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DETERMINATION OF CONVECTION TERMS AND QUASI-LINEARITIES APPEARING IN DIFFUSION EQUATIONS

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Abstract. We consider the highly nonlinear and ill-posed inverse problem of determining some general expression $F(x,t,u,\nabla_x u)$ appearing in the diffusion equation $\partial_t u-\Delta_x u+F(x,t,u,\nabla_x u)=0$ on $\Omega\times(0,T)$, with $T>0$ and Ω a bounded open subset of $\mathbb{R}^n$, $n\geqslant 2$, from measurements of solutions on the lateral boundary $\partial\Omega\times(0,T)$. We consider both linear and nonlinear expression of $F(x,t,u,\nabla_x u)$. In the linear case, the equation is a convection-diffusion equation and our inverse problem corresponds to the unique recovery, in some suitable sense, of a time evolving velocity field associated with the moving quantity as well as the density of the medium in some rough setting described by non-smooth coefficients on a Lipschitz domain. In the nonlinear case, we prove the recovery of more general quasilinear expression appearing in a non-linear parabolic equation. Our result give a positive answer to the unique recovery of a general vector valued first order coefficient, depending on both time and space variable, and to the unique recovery inside the domain of quasilinear terms, from measurements restricted to the lateral boundary, for diffusion equations.

Keywords: Inverse problem, convection-diffusion equation, non-smooth coefficients, uniqueness, nonlinear equation, Carleman estimates.

Mathematics subject classification 2010 : 35R30, 35K20, 35K59, 35K60.

1. Introduction

1.1. **Statement of the problem.** Let Ω be a Lipschitz bounded domain of $\mathbb{R}^n$, $n\geqslant 2$, such that $\mathbb{R}^n\setminus\Omega$ is connected. We set $Q=\Omega\times(0,T)$, $\Sigma=\partial\Omega\times(0,T)$, with $T>0$, $\Omega^s:=\Omega\times\{s\}$, $s=0,T$. In this paper, we study the inverse problems associated with an initial boundary value problem (IBVP in short) taking the form

$$\begin{cases}\partial_t u-\Delta_x u+F(x,t,u,\nabla_x u)=0, & \text{in } Q,\\ u(\cdot,0)=u_0, & \text{in } \Omega,\\ u=g, & \text{on } \Sigma.\end{cases}\tag{1.1}$$

Our goal is to prove the recovery of the term $F(x,t,u,\nabla_x u)$ appearing, in the above diffusion equation, from measurements of its solutions on the lateral boundary Σ. We consider both linear expressions of the form $F(x,t,u,\nabla_x u)=A(x,t)\cdot\nabla_x u+\nabla_x\cdot[B(x,t)]u+q(x,t)u$, and more general quasi-linear expressions.

For the linear problem the IBVP takes the form

$$\begin{cases}\partial_t u-\Delta_x u+A(x,t)\cdot\nabla_x u+\nabla_x\cdot[B(x,t)]u+q(x,t)u=0, & \text{in } Q,\\ u(\cdot,0)=0, & \text{in } \Omega,\\ u=g, & \text{on } \Sigma.\end{cases}\tag{1.2}$$

For $A,B\in L^\infty(Q)^n$ and $q\in L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))$, we consider the Dirichlet-to-Neumann (DN in short) map associated with this problem given by

$$\Lambda_{A,B,q}:g\mapsto N_{A,B,q}u,$$

where u solves (1.2). Here the convection term A takes values in $\mathbb{R}^n$. We define $N_{A,q}u$ in such a way that for $w\in H^1(Q)$ satisfying $w_{|\Omega^T}=0$ we have

$$\left\langle N_{A,B,q}u,w_{|\Sigma}\right\rangle:=\int_Q[-u\partial_t w+\nabla_x u\cdot\nabla_x w+A\cdot\nabla_x uw-B\cdot\nabla_x(uw)+quw]dxdt.\tag{1.3}$$

We refer to Section 2 for more detail and a rigorous definition of this map and we mention that for g, A, B, q and Ω sufficiently smooth, we have

$$N_{A,B,q}u = [\partial_\nu u - (B\cdot\nu)u]_{|\Sigma},$$

with ν the outward unit normal vector to $\partial\Omega$. This means that $N_{A,B,q}$ and $\Lambda_{A,B,q}$ are the natural extension of, respectively, the normal derivative of the solution of (1.2) and the DN map of (1.2) to non-smooth setting. In this paper we study the inverse problem of determining in some suitable sense the coefficient (A,B,q) from the full DN map $\Lambda_{A,B,q}$ or from partial knowledge of this map to some parts of Σ.

1.2. **Motivations.** Let us observe that the IBVP (1.2) is associated with a convection-diffusion equation which corresponds to a combination of diffusion and convection equations. These equations describe the transfer of mass or heat, due to both diffusion and convection process, of different physical quantities (particles, energy,...) inside a physical system (see for instance [61]). The problem (1.2) can also describe the velocity of a particle (Fokker-Planck equations) or the price evolution of a European call (Black-Scholes equations). Here the coefficient A corresponds to the velocity field associated with the moving quantity and our inverse problem corresponds to the recovery of this field from measurement given by an application of source and measurement of the flux at the boundary of the domain. Actually we manage to prove the simultaneous recovery, in some suitable sense, of the the coefficient A, B and q, where the zero order coefficient q can be associated with a time-evolving density of an inhomogeneous medium. By allowing our coefficients to depend both on time and space we can apply our inverse problem to several context where the evolution in time of these physical phenomena can not be omitted. We mention also that the general setting of our problem allows to cover different types of unstable physical phenomenon associated with singular coefficients and a non-smooth domain.

The quasi-linear problem (1.1) corresponds to more complex model where the linear expression

$$F(x,t,u,\nabla_x u) = A(x,t)\cdot\nabla_x u + \nabla_x\cdot[B(x,t)]u + q(x,t)u$$

is replaced by a more general nonlinear term. Here the goal of the inverse problem is to prove the recovery of this nonlinear expression $F(x,t,u,\nabla_x u)$ describing the underlying physical law of the system. This inverse problem can be associated with different models like physics of high temperatures, chemical kinetics and aerodymanics.

1.3. **Obstruction to uniqueness.** We recall that there is an obstruction to uniqueness for our inverse problem given by a gauge invariance. More precisely, we fix $p_1\in[1,+\infty)$ such that we have

$$p_1 := \begin{cases} n & \text{for } n\geqslant 3 \\ 2+\varepsilon & \text{for } n=2, \end{cases} \tag{1.4}$$

with $\varepsilon\in(0,1)$. Let $A_1,B_1\in L^\infty(Q)^n$, $q_1\in L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))$ and

$$\varphi\in L^\infty(0,T;W^{1,\infty}(\Omega))\cap W^{1,\infty}(0,T;L^{p_1}(\Omega))\cap L^\infty(0,T;H_0^1(\Omega))\setminus\{0\}.$$

Now consider $A_2\in L^\infty(Q)^n$, $q_2\in L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))$ given by

$$A_2 = A_1 + 2\nabla_x\varphi,\quad B_2 = B_1+\nabla_x\varphi,\quad q_2 = q_1 - \partial_t\varphi - |\nabla_x\varphi|^2 - A_1\cdot\nabla_x\varphi. \tag{1.5}$$

Then, assuming that $u_1\in H^1(0,T;H^{-1}(\Omega))\cap L^2(0,T;H^1(\Omega))$ solves (1.2) with $A=A_1$, $B=B_1$ and $q=q_1$, and fixing $u_2=e^{\varphi}u_1$ we find

$$\begin{aligned}
&(\partial_t-\Delta_x+A_2\cdot\nabla_x+\nabla_x\cdot(B_2)+q_2)u_2\\
&=e^{\varphi}[\partial_t-\Delta_x+(A_2-2\nabla_x\varphi)\cdot\nabla_x+\nabla_x\cdot(B_2-\nabla_x\varphi)+(q_2+\partial_t\varphi+|\nabla_x\varphi|^2+A_1\cdot\nabla_x\varphi)]u_1\\
&=e^{\varphi}(\partial_t-\Delta_x+A_1\cdot\nabla_x+\nabla_x\cdot(B_1)+q_1)u_1=0.
\end{aligned}$$

This and the fact that $\varphi_{|\Sigma}=0$ proves that u_2 solves (1.2) with $A=A_2$, $B=B_2$ and $q=q_2$. Then, for any $w\in H^1(Q)$ satisfying $w_{|\Omega^T}=0$, we get

$$\begin{aligned}\left\langle N_{A_2,B_2,q_2}u_2,w_{|\Sigma}\right\rangle &= \int_Q[-u_2\partial_t w+\nabla_x u_2\cdot\nabla_x w+(A_2\cdot\nabla_x u_2)w-B_2\cdot\nabla(u_2w)+q_2u_2w]dxdt\\ &= \int_Q e^{\varphi}[-u_1\partial_t w+\nabla_x u_1\cdot\nabla_x w+(\nabla_x\varphi\cdot\nabla_x u_1)w+((A_1-B_1)\cdot\nabla_x u_1)w-u_1(B_1\cdot\nabla_x w)]dxdt\\ &\quad+\int_Q e^{\varphi}(q_2+(A_1-B_1)\cdot\nabla_x\varphi+|\nabla_x\varphi|^2)uwdxdt\\ &= \int_Q[-u_1\partial_t(e^{\varphi}w)+\nabla_x u_1\cdot\nabla_x(e^{\varphi}w)+(A_1\cdot\nabla_x u_1)(e^{\varphi}w)-B_1\cdot\nabla_x(u_1e^{\varphi}w)]dxdt\\ &\quad+\int_Q e^{\varphi}(q_2+\partial_t\varphi+A_1\cdot\nabla_x\varphi+|\nabla_x\varphi|^2)uwdxdt\\ &= \left\langle N_{A_1,B_1,q_1}u_1,e^{\varphi}w_{|\Sigma}\right\rangle.\end{aligned}$$

Combining this with the fact that $(1-e^{\varphi})w\in L^2(0,T;H_0^1(\Omega))$ and the fact that, by the Sobolev embedding theorem, we have $e^{\varphi}w\in H^1(Q)$, we deduce that $N_{A_1,B_1,q_1}u_1=N_{A_2,B_2,q_2}u_2$. It follows that, for any

$$\varphi\in\{h\in L^{\infty}(0,T;W^{1,\infty}(\Omega))\cap W^{1,\infty}(0,T;L^{p_1}(\Omega)):\ h_{|\Sigma}=0\},$$

the DN map of problem (1.2) satisfies the following gauge invariance

$$\Lambda_{A,B,q}=\Lambda_{A+2\nabla_x\varphi,B+\nabla_x\varphi,q-\partial_t\varphi-|\nabla_x\varphi|^2-A\cdot\nabla_x\varphi}.$$

According to this obstruction, the best result that one can expect is the recovery of the gauge class of the coefficients (A,B,q) given by the relation (1.5). Note also that, without additional information about the coefficient B it is even impossible to recover the gauge class of (A,B,q) given by (1.5). Indeed, for any $\varphi\in W^{2,\infty}(Q)$ satisfying $\varphi_{|\Sigma}=\partial_\nu\varphi_{|\Sigma}=0$, the DN map of problem (1.2) satisfies the following gauge invariance

$$\Lambda_{A,B,q}=\Lambda_{A+2\nabla_x\varphi,B,q-\partial_t\varphi-|\nabla_x\varphi|^2-A\cdot\nabla_x\varphi+\Delta_x\varphi}.$$

This means that for

$$A_2=A_1+2\nabla_x\varphi,\quad B_2=B_1,\quad q_2=q_1-\partial_t\varphi-|\nabla_x\varphi|^2-A_1\cdot\nabla_x\varphi+\Delta_x\varphi,$$

with $\varphi\in W^{2,\infty}(Q)$ satisfying $\varphi_{|\Sigma}=\partial_\nu\varphi_{|\Sigma}=0$ and $\varphi\neq 0$, we have $\Lambda_{A_1,B_1,q_1}=\Lambda_{A_2,B_2,q_2}$ but condition (1.5) is not fulfilled. In light of these obstructions, in the present paper we consider the recovery of some information about the gauge class of the coefficient (A,B,q) given by (1.5) from the DN map $\Lambda_{A,B,q}$.

1.4. **State of the art.** The recovery of coefficients appearing in parabolic equations has attracted many attention these last decades. We refer to [14, 34] for an overview of such problems. While numerous authors considered the recovery of the zero order coefficient q, only few authors studied the determination of the convection term A. We can mention the work of [19, 63] for the treatment of this problem in the 1 dimensional case as well as the work of [12] dealing with the unique recovery of a time-independent convection term for $n=2$ from a single boundary measurement.

Recall that, for time-independent coefficients (A,B,q) and with suitable regularity assumptions, one can apply the analyticity in time of solutions of (1.2), with suitable boundary conditions g, and the Laplace transform with respect to the time variable in order to transform our inverse problem into the recovery of coefficients appearing in a steady state convection-diffusion equation (see for instance [36] for more details about this transformation of the inverse problem). This last inverse problem has been studied by [11, 13, 46, 53] and it is strongly connected to the recovery of magnetic Schrödinger operator from boundary measurements which has been intensively studied these last decades. Without being exhaustive, we refer to the work of [9, 20, 40, 54, 55, 57, 59]. In particular, we mention the work of [45] where the recovery of magnetic Schrödinger operators has been addressed for bounded electromagnetic potentials which is the

weakest regularity assumption so far for general bounded domains. Let us also observe that there is a strong connection between this problem and the so called Calderón problem studied by [6, 7, 8, 21, 37, 62] and extended to the non-smooth setting in [1, 10, 25, 26].

Several authors considered also the determination of time-dependent coefficients appearing in parabolic equations. In [30], the author extended the construction of complex geometric optics solutions, introduced by [62], to various PDEs including hyperbolic and parabolic equations to prove density of products of solutions. From the results of [30] one can deduce the unique determination of a coefficient q depending on both space and time variables, when $A = B = 0$, from measurements on the lateral boundary Σ with additional knowledge of all solutions on Ω^0 and Ω^T. In Subsection 3.6 of [14], the author extended the uniqueness result of [30] to a log-type stability estimate. In the special case of cylindrical domain, [22] proved recovery of a time-dependent coefficient, independent of one spatial direction, from single boundary measurements. In [15] the authors addressed recovery of a parameter depending only on the time variable from single boundary measurements. More recently, [16] proved that the result of [30] remains true from measurements given by $\Lambda_{A,B,q}$ when $A = B = 0$. More precisely, [16] proved, what seems to be, the first result of stability in the determination of a coefficient, depending on the space variable, appearing in a parabolic equation with measurements restricted to the lateral boundary Σ. We recall also the works of [3, 5, 29, 38, 39, 42] related to the recovery of time-dependent coefficients for hyperbolic equations and the stable recovery of coefficients appearing in Schrödinger equations established by [17, 43].

For the recovery of nonlinear terms, we mention the series of works [31, 32, 33] of Isakov dedicated to this problem for elliptic and parabolic equations. In [31, 32] the author considered the recovery of a semi-linear term of the form $F(x, u)$ inside the domain (i.e $F(x, u)$ with $x \in \Omega$, $u \in \mathbb{R}$) or restricted to the lateral boundary (i.e $F(x, u)$ with $x \in \partial\Omega$, $u \in \mathbb{R}$) while in [33] he considered the recovery of a quasilinear term of the form $F(u, \nabla_x u)$. In all these works, the approach developed by Isakov is based on a linearization of the inverse problem for nonlinear equations and results based on recovery of coefficients for linear equations. More precisely, in [31] the author used his work [30], related to the recovery of a time-dependent coefficient q on Q, while in [32, 33] he used results of recovery of coefficients on the lateral boundary Σ. The approach of Isakov, which seems to be the most efficient for recovering general nonlinear terms from boundary measurements, has also been considered by [35, 60] for the recovery of more general nonlinear terms appearing in nonlinear elliptic equations and by [40] who proved, for what seems to be the first time, the recovery of a general semi-linear term appearing in a semi-linear hyperbolic equation from boundary measurements. In [16], the authors proved a log-type stability estimate associated with the uniqueness result of [31] but with measurements restricted only to the lateral boundary Σ. Finally, for results stated with single measurements we refer to [18, 44] and for results stated with measurements given by the source-to-solution map associated with semilinear hyperbolic equations we refer to [27, 47, 48, 49].

1.5. **Main result for the linear problem.** Our main result for the linear equation takes the following form.

Theorem 1.1. *For $j = 1, 2$, let $q_j \in L^\infty(0, T; L^p(\Omega)) \cup \mathcal{C}([0, T]; L^{\frac{2n}{3}}(\Omega))$, with $p > 2n/3$, and let $A_j, B_j \in L^\infty(Q)^n$. The condition*

$$\Lambda_{A_1,B_1,q_1} = \Lambda_{A_2,B_2,q_2} \tag{1.6}$$

implies

$$dA_1 = dA_2. \tag{1.7}$$

Here for $A = (a_1, \dots, a_n)$, dA denotes the 2-form given by

$$dA := \sum_{1 \leqslant i < j \leqslant n} (\partial_{x_i} a_j - \partial_{x_j} a_i) dx_i \wedge dx_j.$$

Let us also consider the additional conditions

$$A_1 - A_2 \in W^{1,\infty}(0, T; L^{p_1}(\Omega))^n, \quad \nabla_x \cdot (A_1 - A_2),\ \nabla_x \cdot (B_1 - B_2),\ q_1 - q_2 \in L^\infty(Q), \tag{1.8}$$

$$(B_1 - B_2) \cdot \nu_{|\Sigma} = 2(A_2 - A_1) \cdot \nu_{|\Sigma}, \tag{1.9}$$

where $(B_1-B_2)\cdot\nu$ *(resp.* $(A_1-A_2)\cdot\nu$*) corresponds to the normal trace of* B_1-B_2 *(resp.* (A_1-A_2)*) restricted to* Σ *which is well defined as an element of* $L^\infty(0,T;\mathcal{B}(H^{\frac{1}{2}}(\partial\Omega);H^{-\frac{1}{2}}(\partial\Omega)))$*. Assuming that* (1.8)-(1.9) *are fulfilled,* (1.6) *implies that there exists* $\varphi\in W^{1,\infty}(Q)$ *such that*

$$\begin{cases} A_2 = A_1 + 2\nabla_x\varphi, & \text{in } Q,\\ \nabla_x\cdot B_2 + q_2 = \nabla_x\cdot(B_1+\nabla_x\varphi) + q_1 - \partial_t\varphi - |\nabla_x\varphi|^2 - A_1\cdot\nabla_x\varphi, & \text{in } Q,\\ \varphi = 0, & \text{on } \Sigma. \end{cases} \tag{1.10}$$

This result says that conditions (1.6) implies that (A_1,B_1,q_1) and (A_2,B_2,q_2) are gauge equivalent. A direct consequence of this result is the following corollary.

Corollary 1.1. *Let* Ω *be connected and let* $A_1,A_2\in L^\infty(Q)^n$ *be such that* $\nabla_x\cdot(A_1-A_2)in\in L^\infty(Q)$ *and* $(A_2-A_1)\cdot\nu_{|\Sigma}=0$*. Then, for any* $q\in L^\infty(0,T;L^p(\Omega))\cap\mathcal{C}([0,T];L^{\frac{2n}{3}}(\Omega))$*,* $p\in\left(\frac{2n}{3},n\right)$*, and* $B\in L^\infty(Q)^n$*, we have*

$$\Lambda_{A_1,B,q}=\Lambda_{A_2,B,q}\Rightarrow A_1=A_2.$$

Let us mention that there is another way to formulate convection or advection-diffusion equations given by the following IBVP

$$\begin{cases} \partial_t v - \Delta_x v + \nabla_x\cdot(A(x,t)v) = 0, & \text{in } Q,\\ v(0,\cdot)=0, & \text{in } \Omega,\\ v = g, & \text{on } \Sigma. \end{cases} \tag{1.11}$$

The corresponding inverse problem consists in recovering the velocity field described by the coefficient A from the associated DN map

$$\tilde{\Lambda}_A : g\mapsto \tilde{N}_A v,$$

where, for $v\in L^2(0,T;H^1(\Omega))\cap H^1(0,T;H^{-1}(\Omega))$ solving (1.11) and for $w\in H^1(Q)$ satisfying $w_{|\Omega^T}=0$, $\tilde{N}_A v$ is defined by

$$\left\langle \tilde{N}_A v, w_{|\Sigma}\right\rangle := \int_Q [-u\partial_t w + \nabla_x u\cdot\nabla_x w + \frac{1}{2}(A\cdot\nabla_x u)w - \frac{1}{2}u(A\cdot\nabla_x w) + \frac{1}{2}\nabla_x\cdot(A)uw]dxdt.$$

Using the identity $\tilde{\Lambda}_A = \Lambda_{A,\frac{A}{2},\frac{\nabla_x\cdot(A)}{2}}$ and applying Theorem 1.1 we obtain the following.

Corollary 1.2. *Let* Ω *be connected and let* $A_1,A_2\in L^\infty(Q)^n$ *be such that*

$$\nabla_x\cdot(A_1),\ \nabla_x\cdot(A_2)\in L^\infty(0,T;L^p(\Omega))\cap\mathcal{C}([0,T];L^{\frac{2n}{3}}(\Omega)),\quad p>2n/3.$$

Assume also that (1.8)-(1.9) *are fulfilled, for* $B_j=\frac{A_j}{2}$ *and* $q_j=\frac{\nabla_x\cdot(A_j)}{2}$*,* $j=1,2$*. Then, the condition* $\tilde{\Lambda}_{A_1}=\tilde{\Lambda}_{A_2}$ *implies* $A_1=A_2$*.*

In the spirit of [2], by assuming that the coefficients are known in the neighborhood of Σ, we can improve Theorem 1.1 into the recovery of the coefficients from measurements in an arbitrary portion of the boundary. More precisely, for any open set γ of $\partial\Omega$, we denote by $\mathcal{H}_\gamma$ the subspace of $H^1(Q)$ given by

$$\mathcal{H}_\gamma := \{h_{|\Sigma} :\ h\in H^1(Q),\ h_{|\Omega^T}=0,\ \mathrm{supp}(h_{|\Sigma})\subset\gamma\times[0,T]\}.$$

We fix γ_1,γ_2 two arbitrary open and not empty subsets of $\partial\Omega$. Then, we can consider, for $A,B\in L^\infty(Q)^n$ and $q\in L^\infty(0,T;L^{p_1}(\Omega))$, the partial DN map

$$\Lambda_{A,B,q,\gamma_1,\gamma_2} : \mathcal{H}_+\cap\mathcal{E}'(\gamma_1\times[0,T])\ni g\mapsto N_{A,B,q}u_{|\mathcal{H}_{\gamma_2}},$$

with u the solution of (1.2) and $\mathcal{H}_+$ the space defined in Section 2. Then, we can improve Theorem 1.1 in the following way.

Corollary 1.3. *Let* Ω *be connected. We fix* $q_1,q_2\in L^\infty(0,T;L^{p_1}(\Omega))$*, with* p_1 *given by* (1.4)*, and we consider* $A_j,B_j\in L^\infty(Q)^n$*,* $j=1,2$*, satisfying* $\nabla_x\cdot(A_j),\nabla_x\cdot(B_j)\in L^\infty(0,T;L^{p_1}(\Omega))$*. Assume that there exists an open connected set* $\Omega_*\subset\Omega$*, satisfying* $\partial\Omega\subset\partial\Omega_*$*, such that*

$$A_1(x,t)=A_2(x,t),\quad B_1(x,t)=B_2(x,t),\quad q_1(x,t)=q_2(x,t),\quad (x,t)\in\Omega_*\times(0,T). \tag{1.12}$$

Then the condition

$$\Lambda_{A_1,B_1,q_1,\gamma_1,\gamma_2} = \Lambda_{A_2,B_2,q_2,\gamma_1,\gamma_2} \tag{1.13}$$

implies that $dA_1 = dA_2$. *If in addition* (1.8) *is fulfilled,* (1.13) *implies that there exists* $\varphi \in W^{1,\infty}(Q)$ *satisfying* (1.10).

1.6. **Recovery of quasilinear terms.** In this subsection, we will state our results related to the recovery of general quasi-linear terms $F(x,t,u,\nabla_x u)$ appearing in (1.1). We denote by Σ_p the parabolic boundary of Q defined by $\Sigma_p = \Sigma \cup \Omega^0$. Moreover, for all $\alpha \in (0,1)$, we denote by $\mathcal{C}^{\alpha,\frac{\alpha}{2}}(\overline{Q})$ the space of functions $f \in \mathcal{C}(\overline{Q})$ satisfying

$$[f]_{\alpha,\frac{\alpha}{2}} = \sup\left\{ \frac{|f(x,t)-f(y,s)|}{(|x-y|^2+|t-s|)^{\frac{\alpha}{2}}} : \ (x,t),(y,s) \in \overline{Q},\ (x,t) \neq (y,s) \right\} < \infty.$$

Then we define the space $\mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q})$ as the set of functions f lying in $\mathcal{C}([0,T];\mathcal{C}^2(\overline{\Omega})) \cap \mathcal{C}^1([0,T];\mathcal{C}(\overline{\Omega}))$ such that

$$\partial_t f, \partial_x^\beta f \in \mathcal{C}^{\alpha,\frac{\alpha}{2}}(\overline{Q}), \quad \beta \in \mathbb{N}^n,\ |\beta| = 2.$$

We consider on these spaces the usual norm and we refer to [14, pp. 4] for more details. We consider the nonlinear parabolic equation

$$\begin{cases} (\partial_t u - \Delta u + F(x,t,u,\nabla_x u) = 0 & \text{in } Q, \\ u = G & \text{on } \Sigma_p. \end{cases} \tag{1.14}$$

For $\alpha \in (0,1)$ and Ω a $\mathcal{C}^{2+\alpha}$ bounded domain, $F : (x,t,u,v) \mapsto F(x,t,u,v) \in C^1(\overline{Q} \times \mathbb{R} \times \mathbb{R}^n)$ satisfying (6.1)-(6.3), $G \in \mathcal{X} = \{K_{|\overline{\Sigma}_p};\ \text{for some } K \in C^{2+\alpha,1+\alpha/2}(\overline{Q})\}$, problem (1.14) admits a unique solution $u_{F,G} \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$ (see Section 6 for more detail). Then, for ν the outward unit normal vector to $\partial\Omega$, we can introduce the DN map associated with (1.14) given by

$$\mathcal{N}_F : \mathcal{X} \ni G \mapsto \partial_\nu u_{F,G|\Sigma} \in L^2(\Sigma)$$

and we consider the recovery of F from partial knowledge of $\mathcal{N}_F$. More precisely, we prove in Proposition 6.1 that for $\partial_u F \in \mathcal{C}^1(\overline{Q} \times \mathbb{R}^n \times \mathbb{R};\mathbb{R})$ and $\partial_v F \in \mathcal{C}^1(\overline{Q} \times \mathbb{R}^n \times \mathbb{R};\mathbb{R}^n)$, $\mathcal{N}_F$ is continuously Fréchet differentiable. Then, fixing

$$\mathcal{X}_0 := \{G \in \mathcal{X} :\ G_{|\Omega^0} = 0\}, \quad k_v : x \mapsto x \cdot v, \quad h_{a,v} : x \mapsto x \cdot v + a,$$

where $a \in \mathbb{R}$, $v \in \mathbb{R}^n$, we consider the recovery of F from

$$\mathcal{N}_F'(k_v|_{\Sigma_p})H \quad \text{and} \quad \mathcal{N}_F'(h_{a,v}|_{\Sigma_p})H, \quad H \in \mathcal{X}_0,\ a \in \mathbb{R},\ v \in \mathbb{R}^n,$$

where $\mathcal{N}_F'$ denotes the Fréchet differentiation of $\mathcal{N}_F$.

