Polynomial interpolation in higher dimensions

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Abstract We describe a recent advance in the theory of interpolation in the plane, based on the theory of line arrangements in the complex projective plane.

1 Interpolation in dimensions one and two

1.1 Lagrange interpolation

We denote by $\mathbb{R}$ the field of real numbers and by $\mathbb{C}$ the field of complex numbers. Let p_1,..., p_n be n real numbers, thought of as n points on the real line.

Assume that each point p_i has an associated number $c_i \in \mathbb{R}$, thought of as the result of a measurement effectuated at the point p_i, for $i = 1,2,...,n$, of a physical entity of interest to us, e.g., temperature, pressure, or density of a substance. Let $S = \{p_1, p_2,..., p_n\}$ be the set of these n points, and

$$f : S \to \mathbb{R}$$

the function defined by $f(p_i) = c_i$ for $i = 1,2,...,n$. In order to move from experiment to theory, we would like to find a formula for this function f. The most natural idea is to look for a polynomial $P(x)$ of minimal possible degree such that one has

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$$f(p_i) = P(p_i) \text{ for all } i = 1,2,...,n.$$

The hope is that this polynomial will in fact satisfy $f(t) = P(t)$ for any real number t, and hence our discovered formula would allow us to make predictions as well.

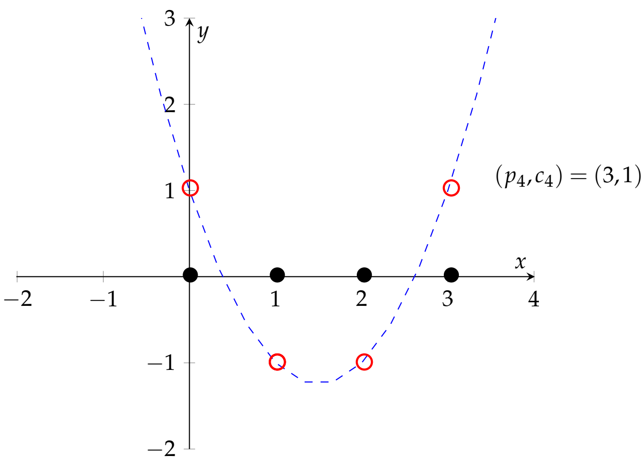

Fig. 1 The $n = 4$ points (p_i, c_i) are on the parabola $y = P(x)$, with $P(x) = x^2 - 3x + 1$.

The following result was first published by Waring in 1779, rediscovered by Euler in 1783, and published by Lagrange in 1795.

Theorem 1. *For any n distinct real numbers p_i, $i = 1,2,...,n$, and any given n values c_i, $i = 1,2,...,n$, there is a unique polynomial $P(x)$ of degree at most $n-1$, such that $P(p_i) = c_i$ for all $i = 1,2,...,n$.*

To give a formula for $P(x)$, consider, for any $i = 1,2,...,n$, the degree $n-1$ polynomial

$$Q_i(x) = \frac{\prod_{j=1,n;j\neq i}(x - p_j)}{\prod_{j=1,n;j\neq i}(p_i - p_j)},$$

and note that $Q_i(p_i) = 1$ and $Q_i(p_k) = 0$ for any $k \neq i$. With this notation one has

$$P(x) = \sum_{i=1,n} c_i Q_i(x).$$

Consider the vector space of polynomials of degree at most d, denoted by $\mathbb{R}[x]_{\leq d}$, and the linear map given by evaluation

$$\epsilon_d : \mathbb{R}[x]_{\leq d} \to \mathbb{R}^S = \mathbb{R}^n, \ \ \epsilon(Q)(p_i) = Q(p_i),$$

for any $i = 1,2,...,n$, where $\mathbb{R}^S$ denotes the vector space of all functions $S \to \mathbb{R}$. The above results say that ϵ_d is injective if and only if $d \leq n-1$, and ϵ_d is surjective if and only if $d \geq n-1$. Such results are true over any field: $\mathbb{R}$, $\mathbb{C}$ or even finite fields.

1.2 Interpolation in dimension 2

Let now $p_i = (x_i, y_i)$ for $i = 1,2,...,n$ be n points in the plane $\mathbb{R}^2$. Assume each point has an associated value $c_i \in \mathbb{R}$, thought of as the result of a measurement at the point p_i. Let $S = \{p_1, p_2, ..., p_n\}$ be the set of these points, consider the associated function $f : S \to \mathbb{R}$ given by $f(p_i) = c_i$ and look for the minimal degree d such that there is a polynomial $Q \in \mathbb{R}[x,y]$ of degree d satisfying

$$f(p_i) = Q(x_i, y_i) \text{ for all } i = 1,2,...,n.$$

This is the old question, but the setting is new: the answer now depends on the position of the points p_i in the plane.

