \otimes B) & \longrightarrow & \mathbf{Parf}(A \otimes_k B[t^{-1}]) \\ \downarrow & & \downarrow \\ \mathbf{Parf}(\widehat{A \otimes_k B}) & \longrightarrow & \mathbf{Parf}(\widehat{A \otimes_k B[-1]}), \end{array}$$

and thus an equivalence of the kernels of the horizontal ∞ -functor. This kernel is precisely $\mathbf{Parf}_D(U)$, and thus generated by $K|_U \otimes_k B$. By descent, we thus have

$$\mathcal{K}(S) \simeq \lim_{U \in \bar{X}_{zar}} \mathbf{Parf}_{D \times S}(U \times S) \simeq \mathbf{Parf}_{D \times S}(\bar{X} \times S).$$

To summarize, let $C = \mathbf{End}(K)$ be the dg-algebra of endomorphisms of the object K . We have an exact sequence of stable dg-categories

$$\mathbf{Parf}(C \otimes_k B) \longrightarrow \mathbf{Parf}(\widehat{\bar{X} \times S}) \longrightarrow \mathbf{Perf}^{ex,pr}(\hat{\partial}X)(S).$$

The ∞ -categories $Parf(X \hat{\times} S)$ can itself be written in terms of the dg-algebra C . Again by descent we can replace \hat{X} by an affine open sub-scheme $U = Spec A$ and K by its restriction $K|_U$ to U (and denote $C_U = End(K|_U)$). It is then easy to see that $Parf(\widehat{A \otimes_k B})$ is naturally equivalent to $PsParf(C \otimes_k B)$, the dg-category of C_U -dg-modules inside $Parf(B)$ (called pseudo-perfect dg-modules relative to B , see [To-Va]). Such an equivalence is produced by sending a perfect dg-module E over $A \otimes_k B$ to $Hom(K|_U, E)$ as dg-module over $End(K|_U)$. We refer to [Ef] for more details.

We thus have an exact sequence of dg-categories

$$Parf(C \otimes_k B) \longrightarrow PsParf(C \otimes_k B) \longrightarrow \mathbf{Perf}^{ex.pr}(\widehat{\partial X})(S).$$

As $Parf(\bar{X} \times S)$ is a smooth and proper dg-category over B , we can thus apply [Ef, Thm. 3.2] to the object $K \otimes_k B \in Parf(\bar{X} \times S)$, which precisely states that the above quotient can be functorially reconstructed from the B -linear dg-category $Parf(\bar{X} \times S) / \langle K \otimes_k B \rangle \simeq Parf(X \times S) \simeq Parf(X) \otimes_k B$, and thus from $Parf(X)$ as a dg-category over k . \square

Corollary 2.1.6 *The derived stack $\mathbf{Perf}^{ex}(\widehat{\partial X})$ does not depend on the choice of the good compactification \bar{X} .*

The above corollary can be made more precise as follows. Suppose that we two good compactifications \bar{X} and \bar{X}' as well as a morphism $\pi : \bar{X}' \rightarrow \bar{X}$ inducing an isomorphism over X . Let $\mathbf{Perf}(\widehat{\partial X})$ and $\mathbf{Perf}(\widehat{\partial X}')$ be the two derived stacks constructed above for \bar{X} and \bar{X}' respectively. There is an obvious pull-back morphism $\pi^* : \mathbf{Perf}(\widehat{\partial X}) \rightarrow \mathbf{Perf}(\widehat{\partial X}')$, and the corollary states that this morphism is an equivalence of derived stacks.

Moreover, for any étale affine $Spec A \rightarrow \bar{X}$, we have a natural morphism of schemes $Spec \hat{A} - V(I) \rightarrow Spec A - V(I)$. Similarly, for any $S = Spec B \in \mathbf{dAff}$ we have a morphism of derived schemes $Spec \widehat{A \otimes_k B} - V(I) \rightarrow (Spec(A) - V(I)) \times S$. When A varies in the étale site of \bar{X} and S inside derived affine schemes, we obtain by base change a natural restriction map $R : \mathbf{Perf}(X) \rightarrow \mathbf{Perf}(\widehat{\partial X})$, where $\mathbf{Perf}(X) := \mathbf{Map}(X, \mathbf{Perf})$ is the derived stack of perfect complexes on X . Similarly, we get a restriction map $R' : \mathbf{Perf}(X) \rightarrow \mathbf{Perf}(\widehat{\partial X}')$. Corollary 2.1.6 can then be refined by stating that we have a commutative triangle of derived stacks

$$\begin{array}{ccc} & & \mathbf{Perf}(\widehat{\partial X}) \\ & \nearrow R & \downarrow \pi^* \\ \mathbf{Perf}(X) & & \mathbf{Perf}(\widehat{\partial X}'), \\ & \searrow R' & \end{array}$$

with π^* an equivalence.

We do not know if the above corollary is also true for the bigger stack $\mathbf{Perf}(\widehat{\partial X})$. Because of [He-Po-Ve, Thm. 7.3] the inclusion $\mathbf{Perf}^{ex}(\widehat{\partial X})(S) \subset \mathbf{Perf}(\widehat{\partial X})(S)$ is an equivalence as soon as S is a smooth variety over k , so the restriction of $\mathbf{Perf}(\widehat{\partial X})$ to smooth varieties does not depend on \bar{X} . We

believe that this remains true for a general derived affine scheme S but we could not find a reference (or prove it). The question is essentially equivalent to proving the analogue of the localization for coherent complexes of [He-Po-Ve] where coherent complexes are replaced by perfect complexes.

2.2 Flat perfect complexes on the formal boundary

In the last section we have seen the derived stack of perfect complexes on the formal boundary of X . In this section we use similar ideas to introduce the derived stack $\mathbf{Perf}^\nabla(\widehat{\partial}X)$ of perfect complexes endowed with flat connections on $\widehat{\partial}X$. When $X = \mathbb{A}^1$ the underived version of $\mathbf{Perf}^\nabla(\widehat{\partial}X)$ has been intensively studied in [Ra] (for vector bundles instead of perfect complexes).

We start with a similar setting as in the previous section, a smooth variety X and a good compactification $X \hookrightarrow \bar{X}$, with $D \subset \bar{X}$ the divisor at infinity. In order to define the derived stack of perfect complexes on $\widehat{\partial}X$ we first define certain sheaves of graded mixed cdga's on X_{et} the small étale site of \bar{X} and will define perfect complexes as certain graded mixed dg-modules.

Fix a commutative smooth k -algebra A of finite type. We consider $\mathbf{DR}(A)$ the de Rham algebra of A (over k), as a graded mixed cdga over k . Concretely, $\mathbf{DR}(A) = \mathit{Sym}_A(\Omega_A^1[1])$ considered as \mathbb{Z} -graded cdga with zero differential and for which Ω_A^1 sits in weight 1. The graded cdga $\mathbf{DR}(A)$ comes equipped with an extra differential, namely the de Rham differential, which we denote here by ϵ . This extra structure makes $\mathbf{DR}(A)$ into a graded mixed cdga in the sense of [PTVV, CPTVV]. When A comes equipped with an ideal $I \subset A$, we denote by $\widehat{\mathbf{DR}}(A)$ the I -adic completion of $\mathbf{DR}(A)$ which is defined by

$$\widehat{\mathbf{DR}}(A) := \lim_n \mathbf{DR}(A/I^n),$$

where the limit is taken in the category of graded mixed cdga's. It is easy to see that the underlying graded cdga of $\widehat{\mathbf{DR}}(A)$ is naturally isomorphic to $\mathit{Sym}_{\widehat{A}}(\widehat{\Omega}_A^1[1])$, the symmetric algebra over the completion of Ω_A^1 . The mixed structure on $\mathit{Sym}_{\widehat{A}}(\widehat{\Omega}_A^1[1])$ simply is the canonical extension of the de Rham differential on A to its completion.

Let $\mathit{Spec} A \rightarrow \bar{X}$ be an étale map, and $I \subset A$ the ideal of definition of the divisor D . We have the completed de Rham graded mixed cdga $\widehat{\mathbf{DR}}(A)$. When $\mathit{Spec} A \rightarrow \bar{X}$ varies in the small étale site of \bar{X} this defines a sheaf of graded mixed cdga's \widehat{DR} on \bar{X}_{et} . This sheaf is set theoretically supported on D and thus defines a sheaf of graded mixed cdga on D_{et} . As before, the sheaf as a version with coefficients in a cdga B over k denoted by $\widehat{\mathbf{DR}}_B$. Its values on an étale $U = \mathit{Spec} A \rightarrow \bar{X}$ is the graded mixed cdga

$$\widehat{\mathbf{DR}}_B(U) := \lim_n (\mathbf{DR}(A/I^n) \otimes_k B).$$

The sheaf $\widehat{\mathbf{DR}}_B$ is now a sheaf of graded mixed B -linear cdga's. Note that the weight zero part of $\widehat{\mathbf{DR}}_B$ is the sheaf $\widehat{\mathcal{O}}_{D,B}$ constructed before. We can therefore invert a local equation of the divisor D to define $\widehat{\mathbf{DR}}_B^o$, another sheaf of graded mixed B -linear cdga. For an étale map $U = \mathit{Spec} A \rightarrow \bar{X}$ on which the divisor D is principal of equation $f \in A$, we have $\widehat{\mathbf{DR}}_B^o(U) := \widehat{\mathbf{DR}}_B(U)[t^{-1}]$. The part of weight zero in $\widehat{\mathbf{DR}}_B^o(U)$ is of course $\widehat{\mathcal{O}}_{D,B}^o$ defined in our last section.

For $S = \mathit{Spec} B \in \mathbf{dAff}_k$, we let $\mathit{Par}f^\nabla(\widehat{\partial}X)(S)$ be the dg-category of sheaves E of graded mixed $\widehat{\mathbf{DR}}_B^o(U)$ -dg-modules which are locally free of weight 0 in the following sense: locally on \bar{X}_{et} , the underlying graded $\widehat{\mathbf{DR}}_B^o$ -dg-module E (obtained by forgetting the mixed structure) is of the form $\widehat{\mathbf{DR}}_B^o \otimes_{\widehat{\mathcal{O}}_{D,B}^o} E(0)$ for a perfect $\widehat{\mathcal{O}}_{D,B}^o$ -module $E(0)$ (considered of weight zero). When $S = \mathit{Spec} B$ varies

in \mathbf{dAff} , the dg-categories $Parf(\widehat{\partial X})(S)$ define an dg-functor $\mathbf{Perf}^\nabla(\widehat{\partial X}) : \mathbf{dAff}^{op} \rightarrow \mathbf{dgCat}$. There is an obvious forgetful map of derived prestacks

$$\mathbf{Perf}^\nabla(\widehat{\partial X}) \longrightarrow \mathbf{Perf}(\widehat{\partial X})$$

sending a graded mixed dg-module to its part of weight 0.

Definition 2.2.1 1. The derived pre-stack of flat perfect complexes on $\widehat{\partial X}$ is $\mathbf{Perf}^\nabla(\widehat{\partial X})$ defined above.

2. The derived pre-stack of extendable flat perfect complexes on $\widehat{\partial X}$ is defined to be the fiber product of derived pre-stacks

$$\mathbf{Perf}^{\nabla,ex}(\widehat{\partial X}) \times_{\mathbf{Perf}(\widehat{\partial X})} \mathbf{Perf}^{ex}(\widehat{\partial X}).$$

By construction, $\mathbf{Perf}^{\nabla,ex}(\widehat{\partial X})$ is a full derived sub-prestack in $\mathbf{Perf}^\nabla(\widehat{\partial X})$ defined by the local condition "the underlying perfect complex is extendable". The main result of this part is the following descent and invariance statements.

Proposition 2.2.2 *With the notations above we have.*

1. The derived pre-stacks $\mathbf{Perf}^\nabla(\widehat{\partial X})$ and $\mathbf{Perf}^{\nabla,ex}(\widehat{\partial X})$ are stacks.
2. The derived stack $\mathbf{Perf}^{\nabla,ex}(\widehat{\partial X})$ only depends on X .

Proof: The key to for the proof of the above proposition is the interpretation of graded mixed structures as actions of the group stack $\mathcal{H} := B\mathbb{G}_a \times \mathbb{G}_m$, the semi-direct product of the classifying stack of the formal additive group with the multiplicative group (see proposition 1.1.1). For a graded mixed cdga Ω , the group stack \mathcal{H} acts on Ω by cdga automorphisms, where the \mathbb{G}_m -action provides the grading and the $B\mathbb{G}_a$ -action induces the mixed structure. This action induces an action of \mathcal{H} on the k -linear dg-category $Parf(\Omega)$ of perfect dg-modules over Ω . The dg-category of graded mixed Ω -modules which are perfect as Ω -dg-modules can be recovered by taking invariants (see 1.1.1)

$$Parf^{gr,\epsilon}(\Omega) \simeq Parf(\Omega)^{\mathcal{H}}.$$

This presentation of graded mixed dg-modules easily implies the above proposition as follows.

For (1), the derived prestack $\mathbf{Perf}^\nabla(\widehat{\partial X})$ is obtained as follows. We start with the prestack $Parf(\widehat{\mathbf{DR}}^o)$ of perfect $\widehat{\mathbf{DR}}^o$ -dg-modules, where $\widehat{\mathbf{DR}}^o$ is simply considered as a sheaf of graded cdga's. This is a derived prestack with values in \mathcal{H} -equivariant dg-categories. It is moreover a stack, by noticing that $\widehat{\mathbf{DR}}^o$ is a cdga inside $\mathbf{Perf}(\widehat{\partial X})$ and by using [He-Po-Ve, Prop. 3.23]. This implies that its fixed points by \mathcal{H} remains a stack (because taking fixed points commutes with taking limits). This stack is denoted by $Parf^{gr,\epsilon}(\widehat{\mathbf{DR}}^o)$, and is the stack of graded mixed $\widehat{\mathbf{DR}}^o$ -dg-modules which are perfect over $\widehat{\mathbf{DR}}^o$. By definition $\mathbf{Perf}^\nabla(\widehat{\partial X})$ is a sub-prestack of $Parf^{gr,\epsilon}(\widehat{\mathbf{DR}}^o)$ defined by a local condition and thus is a stack. The fact that $\mathbf{Perf}^{\nabla,ex}(\widehat{\partial X})$ is also a stack now follows from the fact that it is defined as a fiber product of stacks.

For (2) we use a similar argument. The derived stack $\mathbf{Perf}^{\nabla,ex}(\widehat{\partial X})$ can be expressed as a full sub-stack of the fixed points by \mathcal{H} acting on $\widehat{\mathbf{DR}}^o$ -dg-modules inside $\mathbf{Perf}^{ex}(\widehat{\partial X})$ (note that as a graded cdga $\widehat{\mathbf{DR}}^o$ lives in $\mathbf{Perf}^{ex}(\widehat{\partial X})$). Therefore, to prove that $\mathbf{Perf}^{\nabla,ex}(\widehat{\partial X})$ is independant of the choice of \widehat{X} we have to check that the stack of \mathcal{H} -equivariant dg-categories $Parf(\widehat{\mathbf{DR}}^o)$ only depends on X . This reduces to the following lemma.

Lemma 2.2.3 *Let $\pi : \bar{X}' \rightarrow \bar{X}$ be a morphism between two good compactifications of X . Let $D' = \pi^{-1}(D)$ so that π induces an isomorphism between $\bar{X}' - D'$ and $\bar{X} - D$. Let $\hat{\mathbf{DR}}_{\bar{X}}^{\circ}$ and $\hat{\mathbf{DR}}_{\bar{X}'}^{\circ}$ be the corresponding two sheaves of graded mixed cdga constructed above. Then, for any $\text{Spec } B \in \mathbf{dAff}$, we have.*

1. *There is a pull-back map*

$$f_{\text{pi}} : \pi^{-1}(\hat{\mathbf{DR}}_{\bar{X},B}^{\circ}) \rightarrow \hat{\mathbf{DR}}_{\bar{X}',B}^{\circ}$$

of sheaves of graded mixed cdga on $\bar{X}'_{\text{ét}}$.

