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SPECTRAL ANALYSIS OF THE LAPLACIAN ACTING ON DISCRETE CUSPS AND FUNNELS

NASSIM ATHMOUNI, MARWA ENNACEUR, AND SYLVAIN GOLÉRIA

ABSTRACT. We study perturbations of the discrete Laplacian associated to discrete analogs of cusps and funnels. We perturb the metric and the potential in a long-range way. We establish a propagation estimate and a Limiting Absorption Principle away from the possible embedded eigenvalues. The approach is based on a positive commutator technique.

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1. INTRODUCTION

The spectral theory of discrete Laplacians on graphs has drawn a lot of attention for decades as they are discrete analogs of manifolds. We are especially interested in the nature of the essential spectrum. Without trying to be exhaustive, using positive commutator techniques, [Sa, BoSa] treat the case of \mathbb{Z}^d , [AlFr, GeGo] study the case of binary trees, [MäRiT] investigate some general graphs, and [PaRi] focused on a periodic setting. Some other techniques have been used successfully, e.g., [HiNo] with some geometric approach and [BrKe].

In the context of some manifolds of finite volume, [MoTr, GoMo] prove that the essential spectrum of the (continuous) Laplacian becomes empty under the presence of a magnetic field with compact support. Besides, they establish some

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Weyl asymptotic. Analogously, for some discrete cusps, [GoTr] classify magnetic potentials that lead to the absence of the essential spectrum and compute a kind of Weyl asymptotic for the magnetic discrete Laplacian. Back to [GoMo], one also obtains a refined analysis of the spectral measure (propagation estimate, limiting absorption principle) for long-range perturbation of the metric when the essential spectrum occurs relying on a positive commutator technique. We refer to [GoMo] for further comments and references therein. This part of the analysis was not carried out in [GoTr]. This is the main aim of this article.

To start off, we recall some standard definitions of graph theory. A (non-oriented) *graph* is a triple $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$, where \mathcal{V} is a finite or countable set (the *vertices*), $\mathcal{E} : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}_+$ is symmetric, and $m : \mathcal{V} \rightarrow (0, \infty)$ is a weight. We say that \mathcal{G} is *simple* if $m = 1$ and $\mathcal{E} : \mathcal{V} \times \mathcal{V} \rightarrow \{0, 1\}$.

Given $x, y \in \mathcal{V}$, we say that (x, y) is an *edge* and that x and y are *neighbors* if $\mathcal{E}(x, y) > 0$. Note that in this case, since \mathcal{E} is symmetric, (y, x) is also an edge and y and x are neighbors. We denote this relationship by $x \sim y$ and the set of neighbors of x by $\mathcal{N}_{\mathcal{G}}(x)$. The space of complex-valued functions acting on the set of vertices \mathcal{V} is denoted by $C(\mathcal{V}) := \{f : \mathcal{V} \rightarrow \mathbb{C}\}$. Moreover, $C_c(\mathcal{V})$ is the subspace of $C(\mathcal{V})$ of functions with finite support. We consider the Hilbert space

$$\ell^2(\mathcal{V}, m) := \left\{ f \in C(\mathcal{V}), \sum_{x \in \mathcal{V}} m(x) |f(x)|^2 < \infty \right\}$$

endowed with the scalar product, $\langle f, g \rangle := \sum_{x \in \mathcal{V}} m(x) \overline{f(x)} g(x)$. We define the Laplacian operator

$$(1.1) \quad \Delta_{\mathcal{G}} f(x) := \frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) (f(x) - f(y)),$$

for all $f \in C_c(\mathcal{V})$. $\Delta_{\mathcal{G}}$ is a positive operator since we have $\langle f, \Delta_{\mathcal{G}} f \rangle_{\ell^2(\mathcal{V}, m)} = Q_{\mathcal{G}}(f)$, with

$$Q_{\mathcal{G}}(f) := \frac{1}{2} \sum_{x, y \in \mathcal{V}} \mathcal{E}(x, y) |f(x) - f(y)|^2,$$

for all $f \in C_c(\mathcal{V})$. To simplify, we denote its Friedrichs' extension with the same symbol. We define the *degree* of $x \in \mathcal{V}$ by

$$\deg_{\mathcal{G}}(x) := \frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y)$$

We present a simple version of our model: We consider $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$, where $\mathcal{V}_1 := \mathbb{Z}$, $m_1(n) := e^{-n}$, and $\mathcal{E}(n, n+1) := e^{-(2n+1)/2}$, for all $n \in \mathbb{N}$ and $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$ a connected finite graph such that $|\mathcal{V}_2| = p$, $p \geq 3$, where $|\mathcal{V}_2|$ is the cardinal of the set \mathcal{V}_2 with m_2 constant. Let $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$ be the *twisted cartesian product* $\mathcal{G}_1 \times_{\tau} \mathcal{G}_2$ given by

$$\begin{cases} m(x, y) := m_1(x) \times m_2(y), \\ \mathcal{E}((x, y), (x', y')) := \mathcal{E}_1(x, x') \times \delta_{y, y'} + \delta_{x, x'} \times \mathcal{E}_2(y, y'), \end{cases}$$

for all $x, x' \in \mathcal{V}_1$ and $y, y' \in \mathcal{V}_2$. If $n > 0$, this is a cups side and if $n < 0$, this is a funnel side. We refer to Section 3.1 for more details.

The (twisted cartesian) Laplacian $\Delta_{\mathcal{G}}$ is essentially self-adjoint on $C_c(\mathcal{V})$, see Proposition 3.14. Moreover, it has no singularly continuous spectrum and

$$\sigma_{\text{ac}}(\Delta_{\mathcal{G}}) = \left[\frac{\alpha}{m_2}, \frac{\beta}{m_2} \right],$$

with

$$(1.2) \quad \alpha := e^{1/2} + e^{-1/2} - 2 \quad \text{and} \quad \beta := e^{1/2} + e^{-1/2} + 2.$$

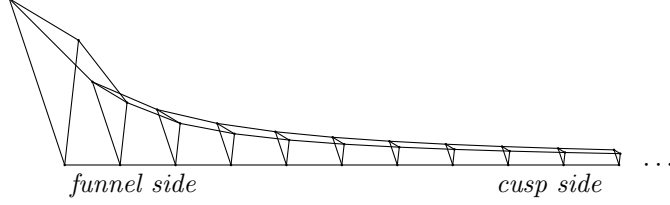


FIGURE 1. Representation of a discrete cusp and funnel side

We turn into perturbation theory. First, we perturb the weights, we consider $\mathcal{G}' := (\mathcal{E}', \mathcal{V}, m')$, where

$$m'(x) := (1 + \mu(x))m(x) \text{ and } \mathcal{E}'(x, y) := (1 + \varepsilon(x, y))\mathcal{E}(x, y),$$

$$(H_0) \quad \begin{cases} \max_{x_2 \in \mathcal{V}_2} |V((x_1, x_2))| \rightarrow 0, & \text{if } |x_1| \rightarrow \infty, \\ \max_{x_2 \in \mathcal{V}_2} |\mu((x_1, x_2))| \rightarrow 0, & \text{if } |x_1| \rightarrow \infty, \\ \max_{x_2 \in \mathcal{V}_2, y \sim (x_1, x_2)} |\varepsilon((x_1, x_2), y)| \rightarrow 0, & \text{if } |x_1| \rightarrow \infty. \end{cases}$$

This ensures that $\Delta_{\mathcal{G}'} + V(\cdot)$ is also essentially self-adjoint on $\mathcal{C}_c(\mathcal{V})$. Here $V(\cdot)$ denotes the operator of multiplication by V . Moreover, (H_0) guarantees the stability of the essential spectrum, see Proposition 4.2. Namely,

$$\sigma_{\text{ess}}(\Delta_{\mathcal{G}'}) = \left[\frac{\alpha}{m_2}, \frac{\beta}{m_2} \right].$$

In order to obtain the absence of singularly continuous spectrum for $\Delta_{\mathcal{G}'}$, we require some additional decay. Let $\epsilon > 0$ and ask:

$$(H_1) \quad \sup_{n \in \mathbb{Z}, y \in \mathcal{V}_2} \langle n \rangle^{1+\epsilon} |V(n-1, y) - V(n, y)| < \infty,$$

$$(H_2) \quad \sup_{n \in \mathbb{Z}, y \in \mathcal{V}_2} \langle n \rangle^{1+\epsilon} |\mu(n-1, y) - \mu(n, y)| < \infty$$

$$(H_3) \quad \sup_{n \in \mathbb{Z}, k \in \mathcal{V}_2} \langle n \rangle^{1+\epsilon} |\varepsilon((n, k), (n+1, k)) - \varepsilon((n-1, k), (n, k))| < \infty,$$

where $\langle \cdot \rangle := \sqrt{1 + |\cdot|^2}$.

Our main result is the following:

Theorem 1.1. *Let $H := \Delta_{\mathcal{G}'} + V(\cdot)$ as above. Suppose that (H_0) holds true. Then, we have the following assertions:*

$$(1) \quad \sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(\Delta_{\mathcal{G}}).$$

Assume furthermore that (H_1) , (H_2) , and (H_3) hold true. Set $\kappa(H) := \sigma_p(H) \cup \{\alpha/m_2, \beta/m_2\}$ with α, β are given in (1.2) and where σ_p denotes the pure point spectrum. Take $s > 1/2$ and $[a, b] \subset \mathbb{R} \setminus \kappa(H)$. We obtain:

(2) *The eigenvalues of H distinct from α/m_2 and β/m_2 are of finite multiplicity and can accumulate only toward α/m_2 and β/m_2 .*

(3) *The singular continuous spectrum of H is empty.*

(4) *The following limit exists and finite:*

$$\lim_{\rho \rightarrow 0} \sup_{\lambda \in [a, b]} \|\langle \Lambda \rangle^{-s} (H - \lambda - i\rho)^{-1} \langle \Lambda \rangle^{-s}\| < \infty,$$

(5) *There exists $c > 0$ such that for all $f \in \ell^2(\mathcal{V}, m')$, we have:*

$$\int_{\mathbb{R}} \|\langle \Lambda \rangle^{-s} e^{-itH} E_{[a, b]}(H) f\|_{\ell^2(\mathcal{V}, m')}^2 dt \leq c \|f\|_{\ell^2(\mathcal{V}, m')}^2.$$

Our approach is based on a positive commutator technique, namely we establish a Mourre estimate. The proof of this theorem is given in Subsection 4.2. We refer to Section 2 for historical references and for an introduction on the subject.

We now describe the structure of the paper. In Section 2, we present the Mourre's theory. The next section is devoted to study the free model. In Subsection 3.1, we present the context and introduce the notion of cusp and funnel. In Subsection 3.2, we start with the Mourre estimate on \mathbb{N} . In Subsections 3.3, and 3.4, we prove the Mourre estimate for the unperturbed Laplacian that acts on a funnel and on a cusp, respectively. Then, in Subsection 3.6, we conclude the Mourre estimate for the whole graph. In Section 4, we perturb the metrics and add a potential. The proofs are more involved than in Section 3.1 as we rely on the optimal class $\mathcal{C}^{1,1}(\mathcal{A})$ of the Mourre theory. This yields the main result.

Notation: We denote by \mathbb{N} the set of non-negative integers. In particular $0 \in \mathbb{N}$. Set $[[a, b]] := [a, b] \cap \mathbb{Z}$. We denote by $\mathbf{1}_X$ the indicator of the set X .

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2. THE MOURRE THEORY

In [Pu], C.R. Putnam used a positive commutator estimate to insure that the spectrum of an operator is purely absolutely continuous. His method was unfortunately not very flexible and did not allow the presence of eigenvalue. In [Mo1, Mo2], E. Mourre had the idea to localise in energy the positive commutator estimate. Thanks to some hypothesis of regularity, he proved that the embedded eigenvalues can accumulated only at some thresholds, that the singularly continuous spectrum is empty and also established a limiting absorption principle, away from the eigenvalues and from the thresholds. Many papers have shown the power of Mourre's commutator theory for a wide class of self-adjoint operators, e.g., [BaFrSi, BoCaHäMi, CaGrHu, DeJa, FrHe, GeGéMø, GeGo, HuSi, JeMoPe, Sa]. We refer to [AmBoGe] for the optimised theory and to [GoJe1, GoJe2, Gé] for recent developments.

Let us now, briefly recall Mourre's commutator theory. The aim is to establish some spectral properties of a given (unbounded) self-adjoint operator H acting in some complex and separable Hilbert space \mathcal{H} with the help of an external unbounded and self-adjoint operator \mathcal{A} . Let $\|\cdot\|$ denote the norm of bounded operators on \mathcal{H} and $\sigma(H)$ the spectrum of H . Recall that the latter is real. We endow $\mathcal{D}(H)$, the domain of H , with its graph norm. We denote by $R(z) := (H - z)^{-1}$ the resolvent of H in z . Take an other Hilbert space \mathcal{K} such that there is a dense and injective embedding from \mathcal{K} to \mathcal{H} , by identifying \mathcal{H} with its antidual \mathcal{H}^* , we have: $\mathcal{K} \hookrightarrow \mathcal{H} \simeq \mathcal{H}^* \hookrightarrow \mathcal{K}^*$, with dense and injective embeddings.

We introduce some regularity classes with respect to \mathcal{A} and follow [AmBoGe, Chapter 6]. Given $k \in \mathbb{N}$, we say that $H \in \mathcal{C}^k(\mathcal{A})$ if for all $f \in \mathcal{H}$, the map $\mathbb{R} \ni t \mapsto e^{it\mathcal{A}}(H + i)^{-1}e^{-it\mathcal{A}}f \in \mathcal{H}$ has the usual \mathcal{C}^k regularity. We say that $H \in \mathcal{C}^{k,u}(\mathcal{A})$ if the map $\mathbb{R} \ni t \mapsto e^{it\mathcal{A}}(H + i)e^{-it\mathcal{A}} \in \mathcal{B}(\mathcal{H})$ is of class $\mathcal{C}^k(\mathbb{R}, \mathcal{B}(\mathcal{H}))$, where $\mathcal{B}(\mathcal{H})$ is endowed with the norm operator topology.

We start with an example, e.g., [GoJe1, Proposition 2.1].

Lemma 2.1. *For $\phi, \varphi \in \mathcal{D}(A)$, the rank one operator $|\phi\rangle\langle\varphi| : \psi \mapsto \langle\varphi, \psi\rangle\phi$ is of class $\mathcal{C}^1(A)$ and*

$$[|\phi\rangle\langle\varphi|, A] = |\phi\rangle\langle A\varphi| - |A\phi\rangle\langle\varphi|.$$

By induction, given $n \in \mathbb{N}$ and $\phi, \varphi \in \mathcal{D}(A^n)$, $|\phi\rangle\langle\varphi|$ is of class $\mathcal{C}^n(A)$.

We turn to a criterion in term of commutator.

Theorem 2.2 ([AmBoGe, p.258]). *Let \mathcal{A} and H be two self-adjoint operators in the Hilbert space \mathcal{H} . The following points are equivalent:*

- (1) $H \in \mathcal{C}^1(\mathcal{A})$.
- (2) For one (then for all) $z \notin \sigma(H)$, there is a finite c such that
$$|\langle \mathcal{A}f, R(z)f \rangle - \langle R(\bar{z})f, \mathcal{A}f \rangle| \leq c\|f\|^2, \text{ for all } f \in \mathcal{D}(\mathcal{A}).$$
- (3) (a) There is a finite c such that for all $f \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(H)$:
$$|\langle \mathcal{A}f, Hf \rangle - \langle Hf, \mathcal{A}f \rangle| \leq c(\|Hf\|^2 + \|f\|^2).$$
- (b) For some (then for all) $z \notin \sigma(H)$, the set
$$\{f \in \mathcal{D}(\mathcal{A}), R(z)f \in \mathcal{D}(\mathcal{A}) \text{ and } R(\bar{z})f \in \mathcal{D}(\mathcal{A})\}$$
is a core for \mathcal{A} .

Note that (2) yields that the commutator $[\mathcal{A}, R(z)]$ extends to a bounded operator in the form sense. We shall denote the extension by $[\mathcal{A}, R(z)]_\circ$. In the same way, from (3a), the commutator $[H, \mathcal{A}]$ extends to a unique element of $\mathcal{B}(\mathcal{D}(H), \mathcal{D}(H)^*)$ denoted by $[H, \mathcal{A}]_\circ$. Note that $\mathcal{D}(H)$ is endowed with the graph norm of H and that $\mathcal{D}(H)^*$ denotes its anti-dual. Moreover, if $H \in \mathcal{C}^1(\mathcal{A})$ and $z \notin \sigma(H)$,

$$[A, (H - z)^{-1}]_\circ = \underbrace{(H - z)^{-1}}_{\mathcal{H} \leftarrow \mathcal{D}(H)^*} \underbrace{[H, \mathcal{A}]_\circ}_{\mathcal{D}(H)^* \leftarrow \mathcal{D}(H)} \underbrace{(H - z)^{-1}}_{\mathcal{D}(H) \leftarrow \mathcal{H}}.$$

Here, we use the Riesz lemma to identify \mathcal{H} with its anti-dual \mathcal{H}^* .

Note that, in practice, the condition (3.b) could be delicate to check. This is addressed by the next lemma.

Lemma 2.3 ([GoMo, Lemma A.2]). *Let \mathcal{D} be a subspace of \mathcal{H} such that $\mathcal{D} \subset \mathcal{D}(H) \cap \mathcal{D}(\mathcal{A})$, \mathcal{D} is a core for A and $H\mathcal{D} \subset \mathcal{D}$. Let $(\chi_n)_{n \in \mathbb{N}}$ be a family of bounded operators such that*

- (1) $\chi_n \mathcal{D} \subset \mathcal{D}$, χ_n tends strongly to 1 as $n \rightarrow \infty$, and $\sup_n \|\chi_n\|_{\mathcal{B}(\mathcal{D}(H))} < \infty$.
- (2) $\mathcal{A}\chi_n f \rightarrow \mathcal{A}f$, for all $f \in \mathcal{D}$, as $n \rightarrow \infty$.
- (3) There is $z \notin \sigma(H)$, such that $\chi_n R(z)\mathcal{D} \subset \mathcal{D}$ and $\chi_n R(\bar{z})\mathcal{D} \subset \mathcal{D}$.

Suppose also that for all $f \in \mathcal{D}$

$$\lim_{n \rightarrow \infty} \mathcal{A}[H, \chi_n]R(z)f = 0 \text{ and } \lim_{n \rightarrow \infty} \mathcal{A}[H, \chi_n]R(\bar{z})f = 0.$$

Finally, suppose that there is a finite c such that

$$|\langle \mathcal{A}f, Hf \rangle - \langle Hf, \mathcal{A}f \rangle| \leq c(\|Hf\|^2 + \|f\|^2), \text{ for all } f \in \mathcal{D}.$$

Then, one has $H \in \mathcal{C}^1(\mathcal{A})$.