We obtain two main results for this problem. In our first main result, we are interested in the recovery of information about general nonlinear terms of the form $F(x,t,u,\nabla_x u)$ form the knowledge of

$$\mathcal{N}_F'(h_{a,v}|_{\Sigma_p})H, \quad H \in \mathcal{X}_0,\ a \in \mathbb{R},\ v \in \mathbb{R}^n.$$

Our first main result can be stated as follows.

Theorem 1.2. *Let* Ω *be a* $\mathcal{C}^{2+\alpha}$ *bounded and connected domain and let* $F_1, F_2 \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q};\mathcal{C}^3(\mathbb{R} \times \mathbb{R}^n))$ *satisfy* (6.1)-(6.3). *Let also, for* $j = 1,2$, $\partial_v F_j \in \mathcal{C}^{1+\frac{\alpha}{2}}([0,T];\mathcal{C}^1(\overline{\Omega} \times \mathbb{R}^n \times \mathbb{R});\mathbb{R}^n)$ *and let*

$$\partial_x^\ell F_j(x,0,u,v) = 0, \quad j = 1,2,\ k = 0,1,\ x \in \partial\Omega,\ u \in \mathbb{R},\ v \in \mathbb{R}^n,\ \ell \in \mathbb{N}^n,\ |\ell| \leqslant 2. \tag{1.15}$$

Then, the condition

$$\mathcal{N}_{F_1}'(h_{a,v}|_{\Sigma_p})H = \mathcal{N}_{F_2}'(h_{a,v}|_{\Sigma_p})H, \quad H \in \mathcal{X}_0,\ a \in \mathbb{R},\ v \in \mathbb{R}^n \tag{1.16}$$

imply that there exists

$$\varphi : Q \times \mathbb{R} \times \mathbb{R}^n \ni (x,t,u,v) \mapsto \varphi(x,t,u,v) \in \mathcal{C}^1([0,T];\mathcal{C}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)) \cap \mathcal{C}^2(\overline{\Omega};\mathcal{C}([0,T] \times \mathbb{R} \times \mathbb{R}^n))$$

such that, for all $(u,v)\in\mathbb{R}\times\mathbb{R}^n$*, we have*

$$\begin{cases} \partial_v(F_2-F_1)(x,0,u,v)=2\partial_x\varphi(x,0,u-x\cdot v,v), & x\in\Omega,\\ \partial_u(F_2-F_1)(x,0,u,v)=-(\partial_t\varphi-|\nabla_x\varphi|^2-\partial_vF_1(x,0,u,v)\partial_x\varphi)(x,0,u-x\cdot v,v), & x\in\Omega,\\ \varphi(x,t,u,v)=0, & (x,t)\in\Sigma. \end{cases}\tag{1.17}$$

From this result, we deduce the following.

Corollary 1.4. *Let the condition of Theorem 1.2 be fulfilled. Assume also that, for all* $x\in\Omega,\ u\in\mathbb{R},\ v\in\mathbb{R}^n$*, the following condition*

$$\sum_{j=1}^n\left[\partial_{x_j}\partial_{v_j}F_1(x,0,u,v)+\partial_u\partial_{v_j}F_1(x,0,u,v)v_j\right]=\sum_{j=1}^n\left[\partial_{x_j}\partial_{v_j}F_2(x,0,u,v)+\partial_u\partial_{v_j}F_2(x,0,u,v)v_j\right]\tag{1.18}$$

is fulfilled. Then condition (1.16) *implies*

$$\partial_vF_1(x,0,u,v)=\partial_vF_2(x,0,u,v),\quad x\in\Omega,\ u\in\mathbb{R},\ v\in\mathbb{R}^n.\tag{1.19}$$

In particular, if there exists $v_0\in\mathbb{R}^n$ *such that*

$$F_1(x,0,u,v_0)=F_2(x,0,u,v_0),\quad x\in\Omega,\ u\in\mathbb{R},\tag{1.20}$$

and (1.24) *are fulfilled, then condition* (1.16) *implies*

$$F_1(x,0,u,v)=F_2(x,0,u,v),\quad x\in\Omega,\ u\in\mathbb{R},\ v\in\mathbb{R}^n.\tag{1.21}$$

Remark 1.1. *The result of Corollary 1.4 can be applied to the unique full recovery of quasilinear terms of the form*

$$F(x,t,u,\nabla_xu)=G_1(x,u,\nabla_xu)+tG_2(x,t,u,\nabla_xu),\tag{1.22}$$

with G_2 *and*

$$H:(x,u,v)\mapsto\sum_{j=1}^n\left[\partial_{x_j}\partial_{v_j}G_1(x,u,v)+\partial_u\partial_{v_j}G_1(x,u,v)v_j\right]$$

two known functions.

For our second main result we consider the full recovery of the nonlinear term $F(x,t,u,\nabla_xu)$ from the data

$$\mathcal{N}'_F(k_v|_{\Sigma_p})H,\quad H\in\mathcal{X}_0,\ v\in\mathbb{R}^n.$$

For this purpose, taking into account the natural invariance for the recovery of such nonlinear terms, described by condition (1.17), our result will require some additional assumptions on the class of nonlinear terms under consideration. Our second main result related to this problem can be stated as follows.

Theorem 1.3. *Let* Ω *be a* $\mathcal{C}^{2+\alpha}$ *bounded and connected domain and let* $F_1,F_2\in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q};\mathcal{C}^3(\mathbb{R}\times\mathbb{R}^n))$ *satisfy* (6.1)-(6.3)*. Let also, for* $j=1,2$*,* $\partial_vF_j\in\mathcal{C}^{1+\frac{\alpha}{2}}([0,T];\mathcal{C}^1(\overline{\Omega}\times\mathbb{R}^n\times\mathbb{R});\mathbb{R}^n)$ *and let* (1.15) *be fulfilled. Then, the conditions*

$$\mathcal{N}'_{F_1}(k_v|_{\Sigma_p})H=\mathcal{N}'_{F_2}(k_v|_{\Sigma_p})H,\quad H\in\mathcal{X}_0,\ v\in\mathbb{R}^n\tag{1.23}$$

and

$$\partial_uF_1(x,t,u,v)=\partial_uF_2(x,t,u,v),\quad (x,t)\in Q,\ u\in\mathbb{R},\ v\in\mathbb{R}^n,\tag{1.24}$$

$$\partial_u^2F_1(x,t,u,v)(x,t)=0,\ \partial_v\partial_uF_1(x,t,u,v)(x,t)=0,\quad (x,t)\in Q,\ u\in\mathbb{R},\ v\in\mathbb{R}^n,\tag{1.25}$$

imply (1.19)*. In addition, if there exists* $v_0\in\mathbb{R}^n$ *such that* (1.20) *and* (1.24) *are fulfilled, then condition* (1.23) *implies* (1.21)*.*

Remark 1.2. *The result of Theorem 1.3 can be applied to the unique full recovery of quasilinear terms of the form*

$$F(x,u,\nabla_xu)=G_1(x,\nabla_xu)+G_2(x,t)u,\tag{1.26}$$

when G_2 *is known.*

A direct consequence of Theorem 1.3 will be a partial data result in the spirit of Corollary 1.3. More precisely, for any open set γ of $\partial\Omega$, we define $\mathcal{N}_{F,\gamma}$ by

$$\mathcal{N}_{F,\gamma}G := \mathcal{N}_F(G)_{|\gamma\times(0,T)}, \quad G \in \mathcal{X}.$$

We denote also by $\mathcal{X}_{0,\gamma}$ the set $\mathcal{X}_{0,\gamma} := \{G \in \mathcal{X}_0 : \ \mathrm{supp}(G_{|\Sigma}) \subset \gamma \times [0,T]\}$. Then, we deduce from Theorem 1.3 the following result.

Corollary 1.5. *Let the condition of Theorem 1.3 be fulfilled. Let Ω_* be an open connected subset of Ω satisfying $\partial\Omega \subset \partial\Omega_*$ and let γ_1, γ_2 be two arbitrary open subset of $\partial\Omega$. We assume that F_1, F_2 fulfill*

$$\partial_v F_1(x,t,u,v) = \partial_v F_2(x,t,u,v) = 0, \quad (x,t) \in \Omega_* \times (0,T), \ u \in \mathbb{R}, \ v \in \mathbb{R}^n \tag{1.27}$$

and (1.24)*. Then the condition*

$$\mathcal{N}'_{F_1,\gamma_2}(k_v|_{\Sigma_p})H = \mathcal{N}'_{F_2,\gamma_2}(k_v|_{\Sigma_p})H, \quad H \in \mathcal{X}_{0,\gamma_1}, \ v \in \mathbb{R}^n, \tag{1.28}$$

implies (1.19)*.*

1.7. **Comments about our results.** Let us first observe that, to our best knowledge, Theorem 1.1 is first result of unique recovery, modulo gauge invariance, of general convection term depending on both time and space variables. Actually, in Theorem 1.1 we prove the simultaneous recovery of the three coefficients A, B, q modulo the gauge invariance given by (1.10). According to the obstruction described in Section 1.3, this is the best one can expect for the simultaneous recovery of the three coefficients A, B, q. Note also that, in contrast to time-independent coefficients, our inverse problem can not be reduced to the recovery of coefficients appearing in a steady state convection-diffusion equation from the associated DN map.

Not only Theorem 1.1 provides, for what seems to be the first time, a result of recovery of general first and zero order time-dependent dependent coefficients appearing in a parabolic equation but it is also stated in a non smooth setting. Indeed, we only require the two vector valued coefficients A, B to be bounded and we allow q to have singularities with respect to the space variable. Moreover, we state our result in a general Lipschitz domain Ω. Such general setting make Theorem 1.1 suitable for many potentials application and the regularity of the coefficients A, B, q can be compared to [45] where one can find the best result known so far, in terms of regularity of the coefficients, about the recovery of similar coefficients for elliptic equations in a general bounded domain (see also [24]). Note that, assuming that A, B are known and $A \in L^\infty(0,T;W^{2,\infty}(\Omega))^n \cap W^{1,\infty}(0,T;L^\infty(\Omega))^n$, $\nabla_x \cdot B \in L^\infty(Q)$, we can prove the recovery of more general zero order coefficient q. Actually, in that context, using our approach, one can prove the recovery of coefficients q lying in $L^\infty(0,T;L^p(\Omega)) \cup \mathcal{C}([0,T];L^{\frac{2n}{3}}(\Omega))$, with $p > 2n/3$. However, like for elliptic equations (see [45]) we can not reduce simultaneously the smoothness assumption for the first and zero order coefficients under consideration. For this reason, we have proved first the recovery of the 2-form dA associated with the convection term A with the weakest regularity that allows our approach for all the coefficients A, B, q. Then, we have proved the recovery of the gauge class of the coefficients (A, B, q), given by (1.10), by increasing the regularity of the unknown part of the coefficients B and q (see (1.8)-(1.9)).

One of the main tools for the proof of Theorem 1.1 are suitable solutions of (1.2) also called geometric optics (GO in short) solutions. Similar type of solutions have already been considered by [16, 30] for the recovery of bounded zero order coefficients q. None of these constructions works with variable coefficients of order 1 or non-bounded coefficient q. Therefore, we introduce a new construction, inspired by the approach of [20, 58, 45] for elliptic equations, in order to overcome the presence of variable coefficients of order 1. More precisely, we derive first a new Carleman estimate stated in Proposition 3.1 from which we obtain Carleman estimates in negative order Sobolev space stated in Proposition 4.1, 4.2. Applying Proposition 4.1, 4.2, we built our GO solutions by a duality argument and an application of the Hahn Banach theorem. In contrast to the analysis of [20, 58, 45] for elliptic equations, we need to consider GO solutions that vanish on the top Ω^T or on the bottom Ω^0 of the space-time cylindrical domain Q. For this purpose, we freeze the time variable and we work only with respect to the space variable for the construction of our GO solutions. Then, using the estimate on Ω^T or Ω^0 of the Carleman estimates of Proposition 4.1 we can apply Proposition

4.1, 4.2 to functions vanishing only at $t = T$ or $t = 0$. This additional constraint on Ω^T or Ω^0, makes an important difference between the construction of the so called complex geometric optics solutions considered by [20, 58, 45] for elliptic equations and our construction of the GO solutions for parabolic equations. Like in [45] for elliptic equations, thanks to the estimate of the Laplacian in Proposition 3.1, we can apply our construction to coefficients with low regularity. Actually, we improve the construction of [45] by extending our approach to unbounded zero order coefficients q. Note also that, quite surprisingly, in Proposition 4.1, 4.1, 4.2, we obtain better estimates with respect to the space variables than what has been proved in [45, 58], for the 3-dimensinal case an averaging procedure provides an equivalent gain to ours, see [24].

From the recovery of the gauge class of (A, B, q), stated in Theorem 1.1, we derive three different results for the linear problem stated in Corollary 1.1, 1.2, 1.3. In all these three results, we use unique continuation results for parabolic equations in order to derive conditions that guaranty $\varphi = 0$ in (1.10) or to obtain a density arguments in norm L^2 on a subdomain of Q. Using such arguments we can prove the full recovery of the convection term A and prove the recovery of the gauge class of (A, B, q) from measurements on an arbitrary portion of $\partial\Omega$ when (A, B, q) are known on neighborhood of Σ.

According to [33, Lemma 8.1], with additional regularity assumptions imposed to the coefficients (A, B, q) and to the domain Ω, the DN map $\Lambda_{A,B,q}$ determines $A\cdot\nu$ on Σ. Therefore, for sufficiently smooth coefficients (A_j, B, q), $j = 1, 2$, and sufficiently smooth domain Ω, the condition (1.9) can be removed from the statement of Corollary 1.1 and 1.2. We believe that the condition (1.9) can also be removed with less regular coefficients and domain. However, we do not treat that issue in the present paper.

To our best knowledge, in Theorem 1.2 and 1.3 we have stated the first results of recovery of a general quasilinear term of the form $F(x, t, u, \nabla_x u)$, $(x, t) \in Q$, that admits variation independent of the solutions inside the domain (i.e we recover the part $F(x, 0, u, v)$ with $x \in \Omega$, $u \in \mathbb{R}$, $v \in \mathbb{R}^n$ of such functions) from measurements restricted to the lateral boundary. Indeed, one can apply our result to the unique full recovery of nonlinear terms of the form (1.22) and (1.26) (see Remark 1.1 and 1.2 for more details). The only other comparable result is the one stated in [33] where the author proved the recovery of quasilinear terms depending only on the solutions (i.e of the form $F(u, \nabla_x u)$) on some suitable sets. Therefore, our result provide a more precise information about the nonlinear term under consideration than [33] and it can be also applied to more general quasilinear terms admitting variation independent of the solution inside the domain, while [33] restrict his analysis to quasilinear terms depending only on the solutions.

We prove Theorem 1.2 and 1.3 by combining Theorem 1.1 and Corollary 1.1 with the linearization procedure described in [16, 32, 33, 34]. More precisely, we transform the recovery of the nonlinear term $F(x, t, u, \nabla_x u)$ into the recovery of time-dependent coefficients $A(x, t) = \partial_v F(x, t, u(x, t), \nabla_x u(x, t))$, where u solves (1.14). Here the variable $v \in \mathbb{R}^n$ corresponds to $\nabla_x u$ in the expression $F(x, t, u, \nabla_x u)$. In contrast to all other results stated for nonlinear parabolic equations (e.g. [16, 31, 32, 33]), we do not need the full map $\mathcal{N}_F$ for proving the recovery of the quasilinear term but only some partial knowledge of its Fréchet derivative $\mathcal{N}'_F$. More precisely, our result requires only the knowledge of $\mathcal{N}'_F$ applied to restriction of linear or affine functions on Σ_p. By taking into account the important amount of information contained in the map $\mathcal{N}_F$, this makes an important restriction on the data used for solving the inverse problem. To do so, in contrast to [31, 32, 33] we need to explicitly derive the Fréchet derivative of $\mathcal{N}_F$. A similar idea has been considered in [40] for the recovery of a semilinear term appearing in nonlinear hyperbolic equations.

The result of Theorem 1.2 is stated for more general quasilinear terms than the one of Theorem 1.3. However, Theorem 1.2 can not be applied directly to the full recovery of the nonlinear term like in Theorem 1.3. Indeed, Theorem 1.2 provide only some knowledge of the nonlinear term $F(x, t, u, \nabla_x u)$ given by the conditions (1.17). On the other hand, with the additional condition (1.18), we can derive form Theorem 1.2 the uniqueness full recovery stated in Corollary 1.4.

Applying Corollary 1.3, we also prove in Corollary 1.5 recovery of nonlinear terms known on the neighborhood of the boundary from measurements on some arbitrary portion of the boundary $\partial\Omega$.

2. Preliminary results

We recall that $\Omega^0 = \Omega \times \{0\} \subset Q$ and $\Omega^T = \Omega \times \{T\} \subset Q$ Let us first consider the space

$$\mathcal{H}_+ := \{v_{|\Sigma} : \ v \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega)),\ v_{|\Omega^0} = 0\},$$

$$\mathcal{H}_- := \{v_{|\Sigma} : \ v \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega)),\ v_{|\Omega^T} = 0\}$$

which is a subspace of $L^2(0,T;H^{\frac{1}{2}}(\partial\Omega))$. We introduce also the spaces

$$S_+ = \{u \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega)) : \ (\partial_t - \Delta_x)u = 0,\ u_{|\Omega^0} = 0\},$$

$$S_- = \{u \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega)) : \ (-\partial_t - \Delta_x)u = 0,\ u_{|\Omega^T} = 0\}.$$

In order to define an appropriate topology on $\mathcal{H}_\pm$ for our problem, we consider the following result.

Proposition 2.1. *For all $f \in \mathcal{H}_\pm$ there exists a unique $u \in S_\pm$ such that $u_{|\Sigma} = f$.*

Proof. Without lost of generality we assume that the functions are real valued. We will only prove the result for $f \in \mathcal{H}_+$, using similar arguments one can extend the result to $f \in \mathcal{H}_-$. Let $f \in \mathcal{H}_+$ and consider $F \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega))$ such that $F_{|\Sigma} = f$ and $F_{|\Omega^0} = 0$. Fix $G = -(\partial_t - \Delta_x)F \in L^2(0,T;H^{-1}(\Omega))$ and w the solution of the IBVP

$$\left\{\begin{array}{rcll} \partial_t w - \Delta_x w & = & G, & (x,t) \in Q, \\ w_{|\Omega^0} & = & 0, & \\ w_{|\Sigma} & = & 0. & \end{array}\right.$$

According to [51, Theorem 4.1, Chapter 3] this IBVP admits a unique solution $w \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega))$. Thus $v = w + F \in S_+$ and clearly $v_{|\Sigma} = w_{|\Sigma} + F_{|\Sigma} = f$. This prove the existence of $u \in S_+$ such that $u_{|\Sigma} = f$. For the uniqueness, let $v_1, v_2 \in S_+$ satisfies $\tau_0 v_1 = \tau_0 v_2$. Then, $v = v_1 - v_2$ solves

$$\left\{\begin{array}{rcll} \partial_t v - \Delta_x v & = & 0, & (x,t) \in Q, \\ v_{|\Omega^0} & = & 0, & \\ v_{|\Sigma} & = & 0. & \end{array}\right.$$

which from the uniqueness of this IBVP implies that $v_1 - v_2 = 0$. □

Following Proposition 2.1, we consider the norm on $\mathcal{H}_\pm$ given by

$$\left\|F_{|\Sigma}\right\|^2_{\mathcal{H}_\pm} = \|F\|^2_{L^2(0,T;H^1(\Omega))} + \|F\|^2_{H^1(0,T;H^{-1}(\Omega))}, \quad F \in S_\pm.$$

We introduce the IBVPs

$$\left\{\begin{array}{ll} \partial_t u - \Delta_x u + A(x,t)\cdot\nabla_x u + [\nabla_x \cdot B(x,t)]u + q(x,t)u = 0, & \text{in } Q, \\ u(0,\cdot) = 0, & \text{in } \Omega, \\ u = g_+, & \text{on } \Sigma, \end{array}\right. \tag{2.1}$$

$$\left\{\begin{array}{ll} -\partial_t u - \Delta_x u - A(x,t)\cdot\nabla_x u + (q(x,t) + [\nabla_x \cdot (B-A)(x,t)])u = 0, & \text{in } Q, \\ u(\cdot,T) = 0, & \text{in } \Omega, \\ u = g_-, & \text{on } \Sigma. \end{array}\right. \tag{2.2}$$

We are now in position to state existence and uniqueness of solutions of these IBVPs for $g_\pm \in \mathcal{H}_\pm$.

Proposition 2.2. *Let $g_\pm \in \mathcal{H}_\pm$, $A, B \in L^\infty(Q)^n$, $q \in L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))$. Then, the IBVP* (2.1) (*respectively* (2.2)) *admits a unique weak solution $u \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega))$ (respectively $u \in H_-$) satisfying*

$$\begin{array}{l} \|u\|_{L^2(0,T;H^1(\Omega))} + \|u\|_{H^1(0,T;H^{-1}(\Omega))} \leqslant C\,\|g_+\|_{\mathcal{H}_+} \\ (\textit{respectively } \|u\|_{L^2(0,T;H^1(\Omega))} + \|u\|_{H^1(0,T;H^{-1}(\Omega))} \leqslant C\,\|g_-\|_{\mathcal{H}_-}), \end{array} \tag{2.3}$$

where C depends on Ω, T and $M \geqslant \|q\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))} + \|A\|_{L^\infty(Q)^n}$.

Proof. Since the proof of the well-posedness result is similar for (2.1) and (2.2), we will only treat (2.1). According to Proposition 2.1, there exists a unique $G \in S_+$ such that $G_{|\Sigma} = g_+$ and

$$\|G\|_{L^2(0,T;H^1(\Omega))} \leqslant \|g_+\|_{\mathcal{H}_+}.$$

We split u into two terms $u = w + G$ where w solves

$$\left\{\begin{array}{rcl} \partial_t w - \Delta_x w + A\cdot\nabla_x w + (\nabla_x\cdot B)w + qw &=& -A\cdot\nabla_x G - (\nabla_x\cdot B)G - qG, \quad (x,t)\in Q,\\ w_{|\Omega^0} &=& 0,\\ w_{|\Sigma} &=& 0. \end{array}\right. \tag{2.4}$$

From the Sobolev embedding theorem we have $-A\cdot\nabla_x G - (\nabla_x\cdot B)G - qG \in L^2(0,T;H^{-1}(\Omega))$ with

$$\begin{aligned} &\|-A\cdot\nabla_x G - (\nabla_x\cdot B)G - qG\|_{L^2(0,T;H^{-1}(\Omega))}\\ &\leqslant C\left(\|A\|_{L^\infty(Q)^n} + \|B\|_{L^\infty(Q)^n} + \|q\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))}\right)\|G\|_{L^2(0,T;H^1(\Omega))}, \end{aligned}$$

with C depending only on Ω. Let $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ and consider the time-dependent sesquilinear form $a(t,\cdot,\cdot)$ with domain V and defined by

$$a(t,h,g) = \int_\Omega \nabla_x h(x)\cdot\overline{\nabla_x g(x)} + (A(x,t)\cdot\nabla_x h(x) + q(x,t)h(x))\overline{g(x)} - B(x,t)\cdot\nabla_x(h\overline{g})(x)dx, \quad h,g\in V.$$

Note that here for all $h,g\in V$, we have $t\mapsto a(t,h,g)\in L^\infty(0,T)$ and an application of the Sobolev embedding theorem implies

$$|a(t,h,g)| \leqslant C\|h\|_{H^1(\Omega)}\|g\|_{H^1(\Omega)}, \tag{2.5}$$

with $C>0$ depending on $\|A\|_{L^\infty(Q)}$, $\|B\|_{L^\infty(Q)}$ and $\|q\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))}$. In addition, there exists $\lambda, c>0$ such that, for any $h\in V$, we have

$$\Re a(t,h,h) + \lambda\|h\|^2_{L^2(\Omega)} \geqslant c\|h\|^2_{H^1(\Omega)}, \quad t\in(0,T). \tag{2.6}$$

Indeed, for $h\in V$, $t\in(0,T)$ and $\varepsilon_1\in(0,1)$, we have

$$\begin{aligned} \Re a(t,h,h) &\geqslant \int_\Omega |\nabla_x h|^2 dx - (\|A\|_{L^\infty(Q)^n} + 2\|B\|_{L^\infty(Q)^n})\int_\Omega |\nabla_x h||h|dx - \int_\Omega |q(t,\cdot)||h|^2 dx\\ &\geqslant (1-\varepsilon_1)\int_\Omega |\nabla_x h|^2 dx - \left(\frac{\left(\|A\|_{L^\infty(Q)^n} + 2\|B\|_{L^\infty(Q)^n}\right)^2}{\varepsilon_1}\right)\int_\Omega |h|^2 dx - \int_\Omega |q||h|^2 dx. \end{aligned}$$

In addition applying the Hölder inequality, the Sobolev embedding theorem and an interpolation between Sobolev spaces, for all $t\in(0,T)$, we get

$$\begin{aligned} \int_\Omega |q(t,\cdot)||h|^2 dx &\leqslant \|q\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))}\|h\|^2_{L^{\frac{2n}{n-\frac{3}{2}}}(\Omega)}\\ &\leqslant C\|q\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))}\|h\|^2_{H^{\frac{3}{4}}(\Omega)}\\ &\leqslant C\|q\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))}\left(\|h\|^2_{H^1(\Omega)}\right)^{\frac{3}{4}}\left(\|h\|^2_{L^2(\Omega)}\right)^{\frac{1}{4}}\\ &\leqslant C\|q\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))}\left(\varepsilon_1\|h\|^2_{H^1(\Omega)}\right)^{\frac{3}{4}}\left(\varepsilon_1^{-3}\|h\|^2_{L^2(\Omega)}\right)^{\frac{1}{4}}\\ &\leqslant C\|q\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))}\left(\frac{3\varepsilon_1}{4}\|h\|^2_{H^1(\Omega)} + \frac{\varepsilon_1^{-3}}{4}\|h\|^2_{L^2(\Omega)}\right), \end{aligned}$$

with $C>0$ depending only on Ω. Therefore, choosing

$$\varepsilon_1 = \left(C\|q\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))} + 2\right)^{-1}, \quad c = 1 - \varepsilon_1(C\|q\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))} + 1),$$

$$\lambda = \frac{\left(\|A\|_{L^\infty(Q)^n} + 2\|B\|_{L^\infty(Q)^n}\right)^2}{\varepsilon_1} + C\|q\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))}\varepsilon_1^{-3} + 2,$$

we get (2.6). Combining (2.5)-(2.6) with the fact that

$$a(t,h,g) = \langle -\Delta_x h + A(\cdot,t)\cdot\nabla_x h + (\nabla_x\cdot B(\cdot,t))h + q(\cdot,t)h, g\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad t\in(0,T),$$

we deduce from [51, Theorem 4.1, Chapter 3] that problem (2.4) admits a unique solution $w\in H^1(0,T;H^{-1}(\Omega))\cap L^2(0,T;H^1(\Omega))$ satisfying

$$\begin{aligned}\|w\|_{L^2(0,T;H^1(\Omega))} + \|w\|_{H^1(0,T;H^{-1}(\Omega))} &\leqslant C\|-A\cdot\nabla_x G - (\nabla_x\cdot B)G - qG\|_{L^2(0,T;H^{-1}(\Omega))}\\ &\leqslant C\|G\|_{L^2(0,T;H^1(\Omega))} \leqslant C\|g_+\|_{\mathcal{H}_+},\end{aligned}$$

where C depends on Ω, T and M. Therefore, $u = w + G$ is the unique solution of (2.1) and the above estimate implies (2.3). □

