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SPECTRAL ANALYSIS OF THE LAPLACIAN ACTING ON DISCRETE CUSPS AND FUNNELS

NASSIM ATHMOUNI, MARWA ENNACEUR, AND SYLVAIN GOLÉLIA

ABSTRACT. We study the Laplacian acting on a discrete cusp and a discrete funnel. We perturb the metric in a long-range way. Then, we establish a Limiting Absorption Principle away the possible embedded eigenvalues. The approach is based on a positive commutator technique.

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1. INTRODUCTION

The spectral theory of discrete Laplacians on graphs has drawn a lot of attention for decades. The spectral analysis of the Laplacian associated to a graph is strongly related to the geometry of the graph. It is well-known that the graph is the discrete version of the manifolds. The nature of the spectral measure of discrete Laplacians and its perturbations has been done by many authors. For instance in [Sa, BoSa], one treats the case of \mathbb{Z}^d , in [AlFr] and [GeGo] the case of binary trees with the help of some positive commutator techniques, see also [MăRiT, PaRi]. Without trying to be exhaustive, some other techniques have been used successfully, eg. [HiNo] with some geometric approach and [BrKe].

In [MoTr, GoMo], it is shown that for a manifold with cusp, adding a magnetic field can drastically destroy the essential spectrum of the Laplacian. In [GoTr],

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the discrete analogous is considered for the same question. In the case of the non-magnetic case, where the essential spectrum exists, they do not analyse the refine properties of the spectral measure. This is the main aim of this article.

We recall some standard definitions of graph theory. A *graph* is a triple $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$, where \mathcal{V} is a countable set (the *vertices*), $\mathcal{E} : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}_+$ is symmetric, and $m : \mathcal{V} \rightarrow (0, \infty)$ is a weight. We say that \mathcal{G} is *simple* if $m = 1$ and $\mathcal{E} : \mathcal{V} \times \mathcal{V} \rightarrow \{0, 1\}$.

Given $x, y \in \mathcal{V}$, we say that (x, y) is an edge (or x and y are *neighbors*) if $\mathcal{E}(x, y) > 0$. We denote this relationship by $x \sim y$ and the set of neighbors of x by $N_{\mathcal{G}}(x)$. The space of complex-valued functions acting on the set of vertices \mathcal{V} is denoted by $C(\mathcal{V}) := \{f : \mathcal{V} \rightarrow \mathbb{C}\}$. Moreover, $C_c(\mathcal{V})$ is the subspace of $C(\mathcal{V})$ of functions with finite support. We consider the Hilbert space

$$\ell^2(\mathcal{V}, m) := \left\{ f \in C(\mathcal{V}), \sum_{x \in \mathcal{V}} m(x) |f(x)|^2 < \infty \right\}$$

with the scalar product, $\langle f, g \rangle := \sum_{x \in \mathcal{V}} m(x) \overline{f(x)} g(x)$. We define the Laplacian operator

$$(1) \quad \Delta_{\mathcal{G}} f(x) := \frac{1}{m(x)} \sum_{x \in \mathcal{V}} \mathcal{E}(x, y) (f(x) - f(y)),$$

for all $f \in C_c(\mathcal{V})$. $\Delta_{\mathcal{G}}$ is a positive operator since we have $\langle f, \Delta_{\mathcal{G}} f \rangle_{\ell^2(\mathcal{V}, m)} = Q_{\mathcal{G}}(f)$, with

$$Q_{\mathcal{G}}(f) := \frac{1}{2} \sum_{x, y \in \mathcal{V}} \mathcal{E}(x, y) |f(x) - f(y)|^2,$$

for all $f \in C_c(\mathcal{V})$. We set

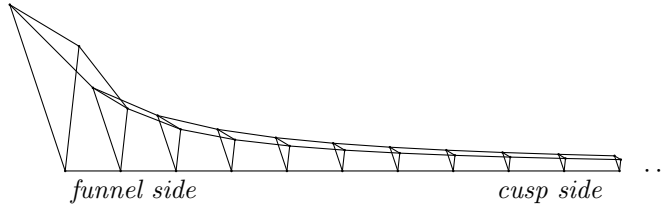
$$\deg_{\mathcal{G}}(x) := \frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y),$$

the *degree* of $x \in \mathcal{V}$.