We define other refined classes of regularity:

$$\text{We say that } H \in \mathcal{C}^{0,1}(\mathcal{A}) \text{ if } \int_0^1 \|[(H + i)^{-1}, e^{it\mathcal{A}}]\| \frac{dt}{t} < \infty.$$

$$\text{We say that } H \in \mathcal{C}^{1,1}(\mathcal{A}) \text{ if } \int_0^1 \|[[(H + i)^{-1}, e^{it\mathcal{A}}], e^{it\mathcal{A}}]\| \frac{dt}{t^2} < \infty.$$

Thanks to [AmBoGe, p. 205], it turns out that

Given an interval open interval \mathcal{I} , we denote by $E_{\mathcal{I}}(H)$ the spectral projection of H above \mathcal{I} . We say that the *Mourre estimate* holds true for H on \mathcal{I} if there exist $c > 0$ and a compact operator K such that

$$(2.1) \quad E_{\mathcal{I}}(H)[H, i\mathcal{A}]_\circ E_{\mathcal{I}}(H) \geq E_{\mathcal{I}}(H)(c + K)E_{\mathcal{I}}(H),$$

when the inequality is understood in the form sense. We say that we have a *strict Mourre estimate* holds for H on the open interval \mathcal{I}' when there exists $c' > 0$ such that

$$(2.2) \quad E_{\mathcal{I}'}(H)[H, i\mathcal{A}]_\circ E_{\mathcal{I}'}(H) \geq c'E_{\mathcal{I}'}(H).$$

Assuming $H \in \mathcal{C}^1(\mathcal{A})$, (2.1), and $\lambda \in \mathcal{I}$ is not an eigenvalue, therefore there exists an open interval \mathcal{I}' that contains λ and $c' > 0$ such that (2.2). The aim of Mourre's

commutator theory is to show a *limiting absorption principle* (LAP), see [AmBoGe, Theorem 7.6.8].

Theorem 2.4. *Let H be a self-adjoint operator, with $\sigma(H) \neq \mathbb{R}$. Assume that $H \in \mathcal{C}^1(\mathcal{A})$ and the Mourre estimate (2.1) holds true for H on \mathcal{I} . Then*

- (1) *The number of eigenvalues (counted with multiplicity) of H , that are in \mathcal{I} , is finite.*

Assuming furthermore that $K = 0$ in (2.1), it yields:

- (2) *H has no eigenvalues in \mathcal{I} .*
(3) *If $H \in \mathcal{C}^{1,1}(\mathcal{A})$ and $K = 0$, $s > 1/2$ and \mathcal{I}' a compact sub-interval of \mathcal{I} , then*

$$\sup_{\Re(z) \in \mathcal{I}', \Im(z) \neq 0} \|\langle \mathcal{A} \rangle^{-s} (H - z)^{-1} \langle \mathcal{A} \rangle^{-s}\| \text{ exists and finite.}$$

Moreover, in the norm topology of bounded operators, the boundary values of the resolvent:

$$\mathcal{I}' \ni \lambda \mapsto \lim_{\rho \rightarrow 0^\pm} \langle \mathcal{A} \rangle^{-s} (H - \lambda - i\rho)^{-1} \langle \mathcal{A} \rangle^{-s} \text{ exists and continuous.}$$

For more details and deeper results, see [AmBoGe, Proposition 7.2.10, Corollary 7.2.11, Theorem 7.5.2].

3. THE FREE MODEL

3.1. Construction of the graph. We discuss two different product of graphs. To start off, given $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$ and $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$, the *Cartesian product of \mathcal{G}_1 by \mathcal{G}_2* is defined by $\mathcal{G}^\diamond := (\mathcal{E}^\diamond, \mathcal{V}^\diamond, m^\diamond)$, where $\mathcal{V}^\diamond := \mathcal{V}_1 \times \mathcal{V}_2$,

$$\begin{cases} m^\diamond(x, y) := m_1(x) \times m_2(y), \\ \mathcal{E}^\diamond((x, y), (x', y')) := \mathcal{E}_1(x, x') \times \delta_{y, y'} m_2(y) + m_1(x) \delta_{x, x'} \times \mathcal{E}_2(y, y'). \end{cases}$$

We denote it by $\mathcal{G}_1 \times \mathcal{G}_2 := \mathcal{G}^\diamond$. This definition generalises the unweighted Cartesian product, e.g., [Ha]. It is used in several places in the literature, e.g., see [Ch, Section 2.6] and see [BoKeGoLiMü] for a generalisation.

The terminology is motivated by the following decomposition:

$$\Delta_{\mathcal{G}^\diamond} = \Delta_{\mathcal{G}_1} \otimes 1 + 1 \otimes \Delta_{\mathcal{G}_2},$$

where $\ell^2(\mathcal{V}, m) \simeq \ell^2(\mathcal{V}_1, m_1) \otimes \ell^2(\mathcal{V}_2, m_2)$. Note that

$$e^{it\Delta_{\mathcal{G}^\diamond}} = e^{it\Delta_{\mathcal{G}_1}} \otimes e^{it\Delta_{\mathcal{G}_2}}, \quad \forall t \in \mathbb{R}.$$

We refer to [ReSi, Section VIII.10] for an introduction to the tensor product of self-adjoint operators.

We now introduce a *twisted Cartesian product*. We refer to [GoTr, Section 2.2] for motivations, its link with hyperbolic geometry and generalisations. Given $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$ and $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$, we define the *product of \mathcal{G}_1 by \mathcal{G}_2* by $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$, where $\mathcal{V} := \mathcal{V}_1 \times \mathcal{V}_2$ and

$$\begin{cases} m(x, y) := m_1(x) \times m_2(y), \\ \mathcal{E}((x, y), (x', y')) := \mathcal{E}_1(x, x') \times \delta_{y, y'} + \delta_{x, x'} \times \mathcal{E}_2(y, y'), \end{cases}$$

for all $x, x' \in \mathcal{V}_1$ and $y, y' \in \mathcal{V}_2$. We denote \mathcal{G} by $\mathcal{G}_1 \times_\tau \mathcal{G}_2$. If $m = 1$, note that $\mathcal{G}_1 \times_\tau \mathcal{G}_2 = \mathcal{G}_1 \times \mathcal{G}_2$.

Under the representation $\ell^2(\mathcal{V}, m) \simeq \ell^2(\mathcal{V}_1, m_1) \otimes \ell^2(\mathcal{V}_2, m_2)$,

$$(3.1) \quad \deg_{\mathcal{G}_1 \times_\tau \mathcal{G}_2}(\cdot) = \deg_{\mathcal{G}_1}(\cdot) \otimes \frac{1}{m_2(\cdot)} + \frac{1}{m_1(\cdot)} \otimes \deg_{\mathcal{G}_2}(\cdot)$$

and

$$(3.2) \quad \Delta_{\mathcal{G}_1 \times_\tau \mathcal{G}_2} = \Delta_{\mathcal{G}_1} \otimes \frac{1}{m_2(\cdot)} + \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2}.$$

If m is non-trivial, we stress that the Laplacian obtained with our product is usually not unitarily equivalent to the Laplacian obtained with the Cartesian product.

A hyperbolic manifold of finite volume is the union of a compact part, of a cusp, and a funnel, e.g., [Th, Theorem 4.5.7]. In this article we study a discrete analog.

In the sequel, we take m_2 constant on \mathcal{V}_2 .

The graph $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$ is divided into three parts: A *cusp* part, a *funnel* part, and a finite part. Set $\mathcal{G}^* := (\mathcal{E}^*, \mathcal{V}^*, m^*)$ be the induced graph of \mathcal{G} over \mathcal{V}^* where $\star \in \{c, f, 0\}$ and \mathcal{V} is a disjoint reunion of \mathcal{V}^c , \mathcal{V}^f , and \mathcal{V}^0 .

We consider $\mathcal{G}_1^c := (\mathcal{E}_1^c, \mathcal{V}_1^c, m_1^c)$, where

$$\mathcal{V}_1^c := \mathbb{N}, \quad m_1^c(n) := \exp(-n), \quad \text{and} \quad \mathcal{E}_1^c(n, n+1) := \exp(-(2n+1)/2),$$

for all $n \in \mathbb{N}$ and $\mathcal{G}_2^c := (\mathcal{E}_2^c, \mathcal{V}_2^c, m_2)$ a possibly disconnected connected finite graph. Set $\mathcal{G}^c := \mathcal{G}_1^c \times_\tau \mathcal{G}_2^c$. This is a *cusp part*. Note it is of finite volume as:

$$\sum_{(x,y) \in \mathcal{V}_1^c \times \mathcal{V}_2^c} m_{\mathcal{G}^c}(x,y) < \infty.$$

We consider $\mathcal{G}_1^f := (\mathcal{E}_1^f, \mathcal{V}_1^f, m_1^f)$, where

$$\mathcal{V}_1^f := \mathbb{N}, \quad m_1^f(n) := \exp(n), \quad \text{and} \quad \mathcal{E}_1^f(n, n+1) := \exp((2n+1)/2),$$

for all $n \in \mathbb{N}$ and $\mathcal{G}_2^f := (\mathcal{E}_2^f, \mathcal{V}_2^f, m_2)$ a connected finite graph. Set $\mathcal{G}^f := \mathcal{G}_1^f \times_\tau \mathcal{G}_2^f$. This is a *funnel part*.

For the *compact part*, we ask that for all $x \in \mathcal{V}^0$, $\text{supp}(\mathcal{E}(x, \cdot))$ is finite and $m^0(x) > 0$.

We now, take advantage of

$$\ell^2(\mathcal{G}) := \ell^2(\mathcal{G}^f) \oplus \ell^2(\mathcal{G}^0) \oplus \ell^2(\mathcal{G}^c).$$

We have that

$$\Delta_{\mathcal{G}} := \Delta_{\mathcal{G}^f} \oplus 0 \oplus \Delta_{\mathcal{G}^c} + K_0,$$

where K_0 is an operator of finite rank with support in $\mathcal{C}^c(\mathcal{V})$.

To analyse the perturbations of operator we shall rely on the following gauge transformation, e.g., [Go, CoToTr, HaKe]. See also [BoGo] for some historical references.

Proposition 3.1. *Let $\mathcal{G} := (\mathcal{V}, \mathcal{E}, m)$ be a weighted graph and $m : \mathcal{V} \rightarrow (0, \infty)$ be a weight. The following map is unitary:*

$$(3.3) \quad \begin{aligned} T_{m \rightarrow m'} f &: \ell^2(\mathcal{V}, m) \rightarrow \ell^2(\mathcal{V}, m') \\ f &\mapsto \left(x \mapsto \sqrt{\frac{m(x)}{m'(x)}} f(x) \right). \end{aligned}$$

We have:

$$(3.4) \quad \Delta_{\mathcal{G}'}^{\mathcal{F}} = T_{m \rightarrow m'} (\Delta_{\tilde{\mathcal{G}}} - W(\cdot))^{\mathcal{F}} T_{m \rightarrow m'}^{-1},$$

where $\mathcal{G}' := (\mathcal{V}, \mathcal{E}', m')$, $\tilde{\mathcal{G}} := (\mathcal{V}, \tilde{\mathcal{E}}, m)$ and,

$$\begin{aligned} \tilde{\mathcal{E}}(x, y) &:= \mathcal{E}'(x, y) \sqrt{\frac{m(x)m(y)}{m'(x)m'(y)}} \\ W(x) &:= \frac{1}{m(x)} \sum_{y \in \mathcal{V}} \tilde{\mathcal{E}}(x, y) \left(1 - \sqrt{\frac{m(x)m'(y)}{m(y)m'(x)}} \right). \end{aligned}$$

Here we emphasised the choice Friedrichs extension with the symbol \mathcal{F} .

3.2. Mourre estimate on \mathbb{N} . In this section we make a preliminary work on the half axis. We construct a conjugate operator, prove a Mourre estimate for $\Delta_{\mathbb{N}}$ and check the regularity conditions. This is a known result, e.g., [AlFr], see also [GeGo, Mic].

Given $f \in \ell^2(\mathbb{N}, 1)$, we set

$$\forall n \in \mathbb{N}^*, \quad Uf(n) := f(n-1) \text{ and } Uf(0) := 0.$$

Note that $U^*f(n) = f(n+1), \forall n \in \mathbb{N}$. The operator U is an isometry and is not unitary: we have $U^*U = \text{id}$ and $UU^* = \mathbf{1}_{[1, \infty[}(\cdot)$.

We define by Q the operator of multiplication by n in $\ell^2(\mathbb{N}, 1)$. Namely, it is the closure of the operator given by $(Qf)(n) = nf(n)$ for all $n \in \mathbb{N}$ and $f \in \mathcal{C}_c(\mathbb{N})$. It is essentially self-adjoint on $\mathcal{C}_c(\mathbb{N})$. In [GeGo], one finds the following elementary relations:

$$(3.5) \quad QU = U(Q+1), \quad U^*Q = (Q+1)U^* \text{ and } UQU = U^2(Q+1) \quad \text{on } \mathcal{D}(Q).$$

The operator $\Delta_{\mathbb{N}}$ is defined by (1.1), where $\mathbb{N} \simeq (\mathbb{N}, \mathcal{E}_{\mathbb{N}}, m)$, with $\mathcal{E}_{\mathbb{N}}(n, n+1) = 1$ and $m(n) = 1$ for all $n \in \mathbb{N}$. Explicitly, we have

$$\Delta_{\mathbb{N}}f(n) := \begin{cases} 2f(n) - f(n-1) - f(n+1) & \text{if } n \geq 1, \\ f(n) - f(n+1) & \text{if } n = 0, \end{cases} \quad \forall f \in \ell^2(\mathbb{N}, 1).$$

We can express it with the help of U . Namely, we have:

$$\Delta_{\mathbb{N}} = 2 - (U + U^*) - \mathbf{1}_{\{0\}}(\cdot).$$

A standard result is :

$$\sigma_{\text{ess}}(\Delta_{\mathbb{N}}) = [0, 4] \quad \text{and} \quad \sigma_{\text{sc}}(\Delta_{\mathbb{N}}) = \emptyset.$$

We construct the conjugate operator in $\ell^2(\mathbb{N}, 1)$. On the space $\mathcal{C}_c(\mathbb{N})$, we define

$$\begin{aligned} \mathcal{A}_{\mathbb{N}}|_{\mathcal{C}_c(\mathbb{N})} &:= \frac{1}{2}(SQ + QS), \quad \text{where } S := \frac{U - U^*}{2i} \\ &= \frac{i}{2} \left(U \left(Q + \frac{1}{2} \right) - U^* \left(Q - \frac{1}{2} \right) \right) \\ (3.6) \quad &= -\frac{i}{2} \left(\frac{1}{2}(U^* + U) + Q(U^* - U) \right). \end{aligned}$$

We denote by $\mathcal{A}_{\mathbb{N}}$ its closure.

Lemma 3.2. *The operator $\mathcal{A}_{\mathbb{N}}$ is essentially self-adjoint on $\mathcal{C}_c(\mathbb{N})$ and*

$$\mathcal{D}(\mathcal{A}_{\mathbb{N}}) = \mathcal{D}(QS) := \{f \in \ell^2(\mathbb{N}), Sf \in \mathcal{D}(Q)\}.$$

We refer to [GeGo] and [Mic, Lemma 5.7] for the essential self-adjointness and [GeGo, Lemma 3.1] for the domain.

We give a first technical lemma.

Lemma 3.3. *On $\mathcal{C}_c(\mathbb{N})$, we have*

$$\begin{aligned} (U^* + U)\mathcal{A}_{\mathbb{N}} &= -\frac{i}{2} \left((U^{2*} - U^2)Q - \mathbf{1} - \frac{1}{2}(\mathbf{1}_{\{0\}}(\cdot) + U^2 + U^{*2}) \right), \\ \mathcal{A}_{\mathbb{N}}(U^* + U) &= \frac{i}{2} \left((U^2 - U^{2*})Q - \frac{1}{2}(U^2 + U^{*2}) - \mathbf{1} - \frac{1}{2}\mathbf{1}_{\{0\}}(\cdot) \right). \end{aligned}$$

Proof. We compute on $\mathcal{C}_c(\mathbb{N})$. The statement follows easily from

$$\begin{aligned} U\mathcal{A}_{\mathbb{N}} &= -\frac{i}{2} \left((\mathbf{1}_{\mathbb{N}} - U^2)Q - \frac{1}{2}(\mathbf{1}_{\mathbb{N}} + U^2) \right), \\ U^*\mathcal{A}_{\mathbb{N}} &= -\frac{i}{2} \left((U^* - \mathbf{1})Q - \frac{1}{2}(U^{*2} - \frac{1}{2}) \right). \end{aligned}$$

by taking the adjoint. □

We can compute the first comutator.

Lemma 3.4. *The operator $\Delta_{\mathbb{N}}$ is $\mathcal{C}^1(\mathcal{A}_{\mathbb{N}})$ and we have:*

$$(3.7) \quad [\Delta_{\mathbb{N}}, \mathbf{i}\mathcal{A}_{\mathbb{N}}]_{\circ} = \frac{1}{2}\Delta_{\mathbb{N}}(4 - \Delta_{\mathbb{N}}) + K_1,$$

with K_1 a finite rank operator belonging to $\mathcal{C}^{\infty}(A)$.

This lemma is essentially given in [GeGo], see also [AlFr] for another type of presentation. For the convenience of the reader we reproduce it.

Proof. First, since $\delta_{\{0\}} \in \mathcal{D}(\mathcal{A}^n)$ for all $n \in \mathbb{N}$, $\delta_{\{0\}}$ and $K_1 := [\delta_{\{0\}}, \mathbf{i}\mathcal{A}_{\mathbb{N}}]_{\circ}$ belong to $\mathcal{C}^1(\mathcal{A}_{\mathbb{N}})$ by Lemma 2.1. Next, we turn to the other part and work in the form sense and by density. Let $f \in \mathcal{C}_c(\mathbb{N})$. Since $\Delta_{\mathbb{N}}f \in \mathcal{C}_c(\mathbb{N})$ and using Lemma 3.3, we obtain:

$$\begin{aligned} \langle f, [\Delta_{\mathbb{N}}, \mathbf{i}\mathcal{A}_{\mathbb{N}}]f \rangle &:= \langle \Delta_{\mathbb{N}}f, \mathbf{i}\mathcal{A}_{\mathbb{N}}f \rangle - \langle -\mathbf{i}\mathcal{A}_{\mathbb{N}}f, \Delta_{\mathbb{N}}f \rangle \\ &= \mathbf{i}\langle f, \mathcal{A}_{\mathbb{N}}(U^* + U) - (U^* + U)\mathcal{A}_{\mathbb{N}}f \rangle + \langle f, [\delta_{\{0\}}, \mathbf{i}\mathcal{A}_{\mathbb{N}}]f \rangle \\ &= \frac{1}{2}\langle f, \Delta_{\mathbb{N}}(4 - \Delta_{\mathbb{N}})f \rangle + \langle f, [\delta_{\{0\}}, \mathbf{i}\mathcal{A}_{\mathbb{N}}]_{\circ}f \rangle. \end{aligned}$$

Since $\Delta_{\mathbb{N}}(4 - \Delta_{\mathbb{N}})$ and $[\delta_{\{0\}}, \mathbf{i}\mathcal{A}_{\mathbb{N}}]_{\circ}$ are bounded operators and since $\mathcal{C}_c(\mathbb{N})$ is a core for $\mathcal{A}_{\mathbb{N}}$, there is a constant c such that

$$|\langle \Delta_{\mathbb{N}}f, \mathbf{i}\mathcal{A}_{\mathbb{N}}f \rangle - \langle -\mathbf{i}\mathcal{A}_{\mathbb{N}}f, \Delta_{\mathbb{N}}f \rangle| \leq c\|f\|^2, \text{ for all } f \in \mathcal{D}(\mathcal{A}).$$

Hence, it is $\mathcal{C}^1(\mathcal{A}_{\mathbb{N}})$. By density, we also obtain (3.7). \square

By induction, we infer:

Corollary 3.5. $\Delta_{\mathbb{N}} \in \mathcal{C}^{\infty}(\mathcal{A}_{\mathbb{N}})$.

