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Polynomial interpolation in higher dimensions

Alexandru Dimca

Abstract We describe a recent advance in the theory of interpolation in the plane, based on the theory of line arrangements in the complex projective plane.

1 Interpolation in dimensions one and two

1.1 Lagrange interpolation

We denote by \mathbb{R} the field of real numbers and by \mathbb{C} the field of complex numbers. Let p_1, \dots, p_n be n real numbers, thought of as n points on the real line.

Assume that each point p_i has an associated number $c_i \in \mathbb{R}$, thought of as the result of a measurement effectuated at the point p_i , for $i = 1, 2, \dots, n$, of a physical entity of interest to us, e.g., temperature, pressure, or density of a substance. Let $S = \{p_1, p_2, \dots, p_n\}$ be the set of these n points, and

$$f : S \rightarrow \mathbb{R}$$

the function defined by $f(p_i) = c_i$ for $i = 1, 2, \dots, n$. In order to move from experiment to theory, we would like to find a formula for this function f . The most natural idea is to look for a polynomial $P(x)$ of minimal possible degree such that one has

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$$f(p_i) = P(p_i) \text{ for all } i = 1, 2, \dots, n.$$

The hope is that this polynomial will in fact satisfy $f(t) = P(t)$ for any real number t , and hence our discovered formula would allow us to make predictions as well.

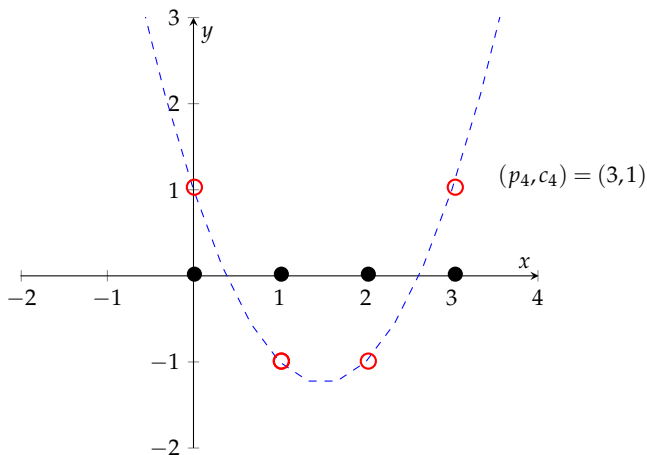


Fig. 1 The $n = 4$ points (p_i, c_i) are on the parabola $y = P(x)$, with $P(x) = x^2 - 3x + 1$.

The following result was first published by Waring in 1779, rediscovered by Euler in 1783, and published by Lagrange in 1795.

Theorem 1. For any n distinct real numbers p_i , $i = 1, 2, \dots, n$, and any given n values c_i , $i = 1, 2, \dots, n$, there is a unique polynomial $P(x)$ of degree at most $n - 1$, such that $P(p_i) = c_i$ for all $i = 1, 2, \dots, n$.

To give a formula for $P(x)$, consider, for any $i = 1, 2, \dots, n$, the degree $n - 1$ polynomial

$$Q_i(x) = \frac{\prod_{j=1, n; j \neq i} (x - p_j)}{\prod_{j=1, n; j \neq i} (p_i - p_j)},$$

and note that $Q_i(p_i) = 1$ and $Q_i(p_k) = 0$ for any $k \neq i$. With this notation one has

$$P(x) = \sum_{i=1, n} c_i Q_i(x).$$

Consider the vector space of polynomials of degree at most d , denoted by $\mathbb{R}[x]_{\leq d}$, and the linear map given by evaluation

$$\epsilon_d : \mathbb{R}[x]_{\leq d} \rightarrow \mathbb{R}^S = \mathbb{R}^n, \quad \epsilon(Q)(p_i) = Q(p_i),$$

for any $i = 1, 2, \dots, n$, where \mathbb{R}^S denotes the vector space of all functions $S \rightarrow \mathbb{R}$. The above results say that ϵ_d is injective if and only if $d \leq n - 1$, and ϵ_d is surjective if and only if $d \geq n - 1$. Such results are true over any field: \mathbb{R} , \mathbb{C} or even finite fields.

1.2 Interpolation in dimension 2

Let now $p_i = (x_i, y_i)$ for $i = 1, 2, \dots, n$ be n points in the plane \mathbb{R}^2 . Assume each point has an associated value $c_i \in \mathbb{R}$, thought of as the result of a measurement at the point p_i . Let $S = \{p_1, p_2, \dots, p_n\}$ be the set of these points, consider the associated function $f : S \rightarrow \mathbb{R}$ given by $f(p_i) = c_i$ and look for the minimal degree d such that there is a polynomial $Q \in \mathbb{R}[x, y]$ of degree d satisfying

$$f(p_i) = Q(x_i, y_i) \text{ for all } i = 1, 2, \dots, n.$$

This is the old question, but the setting is new: the answer now depends on the position of the points p_i in the plane.