We present a simple version of our model: We consider $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$, where $\mathcal{V}_1 := \mathbb{Z}$, $m_1(n) := e^n$, and $\mathcal{E}(n, n+1) := e^{(2n+1)/2}$, for all $n \in \mathbb{N}$ and $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$ a connected finite graph such that $|\mathcal{V}_2| = n$, $n \geq 3$. Let $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$ be the *twisted cartesian product* $\mathcal{G}_1 \times_{\mathcal{V}_2} \mathcal{G}_2$ given by

$$\begin{cases} m(x, y) := m_1(x) \times m_2(y), \\ \mathcal{E}((x, y), (x', y')) := \mathcal{E}_1(x, x') \times \delta_{y, y'} + \delta_{x, x'} \times \mathcal{E}_2(y, y'), \end{cases}$$

for all $x, x' \in \mathcal{V}_1$ and $y, y' \in \mathcal{V}_2$.



(2) *Representation of a discrete cusp and funnel side*

If $n > 0$, this is a funnel side and if $n < 0$, this is a cusp side.

The Laplacian obtained with our product is essentially self-adjoint on $C_c(\mathcal{V})$. Moreover, $\Delta_{\mathcal{G}}$ has no singularly continuous spectrum and

$$\sigma_{ac}(\Delta_{\mathcal{G}}) = [e^{1/2} + e^{-1/2} - 2, e^{1/2} + e^{-1/2} + 2].$$

Now, we perturb the metrics of the previous case which will be small to infinity and aim to reach the same results. Let $H := \Delta_{\mathcal{G}'} + V(Q)$, where V is a real bounded function on \mathcal{V} such that $V(x) \rightarrow 0$ if $|x| \rightarrow \infty$, and $\Delta_{\mathcal{G}'}$ be the closure of

$$\begin{aligned} \Delta_{\mathcal{G}'} : \mathcal{C}_c(\mathcal{V}, m') &\rightarrow \mathcal{C}_c(\mathcal{V}, m') \\ f &\mapsto \Delta_{\mathcal{G}'} f(x) := \frac{1}{m'(x)} \sum_{y \sim x} \mathcal{E}'(x, y)(f(x) - f(y)), \end{aligned}$$

with $\mathcal{G}' := (\mathcal{E}', \mathcal{V}, m')$, $m'(x) := (1 + \varepsilon(x))m(x)$, and $\mathcal{E}'(x, y) := (1 + \theta(x, y))\mathcal{E}(x, y)$, where $\varepsilon(x) \rightarrow 0$ if $|x| \rightarrow \infty$ and $\theta(x, y) \rightarrow 0$ if $|x|, |y| \rightarrow \infty$. m' and \mathcal{E}' are a long-range perturbations of m and \mathcal{E} . The operator H is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V})$. In fact, it is a kind of compact perturbation of $\Delta_{\mathcal{G}}$, see Proposition 4.4 for a precise statement.

Our main result is the following:

Theorem 1.1. *Let $V : \mathbb{Z} \rightarrow \mathbb{R}$ be a function and $\varepsilon > 0$. We assume that*

$$\sup_{n \in \mathbb{Z}} \langle n \rangle^{\varepsilon+1} |\varepsilon(n-1) - \varepsilon(n)| < \infty,$$

$$\sup_{n \sim n'} \langle n \rangle^{\varepsilon+1} |\theta(n, n') - \theta(n-1, n')| < \infty$$

and

$$\sup_{n \in \mathbb{Z}} \left(\left| V(n-1) - V(n) \right| \langle n \rangle^{1+\varepsilon} \right) < \infty,$$

where $\theta(n, n') \rightarrow 0$ if $|n|, |n'| \rightarrow \infty$, $\varepsilon(n) \rightarrow 0$ if $|n| \rightarrow \infty$ and $V(n) \rightarrow 0$ if $|n| \rightarrow \infty$. Then, we have the following assertions:

- (1) $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(\Delta_{\mathcal{G}})$.
- (2) The eigenvalues of H distinct from c_1 and c_2 are of finite multiplicity and can accumulate only toward c_1 and c_2 , with $c_1 := e^{1/2} + e^{-1/2} - 2$ and $c_2 := e^{1/2} + e^{-1/2} + 2$.
- (3) Let \mathcal{I} a bounded interval of (c_1, c_2) . The singular continuous spectrum of H is empty and if $s > 1/2$, \mathcal{I}' a compact sub-interval of \mathcal{I} and $\lambda \notin \kappa(H) := \sigma_p(H) \cup \{c_1, c_2\}$, then $\lim_{\mu \rightarrow 0} \|\langle Q \rangle^{-s} (H - \lambda - i\mu)^{-1} \langle Q \rangle^{-s}\|$ exists and finite, locally uniformly in $\lambda \in \mathbb{R} \setminus \kappa(H)$. In particular, for all $f \in \mathcal{H}$, there exists $c > 0$, such that

$$\int_{\mathbb{R}} \|\langle Q \rangle^{-s} e^{-itH} E_{\mathcal{I}'}(H) f\|^2 dt \leq c \|f\|^2.$$