We mention [Mic] for an anisotropic use on \mathbb{Z} based on the Mourre theory of $\Delta_{\mathbb{N}}$.

3.3. The funnel side. In this section we construct a conjugate operator for $\Delta_{\mathcal{G}^f}$ and establish a Mourre estimate.

3.3.1. A first step into the analysis. As seen above, under the identification

$$(3.8) \quad \ell^2(\mathcal{V}^f, m) = \ell^2(\mathbb{N}, m_1^f) \otimes \ell^2(\mathcal{V}_2^f, m_2^f).$$

We have

$$(3.9) \quad \Delta_{\mathcal{G}^f} := \Delta_{\mathcal{G}_1^f} \otimes \frac{1}{m_2^f} + \frac{1}{m_1^f(\cdot)} \otimes \Delta_{\mathcal{G}_2^f}.$$

Recall here that m_2 is a constant. The first remark is that

Lemma 3.6.

$$\frac{1}{m_1^f(\cdot)} \otimes \Delta_{\mathcal{G}_2^f} \in \mathcal{K}(\ell^2(\mathcal{V}^f)).$$

Proof. Note that $\Delta_{\mathcal{G}_2^f}$ is of finite rank since \mathcal{V}_2 is finite and that $\frac{1}{m_1^f(\cdot)}$ is a compact operator since $m_1^f(n) \rightarrow \infty$, as $n \rightarrow \infty$. \square

Since m_2 is constant and $\deg_{\mathcal{G}_1^f}$ is bounded, we obtain:

Proposition 3.7. *We have $\Delta_{\mathcal{G}^f} \in \mathcal{B}(\ell^2(\mathcal{V}^f), m^f)$.*

Recalling the Proposition 3.1, we obtain:

$$T_{1 \rightarrow m_1^f}^{-1} \Delta_{\mathcal{G}_1^f} T_{1 \rightarrow m_1^f} = \Delta_{\mathbb{N}} + (e^{1/2} - 1)\mathbf{1}_{\{0\}} + e^{1/2} + e^{-1/2} - 2$$

Recalling Lemma 3.6, we infer immediately

$$\sigma_{\text{ess}}(\Delta_{\mathcal{G}^f}) = \left[\frac{\alpha}{m_2}, \frac{\beta}{m_2} \right] \quad \text{and} \quad \sigma_{\text{sc}}(\Delta_{\mathcal{G}_1^f}) = \emptyset,$$

with α and β are given in (1.2).

3.3.2. *Construction of the conjugate operator.* In order to get also $\sigma_{\text{sc}}(\Delta_{\mathcal{G}^f}) = \emptyset$, we rely on the Mourre theory and construct a conjugate operator for $\Delta_{\mathcal{G}^f}$. Recalling (3.6) and with respect to (3.8), we set

$$(3.10) \quad \mathcal{A}_{\mathcal{G}^f} := \mathcal{A}_{m_1^f} \otimes 1_{\mathcal{V}_2^f} := T_{1 \rightarrow m_1^f} \mathcal{A}_{\mathbb{N}} T_{1 \rightarrow m_1^f}^{-1} \otimes 1_{\mathcal{V}_2^f}.$$

It is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V}^f)$ and on $\mathcal{C}_c(\mathbb{N}) \otimes \ell^2(\mathcal{V}^f)$ by Lemma 3.2. It acts as follows:

Proposition 3.8. *On $\mathcal{C}_c(\mathbb{N})$, we have*

$$\mathcal{A}_{m_1^f} = \frac{i}{2} \left(e^{1/2}(Q - 1/2)U - e^{-1/2}(Q + 1/2)U^* \right).$$

Proof. Let $f \in \mathcal{C}_c(\mathbb{N})$,

$$\begin{aligned} \mathcal{A}_{m_1^f} f(n) &= -\frac{i}{2\sqrt{m_1^f(n)}} \left(\frac{1}{2}(U + U^*) + Q(U^* - U) \right) T_{1 \rightarrow m_1^f}^{-1} f(n) \\ &= \frac{i}{2} \left(\left(n - \frac{1}{2} \right) \sqrt{\frac{m_1^f(n-1)}{m_1^f(n)}} f(n-1) - \left(n + \frac{1}{2} \right) \sqrt{\frac{m_1^f(n+1)}{m_1^f(n)}} f(n+1) \right) \\ &= \frac{i}{2} \left(e^{1/2} \left(n - \frac{1}{2} \right) U f(n) - e^{-1/2} \left(n + \frac{1}{2} \right) U^* f(n) \right). \end{aligned}$$

This concludes the proof. \square

We turn to the regularity. In order to lighten the computation, given a graph $\mathcal{G} = (\mathcal{E}, \mathcal{V}, m)$, we write

$$T_1 \simeq T_2 \text{ if there is } K : \mathcal{C}_c(\mathcal{V}) \rightarrow \mathcal{C}_c(\mathcal{V}) \text{ of finite rank such that } T_1 = T_2 + K$$

Thanks to Lemma 2.1 and Proposition 3.8, we obtain immediately:

Lemma 3.9. *Assume that $T_1 \simeq T_2$. Then for all $n \in \mathbb{N}$,*

$$T_1 \in \mathcal{D}(\mathcal{A}_{\mathcal{G}^f}^n) \Leftrightarrow T_2 \in \mathcal{D}(\mathcal{A}_{\mathcal{G}^f}^n).$$

We have:

Lemma 3.10. *We have $\Delta_{\mathcal{G}^f} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}^f})$ and*

$$(3.11) \quad [\Delta_{\mathcal{G}^f}, i\mathcal{A}_{\mathcal{G}^f}]_o = w(\Delta_{\mathcal{G}^f}) + K,$$

where

$$w^f(x) := \frac{m_2}{2} \left(x - \frac{\alpha}{m_2} \right) \left(\frac{\beta}{m_2} - x \right),$$

with α and β as in (1.2) and K is a compact operator.

Proof. We prove that $[\Delta_{\mathcal{G}^f}, i\mathcal{A}_{\mathcal{G}^f}]_o \in \mathcal{B}(\ell^2(\mathcal{V}^f, m^f))$. As in Lemma 3.4 and working in the form sense on $\mathcal{C}_c(\mathbb{N}) \otimes \ell^2(\mathcal{V}_2^f)$, a straightforward computation leads to

$$\begin{aligned} \left[\Delta_{\mathcal{G}_1^f} \otimes \frac{1}{m_2}, i\mathcal{A}_{\mathcal{G}^f} \right] &\simeq \frac{1}{2} (\Delta_{\mathcal{G}_1^f} - \alpha)(\beta - \Delta_{\mathcal{G}_1^f}) \otimes \frac{1}{m_2} \\ &\simeq w^f(\Delta_{\mathcal{G}^f}) - \frac{m_2}{2} \left(\frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2^f} \right) \left(\frac{\beta}{m_2} - \Delta_{\mathcal{G}_1^f} \otimes \frac{1}{m_2} - \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2^f} \right) \\ (3.12) \quad &+ \frac{m_2}{2} \left(\Delta_{\mathcal{G}_1^f} \otimes \frac{1}{m_2} - \frac{\alpha}{m_2} \right) \left(\frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2^f} \right) \\ &= w^f(\Delta_{\mathcal{G}^f}) + K' \end{aligned}$$

where K' is a compact operator coming from Lemma 3.6 and Lemma 2.1. We turn to the second part of $\Delta_{\mathcal{G}^f}$.

$$(3.13) \quad \begin{aligned} \left[\frac{1}{m_1^f(\cdot)}, i\mathcal{A}_{m_1^f} \right] \otimes \Delta_{\mathcal{G}_2^f} &= T_{1 \rightarrow m_1^f} \left[\frac{1}{m_1^f(\cdot)}, i\mathcal{A}_{\mathbb{N}} \right] T_{1 \rightarrow m_1^f}^{-1} \otimes \Delta_{\mathcal{G}_2^f} \\ &= T_{1 \rightarrow m_1^f} \left(\frac{1}{2}(e-1)e^{-Q} \left(Q - \frac{1}{2} \right) U \right) T_{1 \rightarrow m_1^f}^{-1} \otimes \Delta_{\mathcal{G}_2^f}, \end{aligned}$$

in the form sense on $\mathcal{C}_c(\mathbb{N}) \otimes \ell^2(\mathcal{V}_2^f)$. The operator is a compact since U is bounded and $\lim_{n \rightarrow \infty} e^{-n}(n-1/2) = 0$.

This implies that $[\Delta_{\mathcal{G}^f}, i\mathcal{A}_{\mathcal{G}^f}]_{\circ} \in \mathcal{B}(\ell^2(\mathcal{V}^f, m^f))$ and that (3.11) holds true. Finally, since $\mathcal{C}_c(\mathbb{N}) \otimes \ell^2(\mathcal{V}_2^f)$ is a core for $\mathcal{A}_{\mathcal{G}^f}$, we deduce that $\Delta_{\mathcal{G}^f} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}^f})$. \square

Lemma 3.11. *We have $\Delta_{\mathcal{G}^f} \in \mathcal{C}^2(\mathcal{A}_{\mathcal{G}^f})$.*

Proof. As above, since $\mathcal{C}_c(\mathbb{N}) \otimes \ell^2(\mathcal{V}_2^f)$ is a core for $\mathcal{A}_{\mathcal{G}^f}$ it is enough to prove that $[[\Delta_{\mathcal{G}^f}, i\mathcal{A}_{\mathcal{G}^f}]_{\circ}, i\mathcal{A}_{\mathcal{G}^f}]$, defined initially in the form sense on $\mathcal{C}_c(\mathbb{N}) \otimes \ell^2(\mathcal{V}_2^f)$, extends to an element of $\mathcal{B}(\ell^2(\mathcal{V}^f, m^f))$.

We prove that the right hand side of (3.11) belongs to $\mathcal{C}^1(\mathcal{A}_{\mathcal{G}^f})$. It is composed of $w(\Delta_{\mathcal{G}^f})$ which is $\mathcal{C}^1(\mathcal{A}_{\mathcal{G}^f})$ (as product of bounded operators belonging to $\mathcal{C}^1(\mathcal{A}_{\mathcal{G}^f})$), terms with finite support that are also in $\mathcal{C}^1(\mathcal{A}_{\mathcal{G}^f})$ by Lemma 2.1 and terms similar to (3.13). Therefore $[[\Delta_{\mathcal{G}_1^f} \otimes \frac{1}{m_2}, i\mathcal{A}_{\mathcal{G}^f}]_{\circ}, i\mathcal{A}_{\mathcal{G}^f}]$ extends to a bounded operator.

We turn to the second part. It remains to show that the left hand side of (3.13) belongs to $\mathcal{C}^1(\mathcal{A}_{\mathcal{G}^f})$. Repeating the computation done in (3.13), we see that since $\lim_{n \rightarrow \infty} e^{-n}\langle n \rangle^2 = 0$, $[[\frac{1}{m_1^f(\cdot)}, i\mathcal{A}_{m_1^f}]_{\circ}, \mathcal{A}_{m_1^f}]$ extends to a compact operator. \square

Remark 3.12. *By induction, we can prove that $\Delta_{\mathcal{G}^f} \in \mathcal{C}^{\infty}(\mathcal{A}_{\mathcal{G}^f})$.*

Finally, we establish the Mourre estimate.

Proposition 3.13. *We have $\Delta_{\mathcal{G}^f} \in \mathcal{C}^2(\mathcal{A}_{\mathcal{G}^f})$. Given a compact interval $\mathcal{I} \subset (\alpha/m_2, \beta/m_2)$, there are $c > 0$, a compact operator K such that*

$$(3.14) \quad E_{\mathcal{I}}(\Delta_{\mathcal{G}^f})[\Delta_{\mathcal{G}^f}, i\mathcal{A}_{\mathcal{G}^f}]_{\circ} E_{\mathcal{I}}(\Delta_{\mathcal{G}^f}) \geq cE_{\mathcal{I}}(\Delta_{\mathcal{G}^f}) + K,$$

in the form sense. In particular, $\sigma_{\text{sc}}(\Delta_{\mathcal{G}^f}) = \emptyset$.

Proof. Lemma 3.11 gives that $\Delta_{\mathcal{G}^f} \in \mathcal{C}^2(\mathcal{A}_{\mathcal{G}^f})$. By (3.11), we obtain

$$\begin{aligned} E_{\mathcal{I}}(\Delta_{\mathcal{G}^f})[\Delta_{\mathcal{G}^f}, i\mathcal{A}_{\mathcal{G}^f}]_{\circ} E_{\mathcal{I}}(\Delta_{\mathcal{G}^f}) &= E_{\mathcal{I}}(\Delta_{\mathcal{G}^f})w(\Delta_{\mathcal{G}^f})E_{\mathcal{I}}(\Delta_{\mathcal{G}^f}) + K \\ &\geq cE_{\mathcal{I}}(\Delta_{\mathcal{G}^f}) + K, \end{aligned}$$

where K is a compact operator and

$$c := \frac{m_2}{2} \inf_{x \in \mathcal{I}} \left(x - \frac{\alpha}{m_2} \right) \left(\frac{\beta}{m_2} - x \right) > 0.$$

The absence of singular continuous spectrum follows from the general theory. \square

To lighten the text we did not expand more consequences of the Mourre theory in this case and refer to Theorem 4.3 for them.

3.4. The cusps side. In this section we construct a conjugate operator for $\Delta_{\mathcal{G}^c}$ and establish a Mourre estimate. By contrast with the funnel side, we shall refine the tensor product decomposition.

3.4.1. *The model and the low/high energy decomposition.* Again we rely on the decomposition

$$(3.15) \quad \ell^2(\mathcal{V}^c, m) = \ell^2(\mathbb{N}, m_1^c) \otimes \ell^2(\mathcal{V}_2^c, m_2).$$

We have

$$(3.16) \quad \Delta_{\mathcal{G}^c} := \Delta_{\mathcal{G}_1^c} \otimes \frac{1}{m_2} + \frac{1}{m_1^c(\cdot)} \otimes \Delta_{\mathcal{G}_2^c}.$$

Recall that m_2 is a constant. Unlike with the treatment of $\Delta_{\mathcal{G}^f}$, we refine the tensor product decomposition. In the spirit of [GoMo, GoTr], we denote by P^{le} the projection on $\ker(\Delta_{\mathcal{G}_2})$ and by P^{he} is the projection on $\ker(\Delta_{\mathcal{G}_2})^\perp$. Here, *le* stands for *low energy* and *he* for *high energy*. We shall take advantage of

$$(3.17) \quad \begin{aligned} \ell^2(\mathcal{V}^c, m) &:= \mathcal{H}^{\text{le}} \oplus \mathcal{H}^{\text{he}} \\ &:= \ell^2(\mathbb{N}, m_1^c) \otimes \ker(\Delta_2) \oplus \ell^2(\mathbb{N}, m_1^c) \otimes \ker(\Delta_2)^\perp. \end{aligned}$$

The main idea is the continuous spectrum comes from the low energy part of the space whereas the discrete spectrum arises from the high energy part.

We have that $\Delta_{\mathcal{G}^c} := \Delta_{\mathcal{G}^c}^{\text{le}} \oplus \Delta_{\mathcal{G}^c}^{\text{he}}$, where

$$(3.18) \quad \Delta_{\mathcal{G}^c}^{\text{le}} := \Delta_{\mathcal{G}_1^c} \otimes \frac{1}{m_2} P^{\text{le}},$$

on $(1 \otimes P^{\text{le}})\ell^2(\mathcal{V}^c, m^c)$, and

$$(3.19) \quad \Delta_{\mathcal{G}^c}^{\text{he}} := \Delta_{\mathcal{G}_1^c} \otimes \frac{1}{m_2} P^{\text{he}} + \frac{1}{m_1^c(\cdot)} \otimes P^{\text{he}} \Delta_{\mathcal{G}_2^c},$$

on $(1 \otimes P^{\text{he}})\ell^2(\mathcal{V}^c, m^c)$. We stress that m_2 is constant.

Unlike $\Delta_{\mathcal{G}^f}$, $\Delta_{\mathcal{G}^c}$ is unbounded. More precisely we have:

Proposition 3.14. *The operator $\Delta_{\mathcal{G}^c}$ is essentially self-adjoint on $\mathcal{C}_c(\mathbb{N}) \otimes \ell^2(\mathcal{V}_2)$ and on $\mathcal{C}_c(\mathcal{V}^c)$. Its domain is given by $\mathcal{D}\left(\frac{1}{m_1^c(\cdot)} \otimes \Delta_{\mathcal{G}_2^c}\right)$.*

Proof. First $m_1^c(\cdot)$ is essentially self-adjoint of $\mathcal{C}_c(\mathbb{N})$. Since $\Delta_{\mathcal{G}_2^c}$ is bounded, we infer that $\frac{1}{m_1^c(\cdot)} \otimes \Delta_{\mathcal{G}_2^c}$ is essentially self-adjoint on $\mathcal{C}_c(\mathbb{N}) \otimes \ell^2(\mathcal{V}_2)$. Next, since $\Delta_{\mathcal{G}_1^c} \otimes \frac{1}{m_2}$ is bounded, $\Delta_{\mathcal{G}^c}$ is essentially self-adjoint on $\mathcal{C}_c(\mathbb{N}) \otimes \ell^2(\mathcal{V}_2)$ by the Kato-Rellich Theorem, e.g., [ReSi, Theorem X.12]. The statement with $\mathcal{C}_c(\mathcal{V}^c)$ follows by standard approximations. \square

Using the notation given in (3.3), we see that:

$$T_{m_1^c \rightarrow 1} \Delta_{\mathcal{G}_1^c} T_{m_1^c \rightarrow 1}^{-1} = \Delta_{\mathbb{N}} - (e^{-1/2} - 1)\mathbf{1}_{\{0\}}(\cdot) + e^{1/2} + e^{-1/2} - 2 \text{ in } \ell^2(\mathbb{N}).$$

By using for instance some Jacobi matrices techniques, it is well-known that the essential spectrum of $\Delta_{\mathcal{G}^c}^{\text{le}}$ is purely absolutely continuous and

$$\sigma_{\text{ac}}(\Delta_{\mathcal{G}^c}^{\text{le}}) = [\alpha, \beta],$$

with multiplicity one, e.g., [We]. Recall that α and β are defined in (1.2).