2. *The above map, when forgetting the graded mixed structures, induces an equivalence of dg-categories*

$$\pi^* : \text{Parf}(\hat{\mathbf{DR}}_{\bar{X},B}^{\circ}) \simeq \text{Parf}(\hat{\mathbf{DR}}_{\bar{X}',B}^{\circ}).$$

Before giving a proof of the lemma, let us mention how this finishes the proof of the proposition. The fact that f_{π} exists implies that the dg-functor π^* also exists by simply pulling back graded mixed dg-modules. Moreover, as f_{π} is a morphism of graded mixed cdga's, it is clear that the dg-functor π^* is naturally \mathcal{H} -equivariant. As it is an equivalence it also induces an equivalence on the fixed points dg-categories, and the result follows easily by considering the full sub-dg-categories corresponding to $\mathbf{Perf}^{\nabla, \text{ex}}(\hat{\partial}X)$.

Proof of the lemma: The existence of the map f_{π} simply follows from functoriality of the construction $A \mapsto \mathbf{DR}_B(A)$, $A \mapsto \hat{\mathbf{DR}}_B(A)$ and $A \mapsto \mathbf{DR}_B(A)[t^{-1}]$, as functors from smooth k -algebras of finite type to graded mixed cdga. To prove (2), we observe that $\hat{\mathbf{DR}}_{\bar{X},B}^{\circ}$ and $\hat{\mathbf{DR}}_{\bar{X}',B}^{\circ}$, when considered as sheaves of cdga are perfect over $\hat{\mathcal{O}}_{D,B}^{\circ}$ and $\hat{\mathcal{O}}_{D',B}^{\circ}$ and extendable. They can thus be considered as graded cdga's inside the symmetric monoidal the dg-categories $\mathbf{Perf}^{\text{ex}}(\hat{\partial}X)(B)$ and $\mathbf{Perf}^{\text{ex}}(\hat{\partial}X')(B)$. By corollary 2.1.6 we know that pull-back along π induces an equivalence symmetric monoidal of dg-categories

$$\pi^* : \mathbf{Perf}^{\text{ex}}(\hat{\partial}X)(B) \simeq \mathbf{Perf}^{\text{ex}}(\hat{\partial}X')(B).$$

To finish the proof it remains to show that the symmetric monoidal equivalence above π^* sends the cdga $\hat{\mathbf{DR}}_{\bar{X},B}^{\circ}$ to $\hat{\mathbf{DR}}_{\bar{X}',B}^{\circ}$. There are canonical restriction maps

$$R : \mathbf{Perf}(X) \rightarrow \mathbf{Perf}(\hat{\partial}X) \quad R' : \mathbf{Perf}(X) \rightarrow \mathbf{Perf}(\hat{\partial}X')$$

and we have $\pi^* \circ R' \simeq R$. Moreover, by construction we have

$$\hat{\mathbf{DR}}_{\bar{X},B}^{\circ} \simeq R(\mathbf{DR}_X) \quad \hat{\mathbf{DR}}_{\bar{X}',B}^{\circ} \simeq R'(\mathbf{DR}_X),$$

where $\mathbf{DR}_X = \text{Sym}_{\mathcal{O}_X}(\Omega_X^1[1])$ as a sheaf of perfect cdga's over \mathcal{O}_X . The lemma is proven and this finishes the proof of the proposition 2.2.2. \square

To finish this section, as for the case of perfect complexes, there is a restriction morphism

$$R : \mathbf{Perf}^{\nabla}(X) \rightarrow \mathbf{Perf}^{\nabla, \text{ex}}(\hat{\partial}X) \subset \mathbf{Perf}^{\nabla}(\hat{\partial}X),$$

from the derived stack $\mathbf{Perf}^{\nabla}(X)$ of extendable perfect complexes endowed with flat connections to the derived stack of perfect complexes with flat connections on the formal boundary of X . It is defined

as follows. First of all the derived stack $\mathbf{Perf}^\nabla(X)$ is defined as the derived stack of graded mixed dg-modules over \mathbf{DR}_X , the de Rham algebra of X , which are perfect of weight 0. More precisely, for $S = \mathit{Spec} B \in \mathbf{dAff}$, then $\mathbf{Perf}^\nabla(X)(S)$ is defined to be the ∞ -category of graded mixed $\mathbf{DR}_X \otimes_k B$ -dg-modules E , such that $E \simeq E(0) \otimes_{\mathcal{O}_X} \mathbf{DR}_X$ as a graded dg-modules over B , and where $E(0)$ is perfect over $\mathcal{O}_X \otimes_k B$. The restriction map R is then induced by the natural morphism of sheaves of graded mixed cdga over \bar{X}_{et}

$$\mathbf{DR}_X \otimes_k B \longrightarrow \hat{\mathbf{DR}}_B^o.$$

Locally on an étale affine $\mathit{Spec} A \longrightarrow \bar{X}$ on which D is principal of equation $t \in A$, this morphism is the natural map

$$\mathbf{DR}(A \otimes_k B)[t^{-1}] \longrightarrow \hat{\mathbf{DR}}(A \otimes_k B)[t^{-1}]$$

induced by the completion morphism $A \otimes_k B \longrightarrow \lim_n (A/I^n \otimes_k B)$.

This defines a restriction map $R : \mathbf{Perf}^\nabla(X) \longrightarrow \mathbf{Perf}^\nabla(\hat{\partial}X)$, which covers the restriction map of perfect complexes $R : \mathbf{Perf}(X) \longrightarrow \mathbf{Perf}(\hat{\partial}X)$. As the later map factors through extendable perfect complexes (because any perfect complex on $X \times S$ extends to $\bar{X} \times S$ up to a retract), we find a restriction map $\mathbf{Perf}^\nabla(X) \longrightarrow \mathbf{Perf}^\nabla(\hat{\partial}X)$.

Definition 2.2.4 *The restriction map is the morphism of derived stacks*

$$R : \mathbf{Perf}^\nabla(X) \longrightarrow \mathbf{Perf}^{\nabla, ex}(\hat{\partial}X)$$

defined above.

2.3 De Rham cohomology of the formal boundary and compactly supported cohomology

To finish this part, let us describe the Hom -complexes of the dg-category $\mathbf{Perf}^\nabla(\hat{\partial}X)(B)$ in terms of hypercohomology of certain complexes of sheaves on D and relate this to a notion of compactly supported de Rham cohomology. The notion of de Rham cohomology with compact supports already appeared in [Ba-Ca-Fi], but our treatment here is somehow new as it is based on the theory of Tate objects and their duality (see [He]), which makes the theory also available over any base cdga B . In this part we give the constructions and definitions of compactly supported cohomology. The duality will itself be studied later in our section §4.2.

For this, we fix a good compactification $X \hookrightarrow \bar{X}$ with divisor at infinity D . For any connective cdga B and any object $E \in \mathbf{Perf}^\nabla(\hat{\partial}X)(B)$, we define a sheaves of B -dg-modules on D_{Zar} as follows. By definition, E is a sheaf of graded mixed modules over $\hat{\mathbf{DR}}_B^o$. We define $|E|$ to the sheaf of B -modules $\underline{Hom}_{\mathbf{dg}_k^{gr}}(k, E)$, of graded mixed morphisms from the unit k to E . This is also called the de Rham complex of E completed along D . Note that it is a priori given as a infinite product

$$|E| = \prod_{i \geq 0} E(i)[-2i]$$

where the differential is the sum of the cohomological differential and the mixed structure. However, in our situation this infinite product is in fact a finite product, as $E(i)$ is non-zero only for a finite number of index i (because E is free as a graded module and $\hat{\mathbf{DR}}_B^o$ only has weight in degree $[0, d]$ where $d = \dim_k X$). The sheaf B -dg-modules $|E|$ is also called the *completed de Rham complex* of E . We then set the definition below.

Definition 2.3.1 *With the notation above, the de Rham cohomology of $\widehat{\partial}X$ with coefficients in E is the B -module defined by*

$$\mathbb{H}_{DR}(\widehat{\partial}X, E) := \mathbb{H}(D, |E|) \in B - \mathbf{dg}.$$

With the above notation, let E and F be two objects in $\mathbf{Perf}^\nabla(\widehat{\partial}X)(B)$. The dg-category of graded mixed modules over $\widehat{\mathbf{DR}}_B^o$ has a canonical symmetric monoidal structure, for which the tensor product is given by tensoring the underlying B -dg-modules (see [PTVV, CPTVV]). As perfect complexes of $\widehat{\mathcal{O}}_{D,B}^o$ -modules form a rigid symmetric monoidal dg-category, we deduce easily that $\mathbf{Perf}^\nabla(\widehat{\partial}X)(B)$ is also rigid. We can then form $E^\vee \otimes_{\widehat{\mathcal{O}}_{D,B}^o} F$, which is a new graded mixed $\widehat{\mathbf{DR}}_B^o$ -modules and an object in $\mathbf{Perf}^\nabla(\widehat{\partial}X)(B)$. For sake of simplicity this object will simply be denoted by $E^\vee \otimes F$. We then have a natural quasi-isomorphism

$$\underline{Hom}_{\mathbf{Perf}^\nabla(\widehat{\partial}X)(B)}(E, F) \simeq \mathbb{H}_{DR}^*(\widehat{\partial}X, E^\vee \otimes F),$$

given a wanted interpretation of mapping complexes of $\mathbf{Perf}^\nabla(\widehat{\partial}X)(B)$ in terms of de Rham cohomology of $\widehat{\partial}X$.

Remark 2.3.2 For any connective B , and any object $E \in \mathbf{Perf}^\nabla(\widehat{\partial}X)(B)$ the complex of sheaves $|E|$ on D_{zar} is made of acyclic sheaves on affines. In particular, the hyper-cohomology complex $\mathbb{H}(D, |E|)$ can be computed by a finite limit using an affine cover of D . In particular, if $|E|$ is locally perfect as a B -module on D , then $\mathbb{H}(D, |E|)$ is a perfect B -module.

We now use the formal boundary $\widehat{\partial}X$ in order to define a notion of cohomology with compact supports, both for perfect complexes and perfect complexes with flat connections. We start with a connective cdga B and a perfect complex E on $X \times S$, where $S = Spec B$. As explained in the previous section (before definition 2.2.4) we have its restriction $R(E) \in \mathbf{Perf}(\widehat{\partial}X)(B)$, and by functoriality an induced map on cohomology

$$\mathbb{H}(X, E) = \underline{Hom}(\mathcal{O}_X, E) \longrightarrow \mathbb{H}(\widehat{\partial}X, R(E)) = \underline{Hom}(R(\mathcal{O}_X), R(E)).$$

The *cohomology of X with compact supports and with coefficients in E* is defined to be the fiber of the above map. It is denoted by

$$\mathbb{H}_c(X, E) := fib(\mathbb{H}(X, E) \longrightarrow \mathbb{H}(\widehat{\partial}X, R(E))) \in B - \mathbf{dg}.$$

By construction this is a B -dg-module. This is not quite enough for our purpose, as this B -dg-module turns out to be the realization of a natural pro-object that we will now describe. This pro-structure is going to be very important for us, as it will make possible to state that compactly supported cohomology is dual to cohomology, even if the later is infinite dimensional. For sake of simplicity we assume that E extends to our fixed good compactification as a perfect complex \bar{E} on $\bar{X} \times S$. It is not always possible to find \bar{E} in general although it always exists if $B = k$ (because $K_{-1}(X) = 0$). Moreover, an extension always exists up to a retract, assuming the existence of \bar{E} is thus not a true restriction. Let us also note that in our specific situations of application E will always come with an extension to \bar{X} .

Using the formal gluing, we find that there is a cartesian square of B -dg-modules

$$\begin{array}{ccc} \mathbb{H}(\bar{X}, \bar{E}) & \longrightarrow & \mathbb{H}(X, E) \\ \downarrow & & \downarrow \\ \mathbb{H}(\hat{D}, \hat{E}) & \longrightarrow & \mathbb{H}(\hat{\partial}X, R(E)). \end{array}$$

Here \hat{D} is the formal completion of X along D , and $\mathbb{H}(\hat{D}, \hat{E})$ is defined by

$$\mathbb{H}(\hat{D}, \hat{E}) := \lim_n \mathbb{H}(D_{(n)}, j_n^* \bar{E}),$$

where $j_n : D_{(n)} := \text{Spec } \mathcal{O}_X / I_D^n \rightarrow X$ is the n -th infinitesimal thickening of D inside X . From the diagram above we have that $\mathbb{H}_c(X, E)$ can also be described as the fiber of

$$\mathbb{H}(\bar{X}, \bar{E}) \longrightarrow \lim_n \mathbb{H}(D_{(n)}, j_n^* \bar{E}).$$

This above morphism can itself be considered as a morphism of pro-objects in $\text{Parf}(B)$. Therefore, this allows us to defined a pro-perfect B -module by

$$\tilde{\mathbb{H}}_c(X, E) := \text{fib}(\mathbb{H}(\bar{X}, \bar{E}) \longrightarrow \text{"}\lim_n\text{"} \mathbb{H}(D_{(n)}, j_n^* \bar{E})) \in \text{Pro} - \text{Parf}(B).$$

It is possible to show that the above definition is independent of any choices, either \bar{X} or \bar{E} , but we will not do it here. It will be a consequence of Serre duality with supports that will be studied in our section §4.2, as the dual B -module turns out to be canonical equivalent to $\mathbb{H}(X, E^\vee \otimes_{\mathcal{O}_X} \omega_X)$, which only depends on X and E .

Definition 2.3.3 *With the above notation, the refined cohomology with compact supports of X with coefficients in E , is the pro-perfect B -module $\tilde{\mathbb{H}}_c(X, E)$ defined above.*

One nice aspect of the refined version above is that it is obviously compatible with base changes of B . Let $B \rightarrow B'$ be any morphism of connective cdga's, then the natural map

$$\tilde{\mathbb{H}}_c(X, E) \hat{\otimes}_B B' \longrightarrow \tilde{\mathbb{H}}_c(X, E \otimes_B B')$$

is an equivalence of pro-perfect B' -modules. Here we have denoted by

$$\hat{\otimes}_B B' : \text{Pro} - \text{Parf}(B) \longrightarrow \text{Pro} - \text{Parf}(B')$$

the functor induced on pro-objects by the usual base change $\otimes_B B' : \text{Parf}(B) \longrightarrow \text{Parf}(B')$.

Another aspect is the fiber sequence of B -modules

$$\mathbb{H}_c(X, E) \longrightarrow \mathbb{H}(X, E) \longrightarrow \mathbb{H}(\hat{\partial}X, R(E)).$$

The first map can be easily checked to come from a natural morphism of ind-pro-perfect B -modules $\tilde{\mathbb{H}}_c(X, E) \rightarrow \mathbb{H}(X, E)$, where $\mathbb{H}(X, E)$ is considered as a ind-perfect B -module in the canonical way (using $\text{Ind} - \text{Parf}(B) \simeq B - \mathbf{dg}$). This implies that $\mathbb{H}(\hat{\partial}X, R(E))$ is itself the realization of an ind-pro-perfect module $\mathbb{H}(\hat{\partial}X, R(E)) \in \text{Ind} - \text{Pro} - \text{Parf}(B)$, sitting in a triangle

$$\tilde{\mathbb{H}}_c(X, E) \longrightarrow \mathbb{H}(X, E) \longrightarrow \tilde{\mathbb{H}}(\hat{\partial}X, R(E)).$$

By construction this ind-pro-perfect object is an extension of a pro-perfect by an ind-perfect, and thus by definition is a Tate B -module in the sense of [He].

We now turn to the case of an object $E \in \mathbf{Perf}^\nabla(X)(B)$. The naive cohomology with compact supports is defined as

$$\mathbb{H}_{c,DR}(X, E) := \text{fib}(\mathbb{H}_{DR}(X, E) \longrightarrow \mathbb{H}_{DR}(\widehat{\partial}X, R(E))) \in B - \mathbf{dg}.$$

Again, we will now see that this B -module is the realization of a natural pro-perfect B -module denoted by $\widetilde{\mathbb{H}}_{c,DR}(X)$. We assume again that the underlying perfect complex $E(0)$ of E extends to a perfect complex $\bar{E}(0)$ on $\bar{X} \times S$. Using this, it is easy to see that the sheaf of B -modules $|R(E)|$ on D possesses a natural structure of a sheaf of ind-pro B -modules. Indeed, it is of the form $\oplus_i R(E(i))[-2i]$ with a suitable differential. Each $R(E(i))$ is itself of the form $\bar{E}(0) \otimes_{\mathcal{O}_{\bar{X}}} \Omega_{\bar{X}}^i \otimes_{\mathcal{O}_{\bar{X}}} \hat{\mathcal{O}}_{D,B}^o$. As $\hat{\mathcal{O}}_{D,B}^o$ has a canonical ind-pro structure, and as $\bar{E}(0) \otimes_{\mathcal{O}_X} \Omega_X^i \otimes_{\mathcal{O}_X} -$ commutes with limits and colimits of \mathcal{O}_X -modules, we see that each $R(E(i))$ is the realization of a canonical sheaf of ind-pro B -module. Moreover, as $\hat{\mathcal{O}}_{D,B}^o$ is ind-pro perfect as a B -module, this endows $|E|$ with a natural structure of sheaf of Tate B -modules. This provides a canonical Tate structure on hyper-cohomology of D with coefficients in $|E|$, that is de Rham cohomology of $\widehat{\partial}X$ with coefficients in E . We denote this Tate B -module by $\widetilde{\mathbb{H}}_{DR}(\widehat{\partial}X, R(E))$.