Using these properties, we would like to give a suitable definition of the normal derivative of solutions of (1.2). For this purpose, following [45] we will give a variational sense to the normal derivative for solutions of these problems. For this purpose, we start by considering the spaces

$$H^{1/2}(\sqcup) := \{\tilde{g}|_\Sigma : \tilde{g}\in H^{1/2}(\partial Q),\ \mathrm{supp}(\tilde{g})\subset \partial Q\setminus\Omega^T\}.$$

We use the symbols $\sqcup$ because it turns out to be convenient to keep in mind that the corresponding functions vanish on $\Omega^T := \Omega\times\{T\}$. Note that the norms

$$\|g\|_{H^{1/2}(\sqcup)} := \inf\{\|\tilde{g}\|_{H^{1/2}(\partial Q)} : \tilde{g}|_\Sigma = g,\ \mathrm{supp}(\tilde{g})\subset\partial Q\setminus\Omega^T\}$$

makes $H^{1/2}(\sqcup)$ be a Banach space. We recall that there exists a lifting operator $L : H^{1/2}(\sqcup)\longrightarrow\{w\in H^1(Q) :\ w_{|\Omega^T}=0\}$ such that L is a bounded and

$$Lg_{|\Sigma} = g, \quad g\in H^{1/2}(\sqcup).$$

For any $g_+\in\mathcal{H}_+$ and $u\in H^1(0,T;H^{-1}(\Omega))\cap L^2(0,T;H^1(\Omega))$ the solution of (2.1), we define $N_{A,B,q}u\in H^{1/2}(\sqcup)^*$, where $H^{1/2}(\sqcup)^*$ denotes the dual space of $H^{1/2}(\sqcup)$, by

$$\begin{aligned}&\langle N_{A,B,q}u, g_-\rangle_{H^{1/2}(\sqcup)^*,H^{1/2}(\sqcup)}\\ &= \int_Q [-u\partial_t Lg_- + \nabla_x u\cdot\nabla_x Lg_- + A\cdot\nabla_x u Lg_- - B\cdot\nabla_x(uLg_-) + quLg_-]dxdt.\end{aligned} \tag{2.7}$$

Note that, for $w\in H^1(Q)$ satisfying $w_{|\Omega^T}=0$ and $w_{|\Sigma}=0$, since $u\in H^1(0,T;H^{-1}(\Omega))\cap L^2(0,T;H^1(\Omega))$ solves (2.1), we have

$$\begin{aligned}&\int_Q [-u\partial_t w + \nabla_x u\cdot\nabla_x w + A\cdot\nabla_x uw - B\cdot\nabla_x(uw) + quw]dxdt\\ &= \langle\partial_t u - \Delta u + A\cdot\nabla_x u + (\nabla_x\cdot B)u + qu, w\rangle_{L^2(0,T;H^{-1}(\Omega)),L^2(0,T;H_0^1(\Omega))} = 0.\end{aligned}$$

Therefore, (2.7) is well defined since the right hand side of this identity depends only on g_-. We define the DN map associated with (2.1) by

$$\Lambda_{A,B,q} : \mathcal{H}_+ \ni g_+ \mapsto N_{A,B,q}u\in H^{1/2}(\sqcup)^*$$

and, applying Proposition 2.2 one can check that this map is continuous from $\mathcal{H}_+$ to $H^{1/2}(\sqcup)^*$. By density, we derive the following representation formula

Proposition 2.3. *For $j=1,2$, let A_j, $B_j\in L^\infty(Q)^n$, $q_j\in L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))$. Then, the operator $\Lambda_{A_1,B_1,q_1}-\Lambda_{A_2,B_2,q_2}$ can be extended to a bounded operator from $\mathcal{H}_+$ to $\mathcal{H}_-^*$, where $\mathcal{H}_-^*$ denotes the dual space of $\mathcal{H}_-$. Moreover, for $g_+\in\mathcal{H}_+$, $g_-\in\mathcal{H}_-$, we consider $u_1\in H^1(0,T;H^{-1}(\Omega))\cap L^2(0,T;H^1(\Omega))$*

the solution of (2.1) *with* $A = A_1$, $B = B_1$, $q = q_1$ *and* $u_2 \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega))$ *be the solution of* (2.2) *with* $A = A_2$, $B = B_2$, $q = q_2$. *Then we have*

$$\begin{aligned}&\langle(\Lambda_{A_1,B_1,q_1} - \Lambda_{A_2,B_2,q_2})g_+, g_-\rangle_{\mathcal{H}_-^*,\mathcal{H}_-}\\ &= \int_Q (A_1 - A_2)\cdot\nabla_x u_1 u_2 dxdt - \int_Q (B_1 - B_2)\cdot\nabla_x(u_1u_2)dxdt + \int_Q (q_1 - q_2)u_1u_2 dxdt.\end{aligned} \tag{2.8}$$

Proof. Without loss of generality we assume that all the functions are real valued. We consider $v_2 \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega))$ solving

$$\begin{cases} \partial_t v_2 - \Delta_x v_2 + A_2(x,t)\cdot\nabla_x v_2 + [\nabla_x \cdot B_2(x,t)]v_2 + q_2(x,t)v_2 = 0, & \text{in } Q,\\ v_2(0,\cdot) = 0, & \text{in } \Omega,\\ v_2 = g_+, & \text{on } \Sigma. \end{cases}$$

Then, for any $g_- \in H^{1/2}(\sqcup)$ fixing $w = Lg_- \in H^1(Q)$, we find

$$\begin{aligned}&\langle(\Lambda_{A_1,B_1,q_1} - \Lambda_{A_2,B_2,q_2})g_+, g_-\rangle_{H^{1/2}(\sqcup)^*,H^{1/2}(\sqcup)}\\ &= \langle(N_{A_1,B_1,q_1}u_1 - N_{A_2,B_2,q_2}v_2, g_-\rangle_{H^{1/2}(\sqcup)^*,H^{1/2}(\sqcup)}\\ &= \int_Q [-(u_1 - v_2)\partial_t w + \nabla_x(u_1 - v_2)\cdot\nabla_x w + A_2\cdot\nabla_x(u_1 - v_2)w - B_2\cdot\nabla_x((u_1 - v_2)w) + q_2(u_1 - v_2)w]dxdt\\ &\quad + \int_Q [(A_1 - A_2)\cdot\nabla_x u_1 w - (B_1 - B_2)\cdot\nabla_x(u_1 w) + (q_1 - q_2)u_1 w]dxdt\end{aligned}$$

Now using the fact that $(u_1 - v_2) \in L^2(0,T;H_0^1(\Omega))$, we get

$$\begin{aligned}&\langle(\Lambda_{A_1,B_1,q_1} - \Lambda_{A_2,B_2,q_2})g_+, g_-\rangle_{H^{1/2}(\sqcup)^*,H^{1/2}(\sqcup)}\\ &= -\langle\partial_t w, u_1 - v_2\rangle_{L^2(0,T;H^{-1}(\Omega)),L^2(0,T;H_0^1(\Omega))} + \int_Q \nabla_x(u_1 - v_2)\cdot\nabla_x w dxdt\\ &\quad + \int_Q A_2\cdot\nabla_x(u_1 - v_2)w dxdt - \int_Q B_2\cdot\nabla_x[(u_1 - v_2)w]dxdt + \int_Q q_2(u_1 - v_2)w dxdt\\ &\quad + \int_Q [(A_1 - A_2)\cdot\nabla_x u_1 w - (B_1 - B_2)\cdot\nabla_x(u_1 w) + (q_1 - q_2)u_1 w]dxdt\end{aligned} \tag{2.9}$$

We can choose $\tilde{w}$ to be the unique element of S_- satisfying $\tilde{w}_{|\Sigma} = g_-$, and since $w - \tilde{w} \in L^2(0,T;H_0^1(\Omega))$ and $w - \tilde{w}|_{\Omega^T} = 0$, w can be replaced by $\tilde{w}$ in the identity (2.9). Moreover, we have

$$\begin{aligned}\left|\langle(\Lambda_{A_1,B_1,q_1} - \Lambda_{A_2,B_2,q_2})g_+, g_-\rangle_{H^{1/2}(\sqcup)^*,H^{1/2}(\sqcup)}\right| &\leqslant C\,\|g_+\|_{\mathcal{H}_+}\left(\|\tilde{w}\|_{L^2(0,T;H^1(\Omega))} + \|\tilde{w}\|_{H^1(0,T;H^{-1}(\Omega))}\right)\\ &\leqslant C\,\|g_+\|_{\mathcal{H}_+}\,\|g_-\|_{\mathcal{H}_-}\,,\end{aligned}$$

where C depends only on A_j, B_j, q_j, $j = 1,2$, T and Ω. From this identity, we deduce that the map $\Lambda_{A_1,B_1,q_1} - \Lambda_{A_2,B_2,q_2}$ can be extended continuously to a continuous linear map from $\mathcal{H}_+$ to $\mathcal{H}_-^*$ and the identity (2.9) holds for $g_- \in \mathcal{H}_-$, whose extension w to Q belongs to $H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega))$. Thus, we are allowed to replace w in (2.9) by u_2. Since u_2 satisfies the identity below, the proposition is

proved:

$$
\begin{aligned}
&- \langle \partial_t u_2, u_1 - v_2\rangle_{L^2(0,T;H^{-1}(\Omega)),L^2(0,T;H_0^1(\Omega))} + \int_Q \nabla_x(u_1-v_2)\cdot\nabla_x u_2 dxdt \\
&= \langle -\partial_t u_2 - \Delta u_2, u_1 - v_2\rangle_{L^2(0,T;H^{-1}(\Omega)),L^2(0,T;H_0^1(\Omega))} \\
&= \langle A_2\cdot\nabla_x u_2, u_1-v_2\rangle + \langle \nabla_x\cdot(A_2-B_2),(u_1-v_2)u_2\rangle_{L^2(0,T;H^{-1}(\Omega)),L^2(0,T;H_0^1(\Omega))} - \int_Q q_2(u_1-v_2)u_2 dxdt \\
&= \langle \nabla_x\cdot(u_2A_2) - \nabla_x\cdot(B_2)u_2, u_1-v_2\rangle_{L^2(0,T;(H^{-1}(\Omega)),L^2(0,T;H_0^1(\Omega))} - \int_Q q_2(u_1-v_2)u_2 dxdt \\
&= -\int_Q (A_2\cdot\nabla_x(u_1-v_2)u_2 dxdt + \int_Q B_2\cdot\nabla_x[(u_1-v_2)u_2]dxdt - \int_Q q_2(u_1-v_2)u_2 dxdt.
\end{aligned}
$$

□

3. Carleman estimates

We introduce two parameters $s,\rho\in(1,+\infty)$ and we consider, for $\rho>s>1$, the perturbed weight

$$\varphi_{\pm,s}(x,t) := \pm(\rho^2 t + \rho\omega\cdot x) - s\frac{((x+x_0)\cdot\omega)^2}{2}. \tag{3.1}$$

We define

$$L_{\pm,A} = \pm\partial_t - \Delta_x \pm A\cdot\nabla_x, \quad P_{A,\pm,s} := e^{-\varphi_{\pm,s}} L_{\pm,A} e^{\varphi_{\pm,s}}.$$

Here $x_0\in\mathbb{R}^n$ is chosen in such a way that

$$x_0\cdot\omega = 2 + \sup_{x\in\Omega}|x|. \tag{3.2}$$

The goal of this section is to prove the following Carleman estimates.

Proposition 3.1. *Let $A\in L^\infty(Q)^n$ and Ω be $\mathcal{C}^2$. Then there exist $s_1>1$ and, for $s>s_1$, $\rho_1(s)$ such that for any $v\in\mathcal{C}^2(\overline{Q})$ satisfying the condition*

$$v_{|\Sigma}=0, \quad v_{|\Omega^0}=0, \tag{3.3}$$

the estimate

$$
\begin{aligned}
&\rho\int_{\Sigma_{+,\omega}} |\partial_\nu v|^2 |\omega\cdot\nu| d\sigma(x)dt + s\rho\int_\Omega |v|^2(x,T)dx + s^{-1}\int_Q |\Delta_x v|^2 dxdt + s\rho^2\int_Q |v|^2 dxdt \\
&\leqslant C\left[\|P_{A,+,s}v\|^2_{L^2(Q)} + \rho\int_{\Sigma_{-,\omega}} |\partial_\nu v|^2|\omega\cdot\nu| d\sigma(x)dt\right]
\end{aligned} \tag{3.4}
$$

holds true for $s>s_1$, $\rho\geqslant\rho_1(s)$ with C depending only on Ω, T and $M\geqslant\|A\|_{L^\infty(Q)^n}$. In the same way, there exist $s_2>1$ and, for $s>s_2$, $\rho_2(s)$ such that for any $v\in\mathcal{C}^2(\overline{Q})$ satisfying the condition

$$v_{|\Sigma}=0, \quad v_{|\Omega^T}=0, \tag{3.5}$$

the estimate

$$
\begin{aligned}
&\rho\int_{\Sigma_{-,\omega}} |\partial_\nu v|^2 |\omega\cdot\nu| d\sigma(x)dt + s\rho\int_\Omega |v|^2(x,0)dx + s^{-1}\int_Q |\Delta_x v|^2 dxdt + s\rho^2\int_Q |v|^2 dxdt \\
&\leqslant C\left[\|P_{A,-,s}v\|^2_{L^2(Q)} + \rho\int_{\Sigma_{+,\omega}} |\partial_\nu v|^2|\omega\cdot\nu| d\sigma(x)dt\right]
\end{aligned} \tag{3.6}
$$

holds true for for $s>s_2$, $\rho\geqslant\rho_2(s)$. Here s_1, ρ_1, s_2 and ρ_2 depend only on Ω, T and $M\geqslant\|A\|_{L^\infty(Q)^n}$.

Proof. Without loss of generality we assume that v is real valued. We start with (3.4). For this purpose we will first show that, for $A = 0$ and $q = 0$, there exists c depending only on Ω, s_1 depending on Ω, T such that for any $s > s_1$ we can find $\rho_1(s)$ for which the estimate

$$\begin{aligned}\|P_{A,+,s}v\|^2_{L^2(Q)} \geqslant & \rho\int_{\Sigma_{+,\omega}}|\partial_\nu v|^2|\omega\cdot\nu|d\sigma(x)dt - 8\rho\int_{\Sigma_{-,\omega}}|\partial_\nu v|^2|\omega\cdot\nu|d\sigma(x)dt + cs^{-1}\int_Q|\Delta_x v|^2dxdt\\ &+ s\rho\int_\Omega|v|^2(x,T)dx + s\rho^2\int_Q|v|^2dxdt + 2s\int_Q|\nabla_x v|^2dxdt\end{aligned} \tag{3.7}$$

holds true when the condition $\rho > \rho_1(s)$ is fulfilled. Using this estimate, we will derive (3.4). We decompose $P_{A,+,s}$ into three terms

$$P_{A,+,s} = P_{1,+} + P_{2,+} + P_{3,+},$$

with

$$P_{1,+} = -\Delta_x+\partial_t\varphi_{+,s}-|\nabla_x\varphi_{+,s}|^2+\Delta_x\varphi_{+,s},\quad P_{2,+} = \partial_t-2\nabla_x\varphi_{+,s}\cdot\nabla_x-2\Delta_x\varphi_{+,s},\quad P_{3,+} = A\cdot\nabla_x+A\cdot\nabla_x\varphi_{+,s}+q.$$

Note that

$$\partial_t\varphi_{+,s} = \rho^2,\quad \nabla_x\varphi_{+,s} = [\rho - s(x+x_0)\cdot\omega]\omega,\quad -\Delta\varphi_{+,s} = s$$

and

$$P_{1,+}v = -\Delta_x v + [2\rho s(x+x_0)\cdot\omega - s^2((x+x_0)\cdot\omega)^2 - s]v, \tag{3.8}$$

$$\begin{aligned}P_{2,+}v &= \partial_t v - 2[\rho - s(x+x_0)\cdot\omega](\omega\cdot\nabla_x v) + 2sv,\\ P_{1,+}vP_{2,+}v &= -\Delta_x v\partial_t v + 2\Delta_x v[\rho - s(x+x_0)\cdot\omega](\omega\cdot\nabla_x v) - 2s(\Delta v)v\\ &\quad + [2\rho s(x+x_0)\cdot\omega - s^2((x+x_0)\cdot\omega)^2 - s]v[\partial_t v - 2[\rho - s(x+x_0)\cdot\omega](\omega\cdot\nabla_x v) + 2sv]\end{aligned} \tag{3.9}$$

For the first term on the right hand side of (3.9) we find

$$\int_Q(-\Delta_x v\partial_t v)dxdt = \int_Q\partial_t\nabla_x v\cdot\nabla_x v dxdt = \frac{1}{2}\int_Q\partial_t|\nabla_x v|^2dxdt = \frac{1}{2}\int_\Omega|\nabla_x v(x,T)|^2dx.$$

It follows that

$$\int_Q(-\Delta_x v\partial_t v)dxdt \geqslant 0. \tag{3.10}$$

We have also

$$\begin{aligned}&2\int_Q\Delta_x v[\rho - s(x+x_0)\cdot\omega](\omega\cdot\nabla_x v)dxdt\\ &= 2\int_\Sigma\partial_\nu v[\rho - s(x+x_0)\cdot\omega](\omega\cdot\nabla_x v)d\sigma(x)dt + 2s\int_Q(\omega\cdot\nabla_x v)^2dxdt - 2\int_Q[\rho - s(x+x_0)\cdot\omega][\nabla_x v\cdot\nabla_x(\omega\cdot\nabla_x v)]dxdt\\ &= 2\int_\Sigma\partial_\nu v[\rho - s(x+x_0)\cdot\omega](\omega\cdot\nabla_x v)d\sigma(x)dt + 2s\int_Q(\omega\cdot\nabla_x v)^2dxdt - \int_Q[\rho - s(x+x_0)\cdot\omega]\omega\cdot\nabla_x|\nabla_x v|^2dxdt\end{aligned}$$

and using the fact that $v_{|\Sigma} = 0$, we get

$$\begin{aligned}&2\int_Q\Delta_x v[\rho - s(x+x_0)\cdot\omega](\omega\cdot\nabla_x v)dxdt\\ &= 2\int_\Sigma[\rho - s(x+x_0)\cdot\omega]|\partial_\nu v|^2\omega\cdot\nu d\sigma(x)dt + 2s\int_Q(\omega\cdot\nabla_x v)^2dxdt\\ &\quad - \int_Q[\rho - s(x+x_0)\cdot\omega]|\partial_\nu v|^2\omega\cdot\nu d\sigma(x)dt - s\int_Q|\nabla_x v|^2dxdt\\ &= \int_\Sigma[\rho - s(x+x_0)\cdot\omega]|\partial_\nu v|^2\omega\cdot\nu d\sigma(x)dt - s\int_Q|\nabla_x v|^2dxdt + 2s\int_Q(\omega\cdot\nabla_x v)^2dxdt.\end{aligned}$$

Choosing $\rho \geqslant 2s(1+\sup_{x\in\Omega}|x|)$, we obtain

$$\begin{aligned}&2\int_Q \Delta_x v[\rho - s(x+x_0)\cdot\omega](\omega\cdot\nabla_x v)dxdt\\ &\geqslant \rho\int_{\Sigma_{+,\omega}}|\partial_\nu v|^2|\omega\cdot\nu|d\sigma(x)dt - 4\rho\int_{\Sigma_{-,\omega}}|\partial_\nu v|^2|\omega\cdot\nu|d\sigma(x)dt - s\int_Q|\nabla_x v|^2dxdt.\end{aligned}$$

Combining this with the fact that

$$-2s\int_Q(\Delta v)vdxdt = 2s\int_Q|\nabla_x v|^2dxdt,$$

we find

$$\begin{aligned}&2\int_Q \Delta_x v[\rho - s(x+x_0)\cdot\omega](\omega\cdot\nabla_x v)dxdt - 2s\int_Q(\Delta v)vdxdt\\ &\geqslant \rho\int_{\Sigma_{+,\omega}}|\partial_\nu v|^2|\omega\cdot\nu|d\sigma(x)dt - 4\rho\int_{\Sigma_{-,\omega}}|\partial_\nu v|^2|\omega\cdot\nu|d\sigma(x)dt + s\int_Q|\nabla_x v|^2dxdt.\end{aligned}\tag{3.11}$$

Now let us consider the last term on the right hand side of (3.9). Note first that

$$\begin{aligned}&\int_Q[2\rho s(x+x_0)\cdot\omega - s^2((x+x_0)\cdot\omega)^2 - s]v\partial_t v dxdt\\ &= \frac{1}{2}\int_Q[2\rho s(x+x_0)\cdot\omega - s^2((x+x_0)\cdot\omega)^2 - s]\partial_t|v|^2dxdt\\ &\geqslant \rho s\int_\Omega(x+x_0)\cdot\omega|v|^2(x,T)dx - s^2\left(3+\sup_{x\in\Omega}|x|\right)^2\int_\Omega|v|^2(x,T)dx.\end{aligned}$$

Combining this with (3.2) and choosing $\rho \geqslant s\left(3+\sup_{x\in\Omega}|x|\right)^2$, we find

$$\int_Q[2\rho s(x+x_0)\cdot\omega + s^2((x+x_0)\cdot\omega)^2 - s]v\partial_t v dxdt \geqslant s\rho\int_\Omega|v|^2(x,T)dx.\tag{3.12}$$

In addition, integrating by parts with respect to $x\in\Omega$, we get

$$\begin{aligned}&\textstyle\int_Q[2\rho s(x+x_0)\cdot\omega + s^2((x+x_0)\cdot\omega)^2 - s]v[-2[\rho - s(x+x_0)\cdot\omega](\omega\cdot\nabla_x v)]dxdt\\ &= -\textstyle\int_Q[-s^3((x+x_0)\cdot\omega)^3 - \rho s^2((x+x_0)\cdot\omega)^2 + (2\rho^2 s + s^2)(x+x_0)\cdot\omega - s\rho]\omega\cdot\nabla_x|v|^2dxdt\\ &= \textstyle\int_Q[-3s^3((x+x_0)\cdot\omega)^2 - 2\rho s^2((x+x_0)\cdot\omega) + (2\rho^2 s + s^2)]|v|^2dxdt.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_Q[2\rho s(x+x_0)\cdot\omega - s^2((x+x_0)\cdot\omega)^2 - s]v[-2[\rho - s(x+x_0)\cdot\omega](\omega\cdot\nabla_x v) + 2sv]dxdt\\ &\qquad = \int_Q[-5s^3((x+x_0)\cdot\omega)^2 + 2\rho s^2((x+x_0)\cdot\omega) + (2\rho^2 s - s^2)]|v|^2dxdt.\end{aligned}$$

Then, fixing

$$\rho \geqslant \sqrt{5s^2(2+\sup_{x\in\Omega}|x|)^2 + s},$$

we obtain

$$\int_Q[2\rho s(x+x_0)\cdot\omega + s^2((x+x_0)\cdot\omega)^2 + s]v[-2[\rho - s(x+x_0)\cdot\omega](\omega\cdot\nabla_x v) + 2sv]dxdt \geqslant s\rho^2\int_Q|v|^2dxdt.$$

Combining this estimate with (3.10)-(3.12), we find

$$\begin{aligned}\|P_{1,+}v+P_{2,+}v\|^2_{L^2(Q)} &\geqslant 2\int_Q P_{1,+}vP_{2,+}vdxdt+\|P_{1,+}v\|^2_{L^2(Q)}\\ &\geqslant 2\rho\int_{\Sigma_{+,\omega}}|\partial_\nu v|^2|\omega\cdot\nu|d\sigma(x)dt-8\rho\int_{\Sigma_{-,\omega}}|\partial_\nu v|^2|\omega\cdot\nu|d\sigma(x)dt+2s\int_Q|\nabla_x v|^2dxdt\\ &\quad+2s\rho\int_\Omega|v|^2(x,T)dx+2s\rho^2\int_Q|v|^2dxdt+\|P_{1,+}v\|^2_{L^2(Q)}.\end{aligned}\tag{3.13}$$

Moreover, we have

$$\begin{aligned}\|P_{1,+}v\|^2_{L^2(Q)}&=\left\|-\Delta_x v+[2\rho s(x+x_0)\cdot\omega-s^2((x+x_0)\cdot\omega)^2+s]v\right\|^2_{L^2(Q)}\\ &\geqslant\frac{\|\Delta_x v\|^2_{L^2(Q)}}{2}-\left\|[2\rho s(x+x_0)\cdot\omega-s^2((x+x_0)\cdot\omega)^2+s]v\right\|^2_{L^2(Q)}\\ &\geqslant\frac{\|\Delta_x v\|^2_{L^2(Q)}}{2}-36s^2\rho^2\left(2+\sup_{x\in\Omega}|x|\right)^4\|v\|^2_{L^2(Q)}.\end{aligned}$$

Fixing $c=\left(36\left(2+\sup_{x\in\Omega}|x|\right)^4\right)^{-1}$, we deduce that

$$\|P_{1,+}v\|^2_{L^2(Q)}\geqslant cs^{-1}\|P_{1,+}v\|^2_{L^2(Q)}\geqslant\frac{cs^{-1}}{2}\|\Delta_x v\|^2_{L^2(Q)}-s\rho^2\|v\|^2_{L^2(Q)}$$

and, combining this with (3.13), we obtain (3.7) by fixing

$$\rho_1(s)>s\left(3+\sup_{x\in\Omega}|x|\right)^2+\sqrt{5s^2(2+\sup_{x\in\Omega}|x|)^2+s}.$$

Using (3.7), we will complete the proof of the lemma. For this purpose, we remark first that, for $\rho>\rho_1(s)$, we have

$$\begin{aligned}\|P_{A,+,s}v\|^2_{L^2(Q)}&\geqslant\frac{\|P_{1,+}v+P_{2,+}v\|^2_{L^2(Q)}}{2}-\|P_3v\|^2_{L^2(Q)}\\ &\geqslant\frac{\|P_{1,+}v+P_{2,+}v\|^2_{L^2(Q)}}{2}-2\|A\|^2_{L^\infty(Q)}\int_Q|\nabla_x v|^2dxdt\\ &\quad-8\rho^2\|A\|^2_{L^\infty(Q)}\int_Q|v|^2dxdt.\end{aligned}$$

Combining these estimates with (3.7), we deduce that for $s_1=32M^2$ and, for $s>s_1$, $\rho>\rho_1(s)$, estimate (3.4) holds true.