Example 1 (3 points in the plane). If the 3 points p_1, p_2 and p_3 are not collinear, then we can take $d = 1$. Indeed, as above, we can construct 3 polynomials Q_1, Q_2 and Q_3 , by taking the equations of lines passing through two of the points p_i . When the 3 points p_1, p_2 and p_3 are collinear, then the minimal degree is $d = 2$. Indeed, there are conics passing through two of these points and avoiding the remaining one.

We discuss now a special type of interpolation node, i.e., a special class of choices for the points p_i 's. Consider a finite family of lines $L_j : \ell_j(x, y) = a_jx + b_jy + c_j = 0$ in the plane \mathbb{R}^2 , for $j = 1, 2, \dots, m$. If these lines are generic, i.e., no two are parallel and no three are concurrent, then we get precisely

$$N = \binom{m}{2}$$

intersection points, which will play the role of our points p_i .

Theorem 2. *For any m generic lines in the plane and any given N values c_i associated to their intersection points p_i , there is a unique polynomial $P(x, y)$ of degree at most $m - 2$, such that $P(p_i) = c_i$ for all $i = 1, 2, \dots, N$. More precisely, the evaluation map*

$$\epsilon_d : \mathbb{R}[x, y]_{\leq d} \rightarrow \mathbb{R}^S = \mathbb{R}^N, \quad \epsilon(Q)(p_i) = Q(x_i, y_i),$$

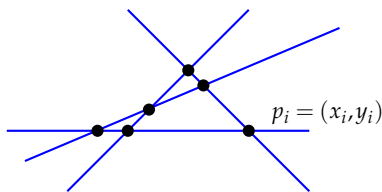


Fig. 2 $m = 4$ generic lines in the plane

is surjective if and only if $d \geq m - 2$ and it is injective for $d \leq m - 2$. A similar claim of surjectivity holds when the lines are replaced by any nodal curve C of degree d and the intersection points by the set of nodes of C .

Note that the degree of P is much smaller than the number of interpolation points p_i , namely $m - 2 < N = m(m - 1)/2$. The injectivity claim is easy, using Bezout Theorem about the intersection of two plane curves. The surjectivity is subtle, the proof uses Hodge theory, see [7]. The case when C is a Chebyshev curve is particularly interesting, see [6].

2 Projective Duality and Interpolation

From now on we move from the real field \mathbb{R} to the complex field \mathbb{C} , and from the affine plane \mathbb{C}^2 to the projective plane \mathbb{P}^2 , with coordinates $(x : y : z)$. A point p in \mathbb{P}^2 is given by 3 homogeneous coordinates

$$p = (a : b : c),$$

where $a, b, c \in \mathbb{C}$ are not all zero. To such a point we can associate a line L_p in \mathbb{P}^2 , given by the equation

$$L_p : ax + by + cz = 0.$$

Hence to a set of points $S = \{p_i : i = 1, 2, \dots, n\}$ in \mathbb{P}^2 , we can associate a line arrangement $\mathcal{A}_S = \{L_{p_i} : i = 1, 2, \dots, n\}$ in \mathbb{P}^2 . The multiplicity of a point p in a line arrangement \mathcal{A} is the number of lines of the arrangement \mathcal{A} passing through p . For more on line arrangements we refer to [4].

2.1 Splitting type of a line arrangement

Why to pass from points to lines ? Because line arrangements have a lot of geometry. In particular, for any line arrangement \mathcal{A} in \mathbb{P}^2 one can define a rank two vector bundle $E = E(\mathcal{A})$ on the projective plane \mathbb{P}^2 , the bundle of logarithmic vector fields along \mathcal{A} . If L is a generic line in \mathbb{P}^2 , then the restriction $E|L$ splits as a direct sum of two line bundles on $L = \mathbb{P}^1$, with first Chern classes given by two negative integers, say $(-a, -b)$, with $0 \leq a \leq b$. The pair (a, b) is called the splitting type of E and satisfies $a + b = |\mathcal{A}| - 1$. For details, see [1,5].