Our approach is based on a positive commutator technique, namely we establish a Mourre estimate. The proof of this theorem is be given in Subsection 4.3. We refer to Section 2 for historical references and for an introduction on the subject.

We now describe the structure of the paper. In Section 2, we present the Mourre's theory. The next section is devoted to study the free model. In Subsection 3.1, we present the context and introduce the notion of cusp and funnel. In Subsection 3.2, we start with the Mourre estimate on \mathbb{N} . In Subsections 3.3, and 3.4, we prove the Mourre estimate for the discrete Laplacian that acts on a funnel and a cusp, respectively. Then, in Subsection 3.6, we conclude the Mourre estimate for the whole graph. In Section 4, we perturb the metrics and add a potential. This yields the main result. The proof are more involved than in Section 3.1 as we rely on the optimal class $\mathcal{C}^{1,1}(\mathcal{A})$ of the Mourre theory.

2. THE MOURRE THEORY

The Mourre Theory allows us to analyze the essential spectrum of different operators (presence of eigenvalues in the essential spectrum and nature of the continuous spectrum) and to obtain a limit absorption principle. In 1956, C.R. Putnam gave a necessary condition for the spectrum of an operator to be purely absolutely continuous. Since its introduction in 1980 (cf., [Mo1, Mo2]), many papers have shown the power of Mourre's commutator theory to study the point and continuous spectra of a quite wide class of self-adjoint operators. Among others, we refer to [BaFrSi, BoCaHäMi, CaGrHu, DeJa, FrHe, GeGéMø, GeGo, HuSi, JeMoPe, Sa] and to the book [AmBoGe] and more recently [GoJe, Gé]. One can also find parameter dependent versions of the theory (a semi-classical one for instance) in [RoTa, Wa, WaZh]. Recently, it has been extended to (non self-adjoint) dissipative operators (eg., [BoGo, Roy]).

Let us Now, briefly recall Mourre's commutator theory. We consider two self-adjoint (unbounded) operators H and \mathcal{A} acting in some complex Hilbert space \mathcal{H} . Let $\|\cdot\|$ denote the norm of bounded operators on \mathcal{H} .

With the help of \mathcal{A} , we study spectral properties of H , the spectrum $\sigma(H)$ of which is included in \mathbb{R} . Let \mathcal{I} be open interval of \mathbb{R} . Given $k \in \mathbb{N}$, we say that $H \in \mathcal{C}^k(\mathcal{A})$ if for all $f \in \mathcal{H}$, the map $\mathbb{R} \ni t \mapsto e^{it\mathcal{A}}(H+i)^{-1}e^{-it\mathcal{A}}f \in \mathcal{H}$ has the usual \mathcal{C}^k regularity. We say that $H \in \mathcal{C}^{1,u}(\mathcal{A})$ if the map $\mathbb{R} \ni t \mapsto e^{it\mathcal{A}}He^{-it\mathcal{A}} \in \mathcal{B}(\mathcal{H})$ is of class $\mathcal{C}^1(\mathbb{R}, \mathcal{B}(\mathcal{H}))$, with $\mathcal{B}(\mathcal{H})$ endowed with the norm operator topology.

Theorem 2.1. *Let \mathcal{A} and H be two self-adjoint operators in the Hilbert space \mathcal{H} . The following points are equivalent:*

- (1) $H \in \mathcal{C}^1(\mathcal{A})$.
- (2) For one (then for all) $z \notin \sigma(H)$, there is a finite c such that
- (3) $|\langle \mathcal{A}f, R(z)f \rangle - \langle R(\bar{z}), \mathcal{A}f \rangle| \leq c\|f\|^2$, for all $f \in \mathcal{D}(\mathcal{A})$.
- (3) a. There is a finite c such that for all $f \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(H)$:
- (4) $|\langle \mathcal{A}f, Hf \rangle - \langle H, \mathcal{A}f \rangle| \leq c(\|Hf\|^2 + \|f\|^2)$.