Now let us consider (3.6). We start by assuming that $A=0$. For this purpose we fix $v\in\mathcal{C}^2(\overline{Q})$ satisfying (3.5) and we consider $w\in\mathcal{C}^2(\overline{Q})$ defined by $w(x,t):=v(x,T-t)$. Clearly $w(x,0)=0$. Moreover, fixing

$$\varphi^*_{+,s}(x,t):=(\rho^2t-\rho\omega\cdot x)-s\frac{((x+x_0)\cdot\omega)^2}{2},$$

which corresponds to $\varphi_{+,s}$ with ω replaced by $-\omega$, one can check that

$$e^{-\varphi^*_{+,s}}(\partial_t-\Delta)e^{\varphi^*_{+,s}}w(x,t)=P_{0,-,s}v(x,T-t),\quad(x,t)\in\overline{Q},$$

with $P_{0,-,s}=P_{A,-,s}$ for $A=0$. Therefore, applying (3.4), with ω replaced by $-\omega$, to w we deduce (3.6). We can extend this result to the case $A\neq0$ by repeating the arguments used at the end of the proof of (3.4). □

4. GO solutions

Armed with the estimates (3.4)-(3.6) we will build suitable GO solutions for our problem. More precisely, for $j=1,2$, fixing the cofficient $(A_j,B_j,q_j)\in L^\infty(Q)^n\times L^\infty(Q)^n\times[L^\infty(0,T;L^p(\Omega))\cap\mathcal{C}([0,T];L^{\frac{2n}{3}}(\Omega))]$ with $p>2n/3$ and $\omega\in\mathbb{S}^{n-1}$, we look for u_j solutions of

$$\begin{cases}\partial_t u_1-\Delta_x u_1+A_1\cdot\nabla_x u_1+(\nabla_x\cdot B_1)u_1+q_1u_1=0,\ (x,t)\in Q,\\ u_1(x,0)=0,\quad x\in\Omega,\end{cases}\tag{4.1}$$

$$\begin{cases}-\partial_t u_2-\Delta_x u_2-A_2\cdot\nabla_x u_2+(q_2+\nabla_x\cdot(B_2-A_2))u_2=0,\ (x,t)\in Q,\\ u_2(x,T)=0,\quad x\in\Omega,\end{cases}\tag{4.2}$$

taking the form

$$u_1(x,t)=e^{\rho^2t+\rho x\cdot\omega}(b_{1,\rho}(x,t)+w_{1,\rho}(x,t)),\quad u_2(x,t)=e^{-\rho^2t-\rho x\cdot\omega}(b_{2,\rho}(x,t)+w_{2,\rho}(x,t)),\quad (x,t)\in Q.\tag{4.3}$$

In these expressions, the term $b_{j,\rho}$, $j=1,2$, are the principal part of our GO solutions and they will be suitably designed for the recovery of the coefficients. The expression $w_{j,\rho}$, $j=1,2$, are the remainder term in this expression that admits a decay with respect to the parameter ρ of the form

$$\lim_{\rho\to+\infty}(\rho^{-1}\left\|w_{j,\rho}\right\|_{L^2(0,T;H^1(\Omega))}+\left\|w_{j,\rho}\right\|_{L^2(Q)})=0.\tag{4.4}$$

We start by considering the principal parts of our GO solutions.

4.1. Principal part of the GO solutions.

In this subsection we will introduce the form of the principal part $b_{j,\rho}$, $j=1,2$, of our GO solutions given by (4.3). For this purpose, we consider $A_j\in L^\infty(Q)^n$, $j=1,2$ and we will consider $b_{j,\rho}$, $j=1,2$, to be an approximation of a solution b_j of the transport equation

$$-2\omega\cdot\nabla_x b_1+(A_1(x,t)\cdot\omega)b_1=0,\quad 2\omega\cdot\nabla_x b_2+(A_2(x,t)\cdot\omega)b_2=0,\quad (x,t)\in Q.\tag{4.5}$$

By replacing the functions b_1, b_2, whose regularity depends on the one of the coefficients A_1 and A_2, with their approximation $b_{1,\rho}$, $b_{2,\rho}$, we can reduce the regularity of the coefficients A_j, $j=1,2$, from $L^\infty(0,T;W^{2,\infty}(\Omega))^n\cap W^{1,\infty}(0,T;L^\infty(\Omega))^n$ to $L^\infty(Q)^n$. This approach, also considered in [4, 39, 45, 57], remove also condition imposed to the coefficients A_j, $j=1,2$, on Σ. Indeed, if in our construction we use the expression b_j instead of $b_{j,\rho}$, $j=1,2$, then we can prove Theorem 1.1 only for coefficients $A_1,A_2\in L^\infty(0,T;W^{2,\infty}(\Omega))^n\cap H^1(0,T;L^\infty(\Omega))^n$ satisfying

$$\partial_x^\alpha A_1(x,t)=\partial_x^\alpha A_2(x,t),\quad (x,t)\in\Sigma,\ \alpha\in\mathbb{N}^n,\ |\alpha|\leqslant 1,$$

where in our case we make no assumption on A_j at Σ for (1.7), and we only assume (1.9) for (1.10).

We start by considering a suitable approximation of the coefficients A_j, $j=1,2$. For all $r>0$ we set $B_r:=\{(x,t)\in\mathbb{R}^{1+n}:\ |(x,t)|<r\}$ and we fix $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^{1+n})$ such that $\chi\geqslant 0$, $\int_{\mathbb{R}^{1+n}}\chi(x,t)dtdx=1$, $\mathrm{supp}(\chi)\subset B_1$. We introduce also χ_ρ given by $\chi_\rho(x,t)=\rho^{\frac{n+1}{3}}\chi(\rho^{\frac13}x,\rho^{\frac13}t)$ and, for $j=1,2$, we fix

$$A_{j,\rho}(x,t):=\int_{\mathbb{R}^{1+n}}\chi_\rho(x-y,t-s)A_j(y,s)dsdy.$$

Here, we assume that $A_j=0$ on $\mathbb{R}^{1+n}\setminus Q$. For $j=1,2$, since $A_j\in L^\infty(\mathbb{R}^{1+n})^n$ is supported in the compact set $\overline{Q}$, we have

$$\lim_{\rho\to+\infty}\left\|A_{j,\rho}-A_j\right\|_{L^1(\mathbb{R}^{1+n})}=\lim_{\rho\to+\infty}\left\|A_{j,\rho}-A_j\right\|_{L^2(\mathbb{R}^{1+n})}=0,\tag{4.6}$$

and one can easily check the estimates

$$\left\|A_{j,\rho}\right\|_{W^{k,\infty}(\mathbb{R}^{1+n})}\leqslant C_k\rho^{\frac{k}{3}},\tag{4.7}$$

with C_k independent of ρ. Note that

$$A_\rho(x,t):=\int_{\mathbb{R}^{1+n}}\chi_\rho(x-y,t-s)A(y,s)dsdy=A_{1,\rho}(x,t)-A_{2,\rho}(x,t),$$

with $A = A_1 - A_2$. Then, for $\xi \in \omega^\perp := \{x \in \mathbb{R}^n : \ \omega \cdot x = 0\}$, we fix

$$b_{1,\rho}(x,t) = e^{-i(t\tau + x\cdot\xi)}\left(1 - e^{-\rho^{\frac{1}{3}}t}\right)\exp\left(-\frac{\int_0^{+\infty} A_{1,\rho}(x+s\omega,t)\cdot\omega ds}{2}\right), \tag{4.8}$$

$$b_{2,\rho}(x,t) = \left(1 - e^{-\rho^{\frac{1}{3}}(T-t)}\right)\exp\left(\frac{\int_0^{+\infty} A_{2,\rho}(x+s\omega,t)\cdot\omega ds}{2}\right). \tag{4.9}$$

According to (4.6)-(4.7) and to the fact that, for $j = 1, 2$, $\mathrm{supp}(A_{j,\rho}) \subset [-1, T+1] \times B_{R+1}$, we have

$$\|b_{1,\rho}\|_{L^\infty(0,T;W^{k,\infty}(\mathbb{R}^n))} + \|b_{2,\rho}\|_{L^\infty(0,T;W^{k,\infty}(\mathbb{R}^n))} \leqslant C_k \rho^{\frac{k}{3}}, \quad k \geqslant 1 \tag{4.10}$$

$$\|b_{1,\rho}\|_{W^{1,\infty}(0,T;W^{k,\infty}(\mathbb{R}^n))} + \|b_{2,\rho}\|_{W^{1,\infty}(0,T;W^{k,\infty}(\mathbb{R}^n))} \leqslant C_k \rho^{\frac{k+1}{3}}, \quad k \geqslant 1 \tag{4.11}$$

and

$$b_{1,\rho}(x,0) = b_{2,\rho}(x,T) = 0, \quad x \in \Omega. \tag{4.12}$$

Here C_k, $k \in \mathbb{N}$, denotes a constant independent of $\rho > 0$. Moreover, conditions (4.6)-(4.7) and (4.10) imply that, for any open bounded subset $\tilde{\Omega}$ of $\mathbb{R}^n$ and for $\tilde{Q} = \tilde{\Omega} \times (0,T)$, we have

$$\lim_{\rho\to+\infty} \|(2\omega\cdot\nabla_x - (A_1\cdot\omega))b_{1,\rho}\|_{L^2(\tilde{Q})} = \lim_{\rho\to+\infty} \|[(A_{1,\rho} - A_1)\cdot\omega]b_{1,\rho}\|_{L^2(\tilde{Q})} = 0, \tag{4.13}$$

$$\lim_{\rho\to+\infty} \|(2\omega\cdot\nabla_x + (A_2\cdot\omega))b_{2,\rho}\|_{L^2(\tilde{Q})} = \lim_{\rho\to+\infty} \|[(A_2 - A_{2,\rho})\cdot\omega]b_{1,\rho}\|_{L^2(\tilde{Q})} = 0, \tag{4.14}$$

4.2. **Carleman estimates in negative order Sobolev space.** In order to complete the construction of the GO taking the form (4.1)-(4.2) we recall some preliminary tools and we derive two Carleman estimates in Sobolev space of negative order. In a similar way to [39], for all $m \in \mathbb{R}$, we introduce the space $H^m_\rho(\mathbb{R}^n)$ defined by

$$H^m_\rho(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \ (|\xi|^2 + \rho^2)^{\frac{m}{2}}\hat{u} \in L^2(\mathbb{R}^n)\},$$

with the norm

$$\|u\|^2_{H^m_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (|\xi|^2 + \rho^2)^m |\hat{u}(\xi)|^2 d\xi.$$

Here for all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $\hat{u}$ the Fourier transform of u which, for $u \in L^1(\mathbb{R}^n)$, is defined by

$$\hat{u}(\xi) := \mathcal{F}u(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) dx.$$

From now on, for $m \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$, we set

$$\langle \xi, \rho \rangle = (|\xi|^2 + \rho^2)^{\frac{1}{2}}$$

and $\langle D_x, \rho \rangle^m u$ defined by

$$\langle D_x, \rho \rangle^m u = \mathcal{F}^{-1}(\langle \xi, \rho \rangle^m \mathcal{F}u).$$

For $m \in \mathbb{R}$ we define also the class of symbols

$$S^m_\rho = \{c_\rho \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n) : \ |\partial_t^k \partial_x^\alpha \partial_\xi^\beta c_\rho(x,t,\xi)| \leqslant C_{k,\alpha,\beta} \langle \xi, \rho \rangle^{m-|\beta|}, \ \alpha, \beta \in \mathbb{N}^n, \ k \in \mathbb{N}\}.$$

Following [28, Theorem 18.1.6], for any $m \in \mathbb{R}$ and $c_\rho \in S^m_\rho$, we define $c_\rho(x,t,D_x)$, with $D_x = -i\nabla_x$, by

$$c_\rho(x,t,D_x)z(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} c_\rho(x,t,\xi)\hat{z}(\xi) e^{ix\cdot\xi} d\xi, \quad z \in \mathcal{C}^\infty_0(\mathbb{R}^n).$$

For all $m \in \mathbb{R}$, we set also $OpS^m_\rho := \{c_\rho(x,t,D_x) : \ c_\rho \in S^m_\rho\}$. We fix

$$P_{A,B,q,\pm} := e^{\mp(\rho^2 t + \rho x\cdot\omega)}(L_{\pm,A} + \nabla_x \cdot B + q) e^{\pm(\rho^2 t + \rho x\cdot\omega)}$$

and we consider the following Carleman estimate.

Proposition 4.1. *Let $A, B \in L^\infty(Q)^n$ and $q \in L^\infty(0,T;L^p(\Omega)) \cup \mathcal{C}([0,T];L^{\frac{2n}{3}}(\Omega))$ with $p > 2n/3$. Then, there exists $\rho_2' > \rho_2$, depending only on Ω, T and $M \geqslant \|A\|_{L^\infty(Q)^n} + \|B\|_{L^\infty(Q)^n}$, such that for all $v \in \mathcal{C}^1([0,T];\mathcal{C}_0^\infty(\Omega))$ satisfying $v_{|\Omega^T} = 0$ we have*

$$(\rho^{-\frac{1}{2}} \|v\|_{L^2(0,T;H^1(\mathbb{R}^n))} + \|v\|_{L^2(0,T;L^2(\mathbb{R}^n))}) \leqslant C \|P_{A,B,q,-}v\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))}, \quad \rho > \rho_2', \tag{4.15}$$

with $C > 0$ depending on Ω, T and $M \geqslant \|A\|_{L^\infty(Q)^n} + \|B\|_{L^\infty(Q)^n} + \|q\|_{L^\infty(0,T;L^p(\Omega))}$, when $q \in L^\infty(0,T;L^p(\Omega))$ and $M \geqslant \|A\|_{L^\infty(Q)^n} + \|B\|_{L^\infty(Q)^n} + \|q\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))}$ when $q \in \mathcal{C}([0,T];L^{\frac{2n}{3}}(\Omega))$.

Proof. For $\varphi_{\rho,s}$ given by (3.1), we consider

$$P_{A,B,q,\pm,s} := e^{-\varphi_{\pm,s}}(L_{\pm,A} + q + \nabla_x \cdot B)e^{\varphi_{\pm,s}}, \quad P_{A,-,s} = e^{-\varphi_{\pm,s}} L_{\pm,A} e^{\varphi_{\pm,s}}$$

and in a similar way to Proposition 3.1 we decompose $P_{A,B,q,-,s}$ into three terms

$$P_{A,B,q,-,s} = P_{1,-} + P_{2,-} + P_{3,-,A,B,q},$$

with

$$P_{1,-} = -\Delta_x + 2\rho s((x+x_0)\cdot\omega + s^2((x+x_0)\cdot\omega)^2 - s, \quad P_{2,-} = -\partial_t - 2[\rho - s((x+x_0)\cdot\omega)]\omega\cdot\nabla_x + 2s.$$

$$P_{3,-,A,B,q} = A\cdot\nabla_x - (\rho + s((x+x_0)\cdot\omega))A\cdot\omega + \nabla_x\cdot B + q.$$

We pick $\tilde{\Omega}$ a bounded open and smooth set of $\mathbb{R}^n$ such that $\overline{\Omega} \subset \tilde{\Omega}$ and we extend the function A, B, q by zero to $\mathbb{R}^n \times (0,T)$. In order to prove (4.15), we fix $w \in \mathcal{C}^1([0,T];\mathcal{C}_0^\infty(\tilde{\Omega}))$ satisfying $w_{|\Omega^T} = 0$ and we consider the quantity

$$\langle D_x, \rho\rangle^{-1} (P_{1,-} + P_{2,-}) \langle D_x, \rho\rangle w.$$

Here for any $z \in \mathcal{C}^\infty([0,T];\mathcal{C}_0^\infty(\tilde{\Omega}))$ we define

$$\langle D_x, \rho\rangle^m z(x,t) = \mathcal{F}_x^{-1}(\langle \xi, \rho\rangle^m \mathcal{F}_x z(\cdot,t))(x).$$

where the partial Fourier transform $\mathcal{F}_x$ is defined by

$$\mathcal{F}_x z(t,\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} z(x,t)dx.$$

In all the remaining parts of this proof $C > 0$ denotes a generic constant depending on Ω, T, M. Combining the properties of composition of pseudoddifferential operators (e.g. [28, Theorem 18.1.8]) with the fact that $\langle D_x, \rho\rangle^{-1}$ commute with ∂_t, we find

$$\langle D_x, \rho\rangle^{-1} (P_{1,-} + P_{2,-}) \langle D_x, \rho\rangle = P_{1,-} + P_{2,-} + R_\rho(x, D_x), \tag{4.16}$$

where R_ρ is defined by

$$R_\rho(x,\xi) = \nabla_\xi \langle \xi, \rho\rangle^{-1} \cdot D_x(p_{1,-}(x,\xi) + p_{2,-}(x,\xi)) \langle \xi, \rho\rangle + \underset{\langle \xi,\rho\rangle\to+\infty}{o}(1),$$

with

$$p_{1,-}(x,\xi) = |\xi|^2 + 2\rho s(x+x_0)\cdot\omega + s^2((x+x_0)\cdot\omega)^2 - s, \quad p_{2,-}(x,\xi) = -2i[\rho - s((x+x_0)\cdot\omega)]\omega\cdot\xi + 2s.$$

Therefore, we have

$$R_\rho(x,\xi) = \frac{i[2\rho s + 2s^2(x+x_0)\cdot\omega + 2is(\omega\cdot\xi)](\omega\cdot\xi)}{|\xi|^2 + \rho^2} + \underset{\langle \xi,\rho\rangle\to+\infty}{o}(1)$$

and it follows

$$\|R_\rho(x,D_x)w\|_{L^2((0,T)\times\mathbb{R}^n)} \leqslant Cs^2 \|w\|_{L^2((0,T)\times\mathbb{R}^n)}. \tag{4.17}$$

On the other hand, applying (3.6) to w with Q replaced by $\tilde{Q} = (0,T)\times\tilde{\Omega}$, we get

$$\|P_{1,-}w + P_{2,-}w\|_{L^2((0,T)\times\mathbb{R}^n)} \geqslant C\left(s^{-1/2}\|\Delta_x w\|_{L^2((0,T)\times\mathbb{R}^n)} + s^{1/2}\rho\|w\|_{L^2((0,T)\times\mathbb{R}^n)}\right). \tag{4.18}$$

Moreover, using the fact that $\mathrm{supp}(w) \subset \tilde{\Omega}$ and the elliptic regularity of the operator Δ we deduce that

$$\|w\|_{L^2(0,T;H^2(\mathbb{R}^n))} \leqslant C \|\Delta_x w\|_{L^2((0,T)\times\mathbb{R}^n)},$$

where in both of these estimates $C > 0$ depends only on $\tilde{\Omega}$, and by interpolation, we deduce that

$$\begin{aligned} s^{1/2} \|w\|_{L^2(0,T;H^1(\mathbb{R}^n))} &\leqslant \left(s^{-1/2} \|w\|_{L^2(0,T;H^2(\mathbb{R}^n))}\right)^{\frac{1}{2}} \left(s^{3/2} \|w\|_{L^2(0,T;L^2(\mathbb{R}^n))}\right)^{\frac{1}{2}} \\ &\leqslant s^{-1/2} \|w\|_{L^2(0,T;H^2(\mathbb{R}^n))} + s^{1/2}\rho \|w\|_{L^2(0,T;L^2(\mathbb{R}^n))}. \end{aligned}$$

Combining these two estimates with (4.18), we get

$$\|P_{1,-}w + P_{2,-}w\|_{L^2((0,T)\times\mathbb{R}^n)} \geqslant C\left(s^{-1/2} \|w\|_{L^2(0,T;H^2(\mathbb{R}^n))} + s^{1/2} \|w\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))}\right).$$

Combining this estimate with (4.16)-(4.17), for $\frac{\rho}{s^2}$ sufficiently large, we obtain

$$\begin{aligned} &\|(P_{1,-} + P_{2,-}) \langle D_x, \rho\rangle w\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))} \\ &= \left\|\langle D_x, \rho\rangle^{-1} (P_{1,-} + P_{2,-}) \langle D_x, \rho\rangle w\right\|_{L^2((0,T)\times\mathbb{R}^n)} \\ &\geqslant C\left(s^{-1/2} \|w\|_{L^2(0,T;H^2(\mathbb{R}^n))} + s^{1/2} \|w\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))}\right). \end{aligned} \tag{4.19}$$

Moreover, we have

$$\begin{aligned} &\|P_{3,-,A,B,q} \langle D_x, \rho\rangle w\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))} \\ &\leqslant \|A\cdot\nabla_x \langle D_x, \rho\rangle w\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))} + \|(\rho + s((x+x_0)\cdot\omega))A\cdot\omega \langle D_x, \rho\rangle w\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))} \\ &\quad + \|(\nabla_x \cdot B) \langle D_x, \rho\rangle w\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))} + \|q \langle D_x, \rho\rangle w\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))}. \end{aligned} \tag{4.20}$$

For the first term on the right hand side of (4.20), we find

$$\begin{aligned} \|A\cdot\nabla_x \langle D_x, \rho\rangle w\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))} &\leqslant \rho^{-1} \|A\cdot\nabla_x \langle D_x, \rho\rangle w\|_{L^2(0,T;L^2(\mathbb{R}^n))} \\ &\leqslant \|A\|_{L^\infty(Q)} \rho^{-1} \|\nabla_x \langle D_x, \rho\rangle w\|_{L^2(0,T;L^2(\mathbb{R}^n))} \\ &\leqslant C \|A\|_{L^\infty(Q)} \left(\rho^{-1} \|w\|_{L^2(0,T;H^2(\mathbb{R}^n))} + \|w\|_{L^2(0,T;H^1(\mathbb{R}^n))}\right). \end{aligned} \tag{4.21}$$

For the second term on the right hand side of (4.20), we get

$$\begin{aligned} \|(\rho + s((x+x_0)\cdot\omega))A\cdot\omega \langle D_x, \rho\rangle w\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))} &\leqslant \rho^{-1} \|(\rho + s((x+x_0)\cdot\omega))A\cdot\omega \langle D_x, \rho\rangle w\|_{L^2(0,T;L^2(\mathbb{R}^n))} \\ &\leqslant \left(1 + |x_0| + \sup_{x\in\overline{\Omega}} |x|\right) \|A\|_{L^\infty(Q)} \|\langle D_x, \rho\rangle w\|_{L^2(0,T;L^2(\mathbb{R}^n))} \\ &\leqslant C \|A\|_{L^\infty(Q)} \|w\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))}. \end{aligned} \tag{4.22}$$

For the third term on the right hand side of (4.20), we have

$$\begin{aligned} \|(\nabla_x \cdot B) \langle D_x, \rho\rangle w\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))} &\leqslant \|B \langle D_x, \rho\rangle w\|_{L^2(0,T;L^2(\mathbb{R}^n))} + \rho^{-1} \|B\cdot\nabla_x \cdot \langle D_x, \rho\rangle w\|_{L^2(0,T;L^2(\mathbb{R}^n))} \\ &\leqslant C \|B\|_{L^\infty(Q)} \left(\rho^{-1} \|w\|_{L^2(0,T;H^2(\mathbb{R}^n))} + \|w\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))}\right). \end{aligned} \tag{4.23}$$

Finally, for the last term on the right hand side of (4.20), we will prove that there exists $\rho_1''(s) > \rho_1(s)$, with $\rho_1(s)$ given by Proposition 3.1, such that the estimate

$$\|q \langle D_x, \rho\rangle w\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))} \leqslant C[s^{-1} \|w\|_{H^2(\mathbb{R}^n)} + \rho \|w\|_{L^2(\mathbb{R}^n)}] \tag{4.24}$$

holds true for $\rho > \rho_1''(s)$. For this purpose, let us first assume that $n \geqslant 3$ and $q \in \mathcal{C}([0,T]; L^{\frac{2n}{3}}(\Omega))$. We consider

$$q_\rho(x,t) := \int_{\mathbb{R}^{1+n}} \rho^{\frac{n}{4}} h(\rho^{\frac{1}{4}}(x-y)) q(y,t) dy, \tag{4.25}$$

with q extended by zero to $\mathbb{R}^n\times(0,T)$ and with $h\in\mathcal{C}_0^\infty(\mathbb{R}^n;[0,+\infty))$ satisfying $\operatorname{supp}(h)\subset\{x\in\mathbb{R}^n:\ |x|<1\}$ and

$$\int_{\mathbb{R}^n} h(x)dx=1.$$

We have the following result.