We turn to the high energy part. Using [GoTr, Equation (10)],

$$\frac{1}{m_1^c(\cdot)} \otimes \Delta_{\mathcal{G}_2^c} P^{\text{he}} \leq \Delta_{\mathcal{G}^c} (1 \otimes P^{\text{he}}) \leq 2M + \frac{1}{m_1^c(\cdot)} \otimes \Delta_{\mathcal{G}_2^c} P^{\text{he}}.$$

Using the min-max Theorem and since $m_1^c(n) \rightarrow 0$ as $n \rightarrow \infty$, $\Delta_{\mathcal{G}^c} (1 \otimes P^{\text{he}})$ has a compact resolvent. We infer that

$$\sigma_{\text{ac}}(\Delta_{\mathcal{G}^c}) = \left[\frac{\alpha}{m_2}, \frac{\beta}{m_2} \right] \text{ and } \sigma_{\text{sc}}(\Delta_{\mathcal{G}^c}) = \emptyset.$$

3.4.2. *The conjugate operator.* We pursue the analysis of $\Delta_{\mathcal{G}^c}$ in order to apply the Mourre theory to it. We go back to $\ell^2(\mathbb{N}, m_1^c) \otimes \ker(\Delta_2)$. We set:

$$(3.20) \quad \mathcal{A}_{\mathcal{G}^c}^{\text{le}} := T_{m_1^c \rightarrow 1}^{-1} \mathcal{A}_{\mathbb{N}} T_{m_1^c \rightarrow 1} \otimes P^{\text{le}}.$$

It is self-adjoint. Straightforwardly we get

$$\mathcal{A}_{\mathcal{G}^c}^{\text{le}} = -\frac{i}{2} \left(e^{-1/2} \left(Q + \frac{1}{2} \right) U^* + e^{1/2} \left(\frac{1}{2} - Q \right) U \right) \otimes P^{\text{le}}$$

on $\mathcal{C}_c(\mathbb{N}) \otimes \ker(\Delta_2)$. With respect to (3.17), we set

$$\mathcal{A}_{\mathcal{G}^c} := \mathcal{A}_{\mathcal{G}^c}^{\text{le}} \oplus \mathcal{A}_{\mathcal{G}^c}^{\text{he}}, \text{ where } \mathcal{A}_{\mathcal{G}^c}^{\text{he}} := 0.$$

By Lemma 3.2, it is essentially self-adjoint on $\mathcal{C}_c(\mathbb{N}) \otimes \ell^2(\mathcal{V}_2^c, m_2)$ and also on $\mathcal{C}_c(\mathcal{V}^c)$ by standard approximation. Keeping the notation of Lemma 3.9, we obtain:

Lemma 3.15. *We have $\Delta_{\mathcal{G}^c} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}^c})$ and*

$$(3.21) \quad [\Delta_{\mathcal{G}^c}, i\mathcal{A}_{\mathcal{G}^c}]_{\circ} \simeq w^c(\Delta_{\mathcal{G}^c}^{\text{le}}) \oplus 0,$$

with respect to (3.17), with $K \in \mathcal{K}(\ell^2(\mathcal{V}^c, m^c))$ and

$$w^c(x) := \frac{m_2}{2} \left(x - \frac{\alpha}{m_2} \right) \left(\frac{\beta}{m_2} - x \right).$$

In particular, $[\Delta_{\mathcal{G}^c}, i\mathcal{A}_{\mathcal{G}^c}]_{\circ} \in \mathcal{B}(\ell^2(\mathcal{V}^c, m^c))$.

Proof. As in Lemma 3.4, using Lemma 2.1, and working in the form sense on $\mathcal{C}_c(\mathbb{N}) \otimes \ell^2(\mathcal{V}_2^c)$, a straightforward computation leads to

$$(3.22) \quad \begin{aligned} \left[\Delta_{\mathcal{G}_1^c} \otimes \frac{1}{m_2^c}, i\mathcal{A}_{\mathcal{G}^c} \right] &\simeq \frac{1}{2} (\Delta_{\mathcal{G}_1^c} - \alpha) (\beta - \Delta_{\mathcal{G}_1^c}) \otimes \frac{1}{m_2} P^{\text{le}} \\ &\simeq w^c(\Delta_{\mathcal{G}_1^c}^{\text{le}}) \oplus 0. \end{aligned}$$

We turn to the second part of $\Delta_{\mathcal{G}^c}$.

$$(3.23) \quad \left[\frac{1}{m_1^c(\cdot)} \otimes \Delta_{\mathcal{G}_2^c}, i\mathcal{A}_{\mathcal{G}^c} \right] = \left[\frac{1}{m_1^c(\cdot)}, \mathcal{A}_{\mathcal{G}_1^c} \right] \otimes 0 = 0$$

This implies that $[\Delta_{\mathcal{G}^c}, i\mathcal{A}_{\mathcal{G}^c}]_{\circ} \in \mathcal{B}(\ell^2(\mathcal{V}^c, m^c))$ and (3.21).

It remains to prove that $\Delta_{\mathcal{G}^c} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}^c})$. We check the hypotheses of Lemma 2.3. Let $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$ be a family of functions defined on $\mathcal{V}_1^c \times \mathcal{V}_2^c$ as follows:

$$\mathcal{X}_n(x_1, x_2) := \left(\left(1 - \frac{x_1 - n}{n^2 + 1} \right) \vee 0 \right) \wedge 1.$$

Note that $\text{supp}(\mathcal{X}_n) = \llbracket 0, n^2 + n \rrbracket \times \mathcal{V}_2$ and $\forall (x_1, x_2) \in \llbracket 0, n \rrbracket \times \mathcal{V}_2^c$, $\mathcal{X}_n(x_1, x_2) = 1$. We set $\mathcal{D} := \mathcal{C}_c(\mathcal{V}^c)$.

1) We have $\|\mathcal{X}_n\|_{\infty} = 1$ then $\|\mathcal{X}_n(\cdot)\|_{\mathcal{B}(\ell^2(\mathcal{V}^c, m^c))} = 1$. Moreover, $\mathcal{X}_n(\cdot)$ tends strongly to 1 as $n \rightarrow +\infty$. Now, we shall show that $\sup_n \|\mathcal{X}_n(\cdot)\|_{\mathcal{D}(\Delta_{\mathcal{G}^c})} < \infty$. Since

$$\left[\Delta_{\mathcal{G}_1^c} \otimes \frac{1}{m_2} + \frac{1}{m_1^c(\cdot)} \otimes \Delta_{\mathcal{G}_2^c}, \mathcal{X}_n(\cdot) \right] = \underbrace{[\Delta_{\mathcal{G}_1^c}, \mathcal{X}_n(\cdot)] \otimes \frac{1}{m_2}}_{\text{bounded by } 2\|\Delta_{\mathcal{G}_1^c}\|/m_2} + \underbrace{\left[\frac{1}{m_1^c(\cdot)}, \mathcal{X}_n(\cdot) \right]}_{=0} \otimes \Delta_2,$$

then there is $c > 0$ such that, for all $f \in \mathcal{C}_c(\mathcal{V}^c)$ such that $f \in (\Delta_{\mathcal{G}^c} + i)\mathcal{C}_c(\mathcal{V}^c)$ and $n \in \mathbb{N}$,

$$\|(\Delta_{\mathcal{G}^c} + i)\mathcal{X}_n(Q)(\Delta_{\mathcal{G}^c} + i)^{-1}f\| \leq c\|f\|.$$

Since $\Delta_{\mathcal{G}^c}$ is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V}^c)$ and since $-i \notin \sigma(\Delta_{\mathcal{G}^c})$, it holds for all $f \in \ell^2(\mathcal{V}^c, m^c)$. In particular, we derive that $\|(\Delta_{\mathcal{G}^c} + i)\mathcal{X}_n(Q)f\| \leq c\|(\Delta_{\mathcal{G}^c} + i)f\|$, for all $f \in \ell^2(\mathcal{V}^c, m^c)$. In particular, $\sup_n \|\mathcal{X}_n(\cdot)\|_{\mathcal{D}(\Delta_{\mathcal{G}^c})} < \infty$.

2) Given $f \in \mathcal{C}_c(\mathcal{V}^c)$, note that for n large enough $\mathcal{X}_n(\cdot)f = f$. In particular, for all $f \in \mathcal{C}_c(\mathcal{V}^c)$, $\mathcal{A}_{\mathcal{G}^c}\mathcal{X}_n(\cdot)f \rightarrow \mathcal{A}_{\mathcal{G}^c}f$, as $n \rightarrow \infty$.

3) Noticing that $[\Delta_{\mathcal{G}^c}, \mathcal{X}_n(\cdot)] = [\Delta_{\mathcal{G}_1^c}, \mathcal{X}_n(\cdot)] \otimes \frac{1}{m_2}$, a straightforward computation ensures that there exists c such that

$$\|\mathcal{A}_{\mathcal{G}^c}[\Delta_{\mathcal{G}^c}, \mathcal{X}_n(\cdot)]\| \leq \frac{c}{\langle n \rangle}.$$

Finally for all $z \in \mathbb{C} \setminus \mathbb{R}$, the condition $\mathcal{X}_n(\cdot)(\Delta_{\mathcal{G}^c} - z)^{-1} \mathcal{C}_c(\mathcal{V}^c) \subset \mathcal{C}_c(\mathcal{V}^c)$ is immediate as \mathcal{X}_n is with finite support. [GoMo, Lemma A.2] gives that $\Delta_{\mathcal{G}^c}^{\text{le}} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}^c}^{\text{le}})$. \square

Lemma 3.16. *We have $e^{it\mathcal{A}_{\mathcal{G}^c}} \mathcal{D}(\Delta_{\mathcal{G}^c}) \subset \mathcal{D}(\Delta_{\mathcal{G}^c})$ for all $t \in \mathbb{R}$.*

Proof. We have $\Delta_{\mathcal{G}^c} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}^c})$ and $[\Delta_{\mathcal{G}^c}, i\mathcal{A}_{\mathcal{G}^c}]_{\circ}$ is bounded. Therefore [GeGé] gives the result. \square

Lemma 3.17. *We have $\Delta_{\mathcal{G}^c} \in \mathcal{C}^2(\mathcal{A}_{\mathcal{G}^c})$ and*

$$(3.24) \quad [[\Delta_{\mathcal{G}^c}, i\mathcal{A}_{\mathcal{G}^c}]_{\circ}, i\mathcal{A}_{\mathcal{G}^c}]_{\circ} \simeq [[\Delta_{\mathcal{G}^c}^{\text{le}}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}]_{\circ}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}]_{\circ} \oplus 0.$$

Proof. Recalling (3.21) and Lemma 2.1, the result follows from noticing that $w^c(\Delta_{\mathcal{G}^c}^{\text{le}})$ is in $\mathcal{C}^1(\mathcal{A}_{\mathcal{G}^c}^{\text{le}})$ as product of bounded elements of $\mathcal{C}^1(\mathcal{A}_{\mathcal{G}^c}^{\text{le}})$. \square

Concerning the Mourre estimate, we prove the following result:

Proposition 3.18. *We have $\Delta_{\mathcal{G}^c} \in \mathcal{C}^2(\mathcal{A}_{\mathcal{G}^c})$. Given a compact interval $\mathcal{I} \subset \left(\frac{\alpha}{m_2}, \frac{\beta}{m_2}\right)$, there are $c > 0$, a compact operator K such that*

$$(3.25) \quad E_{\mathcal{I}}(\Delta_{\mathcal{G}^c})[\Delta_{\mathcal{G}^c}, i\mathcal{A}_{\mathcal{G}^c}]_{\circ} E_{\mathcal{I}}(\Delta_{\mathcal{G}^c}) \geq cE_{\mathcal{I}}(\Delta_{\mathcal{G}^c}) + K,$$

in the form sense.

Proof. The Lemma 3.17 provides that $\Delta_{\mathcal{G}^c} \in \mathcal{C}^2(\mathcal{A}_{\mathcal{G}^c})$. On \mathcal{H}^{he} , $E_{\mathcal{I}}(\Delta_{\mathcal{G}^c}^{\text{he}})$ is compact since $\Delta_{\mathcal{G}^c}^{\text{he}}$ is with compact resolvent and \mathcal{I} is with compact support. With respect to (3.17), we have $E_{\mathcal{I}}(\Delta_{\mathcal{G}^c}) = E_{\mathcal{I}}(\Delta_{\mathcal{G}^c}^{\text{le}}) \oplus E_{\mathcal{I}}(\Delta_{\mathcal{G}^c}^{\text{he}})$ and

$$\begin{aligned} E_{\mathcal{I}}(\Delta_{\mathcal{G}^c})[\Delta_{\mathcal{G}^c}, i\mathcal{A}_{\mathcal{G}^c}]_{\circ} E_{\mathcal{I}}(\Delta_{\mathcal{G}^c}) &= E_{\mathcal{I}}(\Delta_{\mathcal{G}^c}^{\text{le}})[\Delta_{\mathcal{G}^c}^{\text{le}}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}]_{\circ} E_{\mathcal{I}}(\Delta_{\mathcal{G}^c}^{\text{le}}) \oplus 0 \\ &\geq cE_{\mathcal{I}}(\Delta_{\mathcal{G}^c}^{\text{le}}) \oplus 0 \geq cE_{\mathcal{I}}(\Delta_{\mathcal{G}^c}) + K, \end{aligned}$$

in the form sense, where K is a compact operator and

$$c := \frac{m_2}{2} \inf_{x \in \mathcal{I}} \left(x - \frac{\alpha}{m_2}\right) \left(\frac{\beta}{m_2} - x\right) > 0.$$

This concludes the proof. \square

To lighten the text we did not expand more consequences of the Mourre theory in this case and refer to Theorem 4.3 for them.

3.5. The compact part. We define the conjugate operator on $\ell^2(\mathcal{V}) = \ell^2(\mathcal{V}^{\text{f}}) \oplus \ell^2(\mathcal{V}^0) \oplus \ell^2(\mathcal{V}^c)$ as

$$\begin{aligned} \ell^2(\mathcal{V}) &= \ell^2(\mathcal{V}^{\text{f}}) \oplus \ell^2(\mathcal{V}^0) \oplus \ell^2(\mathcal{V}^c) \\ \mathcal{A} &:= \mathcal{A}_{\mathcal{G}^{\text{f}}} \oplus 0 \oplus \mathcal{A}_{\mathcal{G}^c}. \end{aligned}$$

Since \mathcal{V}^0 is finite, we have a finite rank perturbation and we conclude that \mathcal{A} is self-adjoint and essentially self-adjoint on $\mathcal{C}_c(\mathcal{V})$.

Lemma 3.19. *We have $\Delta_{\mathcal{G}} \in \mathcal{C}^2(\mathcal{A})$.*

Proof. We have $(\Delta_{\mathcal{G}} - \Delta_{\mathcal{G}^{\text{f}}} \oplus 0 \oplus \Delta_{\mathcal{G}^c})$ that are with finite support. Hence it belongs to $\mathcal{C}^2(\mathcal{A})$ by Lemma 2.1. Next recalling Lemma 3.11 and Lemma 3.17 we obtain the result. \square

3.6. The whole graph. In this section, we give the Mourre estimate in the whole graph.

Proposition 3.20. *We have $\Delta_{\mathcal{G}} \in \mathcal{C}^2(\mathcal{A})$. Given a compact interval $\mathcal{I} \subset (\frac{\alpha}{m_2}, \frac{\beta}{m_2})$ Moreover, there are $c > 0$, a compact operator K such that*

$$(3.26) \quad E_{\mathcal{I}}(\Delta_{\mathcal{G}})[\Delta_{\mathcal{G}}, i\mathcal{A}]_o E_{\mathcal{I}}(\Delta_{\mathcal{G}}) \geq cE_{\mathcal{I}}(\Delta_{\mathcal{G}}) + K.$$

Proof. First $\Delta_{\mathcal{G}} \in \mathcal{C}^2(\mathcal{A})$ by Lemma 3.19. Then by collecting (3.25) and (3.14), we obtain

$$\begin{aligned} E_{\mathcal{I}}(\Delta_{\mathcal{G}^f} \oplus 0 \oplus \Delta_{\mathcal{G}^c})[\Delta_{\mathcal{G}^f} \oplus 0 \oplus \Delta_{\mathcal{G}^c}, i\mathcal{A}]_o E_{\mathcal{I}}(\Delta_{\mathcal{G}^f} \oplus 0 \oplus \Delta_{\mathcal{G}^c}) \\ \geq cE_{\mathcal{I}}(\Delta_{\mathcal{G}^f} \oplus 0 \oplus \Delta_{\mathcal{G}^c}) + K. \end{aligned}$$

Since the operators $\Delta_{\mathcal{G}}$ and $\Delta_{\mathcal{G}^f} \oplus 0 \oplus \Delta_{\mathcal{G}^c}$ are in $\mathcal{C}_u^1(\mathcal{A})$ (as in $\mathcal{C}^2(\mathcal{A})$, see [AmBoGe]), [AmBoGe, Theorem 7.2.9] implies (3.26). \square

4. THE PERTURBED MODEL

In this section, we perturb the metrics of the previous case which will be small to infinity. We obtain similar results however the proof is more involved because we rely on the optimal class $\mathcal{C}^{1,1}(\mathcal{A})$ of the Mourre theory.

4.1. Perturbation of the metric. Let $\mathcal{G}_{\varepsilon, \mu} := (\mathcal{V}, \mathcal{E}_{\varepsilon}, m_{\mu})$ where

$$m_{\mu}(x) := (1 + \mu(x))m(x) \text{ and } \mathcal{E}_{\varepsilon}(x, y) := (1 + \varepsilon(x, y))\mathcal{E}(x, y),$$

where $\mu > -1$, $\varepsilon > -1$, and

$$(4.1) \quad \mu(x) \rightarrow 0 \text{ if } |x| \rightarrow \infty \text{ and } \varepsilon(x, y) \rightarrow 0 \text{ if } |x|, |y| \rightarrow \infty.$$

We set

$$m_{\mu}^* := m_{\mu}|_{\mathcal{V}^*}, \quad \mathcal{E}_{\varepsilon}^* := \mathcal{E}_{\varepsilon}|_{\mathcal{V}^* \times \mathcal{V}^*},$$

$\mu^* := \mu^*|_{\mathcal{V}^*}$, and $\varepsilon^* := \varepsilon|_{\mathcal{V}^* \times \mathcal{V}^*}$, with $*$ \in $\{c, f\}$.