b. For some (then for all) $z \notin \sigma(H)$, the set

$$\{f \in \mathcal{D}(\mathcal{A}); R(z)f \in \mathcal{D}(\mathcal{A}) \text{ and } R(\bar{z})f \in \mathcal{D}(\mathcal{A})\} \text{ is a core for } \mathcal{A}.$$

Note that (3) yields that the commutator $[\mathcal{A}, R(z)]$ extends to a bounded operator, in the form sense. We shall denote the extension by $[\mathcal{A}, R(z)]_{\circ}$. In the same way, from (4), the commutator $[H, \mathcal{A}]$ extends to a unique element of $\mathcal{B}(\mathcal{D}(H), \mathcal{D}(H)^*)$ denoted by $[H, \mathcal{A}]_{\circ}$. Moreover, if $H \in \mathcal{C}^1(\mathcal{A})$ and $z \notin \sigma(H)$,

$$[\mathcal{A}, (H-z)^{-1}]_{\circ} = \underbrace{(H-z)^{-1}}_{\mathcal{H} \leftarrow \mathcal{D}(H)^*} \underbrace{[\mathcal{A}, \mathcal{A}]_{\circ}}_{\mathcal{D}(H)^* \leftarrow \mathcal{D}(H)} \underbrace{(H-z)^{-1}}_{\mathcal{D}(H) \leftarrow \mathcal{H}}.$$

Here, we use the Riesz lemma to identify \mathcal{H} with its anti-dual \mathcal{H}^* .

Note that, in practice, the condition (3.b) could be delicate to check. This is addressed in the next lemma.

Lemma 2.2. ([AmBoGe, p.258]) *Let \mathcal{D} be a subspace of H such that $\mathcal{D} \subset \mathcal{D}(H) \cap \mathcal{D}(\mathcal{A})$, \mathcal{D} is a core for \mathcal{A} and $H\mathcal{D} \subset \mathcal{D}$. Let $(\chi_n)_{n \in \mathbb{N}}$ be a family of bounded operators such that*

- (1) $\chi_n \mathcal{D} \subset \mathcal{D}$, χ_n tends strongly to 1 as $n \rightarrow \infty$, and $\sup_n \|\chi_n\|_{\mathcal{B}(\mathcal{D}(H))} < \infty$.
- (2) $\mathcal{A}\chi_n f \rightarrow f$, for all $f \in \mathcal{D}$, as $n \rightarrow \infty$.
- (3) There is $z \notin \sigma(H)$, such that $\chi_n R(z)\mathcal{D} \subset \mathcal{D}$ and $\chi_n R(\bar{z})\mathcal{D} \subset \mathcal{D}$.

Suppose also that for all $f \in \mathcal{D}$

$$\lim_{n \rightarrow \infty} \mathcal{A}[H, \chi_n]R(z)f = 0 \text{ and } \lim_{n \rightarrow \infty} \mathcal{A}[H, \chi_n]R(\bar{z})f = 0.$$

Finally, suppose that there is a finite c such that

$$|\langle \mathcal{A}f, Hf \rangle - \langle Hf, \mathcal{A}f \rangle| \leq c(\|Hf\|^2 + \|f\|^2), \text{ for all } f \in \mathcal{D}.$$

Then, one has $H \in \mathcal{C}^1(\mathcal{A})$.

We define another class of regularity called the $\mathcal{C}^{1,1}$ regularity: we say that $H \in \mathcal{C}^{1,1}(\mathcal{A})$, if

$$\int_0^1 \|[(H + i)^{-1}, e^{it\mathcal{A}}], e^{it\mathcal{A}}\| \frac{dt}{t^2} < \infty.$$

Thanks to [AmBoGe, p. 205], it turns out that

$$\mathcal{C}^2(\mathcal{A}) \subset \mathcal{C}^{1,1}(\mathcal{A}) \subset \mathcal{C}^{1,u}(\mathcal{A}) \subset \mathcal{C}^1(\mathcal{A}).$$

Denote by $E_{\mathcal{I}}(H)$ the spectral measure of H above \mathcal{I} . We say that the *Mourre estimate* holds true for H on \mathcal{I} if there exist $c > 0$ and a compact operator K such that

$$(5) \quad E_{\mathcal{I}}(H)[H, i\mathcal{A}]_o E_{\mathcal{I}}(H) \geq E_{\mathcal{I}}(H)(c + K)E_{\mathcal{I}}(H).$$

The aim of Mourre's commutator theory is to show a *limiting absorption principle* (LAP).