The restriction map R induces a morphism $\mathbb{H}_{DR}(X, E) \longrightarrow \widetilde{\mathbb{H}}_{DR}(\widehat{\partial}X, R(E))$, which is a morphism of ind-pro perfect B -modules if one endows the left hand side with the canonical structure of an ind-perfect B -module structure. It thus lifts to a morphism of Tate B -modules

$$\widetilde{\mathbb{H}}_{DR}(X, E) \longrightarrow \widetilde{\mathbb{H}}_{DR}(\widehat{\partial}X, R(E)).$$

Definition 2.3.4 *With the notation above, the refined de Rham cohomology of X with compact supports with coefficients in E is the Tate B -module defined by*

$$\widetilde{\mathbb{H}}_{c,DR}(X, E) := \text{fib}(\widetilde{\mathbb{H}}_{DR}(X, E) \longrightarrow \widetilde{\mathbb{H}}_{DR}(\widehat{\partial}X, R(E))).$$

We first notice that the ind-pro-perfect B -module structure on $\widetilde{\mathbb{H}}_{c,DR}(X, E)$ is in fact only pro-perfect (in particular it is a Tate B -module in the sense of [He]). This can be seen by reducing to the case of perfect complexes without connections, previously treated. Indeed, the complexes of sheaves $|E|$ and $|R(E)|$ are canonically filtered using their Hodge filtrations. The graded layers of this filtration on $\widetilde{\mathbb{H}}_{c,DR}(X, E)$ are $\widetilde{\mathbb{H}}_c(X, \Omega_X^i \otimes_{\mathcal{O}_X} E(0))[-i]$, and thus as pro-perfect. As this filtration is finite, we deduce that ind-pro object $\widetilde{\mathbb{H}}_{c,DR}(X, E)$ is filtered with associated graded being pro-perfect. This implies that $\widetilde{\mathbb{H}}_{c,DR}(X, E)$ itself is pro-perfect. We thus have proven the following corollary.

Corollary 2.3.5 *With the notations and assumptions above the ind-pro perfect B -module $\widetilde{\mathbb{H}}_{c,DR}(X, E)$ is pro-perfect. Furthermore the ind-pro perfect B -module $\widetilde{\mathbb{H}}_{DR}(\widehat{\partial}X, R(E))$ is a Tate B -module in the sense of [He].*

As for the case of perfect complexes, the formation of $\widetilde{\mathbb{H}}_{c,DR}(X, E)$ commutes with base change over B . For any $B \rightarrow B'$ of connective cda's, the natural morphism

$$\widetilde{\mathbb{H}}_{c,DR}(X, E) \hat{\otimes}_B B' \longrightarrow \widetilde{\mathbb{H}}_{c,DR}(X, E \otimes_B B')$$

is an equivalence of pro-perfect B' -modules.

3 Formal properties of moduli functors

We start by some general reminders on formal properties of derived stacks (see [To-Ve], [Lu2]). Let $F \in \mathbf{dAff}_k$ be a derived stack over k . For any derived affine scheme mapping to F , $u : U = \text{Spec } B \rightarrow F$, and any connective B -dg-module M , we can define the space of derivations of F on U with coefficients in M , as the fiber taken at u of the restriction map

$$F(B \oplus M) \rightarrow F(B),$$

where $B \oplus M$ is the trivial square zero extension of B by M . Let us denote this space by $\text{Der}_u(F, M) \in \mathbb{T}$. For any morphism $B \rightarrow B'$ of connective cdga's and any connective B' -dg-module M' , we have a canonical morphism $B \oplus M' \rightarrow B' \oplus M'$ covering the map $B \rightarrow B'$. Therefore, for any commutative diagram of derived stacks

$$\begin{array}{ccc} U = \text{Spec} & \xrightarrow{f} & U' = \text{Spec} , B' \\ & \searrow u & \swarrow u' \\ & F & \end{array}$$

there is a natural induced morphism on the corresponding spaces of derivations

$$f^* : \text{Der}_u(F, M) \rightarrow \text{Der}_{u'}(F, M').$$

Definition 3.0.1 *With the notation above.*

1. *The derived stack F has a cotangent complex at $u : U = \text{Spec } B \rightarrow F$ if there is an eventually connective B -dg-module $\mathbb{L}_{F,u}$ and functorial equivalences*

$$\text{Map}_{B\text{-mod}}(\mathbb{L}_{u,F}, M) \simeq \text{Der}_u(F, M).$$

2. *We say that F has a (global) cotangent complex if it has cotangent complexes at all maps $u : U = \text{Spec } B \rightarrow F$, and if moreover for commutative diagrams*

$$\begin{array}{ccc} U = \text{Spec} & \xrightarrow{f} & U' = \text{Spec} , B' \\ & \searrow u & \swarrow u' \\ & F & \end{array}$$

the induced morphism $\text{Der}_u(F, M) \rightarrow \text{Der}_{u'}(F, M')$ is an equivalence.

As explained in [To-Ve, Lu2] $\mathbb{L}_{F,u}$, if it exists, is uniquely characterized by the ∞ -functor $\text{Der}_u(F, -)$. As for condition (2), it is equivalent to states that the natural morphism $\mathbb{L}_{u,F} \otimes_B B' \rightarrow \mathbb{L}_{u',F}$ is an equivalence of dg-modules.

Let $\text{Spec } B \in \mathbf{dAff}_k$ be a derived affine scheme and M a connective B -module. Let $d : B \rightarrow M[1]$ be a k -linear derivation, which by definition means a section of $B \oplus M \rightarrow B$ inside the ∞ -category of cdga's over k . We remind that the square zero extension of B by M with respect to d , denoted by $B \oplus_d M$ is defined by the cartesian square of cdga's (see [To-Ve])

$$\begin{array}{ccc} B \oplus_d M & \longrightarrow & B \\ \downarrow & & \downarrow 0 \\ B & \xrightarrow{d} & B \oplus M[1] \end{array}$$

where 0 denotes the natural inclusion of B as a direct factor in the trivial square zero extension $B \oplus M[1]$.

Definition 3.0.2 *Let F be a derived stack.*

1. *We say that F is inf-cartesian if for any B, M and d as above the square*

$$\begin{array}{ccc} F(B \oplus_d M) & \longrightarrow & F(B) \\ \downarrow & & \downarrow 0 \\ F(B) & \xrightarrow{d} & F(B \oplus M[1]) \end{array}$$

is cartesian.

2. *We say that F is nil-complete if for any $\text{Spec } B \in \mathbf{dAff}_k$ with postnikov tower $\{B_{\leq n}\}_n$ the natural morphism*

$$F(B) \longrightarrow \lim_n F(B_{\leq n})$$

is an equivalence.

Suppose now that F is a derived stack which is inf-cartesian. For any $x : \text{Spec } B \rightarrow F$, we have an ∞ -functor

$$\mathbb{T}_{F,x} : B\text{-Mod}^c \rightarrow \mathbb{T},$$

from connective B -modules to spaces, that sends M to the fiber of $F(B \oplus M) \rightarrow F(B)$ at the point x . This ∞ -functor restricts to the full sub- ∞ -category of B -modules of the form $B[i]^n$ for various $i \geq 0$ and various n . Because F is inf-cartesian, the ∞ -functor $\mathbb{T}_{F,x}$ preserves finite products as well as the looping construction Ω_* (i.e. the natural map $\mathbb{T}_{F,x}(M[-1]) \rightarrow \Omega_*(\mathbb{T}_{F,x}(M))$ is an equivalence of spaces). This implies that there exists a unique B -dg-module $\mathbb{T}_{F,x}$ such that $\mathbb{T}_{F,x}(B[i]^n) \simeq \text{Map}_{B\text{-Mod}}(B[-i]^n, \mathbb{T}_{F,x})$ for all $i \geq 0$ and n . We still denote this complex by $\mathbb{T}_{F,x}$, and call it the *tangent complex of F at x* . The following result is an easy criterion for existence of cotangent complexes.

Lemma 3.0.3 *Let F be a derived stack which is inf-cartesian and $x : \text{Spec } B \rightarrow F$. Assume that the two conditions below are satisfied.*

1. *The ∞ -functor $M \mapsto \mathbb{T}_{F,x}(M)$ commutes with arbitrary colimits.*
2. *The B -module $\mathbb{T}_{F,x}$ is perfect.*

Then F has a cotangent complex at x and moreover we have $\mathbb{L}_{F,x}$ is naturally identified with $\mathbb{T}_{F,x}^\vee$, the B -linear dual of $\mathbb{T}_{F,x}$.

Proof: We consider the two ∞ -functors

$$B\text{-Mod}^c \rightarrow \mathbb{T},$$

sending M to either $\text{Map}_{B\text{-Mod}}(B, \mathbb{T}_{F,x} \otimes_B M)$ or $\mathbb{T}_{F,x}(M)$. There is a canonical equivalence

$$\text{Map}_{B\text{-Mod}}(B, \mathbb{T}_{F,x} \otimes_B -) \simeq \mathbb{T}_{F,x}(-)$$

when restricted to the full sub- ∞ -category of objects of the form $B[i]^n$. However, these objects generate $B\text{-Mod}^c$ by colimits, so by condition (1) the above equivalence remains an equivalence of ∞ -functors

defined on the whole ∞ -category $B - Mod^c$. In formula we have for any connective B -module M a natural equivalence

$$Map_{B-Mod}(B, \mathbb{T}_{F,x} \otimes_B M) \simeq \mathbb{T}_{F,x}(M).$$

When $\mathbb{F}_{F,x}$ is moreover perfect, this implies that $\mathbb{T}_{F,x}(M) \simeq Map_{B-Mod}(\mathbb{T}_{F,x}^\vee, M)$, and thus that the cotangent complex of F at x exists and is $\mathbb{L}_{F,x} = \mathbb{T}_{F,x}^\vee$. \square

3.1 Infinitesimal properties of \mathbf{Perf}^∇

We now study the formal properties of the derived moduli functors $\mathbf{Perf}^\nabla(X)$ and $\mathbf{Perf}^\nabla(\widehat{\partial}X)$ constructed in the previous section. The main result is the following.

Proposition 3.1.1 *Let X be a smooth algebraic variety over k and let us fix a good compactification $X \hookrightarrow \bar{X}$. Then, the two derived moduli stacks $\mathbf{Perf}^\nabla(X)$ and $\mathbf{Perf}^\nabla(\widehat{\partial}X)$ are nilcomplete and infinitesimally cartesian.*

Proof: We start by the case of $\mathbf{Perf}^\nabla(X)$. By construction this derived stack is a derived mapping stack and can be written of the form $\mathbf{Perf}^\nabla(X) \simeq \mathbf{Map}_{\mathbf{dSt}_k}(X_{DR}, \mathbf{Perf})$, where X_{DR} is the de Rham functor associated to X (see for example [Ga-Ro] for the relation between \mathcal{D} -modules and sheaves on X_{DR}). We can write $X = \text{colim} \text{Spec } A_i$ as a finite colimit of affine schemes, and thus $X_{DR} \simeq \text{colim}(\text{Spec } A_i)_{DR}$. The derived stack $\mathbf{Perf}^\nabla(X)$ is then the limit of $\mathbf{Perf}^\nabla(\text{Spec } A_i)$. As a limit of nilcomplete (resp. infinitesimally cartesian) derived stacks is again nilcomplete (resp. infinitesimally cartesian), we have reduced the statement to the case where $X = \text{Spec } A$ is furthermore affine. We conclude by the following lemma.

Lemma 3.1.2 *Let F be a nilcomplete (resp. infinitesimally cartesian) derived stack over k . For any affine scheme X , the derived mapping stack $\mathbf{Map}_{\mathbf{dSt}_k}(X, F)$ is again nilcomplete (resp. infinitesimally cartesian).*

Proof of the lemma: Let $X = \text{Spec } A$, and B any connective cdga. Assume first that F is nilcomplete. We consider the Postnikov tower $\{B_{\leq n}\}_n$ of B . The natural map

$$\mathbf{Map}_{\mathbf{dSt}_k}(X, F)(B) \longrightarrow \lim_n \mathbf{Map}_{\mathbf{dSt}_k}(X, F)(B_{\leq n})$$

can be written as

$$F(A \otimes_k B) \longrightarrow \lim_n F(A \otimes_k B_{\leq n}).$$

As k is a field, A is a flat over k , and the tower $\{A \otimes_k B_{\leq n}\}_n$ is a Postnikov tower for $A \otimes_k B$, and thus by the assumption on F the above morphism is an equivalence.

Let us now assume that F is infinitesimally cartesian. Let $B \oplus_d M$ be a square zero extension of B by a connective module M , given by a cartesian square

$$\begin{array}{ccc} B \oplus_d M & \longrightarrow & B \\ \downarrow & & \downarrow d \\ B & \xrightarrow{0} & B \oplus M[1]. \end{array}$$

Again because A is flat over k , tensoring with A induces a pull-back diagram of connective cdga's

$$\begin{array}{ccc} C \oplus_d M_C & \longrightarrow & C \\ \downarrow & & \downarrow d \\ C & \xrightarrow{0} & C \oplus M_C[1], \end{array}$$

where $C := A \otimes_k B$ and $M_C := C \otimes_B M$. As F is assumed infinitesimally cartesian, the image of this diagram by F remains a pull-back. By definition this diagram is equivalent to

$$\begin{array}{ccc} \mathbf{Map}_{\mathbf{dSt}_k}(X, F)(B \oplus_d M) & \longrightarrow & \mathbf{Map}_{\mathbf{dSt}_k}(X, F)(B) \\ \downarrow & & \downarrow \\ \mathbf{Map}_{\mathbf{dSt}_k}(X, F)(B) & \longrightarrow & \mathbf{Map}_{\mathbf{dSt}_k}(X, F)(B \oplus M[1]). \end{array}$$

This shows that $\mathbf{Map}_{\mathbf{dSt}_k}(X, F)$ is infinitesimally cartesian. \square

We now analyse the case of $\mathbf{Perf}^\nabla(\widehat{\partial}X)$. The argument here is slightly different as this is not a derived mapping stack. We start by writing $\bar{X} = \operatorname{colim} \operatorname{Spec} A_i$ as a colimit of open affine sub-schemes. By doing so we can assume that the divisor D is principal on each $\operatorname{Spec} A_i$, defined by an equation $f_i \in A_i$. By the descent result of [He-Po-Ve], we know that $\mathbf{Perf}^\nabla(\widehat{\partial}X) = \lim_i F_i$ is then equivalent to a limit of derived stacks. These derived stacks F_i can be described as follows. For each connective cdga B , we have the completed de Rham algebra of $A_i \otimes_k B$ defined by

$$\widehat{DR}_B(A_i) := \lim_j (DR(A_i/(f_i)^j) \otimes_k B).$$

This is a B -linear graded mixed cdga for which the weight zero part is $\widehat{A_i \otimes_k B} := \lim_j (A_i/(f_i)^j \otimes_k B)$. By inverting the weight zero element f_i we have a new graded mixed cdga

$$\widehat{DR}_B^o(A_i) := \lim_j (DR(A_i/(f_i)^j) \otimes_k B)[f_i^{-1}].$$

The derived stack F_i is then the functor sending B to the space of all graded mixed $\widehat{DR}_B^o(A_i)$ -dgmodules which are perfect as $\widehat{A \otimes_k B}$ -dgmodules. Let us drop the index i and simply write A for one of the A_i (and f for f_i). The derived stacks under considerations naturally carry structures of stacks in dg-categories and will be considered as such in the sequel.