Lemma 4.1. *Let $p_2\in[1,+\infty)$, $q\in\mathcal{C}([0,T];L^{p_2}(\Omega))$ and q_ρ given by (4.25). Then, we have*

$$\lim_{\rho\to+\infty}\|q_\rho-q\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^n))}. \tag{4.26}$$

We will prove this result when finished the present proof. For all $\psi\in L^2(0,T;\mathcal{C}_0^\infty(\mathbb{R}^n))$ we have

$$\left|\langle q\langle D_x,\rho\rangle w,\psi\rangle_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n)),L^2(0,T;H_\rho^1(\mathbb{R}^n))}\right|\leqslant\int_0^T\int_{\mathbb{R}^n}(|q-q_\rho|+|q_\rho|)|\langle D_x,\rho\rangle w||\psi|dxdt.$$

Applying the Hölder inequality, for $n\geqslant 3$, we get

$$\begin{aligned}&\left|\langle q\langle D_x,\rho\rangle w,\psi\rangle_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n)),L^2(0,T;H_\rho^1(\mathbb{R}^n))}\right|\\&\leqslant\|q-q_\rho\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\mathbb{R}^n))}\|\langle D_x,\rho\rangle w\|_{L^2(0,T;L^{\frac{2n}{n-2}}(\mathbb{R}^n))}\|\psi\|_{L^2(0,T;L^{\frac{2n}{n-1}}(\mathbb{R}^n))}\\&\quad+\|q_\rho\|_{L^\infty(Q)}\|\langle D_x,\rho\rangle w\|_{L^2(\mathbb{R}^n\times(0,T))}\|\psi\|_{L^2(\mathbb{R}^n\times(0,T))}\end{aligned}\tag{4.27}$$

For the first term on the right hand side of (4.27), applying the Sobolev embedding theorem, we find

$$\begin{aligned}&\|q-q_\rho\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\mathbb{R}^n))}\|\langle D_x,\rho\rangle w\|_{L^2(0,T;L^{\frac{2n}{n-2}}(\mathbb{R}^n))}\|\psi\|_{L^2(0,T;L^{\frac{2n}{n-1}}(\mathbb{R}^n))}\\&\leqslant\|q-q_\rho\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\mathbb{R}^n))}\|\langle D_x,\rho\rangle w\|_{L^2(0,T;H^1(\mathbb{R}^n)}\|\psi\|_{L^2(0,T;H^{\frac12}(\mathbb{R}^n))}.\end{aligned}$$

Moreover, by interpolation, we obtain

$$\begin{aligned}\|\psi\|_{L^2(0,T;H^{\frac12}(\mathbb{R}^n))}&\leqslant\|\psi\|_{L^2(0,T;H_\rho^{\frac12}(\mathbb{R}^n))}\\&\leqslant\left(\|\psi\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))}\right)^{\frac12}\left(\|\psi\|_{L^2(0,T;L^2(\mathbb{R}^n))}\right)^{\frac12}\\&\leqslant C\rho^{-\frac12}\|\psi\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))}\end{aligned}$$

and we deduce that

$$\begin{aligned}&\|q-q_\rho\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\mathbb{R}^n))}\|\langle D_x,\rho\rangle w\|_{L^2(0,T;L^{\frac{2n}{n-2}}(\mathbb{R}^n))}\|\psi\|_{L^2(0,T;L^{\frac{2n}{n-1}}(\mathbb{R}^n))}\\&\leqslant C\|q-q_\rho\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\mathbb{R}^n))}\left(\rho^{-\frac12}\|\langle D_x,\rho\rangle w\|_{L^2(0,T;H^1(\mathbb{R}^n))}\right)\|\psi\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))}\\&\leqslant C\|q-q_\rho\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\mathbb{R}^n))}\left[\rho^{-\frac12}\|w\|_{L^2(0,T;H^2(\mathbb{R}^n))}+\rho^{\frac12}\|w\|_{L^2(0,T;H^1(\mathbb{R}^n))}\right]\|\psi\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))}\\&\leqslant C\rho^{-\frac12}\|q-q_\rho\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\mathbb{R}^n))}\|w\|_{L^2(0,T;H^2(\mathbb{R}^n))}\\&\quad+C\|q-q_\rho\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\mathbb{R}^n))}(s^{-\frac12}\|w\|_{L^2(0,T;H^2(\mathbb{R}^n))})^{\frac12}(s^{\frac12}\rho\|w\|_{L^2(\mathbb{R}^n\times(0,T))})^{\frac12}\|\psi\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))}\\&\leqslant C\|q-q_\rho\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\mathbb{R}^n))}\left[s^{-\frac12}\|w\|_{L^2(0,T;H^2(\mathbb{R}^n))}+s^{\frac12}\rho\|w\|_{L^2(\mathbb{R}^n\times(0,T))}\right]\|\psi\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))}.\end{aligned}$$

In the same way, for the second term on the right hand side of (4.27), applying the Sobolev embedding theorem, we obtain

$$\begin{aligned}&\|q_\rho\|_{L^\infty(Q)}\|\langle D_x,\rho\rangle w\|_{L^2(Q)}\|\psi\|_{L^2(Q)}\\&\leqslant C\|q_\rho\|_{L^\infty(0,T;W^{2,\frac{2n}{3}}(\mathbb{R}^n))}\|\langle D_x,\rho\rangle w\|_{L^2(\mathbb{R}^n\times(0,T))}\|\psi\|_{L^2(\mathbb{R}^n\times(0,T))}\\&\leqslant C\rho^{\frac12}\|w\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))}(\rho^{-1}\|\psi\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))})\\&\leqslant C\rho^{-\frac12}\left[s^{-\frac12}\|w\|_{L^2(0,T;H^2(\mathbb{R}^n))}+s^{\frac12}\rho\|w\|_{L^2(0,T;H^1(\mathbb{R}^n))}\right]\|\psi\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))})\end{aligned}$$

Combining these two estimates with (4.27), we obtain

$$\begin{aligned}&\left|\langle q\,\langle D_x,\rho\rangle\,w,\psi\rangle_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n)),L^2(0,T;H_\rho^1(\mathbb{R}^n))}\right|\\&\leqslant C[\|q-q_\rho\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\mathbb{R}^n))}+\rho^{-\frac12}]\left[s^{-\frac12}\,\|w\|_{L^2(0,T;H^2(\mathbb{R}^n))}+s^{\frac12}\rho\,\|w\|_{L^2(\mathbb{R}^n\times(0,T))}\right]\|\psi\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))}\end{aligned}$$

and we deduce that

$$\begin{aligned}&\|q\,\langle D_x,\rho\rangle\,w\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))}\\&\leqslant C[\|q-q_\rho\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\mathbb{R}^n))}+\rho^{-\frac12}]\left[s^{-\frac12}\,\|w\|_{L^2(0,T;H^2(\mathbb{R}^n))}+s^{\frac12}\rho\,\|w\|_{L^2(\mathbb{R}^n\times(0,T))}\right].\end{aligned}$$

On the other hand, using the fact that

$$\lim_{\rho\to+\infty}[\|q-q_\rho\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\mathbb{R}^n))}+\rho^{-\frac12}]=0,$$

we can find $\rho_1''(s)>\rho_1(s)$ such that for $\rho>\rho_1''(s)$ we have

$$[\|q-q_\rho\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\mathbb{R}^n))}+\rho^{-\frac12}]\leqslant s^{-\frac12}.$$

Thus, we obtain (4.24). In the same way we can deduce (4.24) for $n=2$ and $q\in\mathcal{C}([0,T];L^{\frac{2n}{3}}(\Omega))$. Now let us show (4.24) for $n\geqslant 3$ and $q\in L^\infty(0,T;L^p(\Omega))$, for $p<n$. In that case, applying the Hölder inequality, we get

$$\begin{aligned}&\left|\langle q\,\langle D_x,\rho\rangle\,w,\psi\rangle_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n)),L^2(0,T;H_\rho^1(\mathbb{R}^n))}\right|\\&\leqslant\|q\|_{L^\infty(0,T;L^p(\Omega))}\,\|\langle D_x,\rho\rangle\,w\|_{L^2\left(0,T;L^{\frac{2n}{n-2}}(\mathbb{R}^n)\right)}\,\|\psi\|_{L^2\left(0,T;L^{\frac{2n}{n-2(\frac{n}{p}-1)}}(\mathbb{R}^n)\right)}.\end{aligned}$$

Using the Sobolev embedding theorem, we have

$$\begin{aligned}&\left|\langle q\,\langle D_x,\rho\rangle\,w,\psi\rangle_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n)),L^2(0,T;H_\rho^1(\mathbb{R}^n))}\right|\\&\leqslant C\,\|q\|_{L^\infty(0,T;L^p(\Omega))}\,\|\langle D_x,\rho\rangle\,w\|_{L^2(0,T;H^1(\mathbb{R}^n))}\,\|\psi\|_{L^2(0,T;H^{\frac{n}{p}-1}(\mathbb{R}^n))}.\end{aligned}$$

On the other hand, by interpolation we find

$$\begin{aligned}\|\psi\|_{L^2(0,T;H^{\frac{n}{p}-1}(\mathbb{R}^n))}&\leqslant\|\psi\|_{L^2(0,T;H_\rho^{\frac{n}{p}-1}(\mathbb{R}^n))}\\&\leqslant\left(\|\psi\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))}\right)^{\frac{n}{p}-1}\left(\|\psi\|_{L^2(0,T;L^2(\mathbb{R}^n))}\right)^{2-\frac{n}{p}}\\&\leqslant C\rho^{\frac{n}{p}-2}\,\|\psi\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))}\end{aligned}$$

and we deduce that

$$\begin{aligned}&\|q\,\langle D_x,\rho\rangle\,w\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))}\\&\leqslant C\,\|q\|_{L^\infty(0,T;L^p(\Omega)}\,\rho^{\frac{n}{p}-2}\,\|\langle D_x,\rho\rangle\,w\|_{L^2(0,T;H^1(\mathbb{R}^n))}\\&\leqslant C\,\|q\|_{L^\infty(0,T;L^p(\Omega)}\,\rho^{\frac{n}{p}-\frac32}[s^{-\frac12}\,\|w\|_{H^2(\mathbb{R}^n)}+s^{\frac12}\rho\,\|w\|_{L^2(\mathbb{R}^n)}]\end{aligned}$$

Using the fact that $\frac32>\frac{n}{p}$, we deduce (4.24), for $n\geqslant 3$ and $q\in L^\infty(0,T;L^p(\Omega))$, from this estimate. We prove in the same way, (4.24), for $n=2$ and $q\in L^\infty(0,T;L^p(\Omega))$. Combining (4.20)-(4.24) with (4.19), for $s=C(\|A\|_{L^\infty(Q)}+\|B\|_{L^\infty(Q)})+1$, for some constant $C>0$ depending only on Ω, T but suitably chosen, we find

$$\|P_{A,B,q,-,s}\,\langle D_x,\rho\rangle\,w\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))}\geqslant C\left(\|w\|_{L^2(0,T;H^2(\mathbb{R}^n))}+\|w\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))}\right).\tag{4.28}$$

We fix $\psi_0\in\mathcal{C}_0^\infty(\tilde\Omega)$ satisfying $\psi_0=1$ on $\overline{\Omega_1}$, with Ω_1 an open neighborhood of $\overline{\Omega}$ such that $\overline{\Omega_1}\subset\tilde\Omega$. Then, we fix $w=\psi_0(x)\,\langle D_x,\rho\rangle^{-1}\,v(x,t)$ and for $\psi_1\in\mathcal{C}_0^\infty(\Omega_1)$ satisfying $\psi_1=1$ on Ω, we get $(1-\psi_0)\,\langle D_x,\rho\rangle^{-1}\,v=$

$(1-\psi_0)\langle D_x,\rho\rangle^{-1}\psi_1 v$. According to [28, Theorem 18.1.8], we have $(1-\psi_0)\langle D_x,\rho\rangle^{-1}\psi_1 \in OpS^{-\infty}_{\rho}$ and it follows

$$\begin{aligned}\|v\|_{L^2((0,T)\times\mathbb{R}^n)} &= \left\|\langle D_x,\rho\rangle^{-1} v\right\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))}\\ &\leqslant \|w\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))} + \left\|(1-\psi_0)\langle D_x,\rho\rangle^{-1}\psi_1 v\right\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))}\\ &\leqslant \|w\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))} + \frac{C\|v\|_{L^2((0,T)\times\mathbb{R}^n)}}{\rho^2}.\end{aligned}$$

In addition, by interpolation, we get

$$\begin{aligned}&\rho^{-1}\|v\|^2_{L^2(0,T;H^1(\mathbb{R}^n))}\\ &\leqslant \left\|\langle D_x,\rho\rangle^{-1} v\right\|^2_{L^2(0,T;H^1_\rho(\mathbb{R}^n))} + \left\|\langle D_x,\rho\rangle^{-1} v\right\|^2_{L^2(0,T;H^2(\mathbb{R}^n))} + \rho\left\|\langle D_x,\rho\rangle^{-1} v\right\|^2_{L^2(0,T;H^1(\mathbb{R}^n))}\\ &\leqslant \left\|\langle D_x,\rho\rangle^{-1} v\right\|^2_{L^2(0,T;H^1_\rho(\mathbb{R}^n))} + \left\|\langle D_x,\rho\rangle^{-1} v\right\|^2_{L^2(0,T;H^2(\mathbb{R}^n))} + \rho\left\|\langle D_x,\rho\rangle^{-1} v\right\|_{L^2(0,T;H^2(\mathbb{R}^n))}\left\|\langle D_x,\rho\rangle^{-1} v\right\|_{L^2(0,T;L^2(\mathbb{R}^n))}\\ &\leqslant 2\left\|\langle D_x,\rho\rangle^{-1} v\right\|^2_{L^2(0,T;H^1_\rho(\mathbb{R}^n))} + 2\left\|\langle D_x,\rho\rangle^{-1} v\right\|^2_{L^2(0,T;H^2(\mathbb{R}^n))}\end{aligned}$$

and it follows

$$\begin{aligned}\rho^{-\frac{1}{2}}\|v\|_{L^2(0,T;H^1(\mathbb{R}^n))} &\leqslant 4\left\|\langle D_x,\rho\rangle^{-1} v\right\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))} + 4\left\|\langle D_x,\rho\rangle^{-1} v\right\|_{L^2(0,T;H^2(\mathbb{R}^n))}\\ &\leqslant 4\|w\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))} + \left\|(1-\psi_0)\langle D_x,\rho\rangle^{-1}\psi_1 v\right\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))}\\ &\quad + 4\|w\|_{L^2(0,T;H^2(\mathbb{R}^n))} + \left\|(1-\psi_0)\langle D_x,\rho\rangle^{-1}\psi_1 v\right\|_{L^2(0,T;H^2(\mathbb{R}^n))}\\ &\leqslant 4\|w\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))} + 4\|w\|_{L^2(0,T;H^2(\mathbb{R}^n))} + \frac{C\|v\|_{L^2((0,T)\times\mathbb{R}^n)}}{\rho^2}.\end{aligned}$$

Thus, applying (4.28) for a fixed value of s, we deduce that there exists $\rho_2' > 0$ such that (4.15) is fulfilled. □

Now that the proof of Lemma 4.1 is completed, let us consider the proof of Lemma 4.1.

Proof of Lemma 4.1. We fix $\varepsilon_1 > 0$ and we will prove that

$$\limsup_{\rho\to+\infty}\|q_\rho - q\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^n)} \leqslant 2\varepsilon_1. \tag{4.29}$$

For this purpose, using the fact that $t \mapsto q_\rho(\cdot,t) \in \mathcal{C}([0,T];L^{p_2}(\mathbb{R}^n))$, there exists $\delta > 0$ such that for all $t,t' \in [0,T]$ satisfying $|t-t'| < \delta$ we have

$$\|q(\cdot,t)] - q(\cdot,t')\|_{L^{p_2}(\mathbb{R}^n)} \leqslant \varepsilon_1. \tag{4.30}$$

Using the fact that $[0,T]$ is compact, we can find $t_1,\ldots,t_N$ such that

$$[0,T] \subset \bigcup_{j=1}^N (t_j-\delta, t_j+\delta)$$

and, using the fact that

$$\lim_{\rho\to+\infty}\|q_\rho(\cdot,t) - q(\cdot,t)\|_{L^{p_2}(\mathbb{R}^n)} = 0, \quad t\in[0,T],$$

we get

$$\lim_{\rho\to+\infty}\max_{j=1,\ldots,N}\|q_\rho(\cdot,t_j) - q(\cdot,t_j)\|_{L^{p_2}(\mathbb{R}^n)} = 0, \quad j=1,\ldots,N. \tag{4.31}$$

Thus, for all $t \in [0,T]$ there exists $k \in \{1, \dots N\}$ such that $|t - t_k| < \delta$ and, applying (4.30) and the Young inequality, we get

$$\begin{aligned}
&\|q_\rho(\cdot,t) - q(\cdot,t)\|_{L^{p_2}(\mathbb{R}^n)} \\
&\leqslant \|q(\cdot,t) - q(\cdot,t_k)\|_{L^{p_2}(\mathbb{R}^n)} + \|q_\rho(\cdot,t) - q_\rho(\cdot,t_k)\|_{L^{p_2}(\mathbb{R}^n)} + \|q_\rho(\cdot,t_k) - q(\cdot,t_k)\|_{L^{p_2}(\mathbb{R}^n)} \\
&\leqslant 2\|q(\cdot,t) - q(\cdot,t_k)\|_{L^{p_2}(\mathbb{R}^n)} + \max_{j=1,\dots,N} \|q_\rho(\cdot,t_j) - q(\cdot,t_j)\|_{L^{p_2}(\mathbb{R}^n)} \\
&\leqslant 2\varepsilon_1 + \max_{j=1,\dots,N} \|q_\rho(\cdot,t_j) - q(\cdot,t_j)\|_{L^{p_2}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, we have

$$\|q_\rho - q\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^n))} \leqslant 2\varepsilon_1 + \max_{j=1,\dots,N} \|q_\rho(\cdot,t_j) - q(\cdot,t_j)\|_{L^{p_2}(\mathbb{R}^n)}$$

and using (4.31), we obtain (4.29) from which we deduce (4.26). □

In a similar way to Proposition 4.1, combining estimate (3.7) with the arguments of Lemma 4.1, we deduce the following estimate.

Proposition 4.2. *There exists $\rho_4' > \rho_3$ such that for $\rho > \rho_4'$ and for any $v \in \mathcal{C}^1([0,T];\mathcal{C}_0^\infty(\Omega))$ satisfying $v_{|\Omega^0} = 0$, we have*

$$(\rho^{-\frac{1}{2}}\|v\|_{L^2(0,T;H^1(\mathbb{R}^n))} + \|v\|_{L^2(0,T;L^2(\mathbb{R}^n))}) \leqslant C\|P_{A,B,q,+}v\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))}, \quad \rho > \rho_4' \tag{4.32}$$

with $C > 0$ depending on Ω, T and $M \geqslant \|A\|_{L^\infty(Q)^n} + \|B\|_{L^\infty(Q)^n} + \|q\|_{L^\infty(0,T;L^p(\Omega))}$, when $q \in L^\infty(0,T;L^p(\Omega))$ with $p > 2n/3$ and $M \geqslant \|A\|_{L^\infty(Q)^n} + \|B\|_{L^\infty(Q)^n} + \|q\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))}$ when $q \in \mathcal{C}([0,T];L^{\frac{2n}{3}}(\Omega))$.

4.3. **Remainder term.** In this subsection we will complete the construction of exponentially growing solutions $u_1 \in L^2(0,T;H^1(\Omega))$ of the equation (4.1) and exponentially decaying solutions $u_2 \in L^2(0,T;H^1(\Omega))$ of the equation (4.2) taking the form (4.3). We state these results in the following way.

Proposition 4.3. *There exists $\rho_3 > \rho_2$ such that for $\rho > \rho_3$ we can find a solution $u_1 \in L^2(0,T;H^1(\Omega))$ of (4.1) taking the form (4.3) with $w_{1,\rho} \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega))$ satisfying*

$$\lim_{\rho \to +\infty} \rho^{-1}(\|w_{1,\rho}\|_{L^2(0,T;H^1(\Omega))} + \rho\|w_{1,\rho}\|_{L^2(Q)}) = 0, \tag{4.33}$$

with C depending on Ω, T and $M \geqslant \|A_1\|_{L^\infty(Q)^n} + \|B_1\|_{L^\infty(Q)^n} + \|q_1\|_{L^\infty(0,T;L^p(\Omega))}$, when $q_1 \in L^\infty(0,T;L^p(\Omega))$ with $p > 2n/3$ and $M \geqslant \|A_1\|_{L^\infty(Q)^n} + \|B_1\|_{L^\infty(Q)^n} + \|q_1\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))}$ when $q_1 \in \mathcal{C}([0,T];L^{\frac{2n}{3}}(\Omega))$.

Proposition 4.4. *There exists $\rho_4 > \rho_3$ such that for $\rho > \rho_4$ we can find a solution $u_2 \in L^2(0,T;H^1(\Omega))$ of (4.2) taking the form (4.3) with $w_{2,\rho} \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega))$ satisfying*

$$\lim_{\rho \to +\infty} \rho^{-1}(\|w_{2,\rho}\|_{L^2(0,T;H^1(\Omega))} + \rho\|w_{2,\rho}\|_{L^2(Q)}) = 0, \tag{4.34}$$

with C depending on Ω, T, $M \geqslant \|A_2\|_{L^\infty(Q)^n} + \|B_2\|_{L^\infty(Q)^n} + \|q_2\|_{L^\infty(0,T;L^p(\Omega))}$, when $q_2 \in L^\infty(0,T;L^p(\Omega))$ with $p > 2n/3$ and $M \geqslant \|A_2\|_{L^\infty(Q)^n} + \|B_2\|_{L^\infty(Q)^n} + \|q_2\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))}$ when $q_2 \in \mathcal{C}([0,T];L^{\frac{2n}{3}}(\Omega))$.

The proof of these two propositions being similar, we will only consider the one of Proposition 4.3.

Proof of Proposition 4.3. Note first that the condition $L_{A_1,B_1}u_1 + q_1u_1 = 0$ is fulfilled if and only if $w_{1,\rho}$ solves

$$P_{A_1,B_1,q_1,+}w_{1,\rho} = -P_{A_1,B_1,q_1,+}b_{1,\rho} = \rho(2\omega \cdot \nabla_x b_{1,\rho} - A_1 \cdot \omega b_{1,\rho}) - (L_{A_1} + \nabla_x \cdot B_1 + q_1)b_{1,\rho}.$$

Therefore, fixing $\varphi_1 \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, such that $\varphi_1 = 1$ on $\overline{\Omega}$, and

$$F_\rho(x,t) = \varphi_1(x)[\rho(2\omega \cdot \nabla_x b_{1,\rho} - A_1 \cdot \omega b_{1,\rho})(x,t) - L_{A_1,B_1,q_1}b_{1,\rho}(x,t)]$$

we can consider $w_{1,\rho}$ as a solution of

$$P_{A_1,q,+}w_{1,\rho}(x,t) = F_\rho(x,t), \quad (x,t) \in Q. \tag{4.35}$$

In the expression of F_ρ, we assume that A_1, B_1 and q_1 are extended by zero to a function of $\mathbb{R}^n \times (0,T)$. Let us first show that, we have

$$\lim_{\rho\to+\infty} \|F_\rho\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))} = 0. \tag{4.36}$$

For this purpose, note first that, applying (4.10) and fixing $\tilde{Q} = \tilde{\Omega} \times (0,T)$ with $\tilde{\Omega}$ a bounded open set of $\mathbb{R}^n$ such that $\mathrm{supp}(\varphi) \subset \tilde{\Omega}$, we find

$$\begin{aligned}
&\|F_\rho\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))} \\
&\leqslant \|2\omega\cdot\nabla_x b_{1,\rho} - A_1\cdot\omega b_{1,\rho}\|_{L^2(\tilde{Q})} + \rho^{-1}\|L_{A_1}b_{1,\rho}\|_{L^2(\tilde{Q})} + \|(\nabla_x\cdot B_1)b_{1,\rho}\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))} + \|q_1 b_{1,\rho}\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))} \\
&\leqslant \|2\omega\cdot\nabla_x b_{1,\rho} - A_1\cdot\omega b_{1,\rho}\|_{L^2(\tilde{Q})} + C\rho^{-\frac{1}{3}} + \|(\nabla_x\cdot B_1)b_{1,\rho}\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))} + \|q_1 b_{1,\rho}\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))},
\end{aligned} \tag{4.37}$$

with $C > 0$ independent of ρ. Let us first consider the second term on the right hand side of this inequality. We fix $B_{1,\rho}$ given by

$$B_{1,\rho}(x,t) := \int_{\mathbb{R}^{1+n}} \chi_\rho(x-y,t-s)B_1(y,s)dsdy$$

with B_1 extended by zero to a function defined on $\mathbb{R}^{1+n}$. Then, for any $\psi_1 \in L^2(0,T;\mathcal{C}_0^\infty(\mathbb{R}^n))$, we obtain

$$\begin{aligned}
&\left|\langle(\nabla_x\cdot B_1)b_{1,\rho},\psi_1\rangle_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n)),L^2(0,T;H_\rho^1(\mathbb{R}^n))}\right| \\
&\leqslant \left|\langle(B_1\cdot\nabla_x b_{1,\rho},\psi_1\rangle_{L^2(Q)}\right| + \left|\langle b_{1,\rho},(B_1-B_{1,\rho})\cdot\nabla_x\psi_1\rangle_{L^2(\mathbb{R}^n\times(0,T))}\right| + \left|\langle b_{1,\rho},B_{1,\rho}\cdot\nabla_x\psi_1\rangle_{L^2(\mathbb{R}^n\times(0,T))}\right| \\
&\leqslant \left|\langle(B_1\cdot\nabla_x b_{1,\rho},\psi_1\rangle_{L^2(Q)}\right| + \left|\langle b_{1,\rho},(B_1-B_{1,\rho})\cdot\nabla_x\psi_1\rangle_{L^2(\mathbb{R}^n\times(0,T))}\right| + \left|\langle\nabla_x\cdot(b_{1,\rho}B_{1,\rho}),\psi_1\rangle_{L^2(\mathbb{R}^n\times(0,T))}\right|
\end{aligned} \tag{4.38}$$

For the first term on the right hand side of this inequality, applying (4.10), we find

$$\left|\langle(B_1\cdot\nabla_x b_{1,\rho},\psi_1\rangle_{L^2(Q)}\right| \leqslant C\|B_1\|_{L^\infty(Q)}\rho^{\frac{1}{3}}\|\psi\|_{L^2(Q)} \leqslant C\rho^{-\frac{2}{3}}\|\psi\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))} \tag{4.39}$$

with C independent of ρ. For the second term on the right hand side of (4.38), we obtain

$$\begin{aligned}
\left|\langle b_{1,\rho},(B_1-B_{1,\rho})\cdot\nabla_x\psi_1\rangle_{L^2(\mathbb{R}^n\times(0,T))}\right| &\leqslant \|b_{1,\rho}\|_{L^\infty(\mathbb{R}^n\times(0,T))}\|(B_1-B_{1,\rho})\|_{L^2(\mathbb{R}^{1+n})}\|\psi_1\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))} \\
&\leqslant C\|(B_1-B_{1,\rho})\|_{L^2(\mathbb{R}^{1+n})}\|\psi_1\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))}
\end{aligned} \tag{4.40}$$

For the last term on the right hand side of (4.38), we get

$$\begin{aligned}
\left|\langle\nabla_x\cdot(b_{1,\rho}B_{1,\rho}),\psi_1\rangle_{L^2(\mathbb{R}^n\times(0,T))}\right| &\leqslant \|b_{1,\rho}\|_{L^\infty(0,T;W^{1,\infty}(\mathbb{R}^n)}\|B_\rho\|_{L^2(0,T;H^1(\mathbb{R}^n))}\|\psi_1\|_{L^2(\mathbb{R}^n\times(0,T))} \\
&\leqslant C\rho^{-\frac{1}{3}}\|\psi_1\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))}.
\end{aligned}$$

Combining this estimate with (4.37)-(4.40), we obtain

$$\begin{aligned}
&\left|\langle(\nabla_x\cdot B_1)b_{1,\rho},\psi_1\rangle_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n)),L^2(0,T;H_\rho^1(\mathbb{R}^n))}\right| \\
&\leqslant C[\rho^{-\frac{1}{3}} + \|(B_1-B_{1,\rho})\|_{L^2(\mathbb{R}^{1+n})}]\|\psi_1\|_{L^2(0,T;H_\rho^1(\mathbb{R}^n))}
\end{aligned}$$

and, using the fact that

$$\lim_{\rho\to+\infty}\|(B_1-B_{1,\rho})\|_{L^2(\mathbb{R}^{1+n})} = 0,$$

we obtain

$$\lim_{\rho\to+\infty}\|(\nabla_x\cdot B_1)b_{1,\rho}\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))} = 0. \tag{4.41}$$

For the last term on the right hand side of (4.37), fixing $\psi_1 \in L^2(0,T;\mathcal{C}_0^\infty(\mathbb{R}^n))$, we find

$$\begin{aligned}&\left|\langle q_1 b_{1,\rho},\psi_1\rangle_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n)),L^2(0,T;H_\rho^1(\mathbb{R}^n))}\right|\\&\leqslant \|b_{1,\rho}\|_{L^\infty(Q)}\left(\int_Q |q_1||\psi_1|dxdt\right)\leqslant C\left(\int_Q |q_1||\psi_1|dxdt\right).\end{aligned}$$

For $n=2$, we find

$$\int_Q |q_1||\psi_1|dxdt \leqslant C\|q_1\|_{L^\infty(0,T;L^{\frac{4}{3}}(\Omega))}\|\psi_1\|_{L^2(0,T;L^4(\mathbb{R}^n))}.$$

Applying the Sobolev embedding theorem, we get

$$\|\psi_1\|_{L^2(0,T;L^4(\mathbb{R}^n)}\leqslant C\|\psi_1\|_{L^2(0,T;H^{\frac{1}{2}}(\mathbb{R}^n))}\leqslant C\|\psi_1\|^{\frac{1}{2}}_{L^2(0,T;H^1(\mathbb{R}^n))}\|\psi_1\|^{\frac{1}{2}}_{L^2(0,T;L^2(\mathbb{R}^n))}\leqslant C\rho^{-\frac{1}{2}}\|\psi_1\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))}.$$

It follows,

$$\int_Q |q_1||\psi_1|dxdt \leqslant C\rho^{-\frac{1}{2}}\|q_1\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))}\|\psi_1\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))}. \tag{4.42}$$

In the same way, for $n\geqslant 3$, we have

$$\int_Q |q_1||\psi_1|dxdt \leqslant C\|q_1\|_{L^2(Q)}\|\psi_1\|_{L^2(0,T;L^2(\mathbb{R}^n)}\leqslant C\rho^{-1}\|q_1\|_{L^\infty(0,T;L^{\frac{2n}{3}}(\Omega))}\|\psi_1\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))}.$$

Combining these estimates with (4.41)-(4.42), we obtain

$$\lim_{\rho\to+\infty}\|q_1 b_{1,\rho}\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))}=0. \tag{4.43}$$

Putting conditions (4.13), (4.14), (4.37), (4.41) and (4.43) together, we deduce (4.36).