Theorem 2.3. *Let H be a self-adjoint operator, with $\sigma(H) \neq \mathbb{R}$. Assume that $H \in \mathcal{C}^1(\mathcal{A})$ and a strict Mourre estimate holds true for H on \mathcal{I} . Then*

- (1) *If $K = 0$, then H has no eigenvalues in \mathcal{I} .*
- (2) *The eigenvalues of H are of finite multiplicity on \mathcal{I} .*
- (3) *If $H \in \mathcal{C}^{1,1}(\mathcal{A})$, $s > 1/2$ and \mathcal{I}' a compact sub-interval of \mathcal{I} , then*

$$\sup_{\Re(z) \in \mathcal{I}', \Im(z) \neq 0} \|\langle \mathcal{A} \rangle^{-s} (H - z)^{-1} \langle \mathcal{A} \rangle^{-s}\| \text{ exists and finite.}$$

Moreover, in the norm topology of bounded operators, the boundary values of the resolvent:

$$\mathcal{I}' \ni \lambda \mapsto \lim_{\mu \rightarrow 0^\pm} \langle \mathcal{A} \rangle^{-s} (H - \lambda - i\mu)^{-1} \langle \mathcal{A} \rangle^{-s} \text{ exists and continuous.}$$

For more details, see [AmBoGe, Proposition 7.2.10, Corollary 7.2.11, Theorem 7.5.2].

3. THE FREE MODEL

3.1. Construction of the graph. Given $\mathcal{G}_1 := (\mathcal{E}_1, \mathcal{V}_1, m_1)$ and $\mathcal{G}_2 := (\mathcal{E}_2, \mathcal{V}_2, m_2)$, the *Cartesian product* of \mathcal{G}_1 by \mathcal{G}_2 is defined by $\mathcal{G}^\diamond := (\mathcal{E}^\diamond, \mathcal{V}^\diamond, m^\diamond)$, where $\mathcal{V}^\diamond := \mathcal{V}_1 \times \mathcal{V}_2$,

$$\begin{cases} m^\diamond(x, y) := m_1(x) \times m_2(y), \\ \mathcal{E}^\diamond((x, y), (x', y')) := \mathcal{E}_1(x, x') \times \delta_{y, y'} m_2(y) + m_1(x) \delta_{x, x'} \times \mathcal{E}_2(y, y'). \end{cases}$$

We denote by $\mathcal{G}^\diamond := \mathcal{G}_1 \times \mathcal{G}_2$. This definition generalizes the unweighted Cartesian product, e.g., [Ha]. It is used in several places in the literature, e.g., [Ch, Section 2.6] and in [BoKeGoLiMü] for a generalization.

The terminology is motivated by the following decomposition:

$$\Delta_{\mathcal{G}^\diamond} = \Delta_{\mathcal{G}_1} \otimes 1 + 1 \otimes \Delta_{\mathcal{G}_2},$$

where $\ell^2(\mathcal{V}, m) \simeq \ell^2(\mathcal{V}_1, m_1) \otimes \ell^2(\mathcal{V}_2, m_2)$. We refer to [ReSi, Section VIII.10] for an introduction to the tensor product of self-adjoint operators. To analyze the Laplacian on this product one separates the variables and obtains a decomposition which is not of the type of a Cartesian product, e.g., [GoMo, Eq. (5.22)] for some details.

We aim at mimicking this situation and introducing the product $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m) := \mathcal{G}_1 \times_{\mathcal{V}_2} \mathcal{G}_2$, see Graph 2. Under the representation

$$\ell^2(\mathcal{V}, m) \simeq \ell^2(\mathcal{V}_1, m_1) \otimes \ell^2(\mathcal{V}_2, m_2),$$

the Laplacian obtained with our product is given by

$$\Delta_{\mathcal{G}} = \Delta_{\mathcal{G}_1} \otimes \frac{1}{m_2} + \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2}.$$

A hyperbolic manifold of finite volume is the union of a compact part, of a cusp, and a funnel, e.g., [Th, Theorem 4.5.7]. In this article we study a discrete analogous.

The graph $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$ is divided into three parts: A cusp part, a funnel part, and a compact part. Set $\mathcal{G}^* := (\mathcal{E}^*, \mathcal{V}^*, m^*)$ be the induced graph of \mathcal{G} over \mathcal{V}^* where $\star \in \{c, f, 0\}$.