To analyse the spectral properties of $\Delta_{\mathcal{G}_{\varepsilon, \mu}}$, we compare it to $\Delta_{\mathcal{G}}$. As they do not act in the same spaces, we rely on Proposition 3.1. and send $\Delta_{\mathcal{G}_{\varepsilon, \mu}}$ in $\ell^2(\mathcal{V}, m)$ with the help of the unitary transformation. Namely, supposing (4.1). Let

$$\tilde{\Delta}_{\mathcal{G}_{\varepsilon, \mu}} := T_{m_{\mu} \rightarrow m} \Delta_{\mathcal{G}_{\varepsilon, \mu}} T_{m_{\mu} \rightarrow m}^{-1}.$$

A straightforward calculus ensures:

Lemma 4.1. *For all $f \in \mathcal{C}_c(\mathcal{V})$, we have*

$$(4.2) \quad \begin{aligned} (\tilde{\Delta}_{\mathcal{G}_{\varepsilon, \mu}} - \Delta_{\mathcal{G}})f(x) &:= \frac{1}{m(x)} \sum_{y \sim x} \left(\frac{\varepsilon(x, y)}{\sqrt{(1 + \mu(x))(1 + \mu(y))}} \right. \\ &\quad \left. - \frac{\mu(x) + \mu(y) + \mu(x)\mu(y)}{\sqrt{(1 + \mu(x))(1 + \mu(z))}(1 + \sqrt{(1 + \mu(x))(1 + \mu(z))})}} \right) \mathcal{E}(x, z) (f(x) - f(y)) \\ &\quad - \frac{1}{m(x)} \sum_{z \sim x} (1 + \varepsilon(x, z)) \mathcal{E}(x, z) \frac{\mu(z) - \mu(x)}{(1 + \mu(x))\sqrt{1 + \mu(z)}(\sqrt{1 + \mu(z)} + \sqrt{1 + \mu(x)})} f(x). \end{aligned}$$

Proposition 4.2. *Let $V : \mathcal{V} \rightarrow \mathbb{R}$ be a function, obeying $V(x) \rightarrow 0$ if $|x| \rightarrow \infty$. We assume that (4.1) holds true, then $\tilde{\Delta}_{\mathcal{G}_{\varepsilon, \mu}} - \Delta_{\mathcal{G}} \in \mathcal{K}(\ell^2(\mathcal{V}), m)$. In particular*

- (1) $\mathcal{D}(\Delta_{\mathcal{G}_{\varepsilon, \mu}} + V(\cdot)) = \mathcal{D}(T_{m_{\mu} \rightarrow m}^{-1} \Delta_{\mathcal{G}} T_{m_{\mu} \rightarrow m})$,
- (2) $\Delta_{\mathcal{G}_{\varepsilon, \mu}} + V(\cdot)$ is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V})$,
- (3) $\sigma_{\text{ess}}(\Delta_{\mathcal{G}_{\varepsilon, \mu}} + V(\cdot)) = \sigma_{\text{ess}}(\Delta_{\mathcal{G}})$.

Proof. Use Propositions 4.4 and 4.9 and note that the contribution arising from \mathcal{V}^0 is a finite rank perturbation. \square

4.2. Main result. The main result of this section is the following theorem:

Theorem 4.3. *Let $\mathcal{G}_{\varepsilon,\mu}$ a graph satisfies a condition (4.1) and*

$$\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}} := \mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^f} \oplus 0 \oplus \mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^c}$$

be a self-adjoint operator, where $\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^} := T_{m_{\mu^*} \rightarrow m}^{-1} \mathcal{A}_{\mathcal{G}^*} T_{m_{\mu^*} \rightarrow m}$ with $*$ $\in \{f, c\}$. Let $V : \mathcal{V} \rightarrow \mathbb{R}$ be a function such that V, ε , and μ are radial on \mathcal{V}^c (see Definition 4.10). We assume that:*

$$\begin{aligned} (H1) \quad & \sup_{(x_1, x_2) \in \mathcal{V}^*} \langle x_1 \rangle^{1+\varepsilon} |V(x_1 - 1, x_2) - V(x_1, x_2)| < \infty, \\ (H2) \quad & \sup_{(x_1, x_2) \in \mathcal{V}^*} \langle x_1 \rangle^{\varepsilon+1} |\mu^*(x_1 - 1, x_2) - \mu^*(x_1, x_2)| < \infty, \\ (H3) \quad & \sup_{(x_1, x_2) \in \mathcal{V}^*} \langle x_1 \rangle^{\varepsilon+1} |\varepsilon^*((x_1, x_2), (x_1 + 1, x_2)) - \varepsilon^*((x_1 - 1, x_2), (x_1, x_2))| < \infty, \end{aligned}$$

where $V(x) \rightarrow 0$ if $|x| \rightarrow \infty$. Then $\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot) \in \mathcal{C}^{1,1}(\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}})$. Moreover, for all compact interval $I \subset (\frac{\alpha}{m_2}, \frac{\beta}{m_2})$, with α, β are given in (1.2), there are $c > 0$ and a compact operator K such that

$$(4.3) \quad \begin{aligned} E_I(\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot))[\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot), i\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}}] \circ E_I(\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot)) \\ \geq cE_I(\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot)) + K. \end{aligned}$$

in the form sense. Set $\kappa(\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot)) := \sigma_p(\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot)) \cup \{\frac{\alpha}{m_2}, \frac{\beta}{m_2}\}$ where σ_p denotes the pure point spectrum. Take $s > 1/2$ and $[a, b] \subset \mathbb{R} \setminus \kappa(\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot))$. We obtain:

- (2) *The eigenvalues of $\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot)$ distinct from α and β are of finite multiplicity and can accumulate only toward α and β .*
- (3) *The singular continuous spectrum of $\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot)$ is empty.*
- (4) *The following limit exists and finite:*

$$\lim_{\rho \rightarrow 0} \sup_{\lambda \in [a, b]} \|\langle \Lambda \rangle^{-s} (\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot) - \lambda - i\rho)^{-1} \langle \Lambda \rangle^{-s}\| < \infty,$$

- (5) *There exists $c > 0$ such that for all $f \in \ell^2(\mathcal{V}, m_\mu)$, we have:*

$$\int_{\mathbb{R}} \|\langle \Lambda \rangle^{-s} e^{-it(\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot))} E_{[a, b]}(\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot)) f\|^2 dt \leq c\|f\|^2,$$

with $\Lambda := \Lambda^f \oplus 0 \oplus \Lambda^c$.

Proof. First $\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot) \in \mathcal{C}^{1,1}(\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}})$ because $\Delta_{\mathcal{G}_{\varepsilon,\mu}^f} \oplus 0 \oplus \Delta_{\mathcal{G}_{\varepsilon,\mu}^c} \in \mathcal{C}^{1,1}(\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}})$ by the Lemma 4.8, the Lemma 4.14, the Lemma 4.13, the Lemma 4.7 and by Lemma 3.19. In particular, we have that the two operators are in $\mathcal{C}_u^1(\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}})$, see [AmBoGe].

Then, using the Proposition 4.5 and the Proposition 4.11 we obtain

$$\begin{aligned} E_I(\Delta_{\mathcal{G}_{\varepsilon,\mu}^f} \oplus 0 \oplus \Delta_{\mathcal{G}_{\varepsilon,\mu}^c} + V(\cdot))[\Delta_{\mathcal{G}_{\varepsilon,\mu}^f} \oplus 0 \oplus \Delta_{\mathcal{G}_{\varepsilon,\mu}^c} + V(\cdot), i\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}}] \circ \\ E_I(\Delta_{\mathcal{G}_{\varepsilon,\mu}^f} \oplus 0 \oplus \Delta_{\mathcal{G}_{\varepsilon,\mu}^c} + V(\cdot)) \\ \geq cE_I(\Delta_{\mathcal{G}_{\varepsilon,\mu}^f} \oplus 0 \oplus \Delta_{\mathcal{G}_{\varepsilon,\mu}^c} + V(\cdot)) + K. \end{aligned}$$

Since $\Delta_{\mathcal{G}_{\varepsilon,\mu}^f} \oplus 0 \oplus \Delta_{\mathcal{G}_{\varepsilon,\mu}^c} \in \mathcal{C}_u^1(\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}})$, $\Delta_{\mathcal{G}_{\varepsilon,\mu}^f} \oplus 0 \oplus \Delta_{\mathcal{G}_{\varepsilon,\mu}^c} - \Delta_{\mathcal{G}_{\varepsilon,\mu}} \in \mathcal{K}(\ell^2(\mathcal{V}, m_\mu))$, and $V(\cdot) \in \mathcal{C}_u^1(\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}})$ and by [AmBoGe, Theorem 7.2.9], we obtain (4.3). By Lemma 4.7 and Lemma 4.13, $V(\cdot) \in \mathcal{C}^{1,1}(\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}})$. And by using Proposition 4.4 and Proposition 4.9, we have that $(\Delta_{\mathcal{G}_{\varepsilon,\mu}^f} \oplus 0 \oplus \Delta_{\mathcal{G}_{\varepsilon,\mu}^c} + i)^{-1} - (\Delta_{\mathcal{G}_{\varepsilon,\mu}} + i)^{-1} \in \mathcal{K}(\ell^2(\mathcal{V}, m_\mu))$. Finally, we turn to points (4). It is enough to obtain them with $s \in (1/2, 1)$. We apply [AmBoGe, Proposition 7.5.6] and obtain

$$\lim_{\rho \rightarrow 0} \|\langle \mathcal{A} \rangle^{-s} (\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot) - \lambda - i\rho)^{-1} \langle \mathcal{A} \rangle^{-s}\|,$$

exists and finite. Using Propositions 4.6 b) and 4.12 b)

$$\|\langle \mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}} \rangle f\| \leq a \|\langle \Lambda \rangle f\|,$$

for all $f \in \mathcal{D}(\Lambda)$. By Riesz-Thorin interpolation, there is $a_s > 0$ such that

$$\|\langle \mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}} \rangle^s f\| \leq a_s \|\langle \Lambda \rangle^s f\|,$$

for all $f \in \mathcal{D}(\Lambda^s)$. We conclude that $\lim_{\rho \rightarrow 0} \|\langle \Lambda \rangle^{-s} (\Delta_{\mathcal{G}_{\varepsilon,\mu}} + V(\cdot) - \lambda - i\rho)^{-1} \langle \Lambda \rangle^{-s}\|$ exists and finite. The point (5) is an immediate consequence of (4). \square

4.3. The funnel side. We first deal with the question of the essential spectrum.

Proposition 4.4. *Let $V^f : \mathcal{V}^f \rightarrow \mathbb{R}$ be a function obeying $V^f(x) \rightarrow 0$ if $|x| \rightarrow \infty$. We assume that (4.1) holds true then $\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}}^f - \Delta_{\mathcal{G}^f} \in \mathcal{K}(\ell^2(\mathcal{V}^f), m^f)$, where $\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}} := T_{m_\mu \rightarrow m} \Delta_{\mathcal{G}_{\varepsilon,\mu}} T_{m_\mu \rightarrow m}^{-1}$. In particular,*

- (1) $\mathcal{D}(\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}}^f + V(\cdot)) = \mathcal{D}(T_{m_\mu \rightarrow m}^{-1} \Delta_{\mathcal{G}} T_{m_\mu \rightarrow m})$,
- (2) $\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}}^f + V(\cdot)$ is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V})$,
- (3) $\sigma_{\text{ess}}(\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}}^f + V(\cdot)) = \sigma_{\text{ess}}(\Delta_{\mathcal{G}})$.

Proof. We shall show that $\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}}^f - \Delta_{\mathcal{G}^f} \in \mathcal{K}(\ell^2(\mathcal{V}^f), m^f)$, as in (4.2). Let $f \in \mathcal{C}_c(\mathcal{V})$,

$$\begin{aligned} |\langle f, (\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}}^f - \Delta_{\mathcal{G}^f}) f \rangle_{\ell^2(\mathcal{V}^f, m^f)}| &= \left| \sum_{x \in \mathcal{V}^f} m^f(x) \left((\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}}^f - \Delta_{\mathcal{G}^f}) f \right)(x) \overline{f(x)} \right| \\ &\leq \left| \sum_{x \in \mathcal{V}^f} m^f(x) \frac{1}{m^f(x)} \sum_{z \sim x} \frac{\varepsilon^f(x, z)}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(z))}} \mathcal{E}^f(x, z) \right. \\ &\quad \left. \times (f(x) - f(z)) \overline{f(x)} \right| \\ &\quad + \left| \sum_{x \in \mathcal{V}^f} m^f(x) \frac{1}{m^f(x)} \sum_{z \sim x} \frac{1 - \sqrt{(1 + \mu^f(x))(1 + \mu^f(z))}}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(z))}} \mathcal{E}^f(x, z) \right. \\ &\quad \left. \times (f(x) - f(z)) \overline{f(x)} \right| + |\langle f, W^f(\cdot) f \rangle| \\ &\leq 2 \langle f, (\deg_1(\cdot) + \deg_2(\cdot) + |W^f(\cdot)|) f \rangle, \end{aligned}$$

with

$$\deg_1(x) := \frac{1}{m^f(x)} \sum_{z \in \mathcal{V}^f} \frac{\varepsilon^f(x, z)}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(z))}} \mathcal{E}^f(x, z)$$

and

$$\deg_2(x) := \frac{1}{m^f(x)} \sum_{z \in \mathcal{V}^f} \left| \frac{1 - \sqrt{(1 + \mu^f(x))(1 + \mu^f(z))}}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(z))}} \right| \mathcal{E}^f(x, z),$$

for all $x = (x_1, x_2) \in \mathcal{V}^f$. We have

$$\begin{aligned} |\deg_1(x)| &= \left| \frac{1}{m^f(x)} \sum_{z \in \mathcal{V}^f} \frac{\varepsilon^f(x, z)}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(z))}} \mathcal{E}^f(x, z) \right| \\ &\leq \sup_{z \sim x} \left| \frac{\varepsilon^f(x, z)}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(z))}} \right| \deg_{\mathcal{G}^f}(x). \end{aligned}$$

Since \mathcal{V}_2 is a finite set and for all $x_2 \in \mathcal{V}_2$, $\frac{\varepsilon^f((x_1, x_2), (z_1, z_2))}{\sqrt{(1 + \mu^f(x_1, x_2))(1 + \mu^f(z_1, z_2))}} \rightarrow 0$ when $x_1, z_1 \rightarrow \infty$ and since $\deg_{\mathcal{G}^f}(\cdot)$ is bounded then $\deg_1(\cdot)$ is compact. In the same way, using that $\forall x_2, z_2 \in \mathcal{V}_2$, $\frac{1 - \sqrt{(1 + \mu^f(x_1, x_2))(1 + \mu^f(z_1, z_2))}}{\sqrt{(1 + \mu^f(x_1, x_2))(1 + \mu^f(z_1, z_2))}} \rightarrow 0$ if $x_1, z_1 \rightarrow \infty$, we obtain the compactness of $\deg_2(\cdot)$.

Now, we will show that $W^f \in \mathcal{K}(\ell^2(\mathcal{V}^f, m^f))$. For all $x \in \mathcal{V}^f$, we have

$$\begin{aligned} |W^f(x)| &= \left| \frac{1}{m^f(x)} \sum_{z \sim x} (1 + \varepsilon^f(x, z)) \mathcal{E}^f(x, z) \right. \\ &\quad \times \left. \left(\frac{\mu^f(z) - \mu^f(x)}{(1 + \mu^f(x))\sqrt{1 + \mu^f(z)}(\sqrt{1 + \mu^f(z)} + \sqrt{1 + \mu^f(x)})} \right) \right| \\ &\leq \sup_{z \sim x} \left| (1 + \varepsilon^f(x, z)) \left(\frac{\mu^f(z) - \mu^f(x)}{(1 + \mu^f(x))\sqrt{1 + \mu^f(z)}(\sqrt{1 + \mu^f(z)} + \sqrt{1 + \mu^f(x)})} \right) \right| \\ &\quad \times \deg_{\mathcal{G}^f}(x). \end{aligned}$$

Since \mathcal{V}_2 is a finite set and $(1 + \varepsilon^f(x, z))(\mu^f(z) - \mu^f(x)) \rightarrow 0$ when $|x|, |z| \rightarrow \infty$, $\deg_{\mathcal{G}^f}(\cdot)$ is bounded and since $V^f(\cdot)$ is a compact perturbation, we conclude that $\tilde{\Delta}_{\mathcal{G}_{\varepsilon, \mu}^f} - \Delta_{\mathcal{G}^f}$ is compact. The points (1) and (2) follow from Theorem [ReSi, Theorem XIII.14] and (3) from the Weyl's Theorem. \square

We turn to the Mourre estimate.

Proposition 4.5. *Let $V^f : \mathcal{V}^f \rightarrow \mathbb{R}$ be a function. We assume that (H1), (H2), and (H3) hold true, where $\varepsilon^f(x, z) \rightarrow 0$ if $|x|, |z| \rightarrow \infty$, $\mu^f(x) \rightarrow 0$ if $|x| \rightarrow \infty$ and $V^f(x) \rightarrow 0$ if $|x| \rightarrow \infty$. Then $\Delta_{\mathcal{G}_{\varepsilon, \mu}^f} + V^f(\cdot) \in \mathcal{C}^{1,1}(\mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^f})$. Moreover, for all compact interval $I \subset (\frac{\alpha}{m_2}, \frac{\beta}{m_2})$, there are $c > 0$, a compact operator K such that*

$$(4.4) \quad \begin{aligned} E_I(\Delta_{\mathcal{G}_{\varepsilon, \mu}^f} + V^f(\cdot))[\Delta_{\mathcal{G}_{\varepsilon, \mu}^f} + V^f(\cdot), i\mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^f}] \circ E_I(\Delta_{\mathcal{G}_{\varepsilon, \mu}^f} + V^f(\cdot)) \\ \geq cE_I(\Delta_{\mathcal{G}_{\varepsilon, \mu}^f} + V^f(\cdot)) + K, \end{aligned}$$

in the form sense.

Proof. The Proposition 4.8 and Lemma 4.7 give that $\Delta_{\mathcal{G}_{\varepsilon, \mu}^f} + V^f(\cdot) \in \mathcal{C}^{1,1}(\mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^f})$. Since $\tilde{\Delta}_{\mathcal{G}_{\varepsilon, \mu}^f} - \Delta_{\mathcal{G}^f}$ is a compact operator by Proposition 4.4, thanks to (3.14) and by [AmBoGe, Theorem 7.2.9], we obtain (4.4). \square

We start with a technical lemma so as to apply [AmBoGe, Proposition 7.5.7].

Proposition 4.6. *Let $\Lambda^f := (Q + 1/2) \otimes \mathbf{1}_{\mathcal{V}^f}$. It satisfies the following assertions:*

(1) $e^{i\Lambda^f t} \mathcal{D}(\Delta_{\mathcal{G}_{\varepsilon, \mu}^f}) \subset \mathcal{D}(\Delta_{\mathcal{G}_{\varepsilon, \mu}^f})$ and there exists a finite constant c , such that

$$\|e^{i\Lambda^f t}\|_{\mathcal{B}(\mathcal{D}(\Delta_{\mathcal{G}_{\varepsilon, \mu}^f}))} \leq c, \quad \text{for all } t \in \mathbb{R}.$$

(2) $\mathcal{D}(\Lambda^f) \subset \mathcal{D}(\mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^f})$.