We will now apply estimate (4.15) to build a solution $w_{1,\rho}\in L^2(0,T;H^1(\Omega))\cap H^1(0,T;H^{-1}(\Omega))$ to (4.35) satisfying $w_{1,\rho}(0,\cdot)=0$ and (4.33). We fix $\tilde{\Omega}$ a smooth bounded open set of $\mathbb{R}^n$ such that $\overline{\Omega}\subset\tilde{\Omega}$. Applying the Carleman estimate (4.15), we define the linear form $\mathcal{K}_\rho$ on $\{P_{-A_1,B_1-A_1,q_1,-}z:\ z\in\mathcal{C}^\infty([0,T];\mathcal{C}_0^\infty(\tilde{\Omega})),\ z_{|\tilde{\Omega}^T}=0\}$, considered as a subspace of $L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))$ by

$$\mathcal{K}_\rho(P_{-A_1,B_1-A_1,q_1,-}z)=\langle F_\rho,z\rangle_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n)),L^2(0,T;H_\rho^1(\mathbb{R}^n))},\quad z\in\mathcal{C}^\infty([0,T];\mathcal{C}_0^\infty(\tilde{\Omega})),\ z_{|\tilde{\Omega}^T}=0.$$

Then, (4.15) implies that, for all $z\in\mathcal{C}^\infty([0,T];\mathcal{C}_0^\infty(\tilde{\Omega}))$ satisfying $z_{|\tilde{\Omega}^T}=0$, we have

$$\begin{aligned}|\mathcal{K}_\rho(P_{-A_1,B_1-A_1,q_1,-}z)|&\leqslant\rho\|F_\rho\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))}(\rho^{-1}\|z\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))})\\&\leqslant C\rho\|F_\rho\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))}\|P_{-A_1,B_1-A_1,q_1,-}z\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))}\end{aligned}.$$

Thus, by the Hahn Banach theorem we can extend $\mathcal{K}_\rho$ to a continuous linear form on $L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))$ still denoted by $\mathcal{K}_\rho$ and satisfying $\|\mathcal{K}_\rho\|\leqslant C\rho\|F_\rho\|_{L^2(0,T;H^{-1}(\mathbb{R}^n))}$. Therefore, there exists $w_{1,\rho}\in L^2(0,T;H^1_\rho(\mathbb{R}^n))$ such that

$$\langle h,w_{1,\rho}\rangle_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n)),L^2(0,T;H^1_\rho(\mathbb{R}^n))}=\mathcal{K}_\rho(h),\quad h\in L^2(0,T;H_\rho^{-1}(\mathbb{R}^n)).$$

Choosing $h=P_{-A_1,B_1-A_1,q_1,-}z$ with $z\in\mathcal{C}_0^\infty(Q)$ proves that $w_{1,\rho}$ satisfies $P_{A_1,B_1,q_1,+}w_{1,\rho}=F_\rho$ in Q. In particular, we deduce that $w_{1,\rho}\in H^1(0,T;H^{-1}(\Omega))\cap L^2(0,T;H^1(\Omega))$. Moreover, fixing $h=P_{-A_1,B_1-A_1,q_1,-}z$ with $z\in\mathcal{C}^\infty([0,T];\mathcal{C}_0^\infty(\tilde{\Omega}))$, $z_{|\tilde{\Omega}^T}=0$ and allowing $z_{|\tilde{\Omega}^0}$ to be arbitrary proves that $w_{1,\rho}=0$ on Ω^0. In addition, applying (4.36), we get

$$\limsup_{\rho\to+\infty}\rho^{-1}\|w_{1,\rho}\|_{L^2(0,T;H^1_\rho(\mathbb{R}^n))}\leqslant\limsup_{\rho\to+\infty}\rho^{-1}\|\mathcal{K}_\rho\|\leqslant C\limsup_{\rho\to+\infty}\|F_\rho\|_{L^2(0,T;H_\rho^{-1}(\mathbb{R}^n))}=0.$$

Therefore, $w_{1,\rho}$ fulfills (4.35), $w_{1,\rho}(\cdot,0)=0$ and (4.33). This completes the proof of the proposition. □

5. Recovery from the DN map

In this section we will prove Theorem 1.1. For this purpose, applying Proposition 4.3 and 4.4, we fix a solution $u_1 \in L^2(0,T;H^1(\Omega))$ of (4.1) of the form (4.3) and a solution $u_2 \in L^2(0,T;H^1(\Omega))$ of (4.2) given by (4.3), with $w_{j,\rho}$, $j=1,2$, satisfying the decay property (4.4)

5.1. **Recovery of the first order coefficient.** According to (1.6) and (2.8), we have

$$\int_Q A\cdot\nabla_x u_1 u_2 dxdt - \int_Q B\cdot\nabla_x(u_1u_2)dxdt + \int_Q q u_1 u_2 dxdt = 0,$$

with $A=A_1-A_2$, $B=B_1-B_2$ and $q=q_1-q_2$. On the other hand, we find

$$\int_Q A\cdot\nabla_x u_1 u_2 dxdt - \int_Q B\cdot\nabla_x(u_1u_2)dxdt + \int_Q q u_1 u_2 dxdt = \rho\int_Q (A\cdot\omega)b_{1,\rho}b_{2,\rho}dxdt + \int_Q Z_\rho(x,t)dxdt \quad (5.1)$$

with

$$Z_\rho = A\cdot\nabla_x b_{1,\rho}(b_{2,\rho}+w_{2,\rho}) + B\cdot\nabla_x[(b_{1,\rho}+w_{1,\rho})(b_{2,\rho}+w_{2,\rho})] + A\cdot\nabla_x w_{1,\rho}(b_{2,\rho}+w_{2,\rho}) + q(b_{1,\rho}+w_{1,\rho})(b_{2,\rho}+w_{2,\rho}).$$

In view of (4.4) and (4.10)-(4.11), we have

$$\lim_{\rho\to+\infty}\rho^{-1}\left|\int_Q Z_\rho(x,t)dxdt\right| = 0. \quad (5.2)$$

Moreover, we deduce that

$$\begin{aligned}\int_Q (A\cdot\omega)b_{1,\rho}b_{2,\rho}dxdt = &\int_{\mathbb{R}}\int_{\mathbb{R}^n}((A-A_\rho)\cdot\omega)b_{1,\rho}b_{2,\rho}dxdt - \int_0^{+\infty}\int_{\mathbb{R}^n}(A_\rho\cdot\omega)e^{-\rho^{\frac{1}{3}}t}b_{2,\rho}dxdt\\ &- \int_{-\infty}^{T}\int_{\mathbb{R}^n}(A_\rho\cdot\omega)b_{1,\rho}e^{-\rho^{\frac{1}{3}}(T-t)}dxdt\\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{-i(t\tau+x\cdot\xi)}A_\rho(x,t)\cdot\omega\exp\left(-\frac{\int_0^{+\infty}A_\rho(x+s\omega,t)\cdot\omega ds}{2}\right)dxdt.\end{aligned}$$

Combining this with (4.6) and applying Lebesgue dominate convergence theorem, we deduce that

$$\limsup_{\rho\to+\infty}\left|\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{-i(t\tau+x\cdot\xi)}A_\rho(x,t)\cdot\omega\exp\left(-\frac{\int_0^{+\infty}A_\rho(x+s\omega,t)\cdot\omega ds}{2}\right)dxdt\right| = \limsup_{\rho\to+\infty}\left|\int_Q(A\cdot\omega)b_{1,\rho}b_{2,\rho}dxdt\right|.$$

In addition, applying (5.1)-(5.2), we obtain

$$\limsup_{\rho\to+\infty}\left|\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{-i(t\tau+x\cdot\xi)}(A_\rho(x,t)\cdot\omega)\exp\left(-\frac{\int_0^{+\infty}A_\rho(x+s\omega,t)\cdot\omega ds}{2}\right)dxdt\right| = 0 \quad (5.3)$$

On the other hand, decomposing $\mathbb{R}^n$ into the direct sum $\mathbb{R}^n=\mathbb{R}\omega\oplus\omega^\perp$ and applying the Fubini's theorem we get

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{-i(t\tau+x\cdot\xi)}(A_\rho(x,t)\cdot\omega)\exp\left(-\frac{\int_0^{+\infty}A_\rho(x+s\omega,t)\cdot\omega ds}{2}\right)dxdt\\ &= \int_{\mathbb{R}}\int_{\omega^\perp}\left[\int_{\mathbb{R}}(A_\rho(y+s_2\omega,t)\cdot\omega)\exp\left(-\frac{\int_{s_2}^{+\infty}A_\rho(y+s_1\omega,t)\cdot\omega ds_1}{2}\right)ds_2\right]e^{-it\tau-i\xi\cdot y}dydt.\end{aligned} \quad (5.4)$$

Moreover, for all $t \in (0,T)$ and all $y \in \omega^\perp$ we have

$$\begin{aligned}\int_{\mathbb{R}} A_\rho(y+s_2\omega,t)\cdot\omega \exp\left(-\frac{\int_{s_2}^{+\infty} A_\rho(y+s_1\omega,t)\cdot\omega ds_1}{2}\right) ds_2 &= 2\int_{\mathbb{R}} \partial_{s_2} \exp\left(-\frac{\int_{s_2}^{+\infty} A_\rho(y+s_1\omega,t)\cdot\omega ds_1}{2}\right) ds_2 \\ &= 2\left(1-\exp\left(-\frac{\int_{\mathbb{R}} A_\rho(y+s_1\omega,t)\cdot\omega ds_1}{2}\right)\right).\end{aligned}$$

Combining this with (5.4), we find

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{-i(t\tau+x\cdot\xi)} A_\rho(x,t)\exp\left(-\frac{\int_0^{+\infty} A_\rho(x+s\omega,t)\cdot\omega ds}{2}\right) dxdt \\ &= 2\int_{\mathbb{R}}\int_{\omega^\perp}\left(1-\exp\left(-\frac{\int_{\mathbb{R}} A_\rho(y+s_1\omega,t)\cdot\omega ds_1}{2}\right)\right) e^{-i\xi\cdot y-it\tau} dydt.\end{aligned} \tag{5.5}$$

Now let us introduce the Fourier transform $\mathcal{F}_{\mathbb{R}\times\omega^\perp}$ on $\mathbb{R}\times\omega^\perp$ defined by

$$\mathcal{F}_{\mathbb{R}\times\omega^\perp} f(\xi,\tau) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}}\int_{\omega^\perp} f(y,t) e^{-it\tau-iy\cdot\xi} dydt, \quad f\in L^1(\omega^\perp\times\mathbb{R}), \quad \tau\in\mathbb{R}, \quad \xi\in\omega^\perp.$$

We fix

$$G_\rho : \omega^\perp\times\mathbb{R} \ni (y,t) \mapsto \left(1-\exp\left(-\frac{\int_{\mathbb{R}} A_\rho(t,y+s_1\omega)\cdot\omega ds_1}{2}\right)\right)$$

and we remark that for

$$R = \sup_{x\in\overline{\Omega}} |x|$$

we have $\mathrm{supp}(G_\rho) \subset \{x\in\omega^\perp :\ |x| \leqslant R+1\}\times[-1,T+1]$. We fix also

$$G : \omega^\perp\times\mathbb{R} \ni (y,t) \mapsto \left(1-\exp\left(-\frac{\int_{\mathbb{R}} A(y+s_1\omega,t)\cdot\omega ds_1}{2}\right)\right).$$

Using this and applying the mean value theorem and (4.7), for a.e $(x',t)\in\omega^\perp\times\mathbb{R}$, we obtain

$$\begin{aligned}&|G_\rho(x',t)-G(x',t)| \\ &\leqslant \exp\left(\left|\int_{\mathbb{R}} A(x'+s_1\omega,t)\cdot\omega ds_1\right| + \left|\int_{\mathbb{R}} A_\rho(x'+s_1\omega,t)\cdot\omega ds\right|\right)\left|\int_{\mathbb{R}} A_\rho(x'+s_1\omega,t)\cdot\omega ds_1 - \int_{\mathbb{R}} A(x'+s_1\omega,t)\cdot\omega ds_1\right| \\ &\leqslant \exp\left[2(R+1)(\|A\|_{L^\infty(\mathbb{R}^{1+n})^n} + \|A_\rho\|_{L^\infty(\mathbb{R}^{1+n})^n})\right]\int_{\mathbb{R}} |A(x'+s_1\omega,t)-A_\rho(x'+s_1\omega,t)| ds_1 \\ &\leqslant C\left(\int_{\mathbb{R}} |A(x'+s_1\omega,t)-A_\rho(x'+s_1\omega,t)| ds_1\right),\end{aligned}$$

with $C>0$ independent of ρ. Thus, integrating this expression with respect to $x'\in\omega^\perp$ and $t\in\mathbb{R}$ and applying the Fubini theorem, we obtain

$$\int_{\mathbb{R}}\int_{\omega^\perp} |G_\rho(x',t)-G(x',t)| dx'dt \leqslant C\int_{\mathbb{R}}\int_{\omega^\perp}\int_{\mathbb{R}} |A(y+s_1\omega,t)-A_\rho(y+s_1\omega,t)| ds_1 dx'dt \leqslant C' \|A-A_\rho\|_{L^1(\mathbb{R}^{1+n})}.$$

Then, applying (4.6) we get

$$\lim_{\rho\to+\infty} \|G-G_\rho\|_{L^1(\mathbb{R}\times\omega^\perp)} = 0.$$

Combining this with (5.3)-(5.5), we find

$$\begin{aligned}2\int_{\mathbb{R}}\int_{\omega^\perp} G(x',t)e^{-it\tau-ix'\cdot\xi}dx'dt &= \lim_{\rho\to+\infty} 2\int_{\mathbb{R}}\int_{\omega^\perp} G_\rho(x',t)e^{-it\tau-ix'\cdot\xi}dx'dt\\ &= \lim_{\rho\to+\infty}\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{-i(t\tau+x\cdot\xi)}A_\rho(x,t)\exp\left(-\frac{\int_0^{+\infty}A_\rho(x+s\omega,t)\cdot\omega ds}{2}\right)dxdt\\ &= 0.\end{aligned}$$

Allowing $\xi\in\omega^\perp$ and $\tau\in\mathbb{R}$ to be arbitrary, we deduce that $\mathcal{F}_{\mathbb{R}\times\omega^\perp}G=0$. Using the injectivity of $\mathcal{F}_{\mathbb{R}\times\omega^\perp}$, for a.e $(x',t)\in\omega^\perp\times\mathbb{R}$, we deduce that

$$\exp\left(-\frac{\int_{\mathbb{R}}A(y+s_1\omega,t)\cdot\omega ds_1}{2}\right)=1$$

and, using the fact that A takes value in $\mathbb{R}^n$, we obtain

$$\int_{\mathbb{R}} A(x'+s_1\omega,t)\cdot\omega ds_1=0. \tag{5.6}$$

We recall that, here ω can be arbitrary chosen.

Now fixing $(\xi,\tau)\in\mathbb{R}^n\times\mathbb{R}$ with $\xi\neq 0$, we deduce from (5.6), that, for $\omega\in\xi^\perp\cap\mathbb{S}^{n-1}$, we have

$$\int_{\mathbb{R}}\int_{\omega^\perp}\int_{\mathbb{R}} A(x'+s_1\omega,t)\cdot\omega e^{-it\tau-ix'\cdot\xi}ds_1dx'dt=0.$$

Applying Fubini theorem and a change of variable, we get

$$\int_{\mathbb{R}^{1+n}} A(x,t)\cdot\omega e^{-it\tau-ix\cdot\xi}dxdt=\int_{\mathbb{R}}\int_{\omega^\perp}\int_{\mathbb{R}} A(x'+s_1\omega,t)\cdot\omega e^{-it\tau-ix'\cdot\xi}ds_1dx'dt=0.$$

This proves that

$$\mathcal{F}(A)(\xi,\tau)\cdot\omega=0,\quad \tau\in\mathbb{R},\ \xi\in\mathbb{R}^n\setminus\{0\},\ \omega\in\xi^\perp\cap\mathbb{S}^{n-1}. \tag{5.7}$$

Let $j,k\in\{1,\dots,n\}$ be such that $j\neq k$ and consider the set $\mathcal{I}_j:=\{\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n:\ \xi_j\neq 0\}$. Let $\xi\in\mathcal{I}_j$, $\tau\in\mathbb{R}$ and let

$$\omega=\frac{\xi_k e_j-\xi_j e_k}{\sqrt{\xi_j^2+\xi_k^2}},$$

with $e_j=(0,\dots,0,\underbrace{1}_{\text{position } j},0,\dots 0)$, $e_k=(0,\dots,0,\underbrace{1}_{\text{position } k},0,\dots 0)$. Then, for $A=(a_1,\dots,a_n)$ we have

$$\mathcal{F}(\partial_{x_k}a_j-\partial_{x_j}a_k)(\xi,\tau)=i\sqrt{\xi_j^2+\xi_k^2}\mathcal{F}(A)(\xi,\tau)\cdot\omega.$$

Thus, condition (5.7) implies that

$$\mathcal{F}(\partial_{x_k}a_j-\partial_{x_j}a_k)(\xi,\tau)=0,\quad \xi\in\mathcal{I}_j,\ \tau\in\mathbb{R}.$$

In the same way, we prove that

$$\mathcal{F}(\partial_{x_k}a_j-\partial_{x_j}a_k)(\xi,\tau)=0,\quad \xi\in\mathcal{I}_k,\ \tau\in\mathbb{R}$$

and it is clear that

$$\mathcal{F}(\partial_{x_k}a_j-\partial_{x_j}a_k)(\xi,\tau)=i(\xi_k\mathcal{F}(a_j)(\xi,\tau)-\xi_j\mathcal{F}(a_k)(\xi,\tau))=0,\quad \xi\in\mathbb{R}^n\setminus(\mathcal{I}_k\cup\mathcal{I}_j),\ \tau\in\mathbb{R}.$$

Therefore, we have $\mathcal{F}(\partial_{x_k}a_j-\partial_{x_j}a_k)=0$ which implies $\partial_{x_k}a_j-\partial_{x_j}a_k=0$ and by the same way that $dA=0$. This proves (1.7).

5.2. **Recovery of the zero order coefficients.** In this subsection we assume that (1.7)-(1.9) are fulfilled. Our goal is to prove that (1.6) implies (1.10). In this subsection, we denote by A, B and q the functions $A_1 - A_2$, $B_1 - B_2$ and $q_1 - q_2$ extended by zeo to $\mathbb{R}^{1+n}$. We start, with the following intermediate result.

Lemma 5.1. *Let $\mathcal{A} \in L^\infty(\mathbb{R}^{1+n})^n$ be compactly supported and assume that $d\mathcal{A} = 0$ in the sense of distributions taking value in 2-forms. Then, for*

$$\varphi(x,t) := -\int_0^1 \frac{\mathcal{A}(sx,t)\cdot x}{2} ds, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{5.8}$$

we have $\varphi \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\mathbb{R}^n_x))$ and $\nabla_x \varphi = -\frac{\mathcal{A}}{2}$.

We refer to [40, Lemma 4.2] for the proof of this result. From now on we fix $\varphi \in L^\infty_{loc}(\mathbb{R}^{n+1})$ given by (5.8), with $\mathcal{A} = A$, and applying Lemma 5.1 we deduce that $\nabla_x \varphi = -\frac{A}{2}$ and $\varphi \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\mathbb{R}^n_x))$. Moreover, since $A \in W^{1,\infty}(0,T; L^{p_1}(\Omega))^n$, by the Sobolev embedding theorem, we deduce that for any open bounded set $\tilde{\Omega} \subset \mathbb{R}^n$ we have

$$\varphi \in W^{1,\infty}(0,T; W^{1,p_1}(\tilde{\Omega})) \subset W^{1,\infty}(0,T; L^\infty(\tilde{\Omega})).$$

Thus, we have $\varphi \in L^\infty(0,T; W^{1,\infty}(\mathbb{R}^n)) \cap W^{1,\infty}(0,T; L^\infty_{loc}(\mathbb{R}^n))$. Since $\mathbb{R}^n \setminus \Omega$ is connected and $A = 0$ on $[\mathbb{R}^n \setminus \Omega] \times (0,T)$, there exists a function $h \in W^{1,\infty}(0,T)$ such that

$$\varphi(x,t) = h(t), \quad (x,t) \in (\mathbb{R}^n \setminus \Omega) \times (0,T).$$

Therefore, by replacing $\varphi(x,t)$ with $\varphi(x,t) - h(t)$, we may assume without lost of generality that $\varphi = 0$ on $(\mathbb{R}^n \setminus \Omega) \times (0,T)$. In particular, we have $\varphi_{|\Sigma} = 0$. Therefore, we can apply the gauge invariance of the DN map to get

$$\Lambda_{A_1,B_1,q_1} = \Lambda_{A_1+2\nabla_x\varphi, B_1+\nabla_x\varphi, q_1-\partial_t\varphi-|\nabla_x\varphi|^2-A_1\cdot\nabla_x\varphi} = \Lambda_{A_2,B_1+\nabla_x\varphi,q_1-\partial_t\varphi+\Delta_x\varphi-|\nabla_x\varphi|^2-A_1\cdot\nabla_x\varphi}.$$

Then, condition (1.6) implies that

$$\Lambda_{A_2,B_1+\nabla_x\varphi,q_1-\partial_t\varphi-|\nabla_x\varphi|^2-A_1\cdot\nabla_x\varphi} = \Lambda_{A_2,B_2,q_2}. \tag{5.9}$$

We will prove that this condition implies

$$\nabla_x \cdot B_2 + q_2 = \nabla_x \cdot (B_1 + \nabla_x\varphi) + q_1 - \partial_t\varphi - |\nabla_x\varphi|^2 - A_1 \cdot \nabla_x\varphi. \tag{5.10}$$

For this purpose we fix a solution $u_1 \in L^2(0,T; H^1(\Omega))$ of (4.1), with (A_1, B_1, q_1) replaced by $(A_2, B_1 + \nabla_x\varphi, q_1 - \partial_t\varphi - |\nabla_x\varphi|^2 - A_1 \cdot \nabla_x\varphi)$, of the form (4.3) and a solution $u_2 \in L^2(0,T; H^1(\Omega))$ of (4.2) given by (4.3), with $w_{j,\rho}$, $j = 1,2$, satisfying the decay property (4.33)-(4.34). In light of (2.8), we have

$$\int_Q (B_1 + \nabla_x\varphi - B_2)\cdot\nabla_x(u_1u_2)dxdt + \int_Q (q_1 - \partial_t\varphi - |\nabla_x\varphi|^2 - A_1\cdot\nabla_x\varphi - q_2)u_1u_2dxdt = 0. \tag{5.11}$$

For the first term on left hand side of (5.11), applying (1.8)-(1.9) and the Green formula, we get

$$\begin{aligned}
&\int_Q (B_1 + \nabla_x\varphi - B_2)\cdot\nabla_x(u_1u_2)dxdt \\
&= \int_Q (B_1 - \frac{A}{2} - B_2)\cdot\nabla_x(u_1u_2)dxdt \\
&= -\int_Q \nabla_x\cdot(B_1 - \frac{A}{2} - B_2)u_1u_2dxdt + \left\langle [(B_1 - B_2)\cdot\nu - \frac{(A_1 - A_2)\cdot\nu}{2}]u_1, u_2 \right\rangle_{L^2(0,T;H^{-\frac{1}{2}}(\partial\Omega)), L^2(0,T;H^{\frac{1}{2}}(\partial\Omega))} \\
&= -\int_Q \nabla_x\cdot(B_1 - \frac{A}{2} - B_2)b_{1,\rho}b_{2,\rho}dxdt - \int_Q Z_\rho dxdt \\
&= \int_Q (B_1 + \nabla_x\varphi - B_2)\cdot\nabla_x(b_{1,\rho}b_{2,\rho})dxdt - \int_Q Z_\rho dxdt,
\end{aligned}$$

with $Z_\rho = \nabla_x \cdot (B - \frac{A}{2})(b_{1,\rho} w_{2,\rho} + b_{2,\rho} w_{1,\rho} + w_{1,\rho} w_{2,\rho})$. In view of (4.33), it is clear that

$$\lim_{\rho \to +\infty} \int_Q Z_\rho dx dt = 0.$$

Moreover, one can easily check that

$$\begin{aligned} &\int_Q (B_1 + \nabla_x \varphi - B_2) \cdot \nabla_x (b_{1,\rho} b_{2,\rho}) dx dt \\ &= -i \left[\int_Q \left(1 - e^{-\rho^{\frac{1}{3}} t} \right) \left(1 - e^{-\rho^{\frac{1}{3}} (T-t)} \right) e^{-i(t\tau + x \cdot \xi)} (B_1 + \nabla_x \varphi - B_2)(x,t) dx dt \right] \cdot \xi. \end{aligned}$$

Sending $\rho \to +\infty$ and applying the Lebesgue dominate convergence theorem, we find

$$\begin{aligned} \lim_{\rho \to +\infty} \int_Q (B + \nabla_x \varphi) \cdot \nabla_x (b_{1,\rho} b_{2,\rho}) dx dt &= (2\pi)^{\frac{n+1}{2}} [\mathcal{F}(B + \nabla_x \varphi)(\xi, \tau)] \cdot (-i\xi) \\ &= (2\pi)^{\frac{n+1}{2}} \mathcal{F}[\nabla_x \cdot (B + \nabla_x \varphi)](\xi, \tau) \end{aligned}$$

Therefore, we have

$$\lim_{\rho \to +\infty} \int_Q (B + \nabla_x \varphi) \cdot \nabla_x (u_1 u_2) dx dt = (2\pi)^{\frac{n+1}{2}} \mathcal{F}[\nabla_x \cdot (B + \nabla_x \varphi)](\xi, \tau).$$

In the same way, we can prove that

$$\lim_{\rho \to +\infty} \int_Q (q_1 - \partial_t \varphi - |\nabla_x \varphi|^2 - A_1 \cdot \nabla_x \varphi - q_2) u_1 u_2 dx dt = (2\pi)^{\frac{n+1}{2}} \mathcal{F}[(q - \partial_t \varphi - |\nabla_x \varphi|^2 - A_1 \cdot \nabla_x \varphi)](\xi, \tau).$$

Combining this with (5.11), we obtain

$$\mathcal{F}[\nabla_x \cdot (B + \nabla_x \varphi) + q - \partial_t \varphi - |\nabla_x \varphi|^2 - A_1 \cdot \nabla_x \varphi)](\xi, \tau) = 0, \quad (\xi, \tau) \in \mathbb{R}^{1+n}.$$

This proves (5.10) and the proof of (1.10) is completed.