We consider $\mathcal{G}_1^c := (\mathcal{E}_1^c, \mathcal{V}_1^c, m_1^c)$, where

$$\mathcal{V}_1^c := \mathbb{N}, \quad m_1^c(n) := \exp(-n), \quad \text{and} \quad \mathcal{E}_1^c(n, n+1) := \exp(-(2n+1)/2),$$

for all $n \in \mathbb{N}$ and $\mathcal{G}_2^c := (\mathcal{E}_2^c, \mathcal{V}_2^c, m_2)$ a simple connected finite graph such that $|\mathcal{V}_2^c| = n$. Set $\mathcal{G}^c := \mathcal{G}_1^c \times_{\mathcal{V}_2^c} \mathcal{G}_2^c$. This is a *cusp part*.

We consider $\mathcal{G}_1^f := (\mathcal{E}_1^f, \mathcal{V}_1^f, m_1^f)$, where

$$\mathcal{V}_1^f := \mathbb{N}, \quad m_1^f(n) := \exp(n), \quad \text{and} \quad \mathcal{E}_1^f(n, n+1) := \exp((2n+1)/2),$$

for all $n \in \mathbb{N}$ and $\mathcal{G}_2^f := (\mathcal{E}_2^f, \mathcal{V}_2^f, m_2)$ a simple connected finite graph such that $|\mathcal{V}_2^f| = n$. Set $\mathcal{G}^f := \mathcal{G}_1^f \times_{\mathcal{V}_2^f} \mathcal{G}_2^f$. This is a *funnel part*.

For the *compact part*, we ask that for all $x \in \mathcal{V}^0$, $\text{supp}(\mathcal{E}(x, \cdot))$ is finite and $m^0(x) > 0$.

We Now, take advantage of

$$\ell^2(\mathcal{G}) := \ell^2(\mathcal{G}^f) \oplus \ell^2(\mathcal{G}^0) \oplus \ell^2(\mathcal{G}^c).$$

We have that

$$\Delta_{\mathcal{G}} := \Delta_{\mathcal{G}^f} \oplus 0 \oplus \Delta_{\mathcal{G}^c} + K,$$

with K is an operator of finite rank.

3.2. Mourre estimate on \mathbb{N} . We denote by $\mathbf{1}_X$ the indicatrice of the set X . We denote by \mathbb{N} the set of non-negative integers. We define by Q the operator of multiplication by n .

Given $f \in \ell^2(\mathbb{N})$, we set $\forall n \in \mathbb{N}^* \ Uf(n) = f(n-1)$ and $Uf(0) = 0$. Note that $U^*f(n) = f(n+1)$, $\forall n \in \mathbb{N}$. The operator U is an isometry and not unitary: we have $UU^* = \text{id}$ and $U^*U = \mathbf{1}_{[1, \infty[}(Q)$.

Given $f \in \ell_c^2(\mathbb{Z})$, we set $(Qf)(n) = nf(n)$ for all $n \in \mathbb{N}$. It is essentially self-adjoint on $\ell_c^2(\mathbb{Z})$. Note that

$$(6) \quad QU = U(Q+1), \quad U^*Q = (Q+1)U^* \quad \text{and} \quad UQU = U^2(Q+1) \quad \text{on} \quad \mathcal{D}(Q).$$

The operator $\Delta_{\mathbb{N}}$ is defined by (1), where $\mathbb{N} \simeq (\mathbb{N}, \mathcal{E}_{\mathbb{N}}, m)$, with $\mathcal{E}_{\mathbb{N}}(n, n+1) = 1$ and $m(n) = 1$ for all $n \in \mathbb{N}$. Explicitly, we have

$$\Delta_{\mathbb{N}}f(n) := 2f(n) - f(n-1) - f(n+1).$$

We can express it with the help of U . Namely, we have:

$$\Delta_{\mathbb{N}} = 2 - (U + U^*).$$

We denote by $\mathbf{1}_{[1, \infty[}(Q)$, the operator of multiplication by $\mathbf{1}_{[1, \infty[}$. We construct the conjugate operator in $\ell^2(\mathbb{N}, 1)$. On the space $\ell_c^2(\mathbb{N}, 1)$, we define

$$(7) \quad \begin{aligned} \mathcal{A}_{\mathbb{N}} &:= \frac{1}{2}(SQ + QS), \quad \text{where} \quad S := \frac{U - U^*}{2i} \\ &= \frac{i}{2} \left(U \left(Q + \frac{1}{2} \right) - U^* \left(Q + \frac{1}{2} \right) \right) \\ &= -\frac{i}{2} \left(\frac{1}{2}(U^* + U) + Q(U^* - U) \right). \end{aligned}$$