We have a forgetful dg-functor

$$\widehat{DR}_B^o(A) - \mathbf{dg}_e^{gr} \longrightarrow \widehat{DR}_B^o(A) - \mathbf{dg}$$

from graded mixed dg-modules to dg-modules. According to corollary 1.2.3 this exhibits the left hand side as the dg-category of fixed points in $\widehat{DR}_B^o(A) - \mathbf{dg}$ for the natural action by the group \mathcal{H} on the right hand side. Restricting to dg-modules which are perfect over $\widehat{A \otimes_k B}$ on both sides provides a similar forgetful dg-functor

$$F(A) \longrightarrow \operatorname{Parf}(\widehat{DR}_B^o(A)).$$

Lemma 3.1.3 *The ∞ -functor $B \mapsto \operatorname{Parf}(\widehat{DR}_B^o(A))$ is nilcomplete and infinitesimally cartesian (as a derived stack of dg-categories).*

Proof of the lemma: It is easy to check that if $\{B_{\leq n}\}_n$ is the Postnikov tower for B , then $\{\widehat{A \otimes_k B_{\leq n}}\}_n$ is a Postnikov tower for $\widehat{A \otimes_k B}$. In the same manner, $\widehat{A \otimes_k -}$ will transform a square zero extension to a square zero extension. The lemma therefore reduces to the fact that the derived stack \mathbf{Perf} is nilcomplete and infinitesimally cartesian. This last assertion is true as \mathbf{Perf} is locally geometric (see [To-Va]). \square

We finish the proof of the proposition by using lemma 3.1.3 and the fact that taking fixed points by \mathcal{H} preserves limits of dg-categories. \square

It is unclear to us that $\mathbf{Perf}^{\nabla, ex}(\widehat{\partial}X)$ is also nilcomplete and infinitesimally cartesian. Again, we believe that the inclusion $\mathbf{Perf}^{\nabla, ex}(\widehat{\partial}X) \hookrightarrow \mathbf{Perf}^{\nabla}(\widehat{\partial}X)$ is an equivalence, but are unable to prove this at the moment. Note also that we mention the compactification \bar{X} in the statement of the proposition above as we do not know that $\mathbf{Perf}^{\nabla}(\widehat{\partial}X)$ is independent of the choice of compactification (as opposed to $\mathbf{Perf}^{\nabla, ex}(\widehat{\partial}X)$).

3.2 Cotangent complexes

We now turn to the study of cotangent complexes of the derived stacks $\mathbf{Perf}^{\nabla}(X)$ and $\mathbf{Perf}^{\nabla}(\widehat{\partial}X)$. In general, these cotangent complexes do not exist, except when X is proper. We thus introduce the following notion.

Definition 3.2.1 *Let B be a connective cdga. We say that an object $E \in \mathbf{Perf}^{\nabla}(X)(B)$ (resp. $E \in \mathbf{Perf}^{\nabla}(\widehat{\partial}X)(B)$) is End-Fredholm (or simple Fredholm) if the cotangent complex at E exists and is perfect.*

The computation of the tangent complexes of $\mathbf{Perf}^{\nabla}(X)$ and $\mathbf{Perf}^{\nabla, ex}(\widehat{\partial}X)$ are standards and is given by the following proposition.

Proposition 3.2.2 *Let B be a connective cdga and $E \in \mathbf{Perf}^{\nabla}(X)(B)$ with restriction $R(E) \in \mathbf{Perf}^{\nabla, ex}(\widehat{\partial}X)(B)$.*

1. *The ∞ -functor $M \mapsto \mathbb{T}_{\mathbf{Perf}^{\nabla}(X), E}(M)$ is equivalent to $M \mapsto \mathbb{H}_{DR}(X, E^{\vee} \otimes E \otimes_B M)[1]$.*
2. *The ∞ -functor $M \mapsto \mathbb{T}_{\mathbf{Perf}^{\nabla}(\widehat{\partial}X), R(E)}(M)$ is equivalent to $M \mapsto \mathbb{H}_{DR}(\widehat{\partial}X, R(E^{\vee} \otimes E) \otimes_B M)[1]$.*

As a direct consequence of the above proposition and the definition of being Fredholm we have the following direct corollary.

Corollary 3.2.3 *Let B be any connective cdga and $E \in \mathbf{Perf}^{\nabla}(X)(B)$.*

1. *The object E is Fredholm if and only if $\mathbb{H}_{DR}(X, E^{\vee} \otimes E)$ is a perfect B -module.*
2. *The object $R(E)$ is Fredholm if and only if $\mathbb{H}_{DR}(\widehat{\partial}X, R(E^{\vee} \otimes E))$ is a perfect B -module.*

Proof: (1) By definition it is enough to show that the formation of $\mathbb{H}_{DR}(X, E^{\vee} \otimes E)$ is compatible with base changes of B . But this is a consequence of the fact that it does so as a Tate B -module, as

shown in §2.3. The proof of (2) is similar. \square

We will see later that all objects are Fredholm when B is a field (see corollary 4.2.5). More generally, if $E \in \mathbf{Perf}^\nabla(X)(B)$ is any object, we will see that E and $R(E)$ are Fredholm under the condition that both $j_*(E)$ and $j_*(E^\vee)$ are perfect $\mathcal{D}_X \otimes_k B$ -modules in the sense of our appendix (and where $j : X \hookrightarrow \bar{X}$ is a good compactification). We refer to corollary 4.2.4 for this important statement which will be crucial in the proof of the representability theorem.

4 The lagrangian restriction map

In this section we construct a natural shifted lagrangian structure on the restriction morphism

$$R : \mathbf{Perf}^\nabla(X) \longrightarrow \mathbf{Perf}^\nabla(\widehat{\partial}X),$$

which is the de Rham analogue of the well known facts in Betti cohomology (see for instance [Pa-To]). However, the new feature here is that the derived stacks $\mathbf{Perf}^\nabla(X)$ and $\mathbf{Perf}^\nabla(\widehat{\partial}X)$ are not representable, and their tangent complexes can be infinite dimensional. We thus have to be careful with the notion of lagrangian structure itself. The definition of closed forms and isotropic structure is not problematic as it makes sense on general derived stacks. However, the non-degeneracy conditions in the definition of lagrangian structures causes a problem as there are a priori no direct relations between 2-forms on a derived stack F and global sections of $\wedge^2 \mathbb{L}_F$ (even assuming that \mathbb{L}_F exists).

Non-degeneracy in our setting will have to be defined pointwise, on all field valued points. For this we use in a crucial manner that the derived stacks $\mathbf{Perf}^\nabla(X)$ and $\mathbf{Perf}^\nabla(\widehat{\partial}X)$ are nil-complete and infinitesimally cartesian, and moreover that their cotangent complexes exist and are perfect at all field valued points (see proposition 3.1.1 corollary 4.2.5)

4.1 Reminders on forms and symplectic structures

Recall from [PTVV] that for any derived stack F we have a complex of \mathfrak{p} -forms $\mathcal{A}^p(F)$, and a complex of closed \mathfrak{p} -forms $\mathcal{A}^{p,cl}(F)$, together with a forgetful morphism $\mathcal{A}^{p,cl}(F) \longrightarrow \mathcal{A}^p(F)$. When $F = \mathit{Spec} A$ is a derived affine scheme, the complex $\mathcal{A}^p(F) \sim \wedge_A^p \mathbb{L}_A$ simply is the \mathfrak{p} -th wedge power of the cotangent complex of A . In the same manner, $\mathcal{A}^{p,cl}(F) \sim \prod_{i \geq p} (\wedge_A^i \mathbb{L}_A)[-i]$ is the completed derived truncated de Rham complex.

Suppose that F is any derived stack that possesses a cotangent complex $\mathbb{L}_F \in D_{qcoh}(F)$ in the sense recalled in the definition 3.0.1. There is a descent morphism

$$\mathbb{H}(F, \wedge_{\mathcal{O}_F}^p \mathbb{L}_F) \longrightarrow \mathcal{A}^p(F).$$

When F is a derived Artin stack, it is shown in [PTVV] that this morphism is a quasi-isomorphism. In general this descent morphism has no reason to be a quasi-isomorphism. This fact creates complications to define the non-degeneracy conditions of 2-forms [PTVV]. In this paper we overcome this complication by working pointwise on F as follows.

Definition 4.1.1 *A derived stack F is formally good if it is infinitesimally cartesian and for any k -field L and any $x \in F(L)$ the tangent complex $\mathbb{T}_x F$ is perfect over L .*

According to our results proposition 3.1.1 and corollary 4.2.5 the derived stacks $\mathbf{Perf}^\nabla(X)$ and $\mathbf{Perf}^\nabla(\hat{\partial}X)$ are formally good in the sense above.

Let F be a formally good derived stack and $x \in F(L)$ be a field valued point. We can restrict the functor F to the ∞ -category of artinian local augmented L -cdga by sending such a cdga $A \in \mathbf{dgArt}_L^*$ to the fiber of $F(A) \rightarrow F(L)$ taken at x . We denote this restriction by \hat{F}_x which by definition is the formal completion of F at x . As F is assumed to be infinitesimally cartesian the ∞ -functor \hat{F}_x is a formal moduli problem over L in the sense of [Lu3]. It then corresponds to an L -linear dg-lie algebra \mathcal{L}_x whose underlying complex is $\mathbb{T}_x F[-1]$.

By left Kan extension from artinian to whole connective cdga, the ∞ -functor \hat{F}_x can be itself consider as a derived stack. As such it possesses a complex of p -forms $\mathcal{A}^p(\hat{F}_x)$. It turns out that this complex can be computed purely in terms of the dg-Lie algebra \mathcal{L}_x as follows.

Proposition 4.1.2 *Let F be a formal moduli problem over L , associated to a dg-lie algebra \mathcal{L} . There is a canonical quasi-isomorphism*

$$\mathcal{A}^p(F) \simeq \underline{Hom}_{\mathcal{L}\text{-dg}}(k, \wedge^p(\mathcal{L}^\vee[-1])),$$

where \mathcal{L}^\vee is the L -linear dual of \mathcal{L} considered as a dg-module over \mathcal{L} by the coadjoint action.

Proof: We first prove the statement when F is representable, that is $F = \text{Spec } A$ for $A \in \mathbf{dgArt}_L^*$. In this case F has cotangent complex $\mathbb{L}_{A/L} \in D_{qcoh}(F)$. By [Lu3] there is a full embedding $D_{qcoh}(F) \hookrightarrow D(\mathcal{L} - \mathbf{dg})$, and the image of $\mathbb{L}_{A/L}$ is the dg-module $\mathcal{L}^\vee[-1]$, as this can be checked easily by using the universal property of $\mathbb{L}_{A/L}$. Finally, the above full embedding also sends \mathcal{O} to k , which implies the existence of the required equivalence

$$\mathcal{A}^p(F) = \underline{Hom}(\mathcal{O}, \wedge^p \mathbb{L}_{A/L}) \simeq \underline{Hom}_{\mathcal{L}\text{-dg}}(k, \wedge^p(\mathcal{L}^\vee[-1])).$$

This extends easily to the case where $F = \text{colim Spec } A_i$ is now only pro-representable by a pro-object " $\lim_i A_i$ " in \mathbf{dgArt}_L^* .

To deduce the general case we use the existence of smooth hyper-coverings proved in [Lu3], which insure that a general formal moduli problem F can be written as a geometric realization $|F_*|$ of a simplicial object in pro-representables which moreover satisfies the smooth hyper-coverings condition. We can then use the same descent argument as done in the algebraic case in [PTVV]. We consider the formal moduli problem $TF[-1] = \text{Map}(\text{Spec}(k \oplus k[1]), F)$, the shifted tangent of F . Using that F_* is a smooth hyper-covering we observe that $TF[-1]$ is again the realization of $TF_*[-1]$. Passing to the complex of functions we find that the natural morphism

$$\underline{Hom}_{\mathcal{L}\text{-dg}}(k, \wedge^p(\mathcal{L}^\vee[-1])) \longrightarrow \lim_n \underline{Hom}_{\mathcal{L}_n\text{-dg}}(k, \wedge^p(\mathcal{L}_n^\vee[-1]))$$

is a quasi-isomorphism (where we have denoted by \mathcal{L}_n the dg-Lie algebra corresponding to F_n). This last descent statement, together with the already treated case of pro-representable F provides the result. \square

Going back to our formally good stack F and $x \in F(L)$ a field valued point. Using the proposition 4.1.2 we see that there is a natural restriction map

$$\mathcal{A}^p(F) \longrightarrow \mathcal{A}^p(\hat{F}_x) \simeq \underline{Hom}_{\mathcal{L}_x\text{-dg}}(k, \wedge^p(\mathcal{L}_x^\vee[-1])) \longrightarrow \wedge^p(\mathcal{L}_x^\vee[-1]),$$

where the last morphism is obtained by forgetting the \mathcal{L}_x -module structure.

Definition 4.1.3 Let F be a formally good derived stack and $\omega \in H^n(\mathcal{A}^{2,cl}(F))$ be a closed 2-form of degree n on F . We say that ω is non-degenerate if for any field valued point $x \in F(L)$, the image of ω by the morphism

$$\mathcal{A}^{2,cl}(F) \longrightarrow A^2(F) \longrightarrow \wedge^p(\mathcal{L}_x^\vee[-1]) \simeq \wedge^2(\mathbb{T}_{F,x}^\vee)$$

is a non-degenerate pairing of degree n and induces an equivalence $\mathbb{T}_{F,x} \simeq \mathbb{T}_{F,x}^\vee[n]$.

The above definition is easy to generalize to the relative setting as follows. Suppose now that we have a morphism of formally good derived stacks $f : F \longrightarrow F'$, and ω a closed 2-form of degree n on F' . Assume that we are given a homotopy to zero $h : f^*(\omega) \sim 0$ inside $\mathcal{A}^{2,cl}(F)$. By what we have seen, for any field valued point $x \in F(L)$, the form ω and the homotopy h induces an n-cocycle ω_x in $\wedge^2 \mathbb{T}_{F',x}^\vee$ as well as a null homotopy of its image $f^*(\omega_x)$ in $\wedge^2 \mathbb{T}_{F,x}^\vee$. This null homotopy induces a well defined morphism of complexes

$$\mathbb{T}_{F,x} \longrightarrow \mathbb{T}_{F/F',x}^\vee[n-1].$$

Definition 4.1.4 Let $f : F \longrightarrow F'$ be a morphism of formally good derived stacks. Let ω be a closed 2-form of degree n on F' and $h : f^*(\omega) \sim 0$ a co-isotropic structure on f with respect to ω . We say that the isotropic structure is lagrangian if for any field valued point $x \in F(L)$ the induced morphism of complexes

$$\mathbb{T}_{F,x} \longrightarrow \mathbb{T}_{F/F',x}^\vee[n-1]$$

is a quasi-isomorphism.

4.2 Orientation on the formal boundary

In this section we will prove that the conditions for applying the results of [To3] are satisfied for the restriction morphism of derived stacks

$$R : \mathbf{Perf}^\nabla(X) \longrightarrow \mathbf{Perf}^\nabla(\widehat{\partial}X).$$

The main step consists of studying Serre duality on $\widehat{\partial}X$ and the key ingredient is the construction of the integration map

$$or : \mathbb{H}(\widehat{\partial}X, R(\omega_X)) \longrightarrow k[1-d]$$

where d is the dimension of X (we assume that X is connected for sake of simplicity).

As a start, $\omega_X := \Omega_X^d$ is here the canonical sheaf of X . Its restriction to $\widehat{\partial}X$, $R(\omega_X)$ will still be denoted by ω_X for simplicity. We pick a good compactification $X \hookrightarrow \bar{X}$ once for all. By the formal gluing theorem of we have a cartesian square

$$\begin{array}{ccc} \mathbb{H}(\bar{X}, \omega_{\bar{X}}) & \longrightarrow & \mathbb{H}(X, \omega_X) \\ \downarrow & & \downarrow \\ \mathbb{H}(\hat{D}, \hat{\omega}_{\bar{X}}) & \longrightarrow & \mathbb{H}(\widehat{\partial}X, \omega_X). \end{array}$$

The boundary map for this cartesian square produces a morphism of complexes over k $u : \mathbb{H}(\widehat{\partial}X, \omega_X) \longrightarrow \mathbb{H}(\bar{X}, \omega_{\bar{X}})[1]$. Composed with Grothendieck trace morphism $H^d(\bar{X}, \omega_{\bar{X}}) \simeq k$ we find the required morphism of complexes

$$or : \mathbb{H}(\widehat{\partial}X, \omega_X) \longrightarrow k[1-d].$$

The map defined above is a version of the residue morphism, for instance it coincides with the usual residue of forms when X is a curve (and the residues are taken at the points at infinity).

