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# Polynomial interpolation in higher dimensions

Alexandru Dimca

**Abstract** We describe a recent advance in the theory of interpolation in the plane, based on the theory of line arrangements in the complex projective plane.

## 1 Interpolation in dimensions one and two

### 1.1 Lagrange interpolation

We denote by  $\mathbb{R}$  the field of real numbers and by  $\mathbb{C}$  the field of complex numbers. Let  $p_1, \dots, p_n$  be  $n$  real numbers, thought of as  $n$  points on the real line.

Assume that each point  $p_i$  has an associated number  $c_i \in \mathbb{R}$ , thought of as the result of a measurement effectuated at the point  $p_i$ , for  $i = 1, 2, \dots, n$ , of a physical entity of interest to us, e.g., temperature, pressure, or density of a substance. Let  $S = \{p_1, p_2, \dots, p_n\}$  be the set of these  $n$  points, and

$$f : S \rightarrow \mathbb{R}$$

the function defined by  $f(p_i) = c_i$  for  $i = 1, 2, \dots, n$ . In order to move from experiment to theory, we would like to find a formula for this function  $f$ . The most natural idea is to look for a polynomial  $P(x)$  of minimal possible degree such that one has

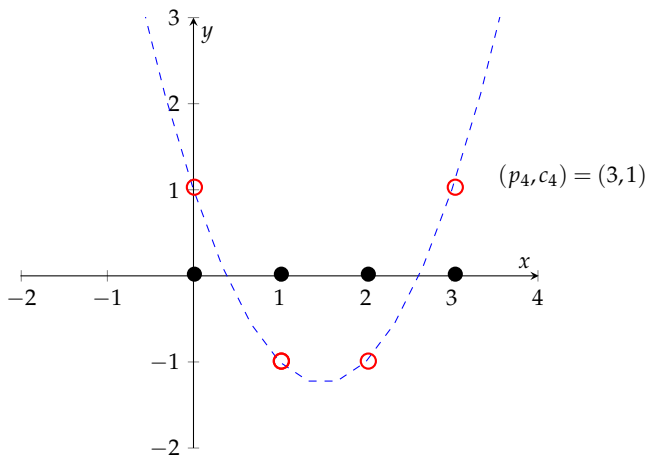
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$$f(p_i) = P(p_i) \text{ for all } i = 1, 2, \dots, n.$$

The hope is that this polynomial will in fact satisfy  $f(t) = P(t)$  for any real number  $t$ , and hence our discovered formula would allow us to make predictions as well.



**Fig. 1** The  $n = 4$  points  $(p_i, c_i)$  are on the parabola  $y = P(x)$ , with  $P(x) = x^2 - 3x + 1$ .

The following result was first published by Waring in 1779, rediscovered by Euler in 1783, and published by Lagrange in 1795.

**Theorem 1.** For any  $n$  distinct real numbers  $p_i$ ,  $i = 1, 2, \dots, n$ , and any given  $n$  values  $c_i$ ,  $i = 1, 2, \dots, n$ , there is a unique polynomial  $P(x)$  of degree at most  $n - 1$ , such that  $P(p_i) = c_i$  for all  $i = 1, 2, \dots, n$ .

To give a formula for  $P(x)$ , consider, for any  $i = 1, 2, \dots, n$ , the degree  $n - 1$  polynomial

$$Q_i(x) = \frac{\prod_{j=1, n; j \neq i} (x - p_j)}{\prod_{j=1, n; j \neq i} (p_i - p_j)},$$

and note that  $Q_i(p_i) = 1$  and  $Q_i(p_k) = 0$  for any  $k \neq i$ . With this notation one has

$$P(x) = \sum_{i=1, n} c_i Q_i(x).$$

Consider the vector space of polynomials of degree at most  $d$ , denoted by  $\mathbb{R}[x]_{\leq d}$ , and the linear map given by evaluation

$$\epsilon_d : \mathbb{R}[x]_{\leq d} \rightarrow \mathbb{R}^S = \mathbb{R}^n, \quad \epsilon(Q)(p_i) = Q(p_i),$$

for any  $i = 1, 2, \dots, n$ , where  $\mathbb{R}^S$  denotes the vector space of all functions  $S \rightarrow \mathbb{R}$ . The above results say that  $\epsilon_d$  is injective if and only if  $d \leq n - 1$ , and  $\epsilon_d$  is surjective if and only if  $d \geq n - 1$ . Such results are true over any field:  $\mathbb{R}$ ,  $\mathbb{C}$  or even finite fields.

### 1.2 Interpolation in dimension 2

Let now  $p_i = (x_i, y_i)$  for  $i = 1, 2, \dots, n$  be  $n$  points in the plane  $\mathbb{R}^2$ . Assume each point has an associated value  $c_i \in \mathbb{R}$ , thought of as the result of a measurement at the point  $p_i$ . Let  $S = \{p_1, p_2, \dots, p_n\}$  be the set of these points, consider the associated function  $f : S \rightarrow \mathbb{R}$  given by  $f(p_i) = c_i$  and look for the minimal degree  $d$  such that there is a polynomial  $Q \in \mathbb{R}[x, y]$  of degree  $d$  satisfying

$$f(p_i) = Q(x_i, y_i) \text{ for all } i = 1, 2, \dots, n.$$

This is the old question, but the setting is new: the answer now depends on the position of the points  $p_i$  in the plane.