The operator is essentially self-adjoint, see [GeGo] and [Mic, Lemma 5.7]. We continue to denote by $\mathcal{A}_{\mathbb{N}}$ its closure. We have that

$$\mathcal{D}(\mathcal{A}_{\mathbb{N}}) = \mathcal{D}(QS) := \{f \in \ell^2(\mathbb{N}), Sf \in \mathcal{D}(Q)\}.$$

see [GeGo, Lemma 3.1].

Lemma 3.1. *On $\ell_c^2(\mathbb{N})$, we have*

$$\begin{aligned} (U^* + U)\mathcal{A}_{\mathbb{N}} &= -\frac{i}{2} \left((U^{2*} - U^2)Q - \mathbf{1} - \frac{1}{2}(\mathbf{1}_{\{0\}} + U^2 + U^{*2}) \right), \\ \mathcal{A}_{\mathbb{N}}(U^* + U) &= \frac{i}{2} \left((U^2 - U^{2*})Q + \frac{3}{2}(U^2 + U^{*2}) - \mathbf{1} - \frac{1}{2}\mathbf{1}_{\{0\}} \right). \end{aligned}$$

Proof. First,

$$\begin{aligned} U\mathcal{A}_{\mathbb{N}} &= -\frac{i}{2} \left(UU^*Q - U^2Q - \frac{1}{2}UU^* - \frac{1}{2}U^2 \right) \\ &= -\frac{i}{2} \left(\mathbf{1}_{\mathbb{N}}Q - U^2Q - \frac{1}{2}\mathbf{1}_{\mathbb{N}} - \frac{1}{2}U^2 \right) \\ &= -\frac{i}{2} \left((\mathbf{1}_{\mathbb{N}} - U^2)Q - \frac{1}{2}(\mathbf{1}_{\mathbb{N}} + U^2) \right). \end{aligned}$$

We also have

$$\begin{aligned} U^*\mathcal{A}_{\mathbb{N}} &= -\frac{i}{2} \left(U^{*2}Q - U^*UQ - \frac{1}{2}U^{*2} - \frac{1}{2}U^*U \right) \\ &= -\frac{i}{2} \left(U^*Q - Q - \frac{1}{2}U^{*2} - \frac{1}{2} \right) \\ &= -\frac{i}{2} \left((U^* - \mathbf{1})Q - \frac{1}{2}(U^{*2} - \frac{1}{2}) \right). \end{aligned}$$

Then,

$$\begin{aligned} (U^* + U)\mathcal{A}_{\mathbb{N}} &= -\frac{i}{2} \left((\mathbf{1}_{\mathbb{N}} - U^2)Q - \frac{1}{2}(\mathbf{1}_{\mathbb{N}} + U^2) + (U^* - \mathbf{1})Q - \frac{1}{2}(U^{*2} - \frac{1}{2}) \right) \\ &= -\frac{i}{2} \left((\mathbf{1}_{\mathbb{N}} - U^2 + U^* - \mathbf{1})Q - \frac{1}{2}(\mathbf{1}_{\mathbb{N}} + U^2 + U^{*2} - \frac{1}{2}) \right) \\ &= -\frac{i}{2} \left((U^{2*} - U^2)Q - \mathbf{1} - \frac{1}{2}(\mathbf{1}_{\{0\}} + U^2 + U^{*2}) \right), \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_{\mathbb{N}}(U^* + U) &= \frac{i}{2} \left(Q(U^2 - U^{*2}) - \mathbf{1} - \frac{1}{2}(\mathbf{1}_{\{0\}} + U^2 + U^{*2}) \right) \\ &= \frac{i}{2} \left((U^2 - U^{2*})Q + \frac{3}{2}(U^2 + U^{*2}) - \mathbf{1} - \frac{1}{2}\mathbf{1}_{\{0\}} \right). \end{aligned}$$

This gives the proof. \square

Lemma 3.2. *The operator $\Delta_{\mathbb{N}}$ is $\mathcal{C}^1(\mathcal{A}_{\mathbb{N}})$ and we have:*

$$(8) \quad [\Delta_{\mathbb{N}}, i\mathcal{A}_{\mathbb{N}}]_{\circ} = \frac{1}{2}\Delta_{\mathbb{N}}(4 - \Delta_{\mathbb{N}}),$$

This lemma is essentially given in [GeGo], see also [AlFr] for another type of presentation.