The morphism or is here defined as a morphism of complexes over k . However, the left hand side is the realization of the ind-pro complex $\tilde{\mathbb{H}}(\widehat{\partial}X, \omega_X)$ as explained in our section §2.3. By construction the above cartesian square obtained by formal gluing lifts canonically to a cartesian square of Tate complexes over k . This implies that the boundary morphism mentioned above u also lifts canonically as a morphism in the ind-pro category. As a result or arises as the realization of a natural morphism of Tate complexes

$$\tilde{or} : \tilde{\mathbb{H}}(\widehat{\partial}X, \omega_X) \longrightarrow k[1-d].$$

By base change (see §2.3) we have an induced morphism for every connective cdga B

$$\tilde{or} : \tilde{\mathbb{H}}(\widehat{\partial}X, \omega_X \otimes_k B) \longrightarrow B[1-d].$$

Assume now that B is a connective cdga and E and F are two perfect complexes over $X \times S$, with $S = \text{Spec } B$. For sake of simplicity we assume that E and F can be extended to perfect complexes on $\bar{X} \times S$ (but this is not strictly necessary for the results below). We have a composition morphism

$$\mathbb{H}(\widehat{\partial}X, R(E)^\vee \otimes R(F)) \otimes_B \mathbb{H}(\widehat{\partial}X, R(F)^\vee \otimes R(E) \otimes \omega_X) \longrightarrow \mathbb{H}(\widehat{\partial}X, R(E)^\vee \otimes R(E) \otimes \omega_X)$$

which we can compose with the trace morphism $R(E)^\vee \otimes R(E) \longrightarrow R(\mathcal{O}_X)$, and with the orientation or in order to get a pairing

$$\mathbb{H}(\widehat{\partial}X, R(E)^\vee \otimes R(F)) \otimes_B \mathbb{H}(\widehat{\partial}X, R(F)^\vee \otimes R(E) \otimes \omega_X) \longrightarrow B[1-d].$$

This pairing also possesses a canonical lift as a pairing of Tate B -modules. Indeed, we already have seen that or has such a lift, and composition and trace also are compatible with the ind-pro structures. We thus have defined a canonical pairing of Tate B -modules

$$\tilde{\mathbb{H}}(\widehat{\partial}X, R(E)^\vee \otimes R(F)) \hat{\otimes}_B \tilde{\mathbb{H}}(\widehat{\partial}X, R(F)^\vee \otimes R(E) \otimes \omega_X) \longrightarrow B[1-d].$$

By rigidity there is no lose of generality to assume $F = \mathcal{O}_X$. The pairing can then be written as

$$\tilde{\mathbb{H}}(\widehat{\partial}X, R(E)^\vee) \hat{\otimes}_B \tilde{\mathbb{H}}(\widehat{\partial}X, R(E) \otimes \omega_X) \longrightarrow B[1-d].$$

By construction, the orientation morphism \tilde{or} canonically vanishes on $\mathbb{H}(X, \omega_X)$, which implies that the above pairing induces a new pairing of Tate B -modules

$$\mathbb{H}(X, E^\vee) \hat{\otimes}_B \tilde{\mathbb{H}}_c(X, E \otimes \omega_X) \longrightarrow B[-d].$$

The following result is Serre duality with compact supports.

Proposition 4.2.1 *The above pairing is non-degenerate. It induces an equivalence of Tate B -modules*

$$\tilde{\mathbb{H}}_c(X, E \otimes \omega_X) \simeq \mathbb{H}(X, E^\vee)^\vee[-d].$$

Proof: The pairing induces a morphism of B -modules $\alpha : \mathbb{H}(X, E^\vee) \longrightarrow \tilde{\mathbb{H}}_c(X, E \otimes \omega_X)^\vee[-d]$. Here, $\tilde{\mathbb{H}}_c(X, E \otimes \omega_X)^\vee$ is the dual of $\tilde{\mathbb{H}}_c(X, E \otimes \omega_X)$ as a Tate module. As $\tilde{\mathbb{H}}_c(X, E \otimes \omega_X)$ is pro-perfect this dual is a genuine B -module. We must show that the above morphism α is an equivalence. For this, we

come back to the formal gluing cartesian square, and the definition of the pairing. We have the exact triangle of Tate B -modules

$$\tilde{\mathbb{H}}_c(X, E \otimes \omega_X) \longrightarrow \mathbb{H}(\bar{X}, E \otimes \omega_{\bar{X}}) \longrightarrow \text{''}lim_n\text{''} \mathbb{H}(D_{(n)}, j_n^*(E \otimes \omega_{\bar{X}})).$$

The right hand side can be written as $\text{''}lim_n\text{''} \mathbb{H}(D_{(n)}, j_n^*(E) \otimes \omega_{D_{(n)}} \otimes \mathcal{L})$, where \mathcal{L} is the conormal sheaf of $D_{(n)} \hookrightarrow X$. By Serre duality on \bar{X} and $D_{(n)}$ each n , the restriction map $\mathbb{H}(\bar{X}, E \otimes \omega_X) \longrightarrow \mathbb{H}(D_{(n)}, j_n^*(E) \otimes \omega_{D_{(n)}} \otimes \mathcal{L})$ is dual to the natural map $\mathbb{H}(D_{(n)}, j_n^*(E^\vee) \otimes \mathcal{L})[d-1] \longrightarrow \mathbb{H}(\bar{X}, E^\vee)[d]$. Passing to the colimit over n these assemble to the natural map

$$\mathbb{H}_D(\bar{X}, E^\vee)[d] \longrightarrow \mathbb{H}(\bar{X}, E^\vee)[d],$$

where the left hand side is cohomology with supports in D . The cofiber of this map is then natural equivalent to $\mathbb{H}(X, E^\vee)[d]$. This constructs a natural equivalence of B -modules

$$\tilde{\mathbb{H}}_c(X, E \otimes \omega_X)^\vee \simeq \mathbb{H}(X, E^\vee)[d].$$

This equivalence can be checked to be the morphism of the proposition induced by the orientation or . \square

Corollary 4.2.2 *The pairing of Tate B -modules*

$$\tilde{\mathbb{H}}(\widehat{\partial}X, R(E)^\vee) \hat{\otimes}_B \tilde{\mathbb{H}}(\widehat{\partial}X, R(E) \otimes \omega_X) \longrightarrow B[1-d]$$

is non-degenerate.

Proof: We have two exact triangles of Tate B -modules

$$\tilde{\mathbb{H}}_c(X, E^\vee) \longrightarrow \mathbb{H}(X, E^\vee) \longrightarrow \tilde{\mathbb{H}}(\widehat{\partial}X, R(E^\vee))$$

$$\tilde{\mathbb{H}}_c(X, E \otimes \omega_X) \longrightarrow \mathbb{H}(X, E \otimes \omega_X) \longrightarrow \tilde{\mathbb{H}}(\widehat{\partial}X, R(E \otimes \omega_X))$$

The dual, inside Tate B -modules, of the second triangle is (up to a rotation and shift by $-d$)

$$\mathbb{H}(X, E \otimes \omega_X)^\vee[-d] \longrightarrow \tilde{\mathbb{H}}_c(X, E \otimes \omega_X)^\vee[-d] \longrightarrow \tilde{\mathbb{H}}(\widehat{\partial}X, R(E \otimes \omega_X))^\vee[1-d].$$

By the construction of the orientation or , the natural pairing produces a commutative diagram of Tate B -modules

$$\begin{array}{ccccc} \tilde{\mathbb{H}}_c(X, E^\vee) & \longrightarrow & \mathbb{H}(X, E^\vee) & \longrightarrow & \tilde{\mathbb{H}}(\widehat{\partial}X, R(E^\vee)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}(X, E \otimes \omega_X)^\vee[d] & \longrightarrow & \tilde{\mathbb{H}}_c(X, E \otimes \omega_X)^\vee[d] & \longrightarrow & \tilde{\mathbb{H}}(\widehat{\partial}X, R(E \otimes \omega_X))^\vee[d-1]. \end{array}$$

The first two vertical morphisms on the left are equivalences by proposition 4.2.1. Therefore the third left vertical morphism on the right is an equivalence. \square

The same orientation morphism can be used in order to prove a duality statement for de Rham cohomology with compact supports. It goes as follows. The complex of sheaves $|\widehat{\mathbf{DR}}_B^o|$ on D , computing $\mathbb{H}_{DR}(\widehat{\partial}X, R(\mathcal{O}_X))$ is bounded of amplitude contained in $[0, d]$. Moreover, its last non-zero term is $R(\omega_X)$. Therefore, there is a canonical map

$$H_{DR}^{2d-1}(\widehat{\partial}X, R(\mathcal{O}_X)) \longrightarrow H^{d-1}(\widehat{\partial}X, R(\omega_X)).$$

Composing with the orientation map $or : H^{d-1}(\widehat{\partial}X, R(\omega_X)) \longrightarrow k[1-d]$ we get an orientation morphism $\mathbb{H}_{DR}(\widehat{\partial}X, R(\mathcal{O}_X)) \longrightarrow k[1-2d]$. As before it extends naturally as a morphism of Tate complexes over k

$$or : \widetilde{\mathbb{H}}_{DR}(\widehat{\partial}X, R(\mathcal{O}_X)) \longrightarrow k[1-2d].$$

For any connective cdga B and any $E \in \mathbf{Perf}^\nabla(X)(B)$, the above orientation defined as before two pairings of Tate B -modules

$$\begin{aligned} \widetilde{\mathbb{H}}_{c,DR}(X, E) \otimes_B \widetilde{H}H_{DR}(X, E^\vee) &\longrightarrow B[-2d] \\ \widetilde{\mathbb{H}}_{DR}(\widehat{\partial}X, R(E)) \otimes_B \widetilde{\mathbb{H}}_{DR}(\widehat{\partial}X, R(E)^\vee) &\longrightarrow B[1-2d]. \end{aligned}$$

Proposition 4.2.3 *Both pairing above are non-degenerate and induces natural equivalences of Tate B -modules*

$$\widetilde{\mathbb{H}}_{c,DR}(X, E) \simeq \widetilde{\mathbb{H}}_{DR}(X, E^\vee)^\vee[1-2d] \quad \widetilde{\mathbb{H}}_{DR}(\widehat{\partial}X, R(E)) \simeq \widetilde{\mathbb{H}}_{DR}(\widehat{\partial}X, R(E)^\vee)^\vee[-2d].$$

Proof: The proof is by using the Hodge filtrations on the various complexes computing these cohomology groups. In terms of graded mixed modules these are the filtrations on $|E|$ given by $\oplus_{i \geq p} E(i)[-2i] \subset \oplus_i E(i)[-2i]$. The associated graded of these filtrations are perfect complexes of the form $E(0) \otimes_{\mathcal{O}_X} \Omega_X^i[-i]$. The pairings of the proposition are compatible with these filtrations and the induced pairings are the one for Serre duality of perfect complexes. Therefore the proposition is a consequence of Serre duality with compact supports (see proposition 4.2.1). \square

One important corollary of the previous results in the following criterion for finiteness of the de Rham cohomology of $\widehat{\partial}X$.

Corollary 4.2.4 *Let $E \in \mathbf{Perf}^\nabla(X)(B)$ be such that $\mathbb{H}_{DR}(X, E)$ and $\mathbb{H}_{DR}(X, E^\vee)$ are both perfect B -modules. Then the Tate B -module $\widetilde{\mathbb{H}}(\widehat{\partial}(X), R(E))$ and $\widetilde{\mathbb{H}}_{c,DR}(X, E)$ are both perfect.*

Proof: Using the exact triangle

$$\widetilde{\mathbb{H}}_{c,DR}(X, E) \longrightarrow \mathbb{H}_{DR}(X, E) \longrightarrow \widetilde{\mathbb{H}}_{DR}(\widehat{\partial}X, R(E))$$

we see that the Tate B -module $\widetilde{\mathbb{H}}_{DR}(\widehat{\partial}X, R(E))$ must be pro-perfect. But corollary 4.2.2 implies that its dual is also pro-perfect. This implies that it must be perfect. \square

One important consequence is the following.

Corollary 4.2.5 *1. Let $E \in \mathbf{Perf}^\nabla(X)(B)$ be such that $\mathbb{H}_{DR}(X, E^\vee \otimes E)$ is perfect over B , then E and $R(E)$ are both Fredholm in the sense of definition 3.2.1.*

2. If $B = k$, any $E \in \mathbf{Perf}^\nabla(X)(k)$ is Fredholm and so is $R(E)$.

Proof: (1) is a direct consequence of corollary 4.2.4 and the fact that both $\mathbb{H}_{DR}(X, E)$ and $\tilde{\mathbb{H}}(\widehat{\partial}(X), R(E))$ are stable by base changes of B . For (2), we have to show that for any $E \in \mathbf{Perf}^\nabla(X)(k)$ the complex $\mathbb{H}_{DR}(X, E^\vee \otimes E)$ is perfect over k . But $E^\vee \otimes E$ is a bounded complex of coherent \mathcal{D}_X -modules with holonomic cohomologies. By Bernstein theorem the holonomic \mathcal{D} -modules are stable by push-forward, and thus $\mathbb{H}_{DR}(X, E^\vee \otimes E)$ is a bounded complex with finite dimensional cohomology and thus perfect. \square

We now are in the position of constructing the lagrangian structure on the restriction morphism $R : \mathbf{Perf}^\nabla(X) \longrightarrow \mathbf{Perf}^\nabla(\widehat{\partial}X)$. For this we use the main result of [To3]. The derived stack $\mathbf{Perf}^\nabla(X)$ is indeed the underlying stack of a derived stack in symmetric monoidal rigid dg-categories. According to [To3] in order to construct a closed 2-form ω on $\mathbf{Perf}^\nabla(\widehat{\partial}X)$, together with an homotopy $h : R^*(\omega) \sim 0$, it is enough to:

1. construct a morphism of complexes of k -modules

$$or : \mathbb{H}_{DR}(\widehat{\partial}X, R(\mathcal{O}_X)) \longrightarrow k[1 - 2d]$$

together with a homotopy to zero of the restriction

$$R^*(or) : \mathbb{H}_{DR}(X, \mathcal{O}_X) \longrightarrow k[1 - 2d]$$

2. prove that for any connective cdga B the induced morphisms

$$\begin{aligned} \mathbb{H}_{DR}(\widehat{\partial}X, R(\mathcal{O}_X)) \otimes_k B &\longrightarrow \mathbb{H}_{DR}(\widehat{\partial}X, R(\mathcal{O}_X) \otimes_k B) \\ \mathbb{H}_{DR}(X, \mathcal{O}_X) \otimes_k B &\longrightarrow \mathbb{H}_{DR}(X, \mathcal{O}_X \otimes_k B) \end{aligned}$$

are equivalences of B -modules.

The statement (2) is true thanks to corollary 4.2.5. The construction of or has been done previously in this section. Let us recall that it comes from the cartesian square

$$\begin{array}{ccc} \mathbb{H}(\bar{X}, \omega_{\bar{X}}) & \longrightarrow & \mathbb{H}(X, \omega_X) \\ \downarrow & & \downarrow \\ \mathbb{H}(\widehat{D}, \hat{j}^*(\omega_X)) & \longrightarrow & \mathbb{H}(\widehat{\partial}X, R(\omega_X)), \end{array}$$

and its boundary map $H^{d-1}(\widehat{\partial}X, R(\omega_X)) \longrightarrow H^d(\bar{X}, \omega_{\bar{X}}) \simeq k$. Precomposing with the canonical map $H_{DR}^{2d-1}(\widehat{\partial}X, R(\mathcal{O}_X)) \longrightarrow H^{d-1}(\widehat{\partial}X, R(\omega_X))$ provides the orientation morphism

$$or : \mathbb{H}_{DR}(\widehat{\partial}X, R(\mathcal{O}_X)) \longrightarrow k[1 - 2d].$$

By construction, the composition $H^{d-1}(X, \omega_X) \longrightarrow H^{d-1}(\widehat{\partial}X, R(\omega_X)) \longrightarrow H^d(\bar{X}, \omega_{\bar{X}})$ is the zero map so a null homotopy of the morphism $R^*(or) : \mathbb{H}_{DR}(X, \mathcal{O}_X) \longrightarrow k[1 - 2d]$ is given by a morphism $H_{DR}^{2d}(X, \mathcal{O}_X) \longrightarrow k$. If X is proper, we take this map to be the natural isomorphism. If X is not proper then $H_{DR}^{2d}(X, \mathcal{O}_X) = 0$ and this map is the zero map.