*Example 1 (3 points in the plane).* If the 3 points  $p_1, p_2$  and  $p_3$  are not collinear, then we can take  $d = 1$ . Indeed, as above, we can construct 3 polynomials  $Q_1, Q_2$  and  $Q_3$ , by taking the equations of lines passing through two of the points  $p_i$ . When the 3 points  $p_1, p_2$  and  $p_3$  are collinear, then the minimal degree is  $d = 2$ . Indeed, there are conics passing through two of these points and avoiding the remaining one.

We discuss now a special type of interpolation node, i.e., a special class of choices for the points  $p_i$ 's. Consider a finite family of lines  $L_j : \ell_j(x, y) = a_jx + b_jy + c_j = 0$  in the plane  $\mathbb{R}^2$ , for  $j = 1, 2, \dots, m$ . If these lines are generic, i.e., no two are parallel and no three are concurrent, then we get precisely

$$N = \binom{m}{2}$$

intersection points, which will play the role of our points  $p_i$ .

**Theorem 2.** *For any  $m$  generic lines in the plane and any given  $N$  values  $c_i$  associated to their intersection points  $p_i$ , there is a unique polynomial  $P(x, y)$  of degree at most  $m - 2$ , such that  $P(p_i) = c_i$  for all  $i = 1, 2, \dots, N$ . More precisely, the evaluation map*

$$\epsilon_d : \mathbb{R}[x, y]_{\leq d} \rightarrow \mathbb{R}^S = \mathbb{R}^N, \quad \epsilon(Q)(p_i) = Q(x_i, y_i),$$

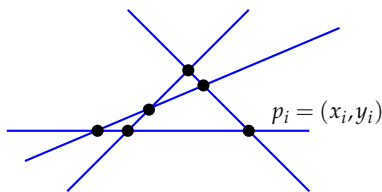


Fig. 2  $m = 4$  generic lines in the plane

is surjective if and only if  $d \geq m - 2$  and it is injective for  $d \leq m - 2$ . A similar claim of surjectivity holds when the lines are replaced by any nodal curve  $C$  of degree  $d$  and the intersection points by the set of nodes of  $C$ .

Note that the degree of  $P$  is much smaller than the number of interpolation points  $p_i$ , namely  $m - 2 < N = m(m - 1)/2$ . The injectivity claim is easy, using Bezout Theorem about the intersection of two plane curves. The surjectivity is subtle, the proof uses Hodge theory, see [7]. The case when  $C$  is a Chebyshev curve is particularly interesting, see [6].

## 2 Projective Duality and Interpolation

From now on we move from the real field  $\mathbb{R}$  to the complex field  $\mathbb{C}$ , and from the affine plane  $\mathbb{C}^2$  to the projective plane  $\mathbb{P}^2$ , with coordinates  $(x : y : z)$ . A point  $p$  in  $\mathbb{P}^2$  is given by 3 homogeneous coordinates

$$p = (a : b : c),$$

where  $a, b, c \in \mathbb{C}$  are not all zero. To such a point we can associate a line  $L_p$  in  $\mathbb{P}^2$ , given by the equation

$$L_p : ax + by + cz = 0.$$

Hence to a set of points  $S = \{p_i : i = 1, 2, \dots, n\}$  in  $\mathbb{P}^2$ , we can associate a line arrangement  $\mathcal{A}_S = \{L_{p_i} : i = 1, 2, \dots, n\}$  in  $\mathbb{P}^2$ . The multiplicity of a point  $p$  in a line arrangement  $\mathcal{A}$  is the number of lines of the arrangement  $\mathcal{A}$  passing through  $p$ . For more on line arrangements we refer to [4].

## 2.1 Splitting type of a line arrangement

Why to pass from points to lines ? Because line arrangements have a lot of geometry. In particular, for any line arrangement  $\mathcal{A}$  in  $\mathbb{P}^2$  one can define a rank two vector bundle  $E = E(\mathcal{A})$  on the projective plane  $\mathbb{P}^2$ , the bundle of logarithmic vector fields along  $\mathcal{A}$ . If  $L$  is a generic line in  $\mathbb{P}^2$ , then the restriction  $E|L$  splits as a direct sum of two line bundles on  $L = \mathbb{P}^1$ , with first Chern classes given by two negative integers, say  $(-a, -b)$ , with  $0 \leq a \leq b$ . The pair  $(a, b)$  is called the splitting type of  $E$  and satisfies  $a + b = |\mathcal{A}| - 1$ . For details, see [1,5].