(3) $(\Lambda^f)^{-2}(\mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^f})^2$ extends to a continuous operator in $\mathcal{D}(\Delta_{\mathcal{G}_{\varepsilon, \mu}^f})$.

Note that $\Delta_{\mathcal{G}_{\varepsilon, \mu}^f}$ is bounded then $\mathcal{D}(\Delta_{\mathcal{G}_{\varepsilon, \mu}^f}) = \ell^2(\mathcal{V}^f, m_{\mu}^f)$.

Proof. With the help of the unitary transformation $T_{m_{\mu} \rightarrow m}$, it is enough to prove the result with $\varepsilon = 0$ and $\mu = 0$.

(1) Since $\Delta_{\mathcal{G}^f}$ is bounded it is verified by a functional calculus.

(2) Let $f \in \mathcal{C}_c(\mathcal{V}^f)$,

$$\begin{aligned} &\|\mathcal{A}_{\mathcal{G}^f} f\|_{\ell^2(\mathcal{V}^f, m^f)}^2 \\ &= \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e^{1/2(Q-1/2)U} \otimes \mathbf{1}_{\mathcal{V}_2^f} - e^{-1/2(Q+1/2)U^*} \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) f(x) \right|^2 \\ &\leq c \sum_{x \in \mathcal{V}^f} m^f(x) \left| (Q+1/2) \otimes \mathbf{1}_{\mathcal{V}_2^f} f(x) \right|^2 \leq c \|\Lambda^f f\|_{\ell^2(\mathcal{V}^f, m^f)}^2. \end{aligned}$$

Since Λ^f is essentially self-adjoint, the result follows.

(3) For all $f \in \mathcal{C}_c(\mathcal{V}^f)$, and by using the relations of Subsection 3.5, we have

$$\begin{aligned} \mathcal{A}_{m_1}^2 f(n) &= \frac{1}{4}(2n^2 + 1/2)f(n) - \frac{1}{4}e(n-1/2)(n-3/2)f(n-2) \\ &\quad - \frac{1}{4}e^{-1}(n+1/2)(n+3/2)f(n+2). \end{aligned}$$

Then for all $f \in \mathcal{C}_c(\mathcal{V}^f)$.

$$\begin{aligned} \|(\Lambda^f)^{-2}(\mathcal{A}_{\mathcal{G}^f})^2 f\|^2 &= \\ &\sum_{(x_1, x_2) \in \mathcal{V}^f} m^f(x_1, x_2) \left| \frac{1}{4} \left(\left((Q+1/2)^{-2}(2Q^2+1/2) \right) \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) f(x_1, x_2) \right. \\ &\quad - \frac{1}{4}e \left(\left((Q+1/2)^{-2}(Q-1/2)(Q-3/2) \right) \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) f(x_1-2, x_2) \\ &\quad \left. - \frac{1}{4}e^{-1} \left(\left((Q+1/2)^{-2}(Q+1/2)(Q+3/2) \right) \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) f(x_1+2, x_2) \right|^2. \end{aligned}$$

Then, there exists $C > 0$ such that for all $f \in \mathcal{C}_c(\mathcal{V}^f)$, $\|(\Lambda^f)^{-2}(\mathcal{A}_{\mathcal{G}^f})^2 f\|^2 \leq C\|f\|^2$. By density, we find the result. \square

The proof of Proposition 4.8 will be long and technical. For the sake of the reader, we have separated the treatment of the potential V^f to present the technical steps.

Lemma 4.7. *Let $V^f : \mathcal{V}^f \rightarrow \mathbb{R}$ be a function. We assume that (H1) holds true, then $V^f(\cdot) \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^f})$ and $[V^f(\cdot), \mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^f}]_{\circ} \in \mathcal{C}^{0,1}(\mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^f})$. In particular, $V^f(\cdot) \in \mathcal{C}^{1,1}(\mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^f})$.*

Proof. First, recalling $[V^f(\cdot), \mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^f}]_{\circ} = T_{m_{\mu^*} \rightarrow m}^{-1} [V^f(\cdot), \mathcal{A}_{\mathcal{G}^f}]_{\circ} T_{m_{\mu^*} \rightarrow m}$, it is enough to deal with $\varepsilon = \mu = 0$. Next, we recall that

$$[V^f(\cdot), \mathcal{A}_{\mathcal{G}^f}]_{\circ} = \frac{e^{-1/2}}{2} \left(Q + \frac{1}{2} \right) [V^f, U^*] \otimes \mathbf{1}_{\mathcal{V}_2^f} + \frac{e^{1/2}}{2} \left(\frac{1}{2} - Q \right) [V^f, U] \otimes \mathbf{1}_{\mathcal{V}_2^f}.$$

By using (H1) at the last step, there is C such that, for all $f \in \mathcal{C}_c(\mathcal{V})$,

$$\begin{aligned} &\| \langle Q + 1/2 \rangle^{\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2^f} [V^f(\cdot), \mathcal{A}_{\mathcal{G}^f}]_{\circ} f \| \\ &\leq \frac{e^{-1/2}}{2} \| \langle Q + 1/2 \rangle^{\varepsilon} (Q + 1/2) [V^f, U^*] \otimes \mathbf{1}_{\mathcal{V}_2^f} f \| \\ &\quad + \frac{e^{1/2}}{2} \| \langle Q + 1/2 \rangle^{\varepsilon} (Q + 1/2) [V^f, U] \otimes \mathbf{1}_{\mathcal{V}_2^f} f \| \\ &\leq \frac{e^{-1/2}}{2} \| \langle \Lambda \rangle^{\varepsilon+1} ([V^f, U^*] \otimes \mathbf{1}_{\mathcal{V}_2^f}) f \| + \frac{e^{1/2}}{2} \| \langle \Lambda \rangle^{\varepsilon+1} ([V^f, U] \otimes \mathbf{1}_{\mathcal{V}_2^f}) f \| \leq C\|f\|. \end{aligned}$$

Finally thanks to Proposition 4.6, we can apply [AmBoGe, Proposition 7.5.7] and the result follows. \square

We conclude this section with the most technical part.

Proposition 4.8. *Assuming (H2) and (H3) hold true, we have $\Delta_{\mathcal{G}_{\varepsilon, \mu}^f} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^f})$. Moreover $[\Delta_{\mathcal{G}_{\varepsilon, \mu}^f}, \mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^f}]_{\circ} \in \mathcal{C}^{0,1}(\mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^f})$. In particular, $\Delta_{\mathcal{G}_{\varepsilon, \mu}^f} \in \mathcal{C}^{1,1}(\mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^f})$.*

Proof. We work in $\ell^2(\mathcal{V}^f, m^f)$. First, using the computation below with $\varepsilon = 0$ and recalling that $\mathcal{A}_{\mathcal{G}^f} \mathcal{C}_c(\mathcal{V}^f) \subset \mathcal{C}_c(\mathcal{V}^f)$, we get there is $c > 0$ such that

$$\| [\tilde{\Delta}_{\mathcal{G}_{\varepsilon, \mu}^f}, \mathcal{A}_{\mathcal{G}^f}] f \|_{\ell^2(\mathcal{V}^f, m^f)} \leq c \| f \|_{\ell^2(\mathcal{V}^f, m^f)},$$

for all $f \in \mathcal{C}_c(\mathcal{V}^f)$. By density, we obtain that $\tilde{\Delta}_{\mathcal{G}_{\varepsilon, \mu}^f} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}^f})$. Next take $\varepsilon > 0$ and $f \in \mathcal{C}_c(\mathcal{V}^f)$. We aim at proving that $[\tilde{\Delta}_{\mathcal{G}_{\varepsilon, \mu}^f}, \mathcal{A}_{\mathcal{G}^f}]_0$ is $\mathcal{C}^{0,1}(\mathcal{A}_{\mathcal{G}^f})$.

$$\begin{aligned}
& \|\langle \Lambda^f \rangle^\varepsilon [\tilde{\Delta}_{\mathcal{G}_{\varepsilon, \mu}^f}, \mathcal{A}_{\mathcal{G}^f}] f\|_{\ell^2(\mathcal{V}^f, m^f)} = \sum_{x \in \mathcal{V}^f} m^f(x) |\langle \Lambda^f \rangle^\varepsilon [\tilde{\Delta}_{\mathcal{G}_{\varepsilon, \mu}^f}, \mathcal{A}_{\mathcal{G}^f}] f(x)|^2 \\
& \leq \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e^{1/2} \langle Q - 1/2 \rangle^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \right. \\
(4.5) \quad & \times \left(\frac{1}{m^f(x)} \sum_{z \sim x} \mathcal{E}^f(x, z) \frac{1 + \varepsilon^f(x, z)}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(z))}} f(x_1 - 1, x_2) \right. \\
& \left. - \frac{1}{m^f(x_1 - 1, x_2)} \sum_{z \sim x} \mathcal{E}^f((x_1 - 1, x_2), z) f(z_1 - 1, z_2) \times \frac{1 + \varepsilon^f((x_1 - 1, x_2), z)}{\sqrt{(1 + \mu^f(x_1 - 1, x_2))(1 + \mu^f(z))}} \right) \Big|^2 \\
& + \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e^{1/2} \langle Q - 1/2 \rangle^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \right. \\
& \times \left(\frac{1}{m^f(x_1 - 1, x_2)} \sum_{z \sim x} \mathcal{E}^f((x_1 - 1, x_2), z) \frac{1 + \varepsilon^f((x_1 - 1, x_2), z)}{\sqrt{(1 + \mu^f(x_1 - 1, x_2))(1 + \mu^f(z))}} f(z) \right. \\
& \left. - \frac{1}{m^f(x)} \sum_{z \sim x} \mathcal{E}^f(x, z) \frac{1 + \varepsilon^f(x, z)}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(z))}} f(z_1 - 1, z_2) \right) \Big|^2 \\
& + \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e^{-1/2} \langle Q + 1/2 \rangle^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \right. \\
& \times \left(\frac{1}{m^f(x_1 + 1, x_2)} \sum_{z \sim x} \mathcal{E}^f((x_1 + 1, x_2), z) \frac{1 + \varepsilon^f((x_1 + 1, x_2), z)}{\sqrt{(1 + \mu^f(x_1 + 1, x_2))(1 + \mu^f(z))}} \right. \\
& \left. - \frac{1}{m^f(x)} \sum_{z \sim x} \mathcal{E}^f(x, z) \frac{1 + \varepsilon^f(x, z)}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(z))}} f(x_1 + 1, x_2) \right) \Big|^2 \\
& + \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e^{-1/2} \langle Q + 1/2 \rangle^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \right. \\
& \times \left(\frac{1}{m^f(x)} \sum_{z \sim x} \mathcal{E}^f(x, z) \frac{1 + \varepsilon^f(x, z)}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(z))}} f(z_1 + 1, z_2) \right. \\
& \left. - \frac{1}{m^f(x_1 + 1, x_2)} \sum_{z \sim x} \mathcal{E}^f((x_1 + 1, x_2), z) \frac{1 + \varepsilon^f((x_1 + 1, x_2), z)}{\sqrt{(1 + \mu^f(x_1 + 1, x_2))(1 + \mu^f(z))}} f(z_1, z_2) \right) \Big|^2 \\
& + \|\langle \Lambda^f \rangle^\varepsilon [W^f(\cdot), \mathcal{A}_{\mathcal{G}^f}] f\|_{\ell^2(\mathcal{V}^f, m^f)},
\end{aligned}$$

with

$$\begin{aligned}
[W^f(\cdot), \mathcal{A}_{\mathcal{G}^f}] f(x) &= \frac{i}{2} e^{1/2} \left((Q - 1/2) \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \frac{1}{m^f(x)} \sum_{z \sim x} (1 + \varepsilon^f(x, z)) \mathcal{E}^f(x, z) U f(x) \\
(4.6) \quad & \times \left(\frac{\mu^f(z) - \mu^f(x)}{(1 + \mu^f(x)) \sqrt{1 + \mu^f(z)} (\sqrt{1 + \mu^f(z)} + \sqrt{1 + \mu^f(x)})} \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{i}{2}e^{1/2}\left((Q-1/2)\otimes\mathbf{1}_{\mathcal{V}_2^f}\right)\frac{1}{m^f(x_1-1,x_2)}\sum_{z\sim x}(1+\varepsilon^f((x_1-1,x_2),(z_1-1,z_2))) \\
& \times\left(\frac{\mu^f(z_1-1,z_2)-\mu^f(x_1-1,x_2)}{(1+\mu^f(x_1-1,x_2))\sqrt{1+\mu^f(z_1-1,z_2)}(\sqrt{1+\mu^f(z_1-1,z_2)}+\sqrt{1+\mu^f(x_1-1,x_2)})}\right) \\
& \quad \times\mathcal{E}^f((x_1-1,x_2),(z_1-1,z_2))Uf(x) \\
& +\frac{i}{2}e^{-1/2}\left((Q+1/2)\otimes\mathbf{1}_{\mathcal{V}_2^f}\right)\frac{1}{m^f(x_1+1,x_2)}\sum_{z\sim x}(1+\varepsilon^f((x_1+1,x_2),(z_1+1,z_2))) \\
& \times\left(\frac{\mu^f(z_1+1,z_2)-\mu^f(x_1+1,x_2)}{(1+\mu^f(x_1+1,x_2))\sqrt{1+\mu^f(z_1+1,z_2)}(\sqrt{1+\mu^f(z_1+1,z_2)}+\sqrt{1+\mu^f(x_1+1,x_2)})}\right) \\
& \quad \times\mathcal{E}^f((x_1+1,x_2),(z_1+1,z_2))U^*f(x) \\
& -\frac{i}{2}e^{-1/2}\left((Q+1/2)\otimes\mathbf{1}_{\mathcal{V}_2^f}\right)\frac{1}{m^f(x)}\sum_{z\sim x}(1+\varepsilon^f(x,z))\mathcal{E}^f(x,z)U^*f(x) \\
& \quad \times\left(\frac{\mu^f(z)-\mu^f(x)}{(1+\mu^f(x))\sqrt{1+\mu^f(z)}(\sqrt{1+\mu^f(z)}+\sqrt{1+\mu^f(x)})}\right).
\end{aligned}$$

We treat the first term of $\|\langle\Lambda^f\rangle^\epsilon[\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}^f},\mathcal{A}_{\mathcal{G}^f}]f\|_{\ell^2(\mathcal{V}^f,m^f)}$ in (4.5).

$$\begin{aligned}
& \sum_{x\in\mathcal{V}^f}m^f(x)\left|\frac{i}{2}\left(e^{1/2}\langle Q-1/2\rangle^{1+\epsilon}\otimes\mathbf{1}_{\mathcal{V}_2^f}\right)\sum_{z\sim x}\left(\frac{\mathcal{E}^f(x,z)(1+\varepsilon^f(x,z))}{m^f(x)\sqrt{(1+\mu^f(x))(1+\mu^f(z))}}\right.\right. \\
& \quad \left.\left.-\frac{\mathcal{E}^f((x_1-1,x_2),z)(1+\varepsilon^f((x_1-1,x_2),z))}{m(x_1-1,x_2)\sqrt{(1+\mu^f(x_1-1,x_2))(1+\mu^f(z))}}\right)f(x_1-1,x_2)\right|^2 \\
& \leq 2\sum_{x\in\mathcal{V}^f}m^f(x)\left|\frac{i}{2}\left(e^{1/2}\langle Q-1/2\rangle^{1+\epsilon}\otimes\mathbf{1}_{\mathcal{V}_2^f}\right)\right. \\
(4.7) \quad & \times\sum_{z_1\sim x_1}\delta_{z_2=x_2}\left(\frac{\mathcal{E}_1^f(x_1,z_1)(1+\varepsilon^f(x,z))}{m(x)\sqrt{(1+\mu^f(x))(1+\mu^f(z))}}\right. \\
& \quad \left.-\frac{\mathcal{E}_1^f(x_1-1,z_1)(1+\varepsilon^f((x_1-1,x_2),z))}{m^f(x_1-1,x_2)\sqrt{(1+\mu^f(x_1-1,x_2))(1+\mu^f(z))}}\right)f(x_1-1,x_2)\Big|^2 \\
& +2\sum_{x\in\mathcal{V}^f}m^f(x)\left|\frac{i}{2}\left(e^{1/2}\langle Q-1/2\rangle^{1+\epsilon}\otimes\mathbf{1}_{\mathcal{V}_2^f}\right)\right. \\
(4.8) \quad & \times\sum_{z_2\sim x_2}\delta_{z_1=x_1}\left(\frac{\mathcal{E}_2^f(x_2,z_2)(1+\varepsilon^f(x,z))}{m(x)\sqrt{(1+\mu^f(x))(1+\mu^f(z))}}\right. \\
& \quad \left.-\frac{\mathcal{E}_2^f(x_2,z_2)(1+\varepsilon^f((x_1-1,x_2),z))}{m_1^f(x_1-1)m_2\sqrt{(1+\mu^f(x_1-1,x_2))(1+\mu^f(z))}}\right)f(x_1-1,x_2)\Big|^2.
\end{aligned}$$

We bound (4.7) as follows:

$$\begin{aligned}
(4.7) &\leq 4 \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e \langle Q - 1/2 \rangle^{1+\epsilon} \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \frac{1}{m_2} \right. \\
(4.9) &\quad \times \left(\frac{\sqrt{1 + \mu^f(x_1 - 1, x_2)} (1 + \varepsilon^f(x, (x_1 + 1, x_2)))}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(x_1 + 1, x_2))(1 + \mu^f(x_1 - 1, x_2))}} \right. \\
&\quad \left. - \frac{\sqrt{1 + \mu^f(x_1 + 1, x_2)} (1 + \varepsilon^f((x_1 - 1, x_2), (x_1, x_2)))}{\sqrt{(1 + \mu^f(x_1 - 1, x_2))(1 + \mu^f(x))(1 + \mu^f(x_1 + 1, x_2))}} \right) f(x_1 - 1, x_2) \Big|^2 \\
&\quad + 4 \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(\langle Q - 1/2 \rangle^{1+\epsilon} \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \frac{1}{m_2} \right. \\
(4.10) &\quad \times \left(\frac{\sqrt{1 + \mu^f(x_1 - 2, x_2)} (1 + \varepsilon^f(x, (x_1 - 1, x_2)))}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(x_1 - 1, x_2))(1 + \mu^f(x_1 - 2, x_2))}} \right. \\
&\quad \left. - \frac{\sqrt{1 + \mu^f(x_1 - 1, x_2)} (1 + \varepsilon^f((x_1 - 1, x_2), (x_1 - 2, x_2)))}{\sqrt{(1 + \mu^f(x_1 - 1, x_2))(1 + \mu^f(x_1 - 2, x_2))(1 + \mu^f(x_1 - 1, x_2))}} \right) f(x_1 - 1, x_2) \Big|^2,
\end{aligned}$$