By the main result of [To3], we have that the derived stack $\mathbf{Perf}^\nabla(\widehat{\partial}X)$ carries a canonical closed 2-form ω of degree $3 - 2d$. Moreover the pull-back form $R^*(\omega)$ comes equipped with a natural null-homotopy $h : R^*(\omega) \sim 0$. We thus have proved the following statement.

Corollary 4.2.6 *The morphism of derived stacks $R : \mathbf{Perf}^\nabla(X) \longrightarrow \mathbf{Perf}^\nabla(\widehat{\partial}X)$ carries a canonical isotropic structure of degree $2 - 2d$.*

As explained in definition 4.1.4, we can evaluate the lagrangian structure to all field valued points of $\mathbf{Perf}^\nabla(X)$. For a given such point $E \in \mathbf{Perf}^\nabla(X)(L)$, for L a k -field, the morphism induced by the isotropic structure

$$\mathbb{T}_{\mathbf{Perf}^\nabla(X), E} \longrightarrow \mathbb{L}_{\mathbf{Perf}^\nabla(X)/\mathbf{Perf}^\nabla(fbX), E}[2 - 2d]$$

becomes, after identifications given by proposition 3.2.2, to the duality morphism

$$\mathbb{H}_{DR}(X, E^\vee \otimes E) \longrightarrow \widetilde{\mathbb{H}}_{c, DR}(X, E^\vee \otimes E)^\vee[-2d].$$

This last morphism is an equivalence because of proposition 4.2.3.

Corollary 4.2.7 *The isotropic structure of corollary 4.2.6 is a lagrangian structure in the sense of definition 4.1.4.*

5 The relative representability theorem

In this last part we prove that the fibers over field valued points of the restriction morphism R are locally representable by algebraic spaces in the sense of our appendix B. We prove this statement restricting ourselves to vector bundles endowed with flat connexions. The extension to the perfect complexes setting can be restricted to this special case by truncation. We omit the details in this work and leave this to the interesting reader.

We first consider the derived substack $\mathbf{Vect}^\nabla(X) \subset \mathbf{Perf}^\nabla(X)$ consisting of all objects whose underlying \mathcal{O}_X -module is a vector bundle. In more explicit terms, for a connective cdga B , an object $E \in \mathbf{Perf}^\nabla(X)(B)$ lies in $\mathbf{Vect}^\nabla(X)(B)$ if the $\mathcal{O}_X \otimes_k B$ -module $E(0)$ is locally free of finite rank. We define similarly $\mathbf{Vect}^\nabla(\widehat{\partial}X)(B) \subset \mathbf{Perf}^\nabla(\widehat{\partial}X)(B)$ as objects E such that $E(0)$ is locally free of finite rank as a $\widehat{\mathcal{O}}_{D, B}$ -module.

We fix once for all $V_\infty \in \mathbf{Vect}^\nabla(\widehat{\partial}X)(k)$, a vector bundle with flat connexion on the formal boundary of X . The fiber of the restriction morphism $R : \mathbf{Vect}^\nabla(X) \longrightarrow \mathbf{Vect}^\nabla(\widehat{\partial}X)$ taken at V_∞ will be denoted by $\mathbf{Vect}_{V_\infty}^\nabla(X)$. It is the derived stack of vector bundles with flat connexions on X whose restriction to $\widehat{\partial}X$ is fixed being V_∞ . When no component of X is proper, the rank of V_∞ fixes the rank of all objects in $\mathbf{Vect}_{V_\infty}^\nabla(X)$. As the proper case of the results is well known we assume that X has no proper component.

Theorem 5.0.1 *With the notations above, the derived stack $\mathbf{Vect}_{V_\infty}^\nabla(X)$ is a derived quasi-algebraic space in the sense of definition B.0.2.*

Proof: We will prove the theorem by applying the version of Artin-Lurie's criterion for quasi-algebraic derived stacks recalled in our appendix B.0.3. By Galois descent we assume that k is algebraically closed. We also assume that the derived stack $\mathbf{Vect}_{V_\infty}^\nabla(X)$ is not empty, or equivalently that V_∞ extends to a flat vector bundle V on the whole X .

By proposition 3.1.1 we know that $\mathbf{Vect}_{V_\infty}^\nabla(X)$ is infinitesimally cartesian and nil-complete, as it is defined to be the fiber between two infinitesimally cartesian and nil-complete derived stacks. Let us

show moreover that it has a global cotangent complex. By definition 3.2.1 this is the content of the following lemma.

Lemma 5.0.2 *Let B be any connective cdga and $E \in \mathbf{Vect}_{V_\infty}^\nabla(X)(B)$ an object. Then the image of E in $\mathbf{Vect}^\nabla(X)(B)$ is Fredholm over B .*

Proof of the lemma: This is consequence of our corollary 3.2.3. Indeed, we have an exact triangle of Tate B -modules

$$\tilde{\mathbb{H}}_{c,DR}(X, E^\vee \otimes E) \longrightarrow \mathbb{H}_{DR}(X, E^\vee \otimes E) \longrightarrow \tilde{\mathbb{H}}_{DR}(\widehat{\partial}X, R(E^\vee \otimes E)).$$

The right hand side is equivalent to $\tilde{\mathbb{H}}_{DR}(\widehat{\partial}X, R(V_\infty^\vee \otimes V_\infty)) \otimes_k B$ and by corollary 4.2.5 is perfect over B . In particular, it is compact and cocompact as an ind-pro-perfect B -module. As $\mathbb{H}_{DR}(X, E^\vee \otimes E)$ is ind-perfect it is cocompact as an ind-pro-perfect B -module. We thus have that $\tilde{\mathbb{H}}_{c,DR}(X, E^\vee \otimes E)$ is also cocompact as an ind-prop-perfect B -module. As it is pro-perfect, it must be perfect. But this implies that $\mathbb{H}_{DR}(X, E^\vee \otimes E)$ is perfect and thus that E is Fredholm by corollary 3.2.3. \square

The previous lemma shows that $\mathbf{Vect}_{V_\infty}^\nabla(X)$ has a global cotangent complex which is furthermore perfect. In order to apply the theorem B.0.3 it remains to prove that $\mathbf{Vect}_{V_\infty}^\nabla(X)$ satisfies the three conditions (2), (5) and (6). These three statements are properties of the restriction of $\mathbf{Vect}_{V_\infty}^\nabla(X)$ to underived k -algebras. Let us denote this restriction by $\mathbf{Vect}_{V_\infty}^\nabla(X)_0$.

We start by studying the diagonal morphism of $\mathbf{Vect}_{V_\infty}^\nabla(X)_0$ in order to check condition (2) of theorem B.0.3.

Lemma 5.0.3 *The diagonal morphism*

$$d : \mathbf{Vect}_{V_\infty}^\nabla(X)_0 \longrightarrow \mathbf{Vect}_{V_\infty}^\nabla(X)_0 \times \mathbf{Vect}_{V_\infty}^\nabla(X)_0$$

is representable by a scheme of finite type over k .

Proof of the lemma: The statement of the lemma is equivalent to state that for any discrete cdga B and any two points E and F in $\mathbf{Vect}_{V_\infty}^\nabla(X)(B)$, the sheaf of isomorphisms $Iso(E, F)$ is representable by a scheme of finite type over $Spec B$. This sheaf is an open sub-sheaf inside the sheaf of morphisms $Hom(E, F)$ from E to F , it is therefore enough to prove that $Hom(E, F)$ is representable by a scheme over finite type over B . The values over a B -algebra B' of this sheaf is given as the fiber at the identity of the restriction map

$$0 \longrightarrow Hom(E, F)(B') \longrightarrow H_{DR}^0(X, E^\vee \otimes F \otimes_B B') \longrightarrow H_{DR}^0(\widehat{\partial}X, V_\infty^\vee \otimes V_\infty) \otimes_k B'.$$

In other words $Hom(E, F)$ is the sheaf of morphisms with compact supports (i.e. restrict to the identity morphism on $\widehat{\partial}X$) from E to F . Because $E^\vee \otimes F$ is automatically Fredholm, the functor sending B' to $H_{DR}^0(X, E^\vee \otimes F \otimes_B B')$ is the H^0 -functor of a perfect complex over B of amplitude $[0, \infty[$, and thus is representable by a scheme of finite type.

Sublemma 5.0.4 *Let K be a perfect complex on a commutative k -algebra B , and suppose that K has amplitude contained in $[0, \infty[$. Then the functor $B' \mapsto H^0(K \otimes_B B')$ is representable by an affine scheme of finite presentation over B' .*

Proof of the sublemma: Because of the amplitude hypothesis K can be presented by a bounded complex of projective modules of finite rank

$$0 \longrightarrow K^0 \longrightarrow K^1 \longrightarrow \dots \longrightarrow K^n$$

for some integer n . The functor under consideration is then the kernel of $K^0 \longrightarrow K^1$, that is the kernel of a morphism between vector bundles over $\text{Spec } B$, and the result follows as affine schemes of finite presentation over B are stable by fiber products. \square

Coming back to the proof of the lemma 5.0.3, the sublemma and the fact that $E^\vee \otimes F$ is automatically Fredholm, imply that both functors

$$B' \mapsto H_{DR}^0(X, E^\vee \otimes F \otimes_B B')$$

$$B' \mapsto H_{DR}^0(\widehat{\partial}X, V_\infty^\vee \otimes V_\infty) \otimes_k B'$$

are representable by affine schemes of finite presentation over $\text{Spec } B$. We thus get that the sheaf $\text{Hom}(E, F)$ is also representable by an affine scheme of finite presentation over $\text{Spec } B$, and thus the lemma. \square

The previous lemma implies that our condition (2) of theorem B.0.3 is also satisfied. Indeed, the diagonal morphism has the property that it is nil-complete, inf-cartesian and possesses a perfect cotangent complex, so the fact that it is representable at the levels of truncations implies that it is representable (see [To-Ve]). The condition (1) of theorem B.0.3 is also satisfied as no components of X are assumed to be proper, so for any $V \in \mathbf{Vect}^\nabla(X)(k)$ the induced morphism

$$H_{DR}^0(X, V^\vee \otimes V) \longrightarrow H_{DR}^0(\widehat{\partial}X, R(V)^\vee \otimes R(V))$$

is injective. It thus remains to check conditions (5) and (6) of B.0.3.

Let us start by showing that condition (5) of theorem B.0.3 is satisfied by $\mathbf{Vect}_{V_\infty}^\nabla(X)$. By [Mo] we can chose a good compactification $X \hookrightarrow \bar{X}$ (possibly stacky) such that the underlying bundle of V_∞ extends to a vector bundle \bar{V} on \bar{X} . We denote by $D \hookrightarrow \bar{X}$ the divisor at infinity. The connection on V_∞ can then be represented by a connection with poles

$$\bar{\nabla} : \bar{V} \longrightarrow \Omega_{\bar{X}}^1(nD) \otimes_{\mathcal{O}_{\bar{X}}} \bar{V}$$

for some integer n . The morphism $\bar{\nabla}$ can also be interpreted as a splitting of the Atiyah extension with poles along D

$$E(V, n) : 0 \longrightarrow \Omega_{\bar{X}}^1(nD) \otimes_{\mathcal{O}_{\bar{X}}} \bar{V} \longrightarrow \mathcal{P}(\bar{V})(nD) \longrightarrow \bar{V} \longrightarrow 0,$$

where $\mathcal{P}(\bar{V})(nD)$ is the vector bundle of principal parts of \bar{V} possibly with poles of order at most n along D .

We consider the (underived) stack of pairs (W, ∇) , consisting of a vector bundle on \bar{X} and a flat connection ∇ on W with poles of order at most n along D . By definition this stack sends a commutative k -algebra B to the groupoid of vector bundles on $\bar{X} \times \text{Spec } B$, together with a splitting ∇ of the exact sequence of bundles on $X \times \text{Spec } B$

$$E(W, n) : 0 \longrightarrow \Omega_{\bar{X}}^1(nD) \otimes_{\mathcal{O}_{\bar{X}}} \bar{W} \otimes_k B \longrightarrow \mathcal{P}(\bar{W})(nD) \otimes_k B \longrightarrow \bar{W} \otimes_k B \longrightarrow 0,$$

satisfying the integrability condition $\nabla^2 = 0$ (as a section of $\Omega_{\bar{X}}^2(2nD) \otimes_{\mathcal{O}_{\bar{X}}} \text{End}(W)$). Let denote this stack by $\mathcal{F}_{\bar{X}}$. This is obviously an Artin stack locally of finite type over k . In the same manner we can define $\mathcal{F}_X := \mathbf{Vect}^\nabla(X)_0$, as well as $\mathcal{F}_{\hat{D}}$, the stack of vector bundles W on the formal completion endowed with flat connections with poles or order at most n along $D \hookrightarrow \hat{D}$. Finally, we have $\mathcal{F}_{\hat{\partial}X} := \mathbf{Vect}^\nabla(\hat{\partial}X)_0$. Formal gluing (see [He-Po-Ve]) implies that there exists a cartesian square of underived stacks

$$\begin{array}{ccc} \mathcal{F}_{\bar{X}} & \longrightarrow & \mathcal{F}_{\hat{D}} \\ \downarrow & & \downarrow \\ \mathcal{F}_X & \longrightarrow & \mathcal{F}_{\hat{\partial}X}. \end{array}$$

The stack $\mathcal{F}_{\hat{D}}$ is a limit of Artin stacks locally of finite type, and thus satisfies the condition (5) of theorem B.0.3. The stack $\mathcal{F}_{\bar{X}}$ satisfies the conditions (5) and (6) of theorem B.0.3. This implies that the fiber of the top horizontal, taken at $(\bar{V}, \bar{\nabla})$, map does satisfy condition (5). But by construction this fiber is the truncated stack $\mathbf{Vect}_{V_\infty}^\nabla(X)_0$. This implies that $\mathbf{Vect}_{V_\infty}^\nabla(X)_0$ satisfies the conditions (5) of the theorem B.0.3, as wanted.

We are now left proving that $\mathbf{Vect}_{V_\infty}^\nabla(X)$ satisfies condition (6) of B.0.3. For this, let $B = \text{colim}_i B_i$ as in (6), and we will assume that each B_i as well as B are noetherian rings. We consider

$$\text{colim}_i \mathbf{Vect}_{V_\infty}^\nabla(X)(B_i) \longrightarrow \mathbf{Vect}_{V_\infty}^\nabla(X)(B).$$

By the lemma 5.0.3 this map is injective and it remains to show it is surjective as well. Let us fix an object in $\mathbf{Vect}_{V_\infty}^\nabla(X)(B)$, represented by a pair (E, α) , of $E \in \mathbf{Vect}^\nabla(X)(B)$ and $\alpha : R(E) \simeq V_\infty \times_k B$ in $\mathbf{Vect}^\nabla(\hat{\partial}X)(B)$. As the stack $\mathbf{Vect}^\nabla(X)$ of flat bundles on X is locally of finite presentation, there is an i and $E_i \in \mathbf{Vect}^\nabla(X)(B_i)$ such that $E_i \otimes_{B_i} B \simeq E$.

We now consider the sheaf \mathcal{I} of isomorphism between $R(E_i)$ and $V_\infty \otimes_k B_i$, which is a sheaf on the big etale site of affine schemes over $S_i = \text{Spec } B_i$. This sheaf is a subsheaf in \mathcal{J} the sheaf of all morphisms from $R(E_i)$ to $V_\infty \otimes_k B_i$.

Lemma 5.0.5 *There exists a non-empty Zariski open $U_i \subset S_i = \text{Spec } B_i$ such that the restriction of the sheaf \mathcal{J} is representable by a scheme of finite type over U_i .*

Proof of the lemma: This is a similar argument of lemma 5.0.3, what we have to prove is, if we set

$$E'_i := R(E_i)^\vee \otimes V_\infty \otimes_k B_i \in \mathbf{Vect}^\nabla(\hat{\partial}X)(B_i),$$

then the Tate object $\widehat{\mathbb{H}}_{DR}(\hat{\partial}X, E')[f^{-1}]$ is a perfect $B_i[f^{-1}]$ -module, for some non-zero localization $B_i[f^{-1}]$. For this we use the criterion corollary 4.2.4 and proposition A.0.4. We let $j : X \hookrightarrow \bar{X}$ the embedding in the good compactification. We first notice that $j_*(E^\vee)$ is a perfect $\mathcal{D}_{\bar{X}, B}$ -module on $\bar{X} \times \text{Spec } B$. This is a local statement on \bar{X} which reduces to the following algebraic fact.