## 2.2 A new look at the 1-dimensional case: a refinement

The fact that the evaluation map

$$\epsilon_d : \mathbb{R}[x]_{\leq d} \rightarrow \mathbb{R}^S = \mathbb{R}^n, \quad \epsilon(Q)(p_i) = Q(p_i),$$

is surjective for  $d \geq n - 1$  is equivalent to the claim that

$$\dim \ker \epsilon_d = \dim \{Q \in \mathbb{R}[x]_{\leq d} : Q(p_i) = 0 \text{ for any } i\} = d + 1 - n,$$

for  $d \geq n - 1$ . Now fix an integer  $k \geq 1$ , consider a new point  $q \in \mathbb{R}$ , but  $q \notin S$ , and define a new evaluation map

$$\epsilon_{d,q,k} : \mathbb{R}[x]_{\leq d} \rightarrow \mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}[x]_{\leq k-1},$$

where  $\epsilon(Q)(p_i) = Q(p_i) \in \mathbb{R}$  and  $\epsilon(Q)(q) = T_{k-1}Q(q) \in \mathbb{R}[x]_{\leq k-1}$  is the  $(k - 1)$ -st Taylor expansion of the polynomial  $Q$  at the point  $q$ .

In particular,  $\epsilon(Q)(q) = T_{k-1}Q(q) = 0$  if and only if the first  $(k - 1)$  derivatives of  $Q$  vanish at  $q$ , namely  $Q^{(j)}(q) = 0$  for all  $0 \leq j \leq k - 1$ . It is easy to show that this new evaluation map is surjective for  $d \geq n + k - 1$ , and hence

$$\dim \ker \epsilon_{d,q,k} = d + 1 - n - k,$$

for  $d \geq n + k - 1$ . The practical interest of this refinement is that, for instance, a zero Taylor expansion of high order means very small values for the polynomial  $Q$  in the neighborhood of the given point  $q$ .

### 2.3 A new 2-dimensional interpolation problem

Starting with a given set of points  $S = \{p_i : i = 1, 2, \dots, n\}$  in  $\mathbb{P}^2$ , we fix an integer  $k \geq 1$ , and define  $I(S)_d$  to be the vector space of homogeneous polynomials  $Q \in \mathbb{C}[x, y, z]_d$  such that  $Q(p_i) = 0$  for any  $i = 1, 2, \dots, n$ . Choose then a generic point  $q \in \mathbb{P}^2$  and consider the vector space  $V(d, S, k, q)$  of homogeneous polynomials  $Q \in I(S)_d$  such that  $T_{k-1}Q(q) = 0$ . The expected dimension of this vector space is

$$\dim_e V(d, S, k, q) := \dim I(S)_d - \binom{k+1}{2},$$

when this number is positive. An important special case is when  $d = k$ , which is also the simplest case to consider. In this setting, we introduce the following notion, see [2].

**Definition 1.** We say that the set  $S$  admits an unexpected curve of degree  $k$  if

$$\dim V(k, S, k, q) > \dim_e V(k, S, k, q) \geq 0.$$

The main result in this direction is the following, see [2].

**Theorem 3.** Let  $S$  be a finite set of  $n$  points in  $\mathbb{P}^2$  and let  $(a_S, b_S)$  be the splitting type of the dual line arrangement  $\mathcal{A}_S$ . Then  $S$  admits an unexpected curve of degree  $k$  if and only if the following hold.

- $a_S + 1 \leq k \leq b_S - 1$ ;
- the multiplicity of any intersection point in  $\mathcal{A}_S$  is at most  $a_S + 1$ .

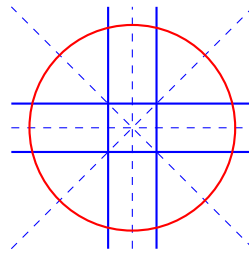
### 2.4 An example: the complete polygonal arrangements

Consider a regular polygon with  $N \geq 3$  edges, and the associated line arrangement  $\mathcal{A}$  consisting of the following  $2N + 1$  lines

- the  $N$  lines determined by the  $N$  edges of the polygon,
- the  $N$  symmetry axes of the polygon, and
- the line at infinity.

This type of line arrangement occurs in the following result, see [3].

**Theorem 4.** For  $N$  even, the complete  $N$ -polygonal arrangement has an unexpected curve of degree  $N$ .



**Fig. 3** Complete 4-polygonal arrangement; the line at infinity is drawn as the red circle.

The proof uses the theory of supersolvable line arrangements to show that  $a = N - 1$  and  $b = N + 1$ . As an example, for  $N = 4$  we get  $a = 3$  and  $b = 5$ . Hence  $\mathcal{A}$  admits an unexpected curve of degree 4 by the result in see [2].

In the case  $N = 4$ , the complete  $N$ -polygonal arrangement is the dual arrangement  $\mathcal{A}_S$ , where the set of points  $S$  consists of the points  $(0 : 0 : 1)$ ,  $(0 : 1 : 0)$ ,  $(1 : 0 : 0)$ ,  $(1 : 1 : 1)$ ,  $(0 : 1 : 1)$ ,  $(1 : 0 : 1)$ ,  $(1 : 1 : 0)$ ,  $(-1 : 1 : 0)$ ,  $(1 : 1 : 2)$ . This situation was considered first by B. Harbourne in [8].

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