Now, we concentrate on (4.9) and in the same way, we deal with (4.10). Since the assertions (H2) and (H3) hold true then there exists an integer c , such that

$$\begin{aligned}
&4 \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e \langle Q - 1/2 \rangle^{1+\epsilon} \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \right. \\
&\quad \times \frac{1}{m_2} \left(\frac{\sqrt{1 + \mu^f(x_1 - 1, x_2)} (1 + \varepsilon^f(x, (x_1 + 1, x_2)))}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(x_1 + 1, x_2))(1 + \mu^f(x_1 - 1, x_2))}} \right. \\
&\quad \left. - \frac{\sqrt{1 + \mu^f(x_1 + 1, x_2)} (1 + \varepsilon^f((x_1 - 1, x_2), (x_1, x_2)))}{\sqrt{(1 + \mu^f(x_1 - 1, x_2))(1 + \mu^f(x))(1 + \mu^f(x_1 + 1, x_2))}} \right) f(x_1 - 1, x_2) \Big|^2 \\
&= 4 \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e \langle Q - 1/2 \rangle^{1+\epsilon} \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \frac{1}{m_2} \right. \\
&\quad \times \left(\frac{(\mu^f(x_1 - 1, x_2) - \mu^f(x_1 + 1, x_2)) \varepsilon^f(x, (x_1 + 1, x_2))}{(\sqrt{1 + \mu^f(x_1 - 1, x_2)} + \sqrt{1 + \mu^f(x_1 + 1, x_2)})} \right. \\
&\quad \times \frac{1}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(x_1 + 1, x_2))(1 + \mu^f(x_1 - 1, x_2))}} \\
&\quad + \frac{\sqrt{1 + \mu^f(x_1 + 1, x_2)} (\varepsilon^f(x, (x_1 + 1, x_2)) - \varepsilon^f((x_1 - 1, x_2), x))}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(x_1 + 1, x_2))(1 + \mu^f(x_1 - 1, x_2))}} \\
&\quad + \frac{\mu^f(x_1 - 1, x_2) - \mu^f(x_1 + 1, x_2)}{(\sqrt{1 + \mu^f(x_1 - 1, x_2)} + \sqrt{1 + \mu^f(x_1 + 1, x_2)})} \\
&\quad \times \left. \frac{1}{\sqrt{(1 + \mu^f(x))(1 + \mu^f(x_1 + 1, x_2))(1 + \mu^f(x_1 - 1, x_2))}} \right) f(x_1 - 1, x_2) \Big|^2 \\
&\leq c \|f\|_{\ell^2(\mathcal{V}^f, m^f)}^2.
\end{aligned}$$

In the same way, we treat (4.8) and $\|\langle \Lambda^f \rangle^\epsilon [W^f(\cdot), \mathcal{A}_{\mathcal{G}^f}] f\|_{\ell^2(\mathcal{V}^f, m^f)}^2$. By density, there exists $c > 0$ such that $\|\langle \Lambda^f \rangle^\epsilon [\tilde{\Delta}_{\mathcal{G}_{\varepsilon, \mu}^f}, \mathcal{A}_{\mathcal{G}^f}] f\|_{\ell^2(\mathcal{V}^f, m^f)}^2 \leq c \|f\|_{\ell^2(\mathcal{V}^f, m^f)}^2$. Finally, by applying [AmBoGe, Proposition 7.5.7] where the hypotheses are verified in Proposition 4.6, we find the result. \square

4.4. The cusp side: Radial metric perturbation. We recall that

$$\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}^c} f(x) := T_{m_\mu \rightarrow m} \Delta_{\mathcal{G}_{\varepsilon,\mu}^c} T_{m_\mu \rightarrow m}^{-1} f(x), \quad \text{for all } f \in \mathcal{C}_c(\mathcal{V}^c).$$

We first deal with the question of the essential spectrum.

Proposition 4.9. *Let $V^c : \mathcal{V}^c \rightarrow \mathbb{R}$ be a function obeying $V^c(x) \rightarrow 0$ if $|x| \rightarrow \infty$. We assume that (4.1) holds true then $\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}^c} - \Delta_{\mathcal{G}^c} \in \mathcal{K}(\ell^2(\mathcal{V}^c, m^c))$. In particular,*

- (1) $\mathcal{D}(\Delta_{\mathcal{G}_{\varepsilon,\mu}^c} + V(\cdot)) = \mathcal{D}(T_{m_\mu \rightarrow m}^{-1} \Delta_{\mathcal{G}} T_{m_\mu \rightarrow m})$,
- (2) $\Delta_{\mathcal{G}_{\varepsilon,\mu}^c} + V(\cdot)$ is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V})$,
- (3) $\sigma_{\text{ess}}(\Delta_{\mathcal{G}_{\varepsilon,\mu}^c} + V(\cdot)) = \sigma_{\text{ess}}(\Delta_{\mathcal{G}})$.

Proof. Let $f \in \mathcal{C}_c(\mathcal{V}^c)$, we have

$$\begin{aligned} & | \langle f, (\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}^c} - \Delta_{\mathcal{G}^c}) f \rangle_{\ell^2(\mathcal{V}^c, m^c)} | = \left| \sum_{x \in \mathcal{V}^c} m^c(x) (\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}^c} - \Delta_{\mathcal{G}^c}) f(x) \overline{f(x)} \right| \\ & \leq \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} \frac{\varepsilon^c(x, (z_1, x_2))}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(z_1, x_2))}} \\ & \quad \times \mathcal{E}_1^c(x_1, z_1) |f(x)|^2 \\ & + 1/2 \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} \frac{\varepsilon^c(x, (z_1, x_2))}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(z_1, x_2))}} \\ & \quad \times \mathcal{E}^c(x, z) (|f(z)|^2 + |f(x)|^2) \\ & + \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} \left| \frac{\mu^c(x) + \mu^c(z_1, x_2) + \mu^c(x)\mu^c(z_1, x_2)}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(z_1, x_2))}} \right| \\ & \quad \times \frac{1}{\sqrt{1 + \mu^c(x) + \sqrt{1 + \mu^c(z_1, x_2)}}} \mathcal{E}_1^c(x_1, z_1) |f(x)|^2 \\ & + 1/2 \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} \left| \frac{\mu^c(x) + \mu^c(z_1, x_2) + \mu^c(x)\mu^c(z_1, x_2)}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(z_1, x_2))}} \right| \\ & \quad \times \frac{1}{\sqrt{1 + \mu^c(x) + \sqrt{1 + \mu^c(z_1, x_2)}}} \mathcal{E}_1^c(x_1, z_1) |f(z_1, x_2)|^2 \\ & + 1/2 \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} \left| \frac{\mu^c(x) + \mu^c(z_1, x_2) + \mu^c(x)\mu^c(z_1, x_2)}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(z_1, x_2))}} \right| \\ & \quad \times \frac{1}{\sqrt{1 + \mu^c(x) + \sqrt{1 + \mu^c(z_1, x_2)}}} \mathcal{E}_1^c(x_1, z_1) |f(x)|^2 \\ & \leq 2 \langle f, (\text{deg}_3(\cdot) + \text{deg}_4(\cdot) + |W^c(\cdot)|) f \rangle, \end{aligned}$$

where

$$\text{deg}_3(x) := \frac{1}{m^c(x)} \sum_{z_1 \in \mathcal{V}_1^c} \frac{\varepsilon^c(x, (z_1, x_2))}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(z_1, x_2))}} \mathcal{E}_1^c(x_1, z_1)$$

and

$$\begin{aligned} \text{deg}_4(x) := & \frac{1}{m^c(x)} \sum_{z_1 \in \mathcal{V}_1^c} \left| \frac{\mu^c(x) + \mu^c(z_1, x_2) + \mu^c(x)\mu^c(z_1, x_2)}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(z_1, x_2))}(\sqrt{1 + \mu^c(x) + \sqrt{1 + \mu^c(z_1, x_2))}} \right| \\ & \times \mathcal{E}_1^c(x_1, z_1). \end{aligned}$$

We have

$$| \text{deg}_3(x) | \leq \sup_{z_1 \sim x_1} \left| \frac{\varepsilon^c(x, (z_1, x_2))}{m_2 \sqrt{(1 + \mu^c(x))(1 + \mu^c(z_1, x_2))}} \right| \text{deg}_{\mathcal{G}_1^c}(x).$$

Since V_2 is a finite set and for all $x_2 \in \mathcal{V}_2$, $\frac{\varepsilon^c((x_1, x_2), (z_1, x_2))}{\sqrt{(1+\mu^c(x_1, x_2))(1+\mu^c(z_1, x_2))}} \rightarrow 0$ when $x_1, z_1 \rightarrow \infty$ and since $\deg_{\mathcal{G}_1^c}(\cdot)$ is bounded then $\deg_3(\cdot)$ is compact. In the same way, using that $\forall x_2 \in \mathcal{V}_2$, $\frac{\mu^c(x) + \mu^c(z_1, x_2) + \mu^c(x)\mu^c(z_1, x_2)}{\sqrt{(1+\mu^c(x_1, x_2))(1+\mu^c(z_1, x_2))}(\sqrt{1+\mu^c(x)} + \sqrt{1+\mu^c(z_1, x_2)})} \rightarrow 0$ if $x_1, z_1 \rightarrow \infty$, we obtain the compactness of $\deg_4(\cdot)$.

Now, we will show that $W^c(\cdot) \in \mathcal{K}(\ell^2(\mathcal{V}^c, m^c))$. For all $x \in \mathcal{V}^c$, we have

$$\begin{aligned} |W^c(x)| &= \left| \frac{1}{m^c(x)} \sum_{z \sim x} \left(\frac{\mu^c(z) - \mu^c(x)}{(1 + \mu^c(x))\sqrt{1 + \mu^c(z)}(\sqrt{1 + \mu^c(z)} + \sqrt{1 + \mu^c(x)})} \right) \right. \\ &\quad \left. \times (1 + \varepsilon^c(x, z))\mathcal{E}^c(x, z) \right| \\ &\leq \sup_{z_1 \sim x_1} (1 + \varepsilon^c(x, (z_1, x_2))) \deg_{\mathcal{G}_1^c}(x) \\ &\quad \times \left| \frac{\mu^c(z_1, x_2) - \mu^c(x)}{(1 + \mu^c(x))\sqrt{1 + \mu^c(z_1, x_2)}(\sqrt{1 + \mu^c(z_1, x_2)} + \sqrt{1 + \mu^c(x)})} \right| \end{aligned}$$

Since \mathcal{V}_2 is a finite set and $\forall x_2 \in \mathcal{V}_2$, $\varepsilon^c((x_1, x_2), (z_1, x_2))(\mu^c(z_1, x_2) - \mu^c(x_1, x_2)) \rightarrow 0$ when $x_1, z_1 \rightarrow \infty$, and since $\deg_{\mathcal{G}_1^c}(\cdot)$ is bounded and since $V^c(\cdot)$ is a compact perturbation. Then, $\tilde{\Delta}_{\mathcal{G}_{\varepsilon, \mu}^c} - \Delta_{\mathcal{G}^c}$ is a compact operator. The points (1) and (2) follow from Theorem [ReSi, Theorem XIII.14] and (3) from the Weyl's Theorem. \square

In order to go into the Mourre theory, we construct the conjugate operator:

$$(4.11) \quad \mathcal{A}_{\mathcal{G}^c} := \mathcal{A}_{m_1} \otimes P^{1e},$$

with

$$\begin{aligned} \mathcal{A}_{m_1} &:= T_{1 \rightarrow m_1} \mathcal{A}_{\mathbb{N}} T_{1 \rightarrow m_1}^{-1} \\ &:= \frac{i}{2} \left(e^{1/2}(Q - 1/2)U - e^{-1/2}(Q + 1/2)U^* \right). \end{aligned}$$

It is self-adjoint and essentially self-adjoint on $\mathcal{C}^c(\mathcal{V}^c)$ by Lemma 3.2. Because of the projection in (4.11), we restrict to radial perturbations.

Definition 4.10. *The perturbations V^c , μ and ε are called **radial** if they do not depend on the second variable, i.e., For all $(x_1, x_2), (z_1, z_2) \in \mathcal{V}^c$, we have $V^c(x_1, x_2) = V^c(x_1, z_2)$, $\mu(x_1, x_2) = \mu(x_1, z_2)$ and $\varepsilon((x_1, x_2), (z_1, z_2)) = \varepsilon((x_1, x_2), (z_1, x_2))$.*

We turn to Mourre estimate.

Proposition 4.11. *Let $\mathcal{G}_{\varepsilon, \mu}^c$ a graph satisfies a condition (4.1). Suppose that $V^c : \mathcal{V}^c \rightarrow \mathbb{R}$, ε and μ are radial and assume (H1), (H2), and (H3) and $V^c(x) \rightarrow 0$ if $|x| \rightarrow \infty$. Then $\Delta_{\mathcal{G}_{\varepsilon, \mu}^c} + V^c(\cdot) \in \mathcal{C}^{1,1}(\mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^c})$. Moreover, for all compact interval $I \subset (\frac{\alpha}{m_2}, \frac{\beta}{m_2})$ there are $c > 0$, a compact operator K such that*

$$(4.12) \quad \begin{aligned} E_I(\Delta_{\mathcal{G}_{\varepsilon, \mu}^c} + V^c(\cdot))[\Delta_{\mathcal{G}_{\varepsilon, \mu}^c} + V^c(\cdot), i\mathcal{A}_{\mathcal{G}_{\varepsilon, \mu}^c}] \circ E_I(\Delta_{\mathcal{G}_{\varepsilon, \mu}^c} + V^c(\cdot)) \\ \geq cE_I(\Delta_{\mathcal{G}_{\varepsilon, \mu}^c} + V^c(\cdot)) + K, \end{aligned}$$

in the form sense.

Proof. The Proposition 4.14 and Lemma 4.13 gives that $\Delta_{\mathcal{G}_{\varepsilon, \mu}^c} \in \mathcal{C}^{1,1}(\mathcal{A}_{\mathcal{G}^c})$. Since $\tilde{\Delta}_{\mathcal{G}_{\varepsilon, \mu}^c} - \Delta_{\mathcal{G}^c}$ is a compact operator by Proposition 4.9, thanks to (3.25) and by [AmBoGe, Theorem 7.2.9] we obtain (4.12). \square

We turn to series of Lemmata. To be able to apply the [AmBoGe, Proposition 7.5.7], we check the next point.

Proposition 4.12. *Let $\Lambda^c := (Q + 1/2) \otimes \mathbf{1}_{\mathcal{V}^c}$, then Λ^c satisfies the following assertions:*

- (1) $e^{i\Lambda^c t} \mathcal{D}(\Delta_{\mathcal{G}_{\varepsilon, \mu}^c}) \subset \mathcal{D}(\Delta_{\mathcal{G}_{\varepsilon, \mu}^c})$ and there exists a finite constant c , such that
- $$\|e^{i\Lambda^c t}\|_{\mathcal{B}(\mathcal{D}(\Delta_{\mathcal{G}_{\varepsilon, \mu}^c}))} \leq c, \quad \text{for all } t \in \mathbb{R}.$$

- (2) $\mathcal{D}(\Lambda^c) \subset \mathcal{D}(\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^c})$.
(3) $(\Lambda^c)^{-2}(\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^c})^2$ extends to a continuous operator in $\mathcal{D}(\Delta_{\mathcal{G}_{\varepsilon,\mu}^c})$.

Proof. With the help of the unitary transformation $T_{m_\mu \rightarrow m}$, it is enough to prove the result with $\varepsilon = 0$ and $\mu = 0$.

(1) We have

$$[\Delta_{\mathcal{G}^c}, e^{i\Lambda^c t}] = [\Delta_{\mathcal{G}_1^c}, e^{i\Lambda^c t}] \otimes \frac{1}{m_2} + [\frac{1}{m_1(\cdot)}, e^{i\Lambda^c t}] \otimes \Delta_2.$$

Since $\frac{1}{m_1(\cdot)}$ and $e^{i\Lambda^c t}$ commute and since $[\Delta_{\mathcal{G}_1^c}, e^{i\Lambda^c t}]$ is uniformly bounded, then there exists $c > 0$ such that for all $f \in \mathcal{C}_c(\mathcal{V}^c)$

$$\|(\Delta_{\mathcal{G}^c} + i)e^{i\Lambda^c t}(\Delta_{\mathcal{G}^c} + i)^{-1}f\|_{\ell^2(\mathcal{V}^c, m^c)} \leq c\|f\|_{\ell^2(\mathcal{V}^c, m^c)}.$$

Hence, there exists $c > 0$ such that for all $f \in \mathcal{C}_c(\mathcal{V}^c)$

$$\|(\Delta_{\mathcal{G}^c} + i)e^{i\Lambda^c t}f\|_{\ell^2(\mathcal{V}^c, m^c)} \leq c\|(\Delta_{\mathcal{G}^c} + i)f\|_{\ell^2(\mathcal{V}^c, m^c)}.$$

Since $\Delta_{\mathcal{G}^c}$ is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V}^c)$ then we find the result.

(2) Let $f \in \mathcal{C}_c(\mathcal{V}^c)$, by using the relations of Subsection 3.5, we have

$$\begin{aligned} \|\mathcal{A}_{\mathcal{G}^c}f\|_{\ell^2(\mathcal{V}^c, m^c)}^2 &\leq \frac{1}{2} \sum_{x \in \mathcal{V}^f} m^f(x) \left| e^{1/2}(Q-1/2)U \otimes P^{le}f(x) \right|^2 \\ &\quad + \left| e^{-1/2}(Q+1/2)U^* \otimes P^{le}f(x) \right|^2 \\ &\leq c \sum_{x \in \mathcal{V}^f} m^f(x) |(Q+1/2) \otimes P^{le}f(x)|^2 \leq c\|\Lambda^c f\|_{\ell^2(\mathcal{V}^f, m^f)}^2. \end{aligned}$$

Since, Λ^c is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V}^c)$. we find the result.