Let A be a smooth k -algebra of finite type and $f \in A$. We consider $\widehat{A \otimes_k B}$, the formal completion of $A \otimes_k B$ at $f \otimes 1$. We denote by $\widehat{\mathcal{D}}_{\bar{X}, B}$ the ring of completed relative differential operators. As a module it is $\widehat{A \otimes_k B}_{A \otimes_k B}(\widehat{\mathcal{D}}_{\bar{X}} \otimes_k B)$, where the ring structure is defined naturally by making $\widehat{\mathcal{D}}_{\bar{X}} \otimes_k B$ acts on the completion $\widehat{A \otimes_k B}$ by extending derivations to the completion. In the same manner we let

$\hat{\mathcal{D}}_{X,B}$ be $\hat{\mathcal{D}}_{\bar{X},B}[f^{-1}]$ Using the formal gluing of [Bh] we have a cartesian square of ∞ -categories

$$\begin{array}{ccc} \mathcal{D}_{qcoh}(\mathcal{D}_{\bar{X},B}) & \longrightarrow & \mathcal{D}_{qcoh}(\mathcal{D}_{X,B}) \\ \downarrow & & \downarrow \\ \mathcal{D}_{qcoh}(\hat{\mathcal{D}}_{\bar{X},B}) & \longrightarrow & \mathcal{D}_{qcoh}(\hat{\mathcal{D}}_{X,B}). \end{array}$$

Therefore, for $j_*(E^\vee) \in \mathcal{D}_{qcoh}(\mathcal{D}_{\bar{X},B})$ to be perfect it is enough that its restriction as $\mathcal{D}_{X,B}$ and $\hat{\mathcal{D}}_{\bar{X},B}$ are both perfect. But, the first of this restriction is E which is perfect over $\mathcal{D}_{X,B}$, and the second of this restriction corresponds to $j_*(V_s^\vee \otimes_k B)$. This is perfect because it is the restriction to $\hat{\partial}X$ of $j_*(V_0^\vee) \otimes_k B \in \mathcal{D}_{qcoh}(\mathcal{D}_{X,B})$, which is perfect because of Bertsein theorem asserting that $j_*(V_0^\vee)$ is a coherent and holonomic complex of \mathcal{D}_X -modules.

We thus have that $j_*(E^\vee)$ is a perfect $\mathcal{D}_{\bar{X},B}$ -module. More is true, it is also holonomic (see appendix A.0.4). Indeed, because $R(E)$ is isomorphic to $j_*(V_s) \otimes_k B$, its characteristic cycle is contained in $\Lambda \times \text{Spec } B \subset T^*X \times \text{Spec } B$, where $\Lambda = \text{Char}(j_*(V_\infty))$. As the ∞ -functor sending B to perfect $\mathcal{D}_{\bar{X},B}$ -module is locally of finite presentation, we can choose i such and $F_i \in \mathcal{D}_{parf}(\mathcal{D}_{\bar{X},B})$ such that $j_*(E^\vee) \simeq F_i \otimes_{B_i} B$. By enlarging i if necessary, we can also assume that the characteristic variety of F_i is contained in $\Lambda \times \text{Spec } B_i$, and thus that F_i is moreover holonomic. Also, we can assume that F_i and $j_*(E_i^\vee)$ are isomorphic as objects in $\text{Vect}^\nabla(X)(B_i)$.

As now both F_i and $j_*(V_\infty) \otimes_k B_i$ are perfect and holonomic, the proposition A.0.4 implies that $F_i \otimes_{\mathcal{O}} (j_*(V_\infty) \otimes_k B_i) \simeq j_*(E_i^\vee)$ remains perfect over $\mathcal{D}_{\bar{X},B_i}[f^{-1}]$ for some non-zero localization of B_i . Working with V_∞^\vee and E^\vee from the start we prove the same manner that $j_*((E_i^\vee)^\vee)$ is also perfect. By corollary 4.2.4 this implies that $\tilde{\mathbb{H}}_{DR}(\hat{\partial}X, E_i^\vee)$ is perfect and thus the lemma. \square

By the above lemma \mathcal{J} is representable by a scheme of finite type. The sheaf \mathcal{I} clearly is an open subsheaf of \mathcal{J} and thus is also representable by a scheme of finite type over an non-empty open on $\text{Spec } B_i$. The canonical isomorphism $\alpha : R(E) \simeq V_\infty \otimes_k B$, which an element in $\mathcal{I}(B)$ is then definable over some $B_i[f^{-1}]$ for some i and non-zero localization, say $\alpha_i : R(E_0)[f^{-1}] \simeq V_\infty \otimes_k B_i[f^{-1}]$. The pair (E_0, α_i) defines an object in $\text{Vect}^\nabla(X)_{V_\infty}(B_i[f^{-1}])$ whose image in $\text{Vect}_{V_\infty}^\nabla(X)(B[f^{-1}])$ is the restriction of our original object E .

This finishes the proof of condition (6) of theorem B.0.3, and thus of theorem 5.0.1. \square

Unfortunately, we do not know if the theorem 5.0.1 can be strengthened to the statement that $\text{Vect}_{V_\infty}^\nabla(X)$ is representable by an algebraic space locally of finite type over k . The only missing condition would be that $\text{Vect}_{V_\infty}^\nabla(X)$ is also locally of finite presentation, a condition that we havent been able to prove or disprove.

A Perfect relative \mathcal{D} -modules

In this section we gather some basic results about \mathcal{D} -modules in the relative settings. Many of these results are already contained in [Ga-Ro], and this part does not claim originality. However, we have been unable to find a reference treating the algebraic situation (possibly with k non-algebraically closed) and also relative to bases $\text{Spec } B$ with B an arbitrary connective cdga.

We start by the important fact concerning the compact generation, and characterization of compact objects inside quasi-coherent relative \mathcal{D} -modules. We fix X a smooth variety over k , and $S = \text{Spec } B$ an affine derived scheme. We have $\mathcal{D}_X \otimes_k B$, which is a sheaf of dg-algebras over X . We can therefore consider the dg-category of all sheaves of $\mathcal{D}_X \otimes_k B$ -modules, whose underlying $\mathcal{O}_X \otimes_k B$ -module is quasi-coherent on $X \times S$. We denote this category by $\mathcal{D}_{qcoh}(\mathcal{D}_{X,B})$, and call it the dg-category of relative \mathcal{D} -modules on $X \times S$ over S . An object $E \in \mathcal{D}_{qcoh}(\mathcal{D}_{X,B})$ will be called *perfect* if it is locally on X given by a perfect dg-module over the dg-algebra $\mathcal{D}_X \otimes_k B$. When B is a regular discrete k -algebra, $\mathcal{D}_X \otimes_k B$ is (locally) a finitely generated algebra of finite homological dimension, and thus is of finite type in the sense of [To-Va]. This implies that the perfect objects in this special case are precisely the bounded coherent $\mathcal{D}_X \otimes_k B$ -modules. In general the two notions do not coincide as being perfect implies in particular being of finite tor dimension over B .

Proposition A.0.1 *The dg-category $\mathcal{D}_{qcoh}(\mathcal{D}_{X,B})$ is compactly generated and its compact objects are the perfect $\mathcal{D}_X \otimes_k B$ -modules.*

Proof: There is a forgetful functor

$$\mathcal{D}_{qcoh}(\mathcal{D}_{X,B}) \longrightarrow \mathcal{D}_{qcoh}(X \times S)$$

to the dg-category of quasi-coherent complexes on $X \times S$. This dg-functor is conservative and continuous. Moreover, it has a left adjoint

$$ind : \mathcal{D}_{qcoh}(X \times S) \longrightarrow \mathcal{D}_{qcoh}(\mathcal{D}_{X,B})$$

which sends a quasi-coherent complex E on $X \times S$ to $\mathcal{D}_X \otimes_{\mathcal{O}_{X \times S}} E$, with its natural $\mathcal{D}_X \otimes_k B$ -module structure. It is known that perfect complexes on $\mathcal{D}_{qcoh}(X \times S)$ are the compact generators, and it is formal to deduce that their image by ind are compact generators of $\mathcal{D}_{qcoh}(\mathcal{D}_{X,B})$. These are obviously perfect $\mathcal{D}_X \otimes_k B$ -modules. Finally, any perfect $\mathcal{D}_X \otimes_k B$ -module is locally compact, and thus compact by quasi-compactness of X . \square

Let now $f : X \longrightarrow Y$ be a morphism between smooth varieties over k . We have a direct image dg-functor

$$f_{*,B} : \mathcal{D}_{qcoh}(\mathcal{D}_{X,B}) \longrightarrow \mathcal{D}_{qcoh}(\mathcal{D}_{Y,B})$$

defined as usual. We will often drop the B in the notation and simply write f_* . On the level of compact generators, f_* acts as follows. Let E be a perfect complex on $X \times S$, and $ind(E) = \mathcal{D}_X \otimes_{\mathcal{O}_X} E$. Then we have a canonical isomorphism $f_*(ind(E)) \simeq ind(f_*(E))$, where $f_*(E)$ is the direct image of E as a quasi-coherent complex on $X \times S$. In particular, when f is proper the dg-functor f_* preserves perfect objects. It is easy to check that the formation of f_* commutes with change of base B : for $B \rightarrow B'$ any morphism of connective cdga, the square

$$\begin{array}{ccc} \mathcal{D}_{qcoh}(\mathcal{D}_{X,B}) & \xrightarrow{\otimes_{B,B'}} & \mathcal{D}_{qcoh}(\mathcal{D}_{X,B'}) \\ f_{*,B} \downarrow & & \downarrow f_{*,B'} \\ \mathcal{D}_{qcoh}(\mathcal{D}_{Y,B}) & \xrightarrow{\otimes_{B,B'}} & \mathcal{D}_{qcoh}(\mathcal{D}_{Y,B'}) \end{array}$$

canonically commutes. We have thus proved the following proposition, which is well known when B is itself a smooth algebra but for which we could not find any general reference.

Proposition A.0.2 *If f is proper, then f_* preserves perfect objects, and its formation commutes with change of bases B .*

We remind also the following notion of holonomicity. For this, we remind that for any coherent $\mathcal{D}_X \otimes_k B$ -module admits a good filtration, and that that support of the associated graded is a well defined algebraic subset inside $T^*X \times S$.

Definition A.0.3 *Let $E \in \mathcal{D}_{qcoh}(\mathcal{D}_{X,B})$ be a quasi-coherent $\mathcal{D}_X \otimes_k B$ -module. We say that E is holonomic if it satisfies the following two conditions.*

1. *It is perfect.*
2. *There exists a conic lagrangian algebraic subset $\Lambda \subset T^*X$ such that the characteristic variety of E is contained in $\Lambda \times S$.*

Contrary to the case of a base field, it is not true that holonomic \mathcal{D} -modules are stable by all operations. However, this holds on a dense open subset in S , and for us the following proposition will be used.

Proposition A.0.4 *Suppose that B is a discrete noetherian k -algebra. Let E and F be two holonomic objects in $\mathcal{D}_{qcoh}(\mathcal{D}_{X,B})$. There exists a non-empty open derived sub-scheme $\text{Spec } B[f^{-1}] \subset \text{Spec } B$ such that their tensor product $E \otimes_{\mathcal{O}} F$ is a perfect $\mathcal{D}_{X,B[f^{-1}]}$ -module on $X \times \text{Spec } B[f^{-1}]$.*

Proof: We start by the observation that a given object $E \in \mathcal{D}_{qcoh}(\mathcal{D}_{X,B})$ is perfect if and only if its restriction to $\mathcal{D}_{qcoh}(\mathcal{D}_{X,B_0})$ is so, where $B_0 = B_{red}$ is the reduced algebra of B . We can use induction on the power annihilating the nil-radical of B to restrict to the case where B is a square zero extension of B_0 by an ideal I . It is easy to see that the functor sending a cdga B to the space of all quasi-coherent $\mathcal{D}_{X,B}$ -module is 1-proximate in the sense of formal deformation theory (see [Lu3]). More precisely, for any discrete noetherian k -algebra B_0 , ideal $I_0 \subset B_0$ and any derivation $d : B_0 \rightarrow I_0[1]$, and $B_0 = B \oplus_d I$ the square-zero extension classified by d , the square of ∞ -categories

$$\begin{array}{ccc} \mathcal{D}_{qcoh}(\mathcal{D}_{X,B}) & \longrightarrow & \mathcal{D}_{qcoh}(\mathcal{D}_{X,B_0}) \\ \downarrow & & \downarrow \\ \mathcal{D}_{qcoh}(\mathcal{D}_{X,B_0}) & \longrightarrow & \mathcal{D}_{qcoh}(\mathcal{D}_{X,B_0 \oplus I}) \end{array}$$

induces a full-embedding from $\mathcal{D}_{qcoh}(\mathcal{D}_{X,B})$ to the fiber product of the three other terms. This easily imply that for a given $E \in \mathcal{D}_{qcoh}(\mathcal{D}_{X,B})$, E is a compact object if its restriction in $\mathcal{D}_{qcoh}(\mathcal{D}_{X,B_0})$ is so.

We are thus lead to the case where B is a reduced noetherian k -algebra. By picking a dense open in an irreducible component we can even assume that B is a smooth domain. Let $K = \text{Frac}(B)$ be its fraction field and we consider $X \times_k K$ as a smooth variety over K . It is known that for any smooth K -variety Z and a smooth sub-variety $j : Y \subset Z$ given by a single equation $f = 0$ on X , and any holonomic coherent $\mathcal{D}_{Z/K}$ -module M , there exists a Bernstein polynomial $b(M)$ for M with respect to the equation f . The polynomial exists as a monic polynomial over a localization $B[f^{-1}]$ of B . Replacing B by $B[f^{-1}]$ we can assume that $b(M)$ exists as a monic polynomial over B . By classical arguments its existence implies that the pull-back $j^*(E)$ is a bounded coherent complex of $\mathcal{D}_{Y,B}$ -modules with coherent cohomology, and thus is perfect as B is smooth.

As the statement of the proposition is local on X , we can apply the above fact to the diagonal $X \subset X \times X$, by writing it as a complete intersection, and to the holonomic $\mathcal{D}_{X \times X, B}$ -module $E \boxtimes F$, exterior product of E and F . The proposition follows. \square

B Derived quasi-algebraic spaces and Artin's representability

In this section we gather some definitions and results on the *derived quasi-algebraic spaces* and the corresponding representability criterion. This is a modification of the the notion of derived algebraic spaces for which atlases only exist generically. These derived stacks are not algebraic in general, but are algebraic as soon as the functor they represent are locally of finite presentation.

For this, we need the following notion of dominant morphism to a derived stack F . For this, we assume that F has a perfect global cotangent complex, is nil-complete and infinitesimally cartesian. We also assume that F is integrable: for any local complete noetherian discrete k -algebra $A = \lim_i A/m^i$, the natural morphism

$$F(A) \longrightarrow \lim_i F(A/m^i)$$

is bijective. For any such F , and any field K which is finitely generated over k , and any point $x : \text{Spec } K \longrightarrow F$, there exists by [Lu2, Thm. 18.2.5.1] a complete local noetherian cdga A with residue field K , and a formally smooth morphism

$$\text{Spf}(A) \longrightarrow F$$

extending the point x . We get this way a morphism from its truncation

$$\text{Spf}(\pi_0(A)) \longrightarrow F,$$

and by integrability a well defined morphism

$$\hat{x} : \text{Spec}(\pi_0(A)) \longrightarrow F.$$

A morphism \hat{x} obtained this way will be called a *formally smooth lift of x* .