2.2 A new look at the 1-dimensional case: a refinement

The fact that the evaluation map

$$\epsilon_d : \mathbb{R}[x]_{\leq d} \rightarrow \mathbb{R}^S = \mathbb{R}^n, \quad \epsilon(Q)(p_i) = Q(p_i),$$

is surjective for $d \geq n - 1$ is equivalent to the claim that

$$\dim \ker \epsilon_d = \dim \{Q \in \mathbb{R}[x]_{\leq d} : Q(p_i) = 0 \text{ for any } i\} = d + 1 - n,$$

for $d \geq n - 1$. Now fix an integer $k \geq 1$, consider a new point $q \in \mathbb{R}$, but $q \notin S$, and define a new evaluation map

$$\epsilon_{d,q,k} : \mathbb{R}[x]_{\leq d} \rightarrow \mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}[x]_{\leq k-1},$$

where $\epsilon(Q)(p_i) = Q(p_i) \in \mathbb{R}$ and $\epsilon(Q)(q) = T_{k-1}Q(q) \in \mathbb{R}[x]_{\leq k-1}$ is the $(k - 1)$ -st Taylor expansion of the polynomial Q at the point q .

In particular, $\epsilon(Q)(q) = T_{k-1}Q(q) = 0$ if and only if the first $(k - 1)$ derivatives of Q vanish at q , namely $Q^{(j)}(q) = 0$ for all $0 \leq j \leq k - 1$. It is easy to show that this new evaluation map is surjective for $d \geq n + k - 1$, and hence

$$\dim \ker \epsilon_{d,q,k} = d + 1 - n - k,$$

for $d \geq n + k - 1$. The practical interest of this refinement is that, for instance, a zero Taylor expansion of high order means very small values for the polynomial Q in the neighborhood of the given point q .

2.3 A new 2-dimensional interpolation problem

Starting with a given set of points $S = \{p_i : i = 1, 2, \dots, n\}$ in \mathbb{P}^2 , we fix an integer $k \geq 1$, and define $I(S)_d$ to be the vector space of homogeneous polynomials $Q \in \mathbb{C}[x, y, z]_d$ such that $Q(p_i) = 0$ for any $i = 1, 2, \dots, n$. Choose then a generic point $q \in \mathbb{P}^2$ and consider the vector space $V(d, S, k, q)$ of homogeneous polynomials $Q \in I(S)_d$ such that $T_{k-1}Q(q) = 0$. The expected dimension of this vector space is

$$\dim_e V(d, S, k, q) := \dim I(S)_d - \binom{k+1}{2},$$

when this number is positive. An important special case is when $d = k$, which is also the simplest case to consider. In this setting, we introduce the following notion, see [2].

Definition 1. We say that the set S admits an unexpected curve of degree k if

$$\dim V(k, S, k, q) > \dim_e V(k, S, k, q) \geq 0.$$

The main result in this direction is the following, see [2].

Theorem 3. Let S be a finite set of n points in \mathbb{P}^2 and let (a_S, b_S) be the splitting type of the dual line arrangement \mathcal{A}_S . Then S admits an unexpected curve of degree k if and only if the following hold.

- $a_S + 1 \leq k \leq b_S - 1$;
- the multiplicity of any intersection point in \mathcal{A}_S is at most $a_S + 1$.

2.4 An example: the complete polygonal arrangements

Consider a regular polygon with $N \geq 3$ edges, and the associated line arrangement \mathcal{A} consisting of the following $2N + 1$ lines

- the N lines determined by the N edges of the polygon,
- the N symmetry axes of the polygon, and
- the line at infinity.

This type of line arrangement occurs in the following result, see [3].

Theorem 4. For N even, the complete N -polygonal arrangement has an unexpected curve of degree N .

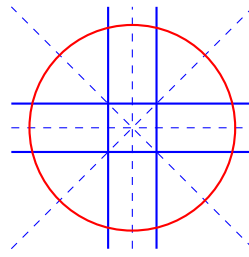


Fig. 3 Complete 4-polygonal arrangement; the line at infinity is drawn as the red circle.

The proof uses the theory of supersolvable line arrangements to show that $a = N - 1$ and $b = N + 1$. As an example, for $N = 4$ we get $a = 3$ and $b = 5$. Hence \mathcal{A} admits an unexpected curve of degree 4 by the result in see [2].

In the case $N = 4$, the complete N -polygonal arrangement is the dual arrangement \mathcal{A}_S , where the set of points S consists of the points $(0 : 0 : 1)$, $(0 : 1 : 0)$, $(1 : 0 : 0)$, $(1 : 1 : 1)$, $(0 : 1 : 1)$, $(1 : 0 : 1)$, $(1 : 1 : 0)$, $(-1 : 1 : 0)$, $(1 : 1 : 2)$. This situation was considered first by B. Harbourne in [8].

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