(3) First for all $f \in \mathcal{C}_c(\mathcal{V}^c)$, we have

$$\begin{aligned} \|(\Lambda^c)^{-2}(\mathcal{A}_{\mathcal{G}^c})^2 f\|_{\ell^2(\mathcal{V}^c, m^c)}^2 &= \sum_{(x_1, x_2) \in \mathcal{V}^c} m^c(x_1, x_2) |\Lambda^{-2}(\mathcal{A}_{\mathcal{G}^c})^2 f(x, y)|^2 \\ &= \sum_{(x_1, x_2) \in \mathcal{V}^c} m^c(x_1, x_2) \left| \frac{1}{4} \left(((Q+1/2)^{-2}(2Q^2+1/2)) \otimes P^{le} \right) f(x_1, x_2) \right. \\ &\quad \left. - \frac{1}{4} e \left(((Q+1/2)^{-2}(Q-1/2)(Q-3/2)) \otimes P^{le} \right) f(x_1-2, x_2) \right. \\ &\quad \left. - \frac{1}{4} e^{-1} \left(((Q+1/2)^{-2}(Q+1/2)(Q+3/2)) \otimes P^{le} \right) f(x_1+2, x_2) \right|^2. \end{aligned}$$

By density, we get $(\Lambda^c)^{-2}(\mathcal{A}_{\mathcal{G}^c})^2$ is a bounded operator. Since Λ^c is a radial operator and $\Delta_{\mathcal{G}_1^c}$ is bounded then there exists $C > 0$ such that, for all $f \in \mathcal{C}_c(\mathcal{V}^c)$,

$$\begin{aligned} &\|[\Delta_{\mathcal{G}^c}, (\Lambda^c)^{-2}(\mathcal{A}_{\mathcal{G}^c})^2]f\|_{\ell^2(\mathcal{V}^c, m^c)} \\ &= \left\| \left([\Delta_{\mathcal{G}_1^c} \otimes \frac{1}{m_2}, (\Lambda^c)^{-2}(\mathcal{A}_{\mathcal{G}^c})^2] + [\frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2^c}, (\Lambda^c)^{-2}(\mathcal{A}_{\mathcal{G}^c})^2] \right) f \right\|_{\ell^2(\mathcal{V}^c, m^c)} \\ &= \left\| \left([\Delta_{\mathcal{G}_1^c} \otimes \frac{1}{m_2}, (\Lambda^c)^{-2}(\mathcal{A}_{\mathcal{G}^c})^2] + (\Lambda^c)^{-2} [\frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2^c}, (\mathcal{A}_{\mathcal{G}^c})^2] \right) f \right\|_{\ell^2(\mathcal{V}^c, m^c)} \\ &= \left\| [\Delta_{\mathcal{G}_1^c} \otimes \frac{1}{m_2}, (\Lambda^c)^{-2}(\mathcal{A}_{\mathcal{G}^c})^2] f \right\|_{\ell^2(\mathcal{V}^c, m^c)} \leq C\|f\|_{\ell^2(\mathcal{V}^c, m^c)}. \end{aligned}$$

We have used $[\frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2^c}, (\mathcal{A}_{\mathcal{G}^c})^2] = 0$ by construction. Conclude by density. \square

The proof of Proposition 4.14 is long and technical. For the sake of the reader, we have separated the treatment of the potential V^c to present the technical steps.

Lemma 4.13. *Let $V^c : \mathcal{V}^c \rightarrow \mathbb{R}$ be a radial function and (H1) holds true, then $[V^c(\cdot), \mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^c}]_0 \in \mathcal{C}^{0,1}(\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^c})$. In particular, $V^c(\cdot) \in \mathcal{C}^{1,1}(\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^c})$.*

Proof. Since V^c is radial, by a slight abuse of notation, we have $V^c := V^c \otimes \mathbf{1}_{\mathcal{V}_2}$. We compute the commutator on $\mathcal{C}^c(\mathcal{V}^c)$ and get

$$\left[V^c(\cdot), i\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^c} \right] = \left(\frac{e^{-1/2}}{2} \left(Q + \frac{1}{2} \right) [V^c, U^*] + \frac{e^{1/2}}{2} \left(\frac{1}{2} - Q \right) [V^c, U] \right) \otimes P^{\text{le}}.$$

By density, we infer that $\left[V^c(\cdot), i\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^c} \right]$ extends to a bounded operator and that $V^c(\cdot) \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^c})$. Next, there exists $C > 0$ so that, for all $f \in \mathcal{C}_c(\mathcal{V}^c)$,

$$\begin{aligned} \left\| \langle \Lambda \rangle^\varepsilon \otimes [V^c(\cdot), i\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^c}] \circ f \right\| &\leq \frac{e^{-1/2}}{2} \left\| \langle (Q + 1/2)^\varepsilon (Q + 1/2) [V^c, U^*] \rangle \otimes P^{\text{le}} f \right\| \\ &\quad + \frac{e^{1/2}}{2} \left\| \langle (Q + 1/2)^\varepsilon (Q + 1/2) [V^c, U] \rangle \otimes P^{\text{le}} f \right\| \\ &\leq \frac{e^{-1/2}}{2} \left\| \langle \Lambda \rangle^{\varepsilon+1} [V^c, U^*] \right\| \otimes P^{\text{le}} f \left\| \right. \\ &\quad \left. + \frac{e^{1/2}}{2} \left\| \langle \Lambda \rangle^{\varepsilon+1} [V^c, U] \right\| \otimes P^{\text{le}} f \right\| \leq C \|f\|, \quad \text{by (H1)}. \end{aligned}$$

Finally, the result follow by applying [AmBoGe, Proposition 7.5.7] where the hypotheses are verified in Proposition 4.12. \square

Here is the most technical part:

Proposition 4.14. *Assuming (H2) and (H3), we have $\Delta_{\mathcal{G}_{\varepsilon,\mu}^c} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^c})$. Moreover $[\Delta_{\mathcal{G}_{\varepsilon,\mu}^c}, \mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^c}] \circ \circ \in \mathcal{C}^{0,1}(\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^c})$. In particular, $\Delta_{\mathcal{G}_{\varepsilon,\mu}^c} \in \mathcal{C}^{1,1}(\mathcal{A}_{\mathcal{G}_{\varepsilon,\mu}^c})$.*

Proof. We work in $\ell^2(\mathcal{V}^c, m^c)$. We first prove that $\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}^c} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}^c})$. By the computation below (with $\varepsilon = 0$), we obtain that there is $c > 0$ such that

$$\| [\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}^c}, \mathcal{A}_{\mathcal{G}^c}] f \|_{\ell^2(\mathcal{V}^c, m^c)} \leq c \|f\|_{\ell^2(\mathcal{V}^c, m^c)}, \quad \forall f \in \mathcal{C}_c(\mathcal{V}^c).$$

Using Lemma 3.16 and [AmBoGe, Theorem 6.3.4], this implies that $\tilde{\Delta}_{\mathcal{G}_{\varepsilon,\mu}^c} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}^c})$.

We turn to the $\mathcal{C}^{0,1}$ property. We assume that (H2) and (H3) are true then

$$\begin{aligned}
& \|\langle \Lambda^c \rangle^\epsilon [\tilde{\Delta} \mathcal{G}_{\varepsilon, \mu}^c, \mathcal{A} \mathcal{G}^c] \circ f\|_{\ell^2(\mathcal{V}^c, m^c)} \leq \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{2} (e^{1/2} \langle Q - 1/2 \rangle^{1+\epsilon} \otimes P^{le}) \\
& \quad \times \left(\frac{1}{m^c(x)} \sum_{z \sim x} \mathcal{E}^c(x, z) \frac{1 + \varepsilon^c(x, z)}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(z))}} \right. \\
(4.13) \quad & \quad - \frac{1}{m^c(x_1 - 1, x_2)} \sum_{z \sim x} \mathcal{E}^c((x_1 - 1, x_2), z) \frac{1 + \varepsilon^c((x_1 - 1, x_2), z)}{\sqrt{(1 + \mu^c(x_1 - 1, x_2))(1 + \mu^c(z))}} \\
& \quad \times f(x_1 - 1, x_2)|^2 \\
& \quad + \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{2} (e^{1/2} \langle Q - 1/2 \rangle^{1+\epsilon} \otimes P^{le}) \\
& \quad \times \left(\frac{1}{m^c(x_1 - 1, x_2)} \sum_{z \sim x} \mathcal{E}^c((x_1 - 1, x_2), z) \frac{1 + \varepsilon^c((x_1 - 1, x_2), z)}{\sqrt{(1 + \mu^c(x_1 - 1, x_2))(1 + \mu^c(z))}} f(z) \right. \\
& \quad - \frac{1}{m^c(x)} \sum_{z \sim x} \mathcal{E}^c(x, z) \frac{1 + \varepsilon^c(x, z)}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(z))}} f(z_1 - 1, z_2)|^2 \\
& \quad + \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{2} (e^{-1/2} \langle Q + 1/2 \rangle^{1+\epsilon} \otimes P^{le}) \\
& \quad \times \left(\frac{1}{m^c(x_1 + 1, x_2)} \sum_{z \sim x} \mathcal{E}^c((x_1 + 1, x_2), z) \frac{1 + \varepsilon^c((x_1 + 1, x_2), z)}{\sqrt{(1 + \mu^c(x_1 + 1, x_2))(1 + \mu^c(z))}} \right. \\
& \quad - \frac{1}{m^c(x)} \sum_{z \sim x} \mathcal{E}^c(x, z) \frac{1 + \varepsilon^c(x, z)}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(z))}} \left. \right) f(x_1 + 1, x_2)|^2 \\
& \quad + \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{2} (e^{-1/2} \langle Q + 1/2 \rangle^{1+\epsilon} \otimes P^{le}) \\
& \quad \times \left(\frac{1}{m^c(x)} \sum_{z \sim x} \mathcal{E}^c(x, z) \frac{1 + \varepsilon^c(x, z)}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(z))}} f(z_1 + 1, z_2) \right. \\
& \quad - \frac{1}{m^c(x_1 + 1, x_2)} \sum_{z \sim x} \mathcal{E}^c((x_1 + 1, x_2), z) \frac{1 + \varepsilon^c((x_1 + 1, x_2), z)}{\sqrt{(1 + \mu^c(x_1 + 1, x_2))(1 + \mu^c(z))}} f(z_1, z_2)|^2 \\
& \quad \left. + \|\langle \Lambda^c \rangle^\epsilon [W^c, \mathcal{A} \mathcal{G}^c] \circ f\|_{\ell^2(\mathcal{V}^c, m^c)}.
\end{aligned}$$

We treat the first term of $\|\langle \Lambda^c \rangle^\epsilon [\tilde{\Delta} \mathcal{G}_{\varepsilon, \mu}^c, \mathcal{A} \mathcal{G}^c] \circ f\|_{\ell^2(\mathcal{V}^c, m^c)}$ in (4.13)

$$\begin{aligned}
& \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{2} \left(e^{1/2} \langle Q - 1/2 \rangle^{1+\epsilon} \otimes P^{le} \right) \\
& \quad \times \sum_{z \sim x} \left(\frac{\mathcal{E}^c(x, z)(1 + \varepsilon^c(x, z))}{m^c(x) \sqrt{(1 + \mu^c(x))(1 + \mu^c(z))}} \right. \\
& \quad \quad \left. - \frac{\mathcal{E}^c((x_1 - 1, x_2), z)(1 + \varepsilon^c((x_1 - 1, x_2), z))}{m^c(x_1 - 1, x_2) \sqrt{(1 + \mu^c(x_1 - 1, x_2))(1 + \mu^c(z))}} \right) f(x_1 - 1, x_2)|^2 \\
& \leq 2 \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{2} \left(e^{1/2} \langle Q - 1/2 \rangle^{1+\epsilon} \otimes P^{le} \right) \\
(4.14) \quad & \quad \times \sum_{z_1 \sim x_1} \delta_{z_2 = x_2} \left(\frac{\mathcal{E}_1^c(x_1, z_1)(1 + \varepsilon^c(x, z))}{m(x) \sqrt{(1 + \mu^c(x))(1 + \mu^c(z))}} \right. \\
& \quad \quad \left. - \frac{\mathcal{E}_1^c(x_1 - 1, z_1)(1 + \varepsilon^c((x_1 - 1, x_2), z))}{m^c(x_1 - 1, x_2) \sqrt{(1 + \mu^c(x_1 - 1, x_2))(1 + \mu^c(z))}} \right) f(x_1 - 1, x_2)|^2
\end{aligned}$$

$$\begin{aligned}
& + 2 \sum_{x \in \mathcal{V}^c} m^c(x) \Big| \frac{i}{2} \left(e^{1/2} (Q - 1/2)^{1+\epsilon} \otimes P^{le} \right) \\
(4.15) \quad & \times \sum_{z_2 \sim x_2} \delta_{z_1=x_1} \left(\frac{\mathcal{E}_2(x_2, z_2)(1 + \varepsilon^c(x, z))}{m(x) \sqrt{(1 + \mu^c(x))(1 + \mu^c(z))}} \right. \\
& \left. - \frac{\mathcal{E}_2(x_2, z_2)(1 + \varepsilon^c((x_1 - 1, x_2), z))}{m_1^c(x_1 - 1) m_2 \sqrt{(1 + \mu^c(x_1 - 1, x_2))(1 + \mu^c(z))}} \right) f(x_1 - 1, x_2) \Big|^2.
\end{aligned}$$

We focus on (4.14).

$$\begin{aligned}
(4.14) & \leq 4 \sum_{x \in \mathcal{V}^c} m^c(x) \Big| \frac{i}{2} \left(e (Q - 1/2)^{1+\epsilon} \otimes P^{le} \right) \\
(4.16) \quad & \times \frac{1}{m_2} \left(\frac{\sqrt{1 + \mu^c(x_1 - 1, x_2)}(1 + \varepsilon^c(x, (x_1 + 1, x_2)))}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(x_1 + 1, x_2))(1 + \mu^c(x_1 - 1, x_2))}} \right. \\
& \left. - \frac{\sqrt{1 + \mu^c(x_1 + 1, x_2)}(1 + \varepsilon^c((x_1 - 1, x_2), (x_1, x_2)))}{\sqrt{(1 + \mu^c(x_1 - 1, x_2))(1 + \mu^c(x))(1 + \mu^c(x_1 + 1, x_2))}} \right) f(x_1 - 1, x_2) \Big|^2 \\
& + 4 \sum_{x \in \mathcal{V}^c} m^c(x) \Big| \frac{i}{2} \left((Q - 1/2)^{1+\epsilon} \otimes P^{le} \right) \\
& \times \frac{1}{m_2} \left(\frac{\sqrt{1 + \mu^c(x_1 - 2, x_2)}(1 + \varepsilon^c(x, (x_1 - 1, x_2)))}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(x_1 - 1, x_2))(1 + \mu^c(x_1 - 2, x_2))}} \right. \\
(4.17) \quad & \left. - \frac{\sqrt{1 + \mu^c(x_1 - 1, x_2)}(1 + \varepsilon^c((x_1 - 1, x_2), (x_1 - 2, x_2)))}{\sqrt{(1 + \mu^c(x_1 - 1, x_2))(1 + \mu^c(x_1 - 2, x_2))(1 + \mu^c(x_1 - 1, x_2))}} \right) \\
& \times f(x_1 - 1, x_2) \Big|^2.
\end{aligned}$$

Now, we concentrate on (4.16). (4.17) can be done in the same way. Since the assertions (H2) and (H3) hold true then there exists an integer c , such that

$$\begin{aligned}
& 4 \sum_{x \in \mathcal{V}^c} m^c(x) \Big| \frac{i}{2} \left(e (Q - 1/2)^{1+\epsilon} \otimes P^{le} \right) \\
& \times \frac{1}{m_2} \left(\frac{\sqrt{1 + \mu^c(x_1 - 1, x_2)}(1 + \varepsilon^c(x, (x_1 + 1, x_2)))}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(x_1 + 1, x_2))(1 + \mu^c(x_1 - 1, x_2))}} \right. \\
& \left. - \frac{\sqrt{1 + \mu^c(x_1 + 1, x_2)}(1 + \varepsilon^c((x_1 - 1, x_2), (x_1, x_2)))}{\sqrt{(1 + \mu^c(x_1 - 1, x_2))(1 + \mu^c(x))(1 + \mu^c(x_1 + 1, x_2))}} \right) f(x_1 - 1, x_2) \Big|^2 \\
& = 4 \sum_{x \in \mathcal{V}^c} m^c(x) \Big| \frac{i}{2} \left(e (Q - 1/2)^{1+\epsilon} \otimes P^{le} \right) \frac{1}{m_2} \\
& \times \left(\frac{(\mu^c(x_1 - 1, x_2) - \mu^c(x_1 + 1, x_2)) \varepsilon^c(x, (x_1 + 1, x_2))}{(\sqrt{1 + \mu^c(x_1 - 1, x_2)} + \sqrt{1 + \mu^c(x_1 + 1, x_2)})} \right. \\
& \left. \times \frac{1}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(x_1 + 1, x_2))(1 + \mu^c(x_1 - 1, x_2))}} \right)
\end{aligned}$$

$$\begin{aligned}
& + 4 \sum_{x \in \mathcal{V}^c} m^c(x) \Big| \frac{1}{2} (e\langle Q - 1/2 \rangle^{1+\epsilon} \otimes P^{le}) \frac{1}{m_2} \\
& \quad \times \left(\frac{\sqrt{1 + \mu^c(x_1 + 1, x_2)} (\varepsilon^c(x, (x_1 + 1, x_2)) - \varepsilon^c((x_1 - 1, x_2), x))}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(x_1 + 1, x_2))(1 + \mu^c(x_1 - 1, x_2))}} \right. \\
& \quad + \frac{\mu^c(x_1 - 1, x_2) - \mu^c(x_1 + 1, x_2)}{(\sqrt{1 + \mu^c(x_1 - 1, x_2)} + \sqrt{1 + \mu^c(x_1 + 1, x_2)})} \\
& \quad \left. \times \frac{1}{\sqrt{(1 + \mu^c(x))(1 + \mu^c(x_1 + 1, x_2))(1 + \mu^c(x_1 - 1, x_2))}} \right) |f(x_1 - 1, x_2)|^2 \\
& \leq c \|f\|_{\ell^2(\mathcal{V}^c, m^c)}^2,
\end{aligned}$$

and in the same way, we deal with (4.15) and $\|\langle \Lambda^c \rangle^\epsilon [W^c, \mathcal{A}_{\mathcal{G}^c}] \circ f\|_{\ell^2(\mathcal{V}^c, m^c)}$. By density, we have proven that there exists $c > 0$ such that

$$\|\langle \Lambda^c \rangle^\epsilon [\widetilde{\Delta}_{\mathcal{G}^c, \mu}, \mathcal{A}_{\mathcal{G}^c}] \circ f\|_{\ell^2(\mathcal{V}^c, m^c)}^2 \leq c \|f\|_{\ell^2(\mathcal{V}^c, m^c)}^2.$$

Finally, by applying [AmBoGe, Proposition 7.5.7] with $\mathcal{G} := \mathcal{D}(\Delta_{\mathcal{G}^c, \mu})$ where the hypotheses are verified in Proposition 4.6, we find the result. \square

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UNIVERSITÉ DE GAFSA, CAMPUS UNIVERSITAIRE 2112, TUNISIE
E-mail address: `athmouninassim@yahoo.fr`

UNIVERSITÉ DE SFAX. ROUTE DE LA SOUKRA KM 4 - B.P. N 802 - 3038 SFAX
E-mail address: `ennaceur.marwa27@gmail.com`

UNIV. BORDEAUX, BORDEAUX INP, CNRS, IMB, UMR 5251, F-33400 TALENCE, FRANCE
E-mail address: `sylvain.golenia@math.u-bordeaux.fr`