Definition B.0.1 *For a derived stack F as above and a derived scheme X locally of finite presentation over k , with a morphism $f : X \longrightarrow F$. We say that f is dominant if for any finitely generated k -field K , any point $x : \text{Spec } K \longrightarrow F$ and any formally smooth lift $\hat{x} : \text{Spec}(\pi_0(A)) \longrightarrow F$, the derived scheme $X \times_F \text{Spec}(\pi_0(A))$ is non-empty.*

We note that if F is itself representable by a derived algebraic space locally of finite presentation, then $f : X \longrightarrow F$ is dominant in the sense above if and only if for any étale morphism $\text{Spec } B \longrightarrow F$ we have $X \times_F \text{Spec } B \neq \emptyset$. Indeed, assume that there is an étale map $\text{Spec } B \longrightarrow F$ whose pull-back to X is empty. We pick a point x of $\text{Spec } B$ and consider its formal completion \hat{B}_x . As $\text{Spec } B \longrightarrow F$ is étale the composition

$$\text{Spec } \pi_0(\hat{B}_x) \longrightarrow \text{Spec } B \longrightarrow F$$

is a formally smooth lift of x . By construction the pull-back $\text{Spec } \pi_0(\hat{B}_x) \times_F X$ is empty. This shows that the notion of dominant map above is a generalization of the notion of a morphism with Zariski

dense image.

We can then give the definition of a derived quasi-algebraic spaces as derived stacks with dominant smooth atlases as follows.

Definition B.0.2 *A derived stack F is an derived quasi-algebraic space (locally of presentation with schematic diagonal of finite presentation) if it satisfies the following three conditions.*

1. *The diagonal of the stack $F \rightarrow F \times F$ is representable by a derived scheme of finite presentation.*
2. *The derived stack F possesses a perfect global cotangent complex, is nil complete and infinitesimally cartesian.*
3. *The derived stack F is integrable: for any local complete noetherian discrete k -algebra $A = \lim_i A/m^i$, the natural morphism*

$$F(A) \longrightarrow \lim_i F(A/m^i)$$

is bijective.

4. *There exists a family of cdga's A_i of finite presentation over k and a morphism $p : \coprod \text{Spec } A_i \rightarrow F$ such that*
 - (a) *For each i the morphism $\text{Spec } A_i \rightarrow F$ is smooth.*
 - (b) *The morphism p is dominant in the sense of definition B.0.1 above.*

A derived quasi-algebraic space is algebraic if and only if the functor F is furthermore locally of finite presentation, as this follows from Artin-Lurie's representability theorem [Lu2, Thm. 18.3.0.1].

The derived quasi-algebraic spaces can be characterized by the following version of Artin's representability.

Theorem B.0.3 *A derived stack F is a derived quasi-algebraic space if it satisfies the following conditions.*

1. *For any discrete cdga B the simplicial set $F(B)$ is 0-truncated.*
2. *The diagonal morphism of its truncation $F \rightarrow F \times F$ is representable by a scheme of finite presentation.*
3. *The derived stack F has a perfect global cotangent complex.*
4. *The derived stack F is nil-complete and infinitesimally cartesian.*
5. *For any discrete local k -algebra (A, m) essentially of finite type, with completion $\hat{A} = \lim_i A/m^i$, the morphism $F(\hat{A}) \rightarrow \lim_i F(A/m^i)$ is an equivalence.*
6. *For any filtered system of noetherian discrete commutative k -algebras $B = \text{colim}_i B_i$ and any $x \in F(B)$, there exists an index i and a non-empty Zariski open $U_i \subset \text{Spec } B_i$ with $U = U_i \times_{\text{Spec } B_i} \text{Spec } B$ non-empty, and such that the restriction of x lies in the image of $F(U_i) \rightarrow F(U)$.*

Sketch of a proof: The proof is essentially the same as the usual representability theorem in [Lu2]. We start by considering all fields K which are finitely generated over k . For any morphism

$$x : \text{Spec } K \longrightarrow F$$

we can use [Lu2, Thm. 18.2.5.1] to find a local complete and noetherian cdga (A, m) with residue field $K = A/m$ and a factorisation

$$\text{Spec } K \hookrightarrow \text{Spf}(\hat{A}) \longrightarrow F,$$

where the second map is formally smooth (i.e. its relative cotangent complex is a vector bundle). We write $B = \pi_0(\hat{A})$, which is a complete local discrete k -algebra with residue field K , and consider the induced morphism on the truncation $\hat{x} : \text{Spf}(B) \longrightarrow F$. We can use condition (4) to lift this to a factorization

$$\text{Spec } K \hookrightarrow \text{Spec}(B) \longrightarrow F.$$

As explained in the proof of [Lu2, Thm. 18.2.5.1], there exists a k -algebra of finite type $B' \subset B$, such that if $p = m \cap B'$, then the induced morphism on formal completions

$$\hat{B}'_p \longrightarrow B$$

is surjective (take A' big enough so that it contains generators for K over k as well as generators of the k -vector space m/m^2). We can apply Popescu's theorem to the regular morphism $B' \longrightarrow \hat{B}'_p$ and thus write $\hat{B}'_p = \text{colim}_i B'_i$ as a filtered colimit of smooth B' -algebras. As B is finitely presented as a \hat{B}'_p -algebra, we can find an index i and a B'_i -algebra C'_i of finite presentation such that

$$C \simeq \text{colim}_i (\hat{B}'_p B \otimes_{B'_i} C'_i).$$

We let $C_i := \hat{B}'_p B \otimes_{B'_i} C'_i$, which is a B'_i -algebra of finite presentation, and thus is itself of finite presentation over k .

We now apply condition (6) to the morphism $\text{Spec } B \longrightarrow F$, and get that there exists an integer i and a Zariski open $U_i = \text{Spec } C_i[f^{-1}] \subset \text{Spec } C_i$, with $U = \text{Spec } B[f^{-1}]$ non-empty, and with a commutative diagram

$$\begin{array}{ccc} & U_i & \\ & \uparrow & \searrow \\ & U & F \\ & \downarrow & \nearrow \\ \text{Spec } B & & \end{array}$$

Lemma B.0.4 *With the notation above, and enlarging i is necessary, the morphism $p : U_i \longrightarrow F$ constructed above is formally smooth in the underived sense: $\tau_{\leq -1}(\mathbb{L}_{U_i/F})$ is a vector bundle in degree 0.*

Proof of the lemma: First of all U_i being of finite type together with the fact that the diagonal of F is representable of locally of finite presentation implies that p is representable and locally of finite type in the underived sense. It thus only remains to show that its p is also formally smooth in the underived sense, that its relative 1-truncated cotangent complex $\tau_{\leq 1}(\mathbb{L}_{U_i/F})$ is a vector bundle.

For this we first notice that $\mathbb{L}_{U_i/F}$ is almost perfect (i.e. quasi-isomorphic to a complex of free modules of finite rank over $C_i[f^{-1}]$ concentrated in degree $]-\infty, 0]$). As we will only be interested in its

truncation $\tau_{\leq 1}(\mathbb{L}_{U_i/F})$ we will be able to act as if $\mathbb{L}_{U_i/F}$ is in fact perfect (simply replace it by a perfect complex having the same cohomology in degree $[-n, 0]$ for n big enough). We start by computing the pull-back of \mathbb{L}_{U_i} to $U = \text{Spec } B[f^{-1}]$.

We start by the exact triangle of complexes of B -modules (where \mathbb{L}_A stands for $\mathbb{L}_{A/k}$ for any k -algebra A).

$$\mathbb{L}_{C_i} \otimes_{C_i} B \longrightarrow \mathbb{L}_B \longrightarrow \mathbb{L}_{B/C_i}.$$

As B is complete with respect to its maximal ideal m , for any connective dg-module E over B , we have its completion $\widehat{E} := \lim_i E \otimes_B B/m^i$, which is another connective B -dg-module together with a natural morphism $E \longrightarrow \widehat{E}$. Moreover, when E is almost perfect this morphism is a quasi-isomorphism. We can then complete the triangle above another triangle

$$\widehat{\mathbb{L}_{C_i} \otimes_{C_i} B} \longrightarrow \widehat{\mathbb{L}_B} \longrightarrow \widehat{\mathbb{L}_{B/C_i}}.$$

As \mathbb{L}_{C_i} is almost perfect the left hand side simply is $\mathbb{L}_{C_i} \otimes_{C_i} B$. Moreover, by base change \mathbb{L}_{B/C_i} is naturally equivalent to $\mathbb{L}_{\widehat{B}'/C'_i} \otimes_{B'} B$. As $B' \longrightarrow B$ is a surjective local morphism we see that the base change of $\widehat{\mathbb{L}_{\widehat{B}'/C'_i}}$, considered as a pro-object in connective B' -dg-module, by the map $\widehat{B}' \longrightarrow B$, is the pro-object $\widehat{\mathbb{L}_{B/C_i}}$.

We now use that \widehat{B}' is the completion of B' along the maximal ideal $m \subset B'$, and thus that for all i we have $\mathbb{L}_{\widehat{B}'/B'} \otimes_{B'} B'/m^i \simeq 0$. We thus have an equivalence of pro-object

$$\widehat{\mathbb{L}_{\widehat{B}'/C'_i}} \simeq \widehat{\mathbb{L}_{C_i/B'}} \otimes_{B'} \widehat{B}'[1].$$

As C_i is smooth over B' we therefore conclude that the pro-object $\widehat{\mathbb{L}_{B/C_i}}$ is a vector bundle in degree 1, and thus that its realization as a connective B -dg-module is $\mathbb{L}_{C_i/B'} \otimes_{C_i} B[1]$. Our original triangle can thus be written as

$$\mathbb{L}_{C_i} \otimes_{C_i} B \longrightarrow \widehat{\mathbb{L}_B} \longrightarrow V[1]$$

with V a vector bundle on $\text{Spec } B$. We can now localize this triangle to the open $U = \text{Spec } B[f^{-1}]$ in order to get new triangle on U

$$\mathbb{L}_{C_i[f^{-1}]} \otimes_{C_i[f^{-1}]} B[f^{-1}] \longrightarrow \widehat{\mathbb{L}_B}[f^{-1}] \longrightarrow V[f^{-1}][1].$$

The morphisms $q : U \rightarrow U_i \rightarrow F$ induces a morphism

$$\begin{array}{ccc} q^*(\mathbb{L}_F) & \longrightarrow & \mathbb{L}_{C_i[f^{-1}]} \otimes_{C_i[f^{-1}]} B[f^{-1}] \\ & \searrow & \downarrow \\ & & \mathbb{L}_B[f^{-1}], \end{array}$$

which factors through completions as $q^*(\mathbb{L}_F)$ is a perfect complex by our condition (3)

$$\begin{array}{ccc} q^*(\mathbb{L}_F) & \longrightarrow & \mathbb{L}_{C_i[f^{-1}]} \otimes_{C_i[f^{-1}]} B[f^{-1}] \\ & \searrow & \downarrow \\ & & \widehat{\mathbb{L}_B}[f^{-1}]. \end{array}$$

The induced morphism on the cones sits in an exact triangle

$$\mathbb{L}_{U_i/F} \otimes_{C_i[f^{-1}]} B[f^{-1}] \longrightarrow \widehat{\mathbb{L}_{\text{Spec } B/F}[f^{-1}]} \longrightarrow V[1].$$

Because $\text{Spf } F \rightarrow F$ was chosen formally smooth in the underived sense we have that $\tau_{\leq -1}(\widehat{\mathbb{L}_{\text{Spec } B/F}[f^{-1}]})$ is a vector bundle in degree 0. The conclusion is that $\mathbb{L}_{U_i/F}$ is an almost perfect complex over $C_i[f^{-1}]$ such that its base change to $B[f^{-1}] = \text{colim}(C_i[f^{-1}])$ has vanishing H^{-1} and a vector bundle as H^0 . This implies that the same is true for $\mathbb{L}_{U_i/F} \otimes_{C_i[f^{-1}]} C_j[f^{-1}]$ for some big enough j . \square

Back to the proof of the theorem, we use again [Lu2, Thm. 18.2.5.1] but this time for $U_i \rightarrow F$, which by the lemma can be chosen to be formally smooth in the underived sense. We can thus produce a smooth morphism

$$W_i \rightarrow F$$

where W_i is a derived affine scheme whose truncations coincides with the given map $U_i \rightarrow F$. The derived scheme W_i is itself of finite presentation over k as its truncation is of finite type and its cotangent complex is perfect (because its maps smoothly to F).

Taking the union of all morphisms $W_i \rightarrow F$ constructed as above provides the required generic atlas for F as in definition B.0.2.

References

- [Ba-Ca-Fi] Baldassarri, Francesco; Cailotto, Maurizio; Fiorot, Luisa *Poincaré duality for algebraic de Rham cohomology*. Manuscripta Math. **114** (2004), no. 1, 61–116.
- [Be-Te] Ben-Bassat, Oren; Temkin, Michael *Berkovich spaces and tubular descent*. Adv. Math., **234**:217–238, 2013.
- [Bh] Bhatt, Bhargav *Formal glueing of module categories*. Available at <http://www-personal.umich.edu/~bhattb/math/formalglueing.pdf>
- [Ca] Calaque, Damien *Lagrangian structures on mapping stacks and semi-classical TFTs*. in Stacks and categories in geometry, topology, and algebra, 1–23, Contemp. Math., **643**, Amer. Math. Soc., Providence, RI, 2015.
- [CPTVV] Calaque, Damien; Pantev, Tony; Toën, Bertrand; Vaquié, Michel; Vezzosi, Gabriele *Shifted Poisson structures and deformation quantization*. J. Topol. **10** (2017), no. 2, 483–584.
- [Ef] Efimov, Alexander *Categorical formal punctured neighborhood of infinity, I*. Preprint arXiv:1711.00756
- [Ga-Ro] Gaitsgory, Dennis; Rozembyum, Nick *Crystals and D-modules*. Available at <http://www.math.harvard.edu/~gaitsgde/GL/Crystalstext.pdf>
- [Gi] Ginzburg, Victor *Principal nilpotent pairs in a semisimple Lie algebra. I*. Invent. Math. **140** (2000), no. 3, 511–561.

- [He] Hennion, Benjamin *Tate objects in stable $(\infty, 1)$ -categories*. Homology Homotopy Appl. **19** (2017), no. 2, 373-395.
- [He-Po-Ve] Hennion, Benjamin; Porta, Mauro; Vezzosi, Gabriele *Formal gluing along non-linear flags*. Preprint arXiv:1607.04503
- [Lu1] Lurie, Jacob *Quasi-Coherent Sheaves and Tannaka Duality Theorems*. Available at <http://www.math.harvard.edu/~lurie/papers/DAG-VIII.pdf>
- [Lu2] Lurie, Jacob *Spectral Algebraic Geometry*. Available at <http://www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf>
- [Lu3] Lurie, Jacob *Formal moduli problems*. Available at <http://www.math.harvard.edu/~lurie/papers/DAG-X.pdf>
- [Me-Sa] Melani, Valerio; Safronov, Pavel *Derived coisotropic structures I and II*. arXiv preprints arXiv:1608.01482, arXiv:1704.03201
- [Mo] Mochizuki, Takuro *Good formal structure for meromorphic flat connections on smooth projective surfaces*. Algebraic analysis and around, 223–253, Adv. Stud. Pure Math., 54, Math. Soc. Japan, Tokyo, 2009.
- [Pa-To] Pantev, Tony; Toën, Bertrand *Poisson geometry of the moduli of local systems on smooth varieties*. arXiv:1809.03536
- [PTVV] Pantev, Tony; Toën, Bertrand; Vaquié, Michel; Vezzosi, Gabriele *Shifted symplectic structures*. Publ. Math. Inst. Hautes Etudes Sci. **117** (2013), 271-328.
- [Ra] Raskin, Sam *On the notion of spectral decomposition in local geometric Langlands*. arXiv:1511.01378
- [Sa] Safronov, Pavel *Poisson reduction as a coisotropic intersection*. arXiv preprint arXiv:1509.08081
- [To1] Toën, Bertrand *Derived algebraic geometry*. EMS Surv. Math. Sci. **1** (2014), no. 2, 153-245.
- [To2] Toën, Bertrand *Derived Azumaya algebras and generators for twisted derived categories*. Invent. Math. **189** (2012), no. 3, 581–652.
- [To3] Toën, Bertrand *Structures symplectiques et de Poisson sur les champs en catégories*. arXiv:1804.10444
- [To-Va] Toën, Bertrand; Vaquié, Michel *Moduli of objects in dg-categories*. Ann. Sci. de l'ENS Volume **40** (2007) Issue 3, Pages 387-444.
- [To-Ve] Toën, Bertrand; Vezzosi, Gabriele *Homotopical Algebraic Geometry II: geometric stacks and applications*. Mem. Amer. Math. Soc. **193** (2008), no. 902, x+224 pp.