Example 1 (3 points in the plane). If the 3 points p_1, p_2 and p_3 are not collinear, then we can take $d = 1$. Indeed, as above, we can construct 3 polynomials Q_1, Q_2 and Q_3, by taking the equations of lines passing through two of the points p_i. When the 3 points p_1, p_2 and p_3 are collinear, then the minimal degree is $d = 2$. Indeed, there are conics passing through two of these points and avoiding the remaining one.

We discuss now a special type of interpolation node, i.e., a special class of choices for the points p_i's. Consider a finite family of lines $L_j : \ell_j(x,y) = a_j x + b_j y + c_j = 0$ in the plane $\mathbb{R}^2$, for $j = 1,2,...,m$. If these lines are generic, i.e., no two are parallel and no three are concurrent, then we get precisely

$$N = \binom{m}{2}$$

intersection points, which will play the role of our points p_i.

Theorem 2. *For any m generic lines in the plane and any given N values c_i associated to their intersection points p_i, there is a unique polynomial $P(x,y)$ of degree at most $m-2$, such that $P(p_i) = c_i$ for all $i = 1,2,...,N$. More precisely, the evaluation map*

$$\epsilon_d : \mathbb{R}[x,y]_{\leq d} \to \mathbb{R}^S = \mathbb{R}^N, \;\; \epsilon(Q)(p_i) = Q(x_i, y_i),$$

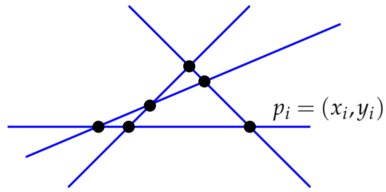

Fig. 2 $m = 4$ generic lines in the plane

is surjective if and only if $d \geq m-2$ and it is injective for $d \leq m-2$. A similar claim of surjectivity holds when the lines are replaced by any nodal curve C of degree d and the intersection points by the set of nodes of C.

Note that the degree of P is much smaller than the number of interpolation points p_i, namely $m-2 < N = m(m-1)/2$. The injectivity claim is easy, using Bezout Theorem about the intersection of two plane curves. The surjectivity is subtle, the proof uses Hodge theory, see [7]. The case when C is a Chebyshev curve is particularly interesting, see [6].

2 Projective Duality and Interpolation

From now on we move from the real field $\mathbb{R}$ to the complex field $\mathbb{C}$, and from the affine plane $\mathbb{C}^2$ to the projective plane $\mathbb{P}^2$, with coordinates $(x:y:z)$. A point p in $\mathbb{P}^2$ is given by 3 homogeneous coordinates

$$p = (a:b:c),$$

where $a,b,c \in \mathbb{C}$ are not all zero. To such a point we can associate a line L_p in $\mathbb{P}^2$, given by the equation

$$L_p : ax + by + cz = 0.$$

Hence to a set of points $S = \{p_i : i = 1,2,...,n\}$ in $\mathbb{P}^2$, we can associate a line arrangement $\mathcal{A}_S = \{L_{p_i} : i = 1,2,...,n\}$ in $\mathbb{P}^2$. The multiplicity of a point p in a line arrangement $\mathcal{A}$ is the number of lines of the arrangement $\mathcal{A}$ passing through p. For more on line arrangements we refer to [4].

2.1 Splitting type of a line arrangement

Why to pass from points to lines ? Because line arrangements have a lot of geometry. In particular, for any line arrangement $\mathcal{A}$ in $\mathbb{P}^2$ one can define a rank two vector bundle $E = E(\mathcal{A})$ on the projective plane $\mathbb{P}^2$, the bundle of logarithmic vector fields along $\mathcal{A}$. If L is a generic line in $\mathbb{P}^2$, then the restriction $E|L$ splits as a direct sum of two line bundles on $L = \mathbb{P}^1$, with first Chern classes given by two negative integers, say $(-a,-b)$, with $0 \leq a \leq b$. The pair (a,b) is called the splitting type of E and satisfies $a+b = |\mathcal{A}| - 1$. For details, see [1,5].

2.2 A new look at the 1-dimensional case: a refinement

The fact that the evaluation map

$$\epsilon_d : \mathbb{R}[x]_{\leq d} \to \mathbb{R}^S = \mathbb{R}^n, \;\; \epsilon(Q)(p_i) = Q(p_i),$$

is surjective for $d \geq n-1$ is equivalent to the claim that

$$\dim \ker \epsilon_d = \dim\{Q \in \mathbb{R}[x]_{\leq d} : Q(p_i) = 0 \text{ for any } i\} = d+1-n,$$

for $d \geq n-1$. Now fix an integer $k \geq 1$, consider a new point $q \in \mathbb{R}$, but $q \notin S$, and define a new evaluation map

$$\epsilon_{d,q,k} : \mathbb{R}[x]_{\leq d} \to \mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}[x]_{\leq k-1},$$

where $\epsilon(Q)(p_i) = Q(p_i) \in \mathbb{R}$ and $\epsilon(Q)(q) = T_{k-1}Q(q) \in \mathbb{R}[x]_{\leq k-1}$ is the $(k-1)$-st Taylor expansion of the polynomial Q at the point q.