5.3. **Proof of Corollary 1.1.** This subsection is devoted to the proof of Corollary 1.1. For this purpose, we assume that $\Lambda_{A_1,B,q} = \Lambda_{A_2,B,q}$. Then Theorem 1.1 implies that there exists $\varphi \in W^{1,\infty}(Q))$ such that

$$\begin{cases} A_2 = A_1 + 2\nabla_x \varphi, & \text{in } Q, \\ \nabla_x \cdot B + q = \nabla_x \cdot (B + \nabla_x \varphi) + q - \partial_t \varphi - |\nabla_x \varphi|^2 - A_1 \cdot \nabla_x \varphi, & \text{in } Q, \\ \varphi = 0, & \text{on } \Sigma. \end{cases}$$

Thus, fixing $A_3 = A_1 + \nabla_x \varphi \in L^\infty(Q)$, we deduce that φ satisfies

$$\begin{cases} \partial_t \varphi - \Delta_x \varphi + A_3 \cdot \nabla_x \varphi = 0, & \text{in } Q, \\ \varphi = 0, & \text{on } \Sigma. \end{cases}$$

Note that since $\varphi \in L^\infty(0,T; W^{1,\infty}(\Omega))$ and $\Delta \varphi = \frac{\nabla_x \cdot (A_2 - A_1)}{2} \in L^\infty(Q)$ we can define $\partial_\nu \varphi$ as an element of $L^\infty(0,T; H^{-\frac{1}{2}}(\partial\Omega))$. Moreover, the conditions $\varphi_{|\Sigma} = 0$ and $(A_2 - A_1) \cdot \nu_{|\Sigma} = 0$ imply that $\varphi_{|\Sigma} = \partial_\nu \varphi_{|\Sigma} = 0$. Thus, fixing $\mathcal{O}$ a set with not empty interior such that $\tilde{\Omega} = \mathcal{O} \cup \Omega$ is an open bounded connected set of $\mathbb{R}^n$ with Lipschitz boundary, we can see that φ extended by zero to $\tilde{\Omega} \times (0,T)$ solves

$$\begin{cases} \partial_t \varphi - \Delta_x \varphi + A_3 \cdot \nabla_x \varphi = 0, & \text{in } \tilde{\Omega} \times (0,T), \\ \varphi = 0, & \text{on } \mathcal{O} \times (0,T), \end{cases}$$

with A_3 extended by zero to $\tilde{\Omega} \times (0,T)$. Then the unique continuation properties for parabolic equations (e.g. [56, Theorem 1.1]) implies that $\varphi = 0$. Note that such results of unique continuation are stated for solutions of parabolic equations lying $H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega))$, but since they follow from Carleman estimates like [56, Theorem 1.2], they can be extended to solutions lying in $H^1(Q)$ and we can apply this result to φ. This proves that $A_1 = A_2$ and the proof of Corollary 1.1 is completed.

5.4. **Proof of Corollary 1.2.** In this section we will prove Corollary 1.2. For this purpose, we first recall that $\tilde{\Lambda}_{A_j} = \Lambda_{A_j, \frac{A_j}{2}, \frac{\nabla_x \cdot (A_j)}{2}}$, $j = 1, 2$. Therefore, Theorem 1.1 implies that there exists $\varphi \in W^{1,\infty}(Q)$ such that

$$\begin{cases} A_2 = A_1 + 2\nabla_x \varphi, & \text{in } Q, \\ \nabla_x \cdot (A_2) = \nabla_x \cdot (A_1) - \partial_t \varphi + \Delta_x \varphi - |\nabla_x \varphi|^2 - A_1 \cdot \nabla_x \varphi, & \text{in } Q, \\ \varphi = 0, & \text{on } \Sigma. \end{cases}$$

Then, fixing $A_3 = -A_1 - \nabla_x \varphi \in L^\infty(Q)$ and applying the fact that $(A_2 - A_1) \cdot \nu_{|\Sigma} = 0$, we deduce that φ satisfies

$$\begin{cases} -\partial_t \varphi - \Delta_x \varphi + A_3 \cdot \nabla_x \varphi = 0, & \text{in } Q, \\ \varphi = \partial_\nu \varphi = 0, & \text{on } \Sigma. \end{cases}$$

Therefore, applying again the unique continuation properties for parabolic equations we deduce that $\varphi = 0$ and the proof of Corollary 1.2 is completed.

5.5. **Proof of Corollary 1.3.** In this subsection we will show Corollary 1.3. Let us first consider the following intermediate result.

Lemma 5.2. *Let Ω be a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary. Then, for every $F \in L^2(Q)$, the problem*

$$\begin{cases} -\partial_t v - \Delta_x v = F, & \text{in } Q, \\ v(\cdot, T) = 0, & \text{in } \Omega, \\ v = 0, & \text{on } \Sigma. \end{cases} \tag{5.12}$$

admits a unique solution $v \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$.

Proof. This result is classical but we prove it for sake of completeness. Applying [51, Theorem 4.1, Chapter 3] we know that (5.12) admits a unique solution $v \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$. So the proof of the lemma will be completed if we show that $v \in H^1(0, T; L^2(\Omega))$. Let $(\lambda_n)_{n \geqslant 1}$ be the non-decreasing sequence of eigenvalues for the operator $H = -\Delta$ with Dirichlet boundary condition and $(\varphi_n)_{n \geqslant 1}$ an associated orthonormal basis of eigenfunctions. We will prove that actually $v \in L^2(0, T; D(H))$ which will complete the proof of the lemma. We fix $v_n(t) = \langle v(\cdot, t), \varphi_n \rangle_{L^2(\Omega)}$, $F_n(t) = \langle F(\cdot, t), \varphi_n \rangle_{L^2(\Omega)}$ and we remark that v_n solves

$$\begin{cases} -\partial_t v_n + \lambda_n v = F_n, \\ v_n(T) = 0. \end{cases}$$

Therefore, we have

$$v_n(t) = \int_0^{T-t} e^{-\lambda_n (T-t-s)} F_n(s) ds = (e^{-\lambda_n t} \mathbb{1}_{(0,+\infty)}) * (F_n \mathbb{1}_{(0,T)})(T-t),$$

with $*$ the convolution product and, for any interval I, $\mathbb{1}_I$ the characteristic function of I. An application of Young inequality yields

$$\|v_n\|_{L^2(0,T)} \leqslant \left(\int_0^\infty e^{-\lambda_n t} dt \right) \|F_n\|_{L^2(0,T)} \leqslant \frac{\|F_n\|_{L^2(0,T)}}{\lambda_n}.$$

Thus, we have

$$\begin{aligned} \int_0^T \left(\sum_{n=1}^\infty \lambda_n^2 |v_n(t)|^2 \right) dt &= \sum_{n=1}^\infty \left(\int_0^T \lambda_n^2 |v_n(t)|^2 \right) dt \\ &\leqslant \sum_{n=1}^\infty \left(\int_0^T |F_n(t)|^2 \right) dt = \int_0^T \left(\sum_{n=1}^\infty |F_n(t)|^2 \right) dt = \|F\|^2_{L^2(Q)}. \end{aligned}$$

This proves that $v = \sum_{n \in \mathbb{N}} v_n \varphi_n \in L^2(0, T; D(H))$ and using the fact that $\partial_t v = Hv - F$ we deduce that $v \in H^1(0, T; L^2(\Omega))$. This completes the proof of the lemma. □

Let us observe that for Ω a $C^{1,1}$ bounded domain, by the elliptic regularity, the result of Lemma 5.2 would correspond to existence of a strong solution $v \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ of (5.12). However, we do not want to assume such regularity for $\partial\Omega$.

From now on, we assume that the conditions of Corollary 1.3 are fulfilled and, for $A, B \in L^\infty(Q)^n$ satisfying $\nabla_x \cdot A, \nabla_x \cdot B \in L^\infty(Q)$ and $q \in L^\infty(0,T;L^{p_1}(\Omega))$, we consider the following spaces

$$S_{+,A,B,q} := \{u \in L^2(0,T;H^1(\Omega)) : \ \partial_t u - \Delta u + A \cdot \nabla_x u + \nabla_x \cdot Bu + qu = 0, \ u_{|\Omega^0} = 0\},$$

$$S_{-,A,B,q} := \{u \in L^2(0,T;H^1(\Omega)) : \ -\partial_t u - \Delta u - A \cdot \nabla_x u + (q + \nabla_x \cdot (B - A))u = 0, \ u_{|\Omega^T} = 0\},$$

$$S_{+,A,B,q,\gamma_1} := \{u \in S_{+,A,B,q} : \ \mathrm{supp}(u_{|\Sigma}) \subset [0,T] \times \gamma_1\},$$

$$S_{-,A,B,q,\gamma_2} := \{u \in S_{-,A,B,q} : \ \mathrm{supp}(u_{|\Sigma}) \subset [0,T] \times \gamma_2\}.$$

Fixing $Q_1 := (\Omega \setminus \overline{\Omega_*}) \times (0,T)$, we can consider the following density result.

Lemma 5.3. *Assume that $\nabla_x \cdot (B), \nabla_x \cdot (A) \in L^\infty(0,T;L^{p_1}(\Omega))$. Then the space S_{+,A,B,q,γ_1} (resp. S_{-,A,B,q,γ_2}) is dense in the space $S_{+,A,B,q}$ (resp. $S_{-,A,B,q}$) with respect to the norm $L^2(Q_1)$.*

Proof. Since the proof of these two results are similar, we prove only the density of S_{+,A,B,q,γ_1} in $S_{+,A,B,q}$. For this purpose, we assume the contrary. Then, an application of Hahn Banach theorem implies that there exist $h \in L^2(Q_1)$ and $u_0 \in S_{+,A,B,q}$ such that

$$\int_{Q_1} h u dx dt = 0, \quad u \in S_{+,A,B,q,\gamma_1}, \tag{5.13}$$

$$\int_{Q_1} h u_0 dx dt = 1. \tag{5.14}$$

Now let us extend h by zero to $h \in L^2(Q)$. According to [51, Theorem 4.1, Chapter 3] there exists a unique solution $w \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ to the IBVP

$$\begin{cases} -\partial_t w - \Delta_x w - A \cdot \nabla_x w + (q + \nabla_x \cdot (B - A))w = h, & \text{in } Q, \\ w(\cdot, T) = 0, & \text{in } \Omega, \\ w = 0, & \text{on } \Sigma. \end{cases}$$

Moreover, fixing $F = A \cdot \nabla_x w - (q + \nabla_x \cdot (B - A))w + h \in L^2(Q)$, we deduce that w solves

$$\begin{cases} -\partial_t w - \Delta_x w = F, & \text{in } Q, \\ w(\cdot, T) = 0, & \text{in } \Omega, \\ w = 0, & \text{on } \Sigma \end{cases}$$

and from Lemma 5.2, we deduce that $w \in H^1(Q)$. In particular, we have $\Delta w \in L^2(0,T;L^2(\Omega))$ which implies that $\partial_\nu w \in L^2(0,T;H^{-\frac{1}{2}}(\partial\Omega))$. In view of (5.13), choosing $u \in S_{+,A,B,q,\gamma_1}$, we get

$$\begin{aligned}
&\langle \partial_\nu w, u \rangle_{L^2(0,T;H^{-\frac{1}{2}}(\partial\Omega)),L^2(0,T;H^{\frac{1}{2}}(\partial\Omega))} \\
&= \int_Q \Delta w u dx dt + \int_Q \nabla_x w \cdot \nabla_x u dx dt + \int_\Sigma (A \cdot \nu) u w d\sigma(x) dt \\
&= \langle \partial_t u, w \rangle_{L^2(0,T;H^{-1}(\Omega)),L^2(0,T;H_0^1(\Omega))} + \int_Q \nabla_x w \cdot \nabla_x u dx dt - \int_Q [-\partial_t w - \Delta w] u dx dt + \int_Q \nabla_x \cdot (uwA) dx dt \\
&= \langle \partial_t u - \Delta_x u + A \cdot \nabla_x u + (\nabla_x \cdot B + q)u, w \rangle_{L^2(0,T;H^{-1}(\Omega)),L^2(0,T;H_0^1(\Omega))} \\
&\quad - \int_Q u[-\partial_t w - \Delta_x w - A \cdot \nabla_x w + (q + \nabla_x (B - A))w] u dx dt \\
&= -\int_{Q_1} h u dx dt = 0.
\end{aligned}$$

Allowing $u \in S_{+,A,B,q,\gamma_1}$ to be arbitrary, we deduce that $\partial_\nu w_{|\gamma_1\times(0,T)} = 0$. Thus, fixing Ω_1 a set with nonempty interior such that $\Omega_1 \cap \partial\Omega \subset \gamma_1$ and $\Omega_2 = \Omega_* \cup \Omega_1$ is a connected open set of $\mathbb{R}^n$, we have

$$\begin{cases} -\partial_t w - \Delta_x w - A\cdot\nabla_x w + (q - \mathrm{div}_x A)w = 0, & \text{in } \Omega_2\times(0,T),\\ w = 0, & \text{on } \Omega_1\times(0,T). \end{cases}$$

Then the unique continuation properties for parabolic equations implies that $w_{|\Omega_2\times(0,T)} = 0$ which implies that $w_{|\Omega_*\times(0,T)} = 0$. Note that here we consider an application of unique continuation to solutions of parabolic equations lying in $H^1(Q)$ and with a zero order coefficient $(q + \nabla_x\cdot(B-A)) \in L^\infty(0,T;L^{p_1}(\Omega_2))$. For this purpose one needs to extend by density Carleman estimates like [56, Theorem 1.2] to such solutions and use Sobolev embedding theorem in order to absorb the multiplication by $(q + \nabla_x\cdot(B-A))$ which corresponds to a bounded operator from $L^2(0,T;H^1(\Omega_2))$ to $L^2(\Omega_2\times(0,T))$. In particular, we have

$$w_{|\partial\Omega_*\times(0,T)} = \partial_\nu w_{|\partial\Omega_*\times(0,T)} = 0$$

and it follows that

$$w_{|\partial(\Omega\setminus\Omega_*)\times(0,T)} = \partial_\nu w_{|\partial(\Omega\setminus\Omega_*)\times(0,T)} = 0.$$

Therefore, we have

$$\int_{Q_1} \Delta w u_0 dxdt + \int_{Q_1} \nabla_x w\cdot\nabla_x u_0 dxdt = 0,$$

$$\int_{Q_1} \nabla_x w\cdot\nabla_x u_0 dxdt = -\left\langle \Delta u_0, w\right\rangle_{L^2(0,T;H^{-1}(\Omega\setminus\Omega_*),L^2(0,T;H_0^1(\Omega\setminus\Omega_*)}\,,$$

$$-\int_{Q_1} \partial_t w u_0 dxdt = \left\langle \partial_t u_0, w\right\rangle_{L^2(0,T;H^{-1}(\Omega\setminus\Omega_*),L^2(0,T;H_0^1(\Omega\setminus\Omega_*)}\,.$$

Thus, we find

$$\begin{aligned} &\int_{Q_1} u_0[-\partial_t w - \Delta_x w - A\cdot\nabla_x w + (q+\nabla_x\cdot(B-A))w]dxdt\\ &= \int_{Q_1} u_0[-\partial_t w - \Delta_x w - A\cdot\nabla_x w + (q+\nabla_x\cdot(B-A))w]dxdt\\ &\quad - \int_{Q_1}[\partial_t u_0 - \Delta_x u_0 + A\cdot\nabla_x u_0 + (q+\nabla_x\cdot(B))u_0]w dxdt\\ &= 0 \end{aligned}$$

According to this last formula, we have

$$\int_{Q_1} u_0 h dxdt = \int_{Q_1} u_0[-\partial_t w - \Delta_x w - A\cdot\nabla_x w + (q+\nabla_x\cdot(B-A))w]dxdt = 0$$

which contradicts (5.14). This proves the required density result. □

Armed with this lemma we are now in position to complete the proof of Corollary 1.3.

Proof of Corollary 1.3. Using arguments similar to those used for the derivation of (2.8), we can prove that, for any $u_1 \in S_{+,A_1,B_1,q_1,\gamma_1}$ and $u_2 \in S_{-,A_2,B_2,q_2,\gamma_2}$, we have

$$\begin{aligned} &\langle(\Lambda_{A_1,B_1,q_1,\gamma_1,\gamma_2} - \Lambda_{A_2,B_2,q_2,\gamma_1,\gamma_2})g_+, g_-\rangle_{\mathcal{H}_-^*,\mathcal{H}_-}\\ &= \int_Q (A_1-A_2)\cdot\nabla_x u_1 u_2 dxdt - \int_Q (B_1-B_2)\cdot\nabla_x(u_1u_2)dxdt + \int_Q (q_1-q_2)u_1u_2dxdt. \end{aligned}$$

with $g_+ = u_1$ and $g_- = u_2$ on Σ. Then, (1.13) implies that, for any $u_1 \in S_{+,A_1,B_1,q_1,\gamma_1}$, $u_2 \in S_{-,A_2,B_2,q_2,\gamma_2}$, we get

$$\int_Q (A_1-A_2)\cdot\nabla_x u_1 u_2 dxdt - \int_Q (B_1-B_2)\cdot\nabla_x(u_1u_2)dxdt + \int_Q (q_1-q_2)u_1u_2dxdt = 0. \tag{5.15}$$

In view of (1.12), we can rewrite (5.15) as

$$\int_{Q_1}(A_1-A_2)\cdot\nabla_x u_1u_2dxdt-\int_{Q_1}(B_1-B_2)\cdot\nabla_x(u_1u_2)dxdt+\int_{Q_1}(q_1-q_2)u_1u_2dxdt=0.$$

Then, using (5.15) and integrating by parts in $x\in\Omega\setminus\Omega_*$, for any $u_1\in S_{+,A_1,B_1,q_1,\gamma_1}$, $u_2\in S_{-,A_2,B_2,q_2,\gamma_2}$, we find

$$\int_{Q_1}(A_1-A_2)\cdot\nabla_x u_1u_2dxdt+\int_{Q_1}\nabla_x\cdot(B_1-B_2)u_1u_2dxdt+\int_{Q_1}(q_1-q_2)u_1u_2dxdt=0. \tag{5.16}$$

Applying the density result of Lemma 5.3, we deduce that (5.16) holds true for any $u_1\in S_{+,A_1,B_1,q_1,\gamma_1}$, $u_2\in S_{-,A_2,B_2,q_2}$. Then, using (5.15) and integrating by parts in $x\in\Omega\setminus\Omega_*$, for any $u_1\in S_{+,A_1,B_1,q_1,\gamma_1}$, $u_2\in S_{-,A_2,B_2,q_2}$, we obtain

$$-\int_{Q_1}\nabla_x\cdot\left[(A_1-A_2)u_2\right]u_1dxdt+\int_{Q_1}\nabla_x\cdot(B_1-B_2)u_1u_2dxdt+\int_{Q_1}(q_1-q_2)u_1u_2dxdt=0. \tag{5.17}$$

Applying again Lemma 5.3, we deduce that (5.17) holds for any $u_1\in S_{+,A_1,B_1,q_1}$, $u_2\in S_{-,A_2,B_2,q_2}$. Integrating again by parts, we deduce that (5.15) holds for any $u_1\in S_{+,A_1,B_1,q_1}$, $u_2\in S_{-,A_2,B_2,q_2}$. Finally, allowing $u_1\in S_{+,A_1,B_1,q_1}$, $u_2\in S_{-,A_2,B_2,q_2}$ to correspond to the exponentially growing and decaying GO solutions used in Theorem 1.1, we can complete the proof of the corollary. □

6. Application to the recovery of nonlinear terms

In this section Ω is of class $\mathcal{C}^{2+\alpha}$ and we denote by Σ_p the parts of ∂Q given by $\Sigma_p=\Sigma\cup(\Omega\times\{0\})$. Consider the quasilinear IBVP (1.14). Following [50], we start by fixing the condition for the well posedness of this problem. We consider, functions $F\in C^1(\overline{Q}\times\mathbb{R}\times\mathbb{R}^n)$ satisfying the following conditions:

There exist three non-negative constants c_0, c_1 and c_2 so that

$$uF(x,t,u,v)\geqslant -c_0|v|^2-c_1|u|^2-c_2,\ \ (x,t,u,v)\in\overline{Q}\times\mathbb{R}\times\mathbb{R}^n. \tag{6.1}$$

$$F(x,0,u,v)=0,\ \ (x,u,v)\in\partial\Omega\times\mathbb{R}\times\mathbb{R}^n. \tag{6.2}$$

Moreover, we assume that for $|u|\leqslant M_1$ and $(x,t)\in\overline{Q}$ there exists a constant $c_3(M_1)>0$, depending only on T, Ω and M_1, such that

$$|F(x,t,u,v)|\leqslant c_3(M_1)(1+|v|)^2. \tag{6.3}$$

Here $M_1\mapsto c_3(M_1)$ is assumed to be monotonically increasing.

Now, for $G\in C^{2+\alpha,1+\alpha/2}(\overline{Q})$ consider the compatibility condition

$$\partial_tG(x,0)=\Delta G(x,0),\quad x\in\partial\Omega. \tag{6.4}$$

We consider the set $\mathcal{X}=\{G_{|\overline{\Sigma}_p};\ \text{for some } G\in C^{2+\alpha,1+\alpha/2}(\overline{Q}) \text{ such that (6.4) is fulfilled}\}$ with the norm

$$\|G\|_{\mathcal{X}}:=\|G_{|\Sigma}\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Sigma})}+\|G_{|\Omega\times\{0\}}\|_{C^{2+\alpha}(\overline{\Omega})}.$$

According to [50, Theorem 6.1, pp. 452], for any $G\in\mathcal{X}$ and for any $F\in C^1(\overline{Q}\times\mathbb{R}\times\mathbb{R}^n)$ satisfying (6.1)-(6.3), problem (1.14) admits a unique solution $u_{F,G}\in C^{2+\alpha,1+\alpha/2}(\overline{Q})$. Moreover, according to [50, Theorem 2.2, pp. 429], [50, Theorem 4.1, pp. 443] and [50, Theorem 5.4, pp. 448], for any $r>0$ and for any $G\in\mathcal{X}$ satisfying

$$\|G\|_{\mathcal{X}}\leqslant r,$$

there exists a constant M_r, depending on Ω, T, c_0, c_1, c_2, c_3 and r such that

$$\|u_{F,G}\|_{C^{\alpha,\alpha/2}(\overline{Q})}+\|\nabla_xu_{F,G}\|_{C^{\alpha,\alpha/2}(\overline{Q})}\leqslant M_r. \tag{6.5}$$

We associate to (1.14) the DN map

$$\mathcal{N}_F:\mathcal{X}\ni G\mapsto\partial_\nu u_{F,G_{|\Sigma}}\in L^2(\Sigma).$$

Since (1.14) is not linear, clearly $\mathcal{N}_F$ is also nonlinear. Therefore, in a similar way to [16, 31, 32, 33], we will start by linearizing this operator by considering the Fréchet derivative of $\mathcal{N}_F$.

6.1. **Linearization procedure.** We fix $F \in \mathcal{C}^1(\overline{Q} \times \mathbb{R}^n \times \mathbb{R})$ satisfying (6.1)-(6.3) such that $\partial_u F \in \mathcal{C}^2(\overline{Q} \times \mathbb{R}^n \times \mathbb{R}; \mathbb{R})$ and $\partial_v F \in \mathcal{C}^2(\overline{Q} \times \mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$. Then, for $H \in \mathcal{X}$, we consider the IBVP

$$\begin{cases} (\partial_t w - \Delta w + A_{F,G}(x,t) \cdot \nabla_x w + q_{F,G}(x,t) w = 0 & \text{in } Q, \\ w = H & \text{on } \Sigma_p, \end{cases} \tag{6.6}$$

with

$$A_{F,G}(x,t) := \partial_v F(x,t,u_{F,G}(x,t), \nabla_x u_{F,G}(x,t)), \quad (x,t) \in Q,$$
$$q_{F,G}(x,t) := \partial_u F(x,t,u_{F,G}(x,t), \nabla_x u_{F,G}(x,t)), \quad (x,t) \in Q.$$

In light of [50, Theorem 5.4, pp. 322] the IBVP (6.6) has a unique solution $w = w_{F,G,H} \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$ satisfying

$$\|w_{F,G,H}\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q})} \leq C \|H\|_{\mathcal{X}}$$

for some constant C depending only on Q, F and G. From now on, for $X = \Omega$ or $X = \partial\Omega$ and $r, s > 0$ we consider the Sobolev spaces

$$H^{r,s}(X \times (0,T)) = H^s(0,T; L^2(X)) \cap L^2(0,T; H^r(X)).$$

Using solutions of (6.6), we will consider the linearization of $\mathcal{N}_F$ in the following way.

Proposition 6.1. *$F \in C^1(\overline{Q} \times \mathbb{R}^n \times \mathbb{R})$ satisfying (6.1)-(6.3) such that $\partial_u F \in \mathcal{C}^1(\overline{Q} \times \mathbb{R}^n \times \mathbb{R}; \mathbb{R})$ and $\partial_v F \in \mathcal{C}^1(\overline{Q} \times \mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$. Then, N_F is Fréchet continuously differentiable and*

$$N_F'(G)H = \partial_\nu w_{F,G,H}, \ \ G, H \in \mathcal{X}.$$

Proof. Since $u \in H^{2,1}(Q) \to \partial_\nu u \in L^2(\Sigma)$ is a bounded linear operator, we only need to show that

$$\mathcal{M}_F : G \in \mathcal{X} \to u_{F,G} \in H^{2,1}(Q)$$

is differentiable with $\mathcal{M}_F'(G)(H) = w_{F,G,H}$, $G, H \in \mathcal{X}$. For this purpose, we fix $G, H \in \mathcal{X}$ with $\|H\|_{\mathcal{X}} \leqslant 1$ and we consider

$$z = u_{F,G+H} - u_{F,G} - w_{F,G,H} \in C^{2+\alpha,1+\alpha/2}(\overline{Q})$$

and set

$$\begin{aligned}
&A(x,t) = \partial_v F(x,t,u_{F,G}(x,t), \nabla_x u_{F,G}(x,t)), \\
&q(x,t) = \partial_u F(x,t,u_{F,G}(x,t), \nabla_x u_{F,G}(x,t)), \\
&A_1(x,t) = \int_0^1 (1-\tau) \partial_v^2 F(x,t,u_{F,G}(x,t), \nabla_x u_{F,G}(x,t) + \tau(\nabla_x u_{F,G+H} - \nabla_x u_{F,G})(x,t)) d\tau, \\
&A_2(x,t) = \int_0^1 (1-\tau) \partial_v \partial_u F(x,t,u_{F,G}(x,t), \nabla_x u_{F,G}(x,t) + \tau(\nabla_x u_{F,G+H} - \nabla_x u_{F,G})(x,t)) d\tau, \\
&q_1(x,t) = \int_0^1 (1-\tau) \partial_u^2 F(x,t,u_{F,G}(x,t) + \tau(u_{F,G+H} - u_{F,G})(x,t), \nabla_x u_{F,G+H}(x,t)) d\tau
\end{aligned}$$

Applying Taylor's formula, we get

$$\begin{aligned}
&F(x,t,u_{F,G}(x,t), \nabla_x u_{F,G+H}(x,t)) - F(x,t,u_{F,G}(x,t), \nabla_x u_{F,G}(x,t)) \\
&= A(x,t) \cdot (\nabla_x u_{F,G+H}(x,t) - \nabla_x u_{F,G}(x,t)) \\
&\quad + A_1(x,t)(\nabla_x u_{F,G+H}(x,t) - \nabla_x u_{F,G}(x,t), \nabla_x u_{F,G+H}(x,t) - \nabla_x u_{F,G}(x,t)).
\end{aligned}$$

$$\begin{aligned}
&F(x,t,u_{F,G+H}(x,t), \nabla_x u_{F,G+H}(x,t)) - F(x,t,u_{F,G}(x,t), \nabla_x u_{F,G+H}(x,t)) \\
&= q(x,t)(u_{F,G+H}(x,t) - u_{F,G}(x,t)) + q_1(x,t)(u_{F,G+H}(x,t) - u_{F,G}(x,t))^2 \\
&\quad + A_2(x,t)(u_{F,G+H}(x,t) - u_{F,G}(x,t))(\nabla_x u_{F,G+H}(x,t) - \nabla_x u_{F,G}(x,t)).
\end{aligned}$$

Thus, fixing

$$\begin{aligned}&K_H(x,t)\\&= q_1(x,t)(u_{F,G+H}(x,t)-u_{F,G}(x,t))^2 + A_2(x,t)(u_{F,G+H}(x,t)-u_{F,G}(x,t))(\nabla_x u_{F,G+H}(x,t)-\nabla_x u_{F,G}(x,t))\\&\quad + A_1(x,t)(\nabla_x u_{F,G+H}(x,t)-\nabla_x u_{F,G}(x,t),\nabla_x u_{F,G+H}(x,t)-\nabla_x u_{F,G}(x,t))\end{aligned}$$

we deduce that z is the solution of the IBVP

$$\begin{cases} \partial_t z - \Delta_x z + A\cdot\nabla_x z + qz = K_H & \text{in } Q,\\ z=0 & \text{on } \Sigma_p.\end{cases}$$

Combining this with (6.5), [51, Theorem 4.1, Chapter 3], [52, Theorem 3.2, Chapter 4] and the fact that $\|H\|_{\mathcal{X}}\leqslant 1$, we deduce that this last problem admits a unique solution $z\in H^{2,1}(Q)$ satisfying

$$\|z\|_{H^{2,1}(Q)}\leqslant C\|K_H\|_{L^2(Q)}\leqslant C\|K_H\|_{L^\infty(Q)} \tag{6.7}$$

with C depending on Ω, T, c_0, c_1, c_2, c_3 and $\|G\|_{\mathcal{X}}$. Moreover, applying again (6.5), we obtain

$$\|K_H\|_{L^\infty(Q)}\leqslant C\left(\|\nabla_x u_{F,G+H}-\nabla_x u_{F,G}\|_{L^\infty(Q)}+\|u_{F,G+H}-u_{F,G}\|_{L^\infty(Q)}\right)^2,$$

with C depending on Ω, T, c_0, c_1, c_2, c_3 and $\|G\|_{\mathcal{X}}$. Combining this with (6.7), we get

$$\|z\|_{H^{2,1}(Q)}\leqslant C\left(\|\nabla_x u_{F,G+H}-\nabla_x u_{F,G}\|_{L^\infty(Q)}+\|u_{F,G+H}-u_{F,G}\|_{L^\infty(Q)}\right)^2 \tag{6.8}$$

with C depending on Ω, T, c_0, c_1, c_2, c_3 and $\|G\|_{\mathcal{X}}$. On the other hand, fixing $y=u_{F,G+H}-u_{F,G}$, one can check that y solves

$$\begin{cases} \partial_t y - \Delta_x y + \tilde{A}(x,t)\cdot\nabla_x y + \tilde{q}(x,t)y = 0 & \text{in } Q,\\ y=H & \text{on } \Sigma_p,\end{cases}$$

with

$$\tilde{A}(x,t)=\int_0^1 \partial_v F(x,t,u_{F,G+H}(x,t),\nabla_x u_{F,G}(x,t)+\tau(\nabla_x u_{F,G+h}(x,t)-\nabla_x u_{F,G}(x,t)))d\tau,$$

$$\tilde{q}(x,t)=\int_0^1 \partial_u F(x,t,u_{F,G}(x,t)+\tau(u_{F,G+h}(x,t)-u_{F,G}(x,t)),\nabla_x u_{F,G}(x,t))d\tau.$$

Applying again (6.5), we deduce that

$$\left\|\tilde{A}\right\|_{C^{\alpha,\alpha/2}(\overline{Q})}+\|\tilde{q}\|_{C^{\alpha,\alpha/2}(\overline{Q})}\leqslant C$$

with C depending on Ω, T, c_0, c_1, c_2, c_3 and $\|G\|_{\mathcal{X}}$. Therefore, applying [50, Theorem 5.3, pp. 320-321] we obtain

$$\|\nabla_x y\|_{L^\infty(Q)}+\|y\|_{L^\infty(Q)}\leqslant C\|H\|_{\mathcal{X}}, \tag{6.9}$$

with C depending on Ω, T, c_0, c_1, c_2, c_3 and $\|G\|_{\mathcal{X}}$. Combining (6.7)-(6.9), we find

$$\|u_{F,G+H}-u_{F,G}-w_{F,G,H}\|_{H^{2,1}(Q)}\leqslant C\|H\|_{\mathcal{X}}^2.$$

From this last estimate one can easily check that $\mathcal{M}_F$ is differentiable at G and $M_F'(G)(H)=w_{F,G,H}$, $H\in\mathcal{X}$. To complete the proof of the proposition, we only need to check the continuity of $\mathcal{X}\ni G\to\mathcal{M}_F'(G)\in\mathscr{B}(\mathcal{X},H^{2,1}(Q))$. For this purpose, we fix $G,K,H\in\mathcal{X}$, we consider $S:=w_{F,G+K,H}-w_{F,G,H}$, with $\|K\|_{\mathcal{X}}\leqslant 1$, and we observe that S solves

$$\begin{cases} \partial_t S - \Delta_x S + \tilde{A}_1(x,t)\cdot\nabla_x S + \tilde{q}_1(x,t)S = R_K & \text{in } Q,\\ S=0 & \text{on } \Sigma_p,\end{cases}$$

where

$$\begin{aligned}\tilde{A}_1(x,t)&:=\partial_v F(x,t,u_{F,G+K}(x,t),\nabla_x u_{F,G+K}(x,t)),\quad (x,t)\in Q,\\ \tilde{q}_1(x,t)&:=\partial_u F(x,t,u_{F,G+K}(x,t),\nabla_x u_{F,G+K}(x,t)),\quad (x,t)\in Q,\\ R_K&:=A_3(\nabla_x u_{F,G}-\nabla_x u_{F,G+k},\nabla_x w_{F,G,H})+q_3(u_{F,G}-u_{F,G+K})w_{F,G,H},\end{aligned}$$

with

$$A_3(x,t)=\int_0^1 \partial_v^2 F(x,t,u_{F,G+K}(x,t),\nabla_x u_{F,G+K}(x,t)+\tau(\nabla_x u_{F,G+K}(x,t)-\nabla_x u_{F,G}(x,t)))d\tau,$$

$$q_3(x,t)=\int_0^1 \partial_u^2 F(x,t,u_{F,G+K}(x,t)+\tau(u_{F,G+K}(x,t)-u_{F,G}(x,t)),\nabla_x u_{F,G+K}(x,t))d\tau.$$

Repeating the above arguments, we find

$$\|S\|_{H^{2,1}(Q)}\leqslant C\,\|R_K\|_{L^2(Q)}\leqslant C\,\|K\|_{\mathcal{X}}\,,$$

with C depending on Ω, T, c_0, c_1, c_2, c_3 and $\|G\|_{\mathcal{X}}+\|H\|_{\mathcal{X}}$. This proves the continuity of $G\mapsto \mathcal{M}_F'(G)\in \mathscr{B}(\mathcal{X},H^{2,1}(Q))$ and it completes the proof of the proposition. □

We will apply this property of the DN map $\mathcal{N}_F$ in order to complete the proof of Theorem 1.3, 1.2 and Corollary 1.4, 1.5.

6.2. **Proof of Theorem 1.3 and Corollary 1.5.** This subsection is devoted to the proof of Theorem 1.3, 1.2 and Corollary 1.4, 1.5. We start by considering the following intermediate result.

Lemma 6.1. *Let $G\in\{K_{|\Sigma_p}:\ K\in\mathcal{C}^\infty(\overline{Q}),\ \partial_t K=0,\ \nabla_x K$ is constant$\}$ and assume that*

$$\partial_x^\ell F(x,0,u,v)=0,\quad x\in\partial\Omega,\ u\in\mathbb{R},\ v\in\mathbb{R}^n,\ \ell\in\mathbb{N}^n,\ |\ell|\leqslant 2. \tag{6.10}$$

Then the problem (1.14) *admits a unique solution $u_{F,G}\in\mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q})$ satisfying $\partial_t u_{F,G}\in\mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q})$.*

Proof. Let $u_{F,G}\in\mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q})$ be the solution of (1.14). We start by fixing $z=\partial_t u_{F,G}$,

$$A(x,t):=\partial_v F(x,t,u_{F,G}(x,t),\nabla_x u_{F,G}(x,t)),\quad q(x,t):=\partial_u F(x,t,u_{F,G}(x,t),\nabla_x u_{F,G}(x,t))$$

and Z defined on Σ_p by

$$Z(x,0):=\Delta_x G(x,0)-F(x,0,G(x,0),\nabla_x G(x,0))=-F(x,0,G(x,0),\nabla_x G(x,0)),\quad x\in\Omega,$$
$$Z(x,t)=\partial_t G(x,t):=0,\quad (x,t)\in\Sigma.$$

Applying (6.10), one can check that $Z\in\mathcal{X}$, and z solves the IBVP

$$\begin{cases}\partial_t z-\Delta_x z+A(x,t)\cdot\nabla_x z+q(x,t)z=R(x,t), & \text{in } Q,\\ z=Z, & \text{on } \Sigma_p,\end{cases} \tag{6.11}$$

with $R:(x,t)\mapsto-\partial_t F(x,t,u_{F,G}(x,t),\nabla_x u_{F,G}(x,t))\in\mathcal{C}^{\alpha,\frac{\alpha}{2}}(\overline{Q})$. Using the fact that $A,q,R\in\mathcal{C}^{\alpha,\frac{\alpha}{2}}(\overline{Q})$, we deduce from [50, Theorem 5.3, pp. 320-321] that $\partial_t u_{F,G}\in\mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q})$. □

Armed with this lemma we will complete the proof of Theorem 1.2 and 1.3.

Proof of Theorem 1.2. For $j=1,2$, $a\in\mathbb{R}$, $v\in\mathbb{R}^n$, $(x,t)\in Q$, we fix

$$A_{j,a,v}(x,t):=\partial_v F_j(x,t,u_{F_j,h_{a,v}}(x,t),\nabla_x u_{F_j,h_{a,v}}(x,t)),\quad q_{j,a,v}(x,t):=\partial_u F_j(x,t,u_{F_j,h_{a,v}}(x,t),\nabla_x u_{F_j,h_{a,v}}(x,t))$$

and $B_{1,a,v}=0$. According to (1.16) and Proposition 6.1, we have

$$\Lambda_{A_{1,v},B_{1,v},q_{1,v}}H_{|\Sigma}=\Lambda_{A_{2,v},B_{1,v},q_{2,v}}H_{|\Sigma},\quad a\in\mathbb{R},\ v\in\mathbb{R}^n,\ H\in\mathcal{X}_0.$$

Combining this with the density of $\{H_{|\Sigma}:\ H\in\mathcal{X}_0\}$ in $\mathcal{H}_+$, we obtain

$$\Lambda_{A_{1,a,v},B_{1,a,v},q_{1,a,v}}=\Lambda_{A_{2,a,v},B_{1,a,v},q_{2,a,v}},\quad a\in\mathbb{R},\ v\in\mathbb{R}^n \tag{6.12}$$

and, in view of (1.15) and Lemma 6.1, we have

$$A_{j,a,v}\in\mathcal{C}^1(\overline{Q}),\quad q_{j,a,v}\in L^\infty(Q),\quad j=1,2,\ a\in\mathbb{R},\ v\in\mathbb{R}^n.$$

Note that, due to (1.15), the fact that $\partial_t h_{a,v}=0$ and the fact that $\nabla_x h_{a,v}=v$, here we are actually in position to apply Lemma 6.1. Moreover, according to [33, Lemma 8.2], (6.12) implies

$$A_{1,a,v}(x,t)=A_{2,a,v}(x,t),\quad (x,t)\in\Sigma,\ a\in\mathbb{R},v\in\mathbb{R}^n. \tag{6.13}$$

Therefore, Theorem 1.1 implies that, for $A_{a,v}=A_{1,a,v}-A_{2,a,v}$ extended by zero to $\mathbb{R}^n\times(0,T)$ and for

$$\varphi_{a,v}(x,t):=-\int_0^1\frac{A_{a,v}(sx,t)\cdot x}{2}ds,\quad (x,t)\in\mathbb{R}^n\times\mathbb{R},\ (a,v)\in\mathbb{R}\times\mathbb{R}^n,$$

we have

$$A_{2,a,v}(x,t)=A_{1,a,v}(x,t)+2\nabla_x\varphi_{a,v}(x,t),\quad (x,t)\in\mathbb{R}^n\times(0,T). \tag{6.14}$$

In view of (6.13), the fact that $F_j\in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q};\mathcal{C}^3(\mathbb{R}\times\mathbb{R}^n))$ and the definition of $A_{j,a,v}$, $j=1,2$, one can easily check that $(x,t,a,v)\mapsto A_{a,v}\in\mathcal{C}^1(\overline{Q};\mathcal{C}(\mathbb{R}\times\mathbb{R}^n))$. Therefore, we have

$$\varphi:(x,t,a,v)\mapsto\varphi_{a,v}(x,t)\in\mathcal{C}^1([0,T];\mathcal{C}(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n))\cap\mathcal{C}^2(\overline{\Omega};\mathcal{C}([0,T]\times\mathbb{R}\times\mathbb{R}^n)).$$

In a similar way to the end of the proof of Theorem 1.1, by eventually subtracting to φ a function y depending only on $(t,a,v)\in(0,T)\times\mathbb{R}\times\mathbb{R}^n$, we may assume that

$$\varphi(x,t,a,v)=0,\quad (x,t)\in\Sigma,\ (a,v)\in\mathbb{R}\times\mathbb{R}^n. \tag{6.15}$$

Therefore, we can apply the gauge invariance of the DN map $\Lambda_{A_{1,a,v},B_{1,a,v},q_{1,a,v}}$ to get

$$\Lambda_{A_{1,a,v},B_{1,a,v},q_{1,a,v}}=\Lambda_{A_{2,a,v},B_{1,a,v}+\nabla_x\varphi(\cdot,a,v),q_{1,a,v}+[-\partial_t\varphi(a,v,+\Delta_x\varphi-|\nabla_x\varphi|^2-A_{1,a,v}\cdot\nabla_x\varphi](\cdot,a,v)}.$$

Combining this with (6.12), we get

$$\Lambda_{A_{2,a,v},B_{1,a,v},q_{2,a,v}}=\Lambda_{A_{2,a,v},B_{1,a,v}+\nabla_x\varphi(\cdot,a,v),q_{1,a,v}+[-\partial_t\varphi(a,v,+\Delta_x\varphi-|\nabla_x\varphi|^2-A_{1,a,v}\cdot\nabla_x\varphi](\cdot,a,v)}.$$

Using (6.13) and repeating the arguments used at the end of the proof of Theorem 1.1 we deduce that, for all $(a,v)\in\mathbb{R}\times\mathbb{R}^n$, we have

$$\begin{cases} A_{2,a,v}(x,t)=A_{1,a,v}(x,t)+2\nabla_x\varphi(x,t,a,v), & (x,t)\in Q,\\ q_{2,a,v}(x,t)=q_{1,a,v}(x,t)+[-\partial_t\varphi(a,v,+\Delta_x\varphi-|\nabla_x\varphi|^2-A_{1,a,v}\cdot\nabla_x\varphi](x,t,a,v), & (x,t)\in Q,\\ \varphi(x,t,a,v)=0, & (x,t)\in\Sigma. \end{cases}$$

Sending $t\to 0$ in the first two above equality, for all $(a,v)\in\mathbb{R}\times\mathbb{R}^n$, we obtain

$$\begin{cases} \partial_v(F_2-F_1)(x,0,x\cdot v+a,v)=2\partial_x\varphi(x,0,a,v), & x\in\Omega,\\ \partial_u(F_2-F_1)(x,0,x\cdot v+a,v)=-(\partial_t\varphi-|\nabla_x\varphi|^2-\partial_vF_1(x,0,x\cdot v+a,v)\partial_x\varphi)(x,0,a,v), & x\in\Omega,\\ \varphi(x,t,a,v)=0, & (x,t)\in\Sigma. \end{cases}$$

Finally, fixing $a=u-x\cdot v$ in the two first above equalities, we obtain (1.17). This completes the proof of the theorem. □

Proof of Theorem 1.3. For $j=1,2$, $v\in\mathbb{R}^n$, $(x,t)\in Q$, we fix

$$A_{j,v}(x,t):=\partial_vF_j(x,t,u_{F_j,k_v}(x,t),\nabla_xu_{F_j,k_v}(x,t)),\quad q_{j,v}(x,t):=\partial_uF_j(x,t,u_{F_j,k_v}(x,t),\nabla_xu_{F_j,k_v}(x,t))$$

and $B_{j,v}=0$. In a similar way to Theorem 1.2, applying (1.23) and Proposition 6.1, we obtain

$$\Lambda_{A_{1,v},B_{1,v},q_{1,v}}=\Lambda_{A_{2,v},B_{1,v},q_{2,v}},\quad v\in\mathbb{R}^n \tag{6.16}$$

and, in view of (1.15) and Lemma 6.1, we have

$$A_{j,v}\in W^{1,\infty}(Q),\quad q_{j,v}\in L^\infty(Q),\quad j=1,2,\ v\in\mathbb{R}^n.$$

Note that, due to (1.15), the fact that $\partial_tk_v=0$ and the fact that $\nabla_xk_v=v$, here we are actually in position to apply Lemma 6.1. Moreover, according to [33, Lemma 8.2], (6.16) implies

$$A_{1,v}(x,t)=A_{2,v}(x,t),\quad (x,t)\in\Sigma,\ v\in\mathbb{R}^n.$$

In addition, from (1.24)-(1.25), we deduce $(x,t,u,v)\mapsto\partial_uF_j(x,t,u,v)$, $j=1,2$, is a function independent of u and v with

$$\partial_uF_1(x,t,0,0)=\partial_uF_1(x,t,u,v)=\partial_uF_2(x,t,u,v)=\partial_uF_2(x,t,0,0),\quad (x,t)\in Q,\ u\in\mathbb{R},\ v\in\mathbb{R}^n.$$

It follows

$$\begin{aligned} q_{1,v}(x,t) &= \partial_u F_1(x,t,u_{F_1,k_v}(x,t),\nabla_x u_{F_1,k_v}(x,t)) \\ &= \partial_u F_1(x,t,0,0) \\ &= \partial_u F_2(x,t,0,0) = \partial_u F_2(x,t,u_{F_1,k_v}(x,t),\nabla_x u_{F_1,k_v}(x,t)) = q_{2,v}(x,t), \quad (x,t)\in Q,\ v\in\mathbb{R}^n. \end{aligned} \tag{6.17}$$

Thus, applying Corollary 1.1, we deduce that

$$A_{1,v}(x,t) = A_{2,v}(x,t), \quad (x,t)\in Q,\ v\in\mathbb{R}^n.$$

Sending $t\to 0$ in this formula, we obtain

$$F_1(x,0,x\cdot v,v) = F_2(x,0,x\cdot v,v), \quad x\in\Omega,\ v\in\mathbb{R}^n. \tag{6.18}$$

On the other hand, according to (1.24)-(1.25) we have

$$F_j(x,t,u,v) = F_j(x,t,0,v) + \partial_u F_j(x,t,0,v)u = F_j(x,t,0,v) + \partial_u F_1(x,t,0,v)u$$

and (6.18) clearly implies (1.19). Assuming that (1.20) is fulfilled, we can easily deduce (1.21) from (1.19). This completes the proof of the theorem. □

Now let us consider Corollary 1.4 and 1.5 which follow from Theorem 1.2 and 1.3.

Proof of Corollary 1.4. Let condition (1.16) be fulfilled. Then Theorem 1.2 implies that there exists $\varphi : Q\times\mathbb{R}\times\mathbb{R}^n \ni (x,t,u,v)\mapsto \varphi(x,t,u,v)\in \mathcal{C}^1([0,T];\mathcal{C}(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n))\cap\mathcal{C}^2(\overline{\Omega};\mathcal{C}([0,T]\times\mathbb{R}\times\mathbb{R}^n))$ such that, for all $(u,v)\in\mathbb{R}\times\mathbb{R}^n$, conditions (1.17) are fulfilled. Note that, for all $x\in\Omega$, $(u,v)\in\mathbb{R}\times\mathbb{R}^n$, we have

$$2\Delta_x\varphi(x,0,u-x\cdot v,v) = \nabla_x\cdot[2\partial_x\varphi(x,0,u-x\cdot v,v)] + 2\partial_u\partial_x\varphi(x,0,u-x\cdot v,v)v.$$

Then, (1.17) implies

$$2\Delta_x\varphi(x,0,u-x\cdot v,v) = \sum_{j=1}^n \left[\partial_{x_j}\partial_{v_j}(F_2-F_1)(x,0,u,v) + \partial_u\partial_{v_j}(F_2-F_1)(x,0,u,v)v_j\right].$$

Applying (1.18), we get

$$\Delta_x\varphi(x,0,u-x\cdot v,v) = 0, \quad x\in\Omega,\ (u,v)\in\mathbb{R}\times\mathbb{R}^n$$

and, replacing u by $u+x\cdot v$ and applying (1.17), we find

$$\begin{cases} \Delta_x\varphi(x,0,u,v) = 0, & x\in\Omega,\ (u,v)\in\mathbb{R}\times\mathbb{R}^n \\ \varphi(x,0,u,v) = 0, & x\in\partial\Omega,\ (u,v)\in\mathbb{R}\times\mathbb{R}^n. \end{cases}$$

From the uniqueness of this boundary value problem, we obtain

$$\varphi(x,0,u,v) = 0, \quad x\in\Omega,\ (u,v)\in\mathbb{R}\times\mathbb{R}^n,$$

which, combined with (1.17), imply (1.19). In addition, assuming that (1.20) is fulfilled, we can easily deduce (1.21) from (1.19). □

Proof of Corollary 1.5. In a similar way to Theorem 1.3, for $j=1,2$, $v\in\mathbb{R}^n$, $(x,t)\in Q$, we fix

$$A_{j,v}(x,t) := \partial_v F_j(x,t,u_{F_j,k_v}(x,t),\nabla_x u_{F_j,k_v}(x,t)), \quad q_{j,v}(x,t) := \partial_u F_j(x,t,u_{F_j,k_v}(x,t),\nabla_x u_{F_j,k_v}(x,t)).$$

Applying (1.24) we deduce (6.17) and from (1.27) we obtain that

$$A_{1,v}(x,t) = 0 = A_{2,v}(x,t), \quad (x,t)\in\Omega_*\times(0,T),\ v\in\mathbb{R}^n.$$

Combining this with Corollary 1.3, we deduce that there exists $\varphi_v\in L^\infty(0,T;W^{2,\infty}(\Omega))\cap W^{1,\infty}(0,T;L^\infty(\Omega))$ such that

$$\begin{cases} A_{2,v} = A_{1,v} + 2\nabla_x\varphi_v, & \text{in } Q, \\ q_{2,v} = q_{1,v} - \partial_t\varphi_v + \Delta_x\varphi_v - |\nabla_x\varphi_v|^2 - A_{1,v}\cdot\nabla_x\varphi_v, & \text{in } Q, \\ \varphi_v = \partial_\nu\varphi_v = 0, & \text{on } \Sigma. \end{cases}$$

Then, using (6.17), we deduce that φ_v satisfies

$$\begin{cases} \partial_t \varphi_v - \Delta_x \varphi_v + A_{3,v} \cdot \nabla_x \varphi_v = 0, & \text{in } Q, \\ \varphi_v = \partial_\nu \varphi_v = 0, & \text{on } \Sigma, \end{cases}$$

with $A_{3,v} = A_{1,v} + \nabla_x \varphi_v \in L^\infty(Q)$. Thus, from the unique continuation properties for parabolic equations, we deduce that $\varphi_v = 0$ and

$$A_{1,v}(x,t) = A_{2,v}(x,t), \quad (x,t) \in Q, \ v \in \mathbb{R}^n.$$

Then in a similar way to the end of the proof of Theorem 1.3 we deduce (1.19). □

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