In particular, $\epsilon(Q)(q) = T_{k-1}Q(q) = 0$ if and only if the first $(k-1)$ derivatives of Q vanish at q, namely $Q^{(j)}(q) = 0$ for all $0 \leq j \leq k-1$. It is easy to show that this new evaluation map is surjective for $d \geq n+k-1$, and hence

$$\dim \ker \epsilon_{d,q,k} = d+1-n-k,$$

for $d \geq n+k-1$. The practical interest of this refinement is that, for instance, a zero Taylor expansion of high order means very small values for the polynomial Q in the neighborhood of the given point q.

2.3 A new 2-dimensional interpolation problem

Starting with a given set of points $S = \{p_i : i = 1,2,...,n\}$ in $\mathbb{P}^2$, we fix an integer $k \geq 1$, and define $I(S)_d$ to be the vector space of homogeneous polynomials $Q \in \mathbb{C}[x,y,z]_d$ such that $Q(p_i) = 0$ for any $i = 1,2,...,n$. Choose then a generic point $q \in \mathbb{P}^2$ and consider the vector space $V(d,S,k,q)$ of homogeneous polynomials $Q \in I(S)_d$ such that $T_{k-1}Q(q) = 0$. The expected dimension of this vector space is

$$\dim_e V(d,S,k,q) := \dim I(S)_d - \binom{k+1}{2},$$

when this number is positive. An important special case is when $d = k$, which is also the simplest case to consider. In this setting, we introduce the following notion, see [2].

Definition 1. We say that the set S admits an unexpected curve of degree k if

$$\dim V(k,S,k,q) > \dim_e V(k,S,k,q) \geq 0.$$

The main result in this direction is the following, see [2].

Theorem 3. *Let S be a finite set of n points in $\mathbb{P}^2$ and let (a_S,b_S) be the splitting type of the dual line arrangement $\mathcal{A}_S$. Then S admits an unexpected curve of degree k if and only if the following hold.*

- $a_S + 1 \leq k \leq b_S - 1$;
- *the multiplicity of any intersection point in $\mathcal{A}_S$ is at most $a_S + 1$.*

2.4 An example: the complete polygonal arrangements

Consider a regular polygon with $N \geq 3$ edges, and the associated line arrangement $\mathcal{A}$ consisting of the following $2N + 1$ lines

- the N lines determined by the N edges of the polygon,
- the N symmetry axes of the polygon, and
- the line at infinity.

This type of line arrangement occurs in the following result, see [3].

Theorem 4. *For N even, the complete N-polygonal arrangement has an unexpected curve of degree N.*

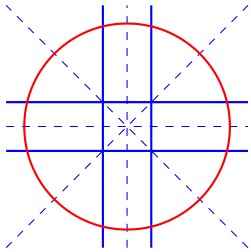

Fig. 3 Complete 4-polygonal arrangement; the line at infinity is drawn as the red circle.

The proof uses the theory of supersolvable line arrangements to show that $a = N - 1$ and $b = N + 1$. As an example, for $N = 4$ we get $a = 3$ and $b = 5$. Hence $\mathcal{A}$ admits an unexpected curve of degree 4 by the result in see [2].

In the case $N = 4$, the complete N-polygonal arrangement is the dual arrangement $\mathcal{A}_S$, where the set of points S consists of the points $(0:0:1)$, $(0:1:0)$, $(1:0:0)$, $(1:1:1)$, $(0:1:1)$, $(1:0:1)$, $(1:1:0)$, $(-1:1:0)$, $(1:1:2)$. This situation was considered first by B. Harbourne in [8].

References

1. Abe T. and Dimca, A: On the splitting types of bundles of logarithmic vector fields along plane curves, Internat. J. Math. 29 (2018), no. 8, 1850055, 20 pp.
2. Cook, D., Harbourne, B., Migliore, J., Nagel, U.: Line arrangements and configurations of points with an unexpected geometric property. Compositio Mathematica, 154(2018), 2150-2194.
3. Di Marca, M., Malara, G., Oneto, A.: Unexpected curves arising from special line arrangements, arXiv:1804.02730.
4. Dimca, A.: Hyperplane Arrangements: An Introduction, Universitext, Springer-Verlag, 2017, 192+viii pages.
5. Dimca, A. and Sernesi, E.: Syzygies and logarithmic vector fields along plane curves, Journal de l'École polytechnique-Mathématiques **1**, 247–267(2014)
6. Dimca, A. and Sticlaru, G.: Chebyshev curves, free resolutions and rational curve arrangements, Math. Proc. Cambridge Phil. Soc. **153**, 385-397 (2012)
7. Dimca, A. and Sticlaru, G.: Koszul complexes and pole order filtrations, Proc. Edinburg. Math. Soc. **58**, 333–354 (2015)
8. Harbourne, B.: Asymptotics of linear systems, with connections to line arrangements, arXiv:1705.09946.