Proof. We work in the form sense and by density. Let $f \in \ell_c^2(\mathbb{N})$. Since $\Delta_{\mathbb{N}}f \in \ell_c^2(\mathbb{N})$ and using Lemma 3.1, we obtain:

$$\begin{aligned} \langle f, [\Delta_{\mathbb{N}}, i\mathcal{A}_{\mathbb{N}}]f \rangle &:= \langle \Delta_{\mathbb{N}}f, i\mathcal{A}_{\mathbb{N}}f \rangle - \langle -i\mathcal{A}_{\mathbb{N}}f, \Delta_{\mathbb{N}}f \rangle \\ &= i\langle f, \mathcal{A}_{\mathbb{N}}(U^* + U) - (U^* + U)\mathcal{A}_{\mathbb{N}}f \rangle \\ &= \frac{1}{2}\langle f, 2 + \mathbf{1}_{\{0\}} - (U^2 + U^{*2})f \rangle = \frac{1}{2}\langle f, \Delta_{\mathbb{N}}(4 - \Delta_{\mathbb{N}})f \rangle. \end{aligned}$$

Since $\ell_c^2(\mathbb{N})$ is a core for \mathcal{A} and since $\Delta_{\mathbb{N}}(4 - \Delta_{\mathbb{N}})$ is a bounded operator, there is a constant c such that $|\langle \Delta_{\mathbb{N}}f, i\mathcal{A}_{\mathbb{N}}f \rangle - \langle -i\mathcal{A}_{\mathbb{N}}f, \Delta_{\mathbb{N}}f \rangle| \leq c\|f\|^2$, for all $f \in \mathcal{D}(\mathcal{A})$. Hence, it is $\mathcal{C}^1(\mathcal{A})$. By density, we also obtain (8). \square

By induction, one easily obtains that:

Corollary 3.3. $\Delta_{\mathbb{N}} \in \mathcal{C}^\infty(\mathcal{A})$.

3.3. The funnel side. Recall that

$$\ell^2(\mathcal{V}^f, m) = \ell^2(\mathbb{N}, m_1^f) \otimes \ell^2(\mathcal{V}_2^f, m_2).$$

We have

$$(9) \quad \Delta_{\mathcal{G}^f} := \Delta_{\mathcal{G}_1^f} \otimes \frac{1}{m_2} + \frac{1}{m_1^f(\cdot)} \otimes \Delta_{\mathcal{G}_2^f}.$$

The first remark is that

Lemma 3.4.

$$\frac{1}{m_1^f(\cdot)} \otimes \Delta_{\mathcal{G}_2^f} \in \mathcal{K}(\ell^2(\mathcal{V}^f)).$$

Proof. Note that $\Delta_{\mathcal{G}_2^f}$ is of finite rank since \mathcal{V}^2 is finite and that $\frac{1}{m_1^f(\cdot)}$ is a compact operator since $m_1^f(n) \rightarrow \infty$, as $n \rightarrow \infty$. \square

Let

$$\begin{aligned} T : \ell^2(\mathbb{N}, m_1^f) &\rightarrow \ell^2(\mathbb{N}, 1) \\ f &\mapsto \left(n \mapsto \sqrt{m_1^f(n)}f(n) \right). \end{aligned}$$

It is a unitary map.

Lemma 3.5. We have $\Delta_{\mathcal{G}^f} \in \mathcal{B}(\ell^2(\mathcal{V}^f), m^f)$.

We Now, introduce the conjugate operator. Recalling (7), we set

$$\mathcal{A}_{\mathcal{G}^f} := T^{-1}\mathcal{A}_{\mathbb{N}}T \otimes 1.$$

We turn to the regularity. We have:

Lemma 3.6. We have $\Delta_{\mathcal{G}^f} \in \mathcal{C}^1(\mathcal{A}^f)$ and

$$(10) \quad [\Delta_{\mathcal{G}^f}, i\mathcal{A}_{\mathcal{G}^f}]_{\circ} = w(\Delta_{\mathcal{G}_1^f}) \otimes \frac{1}{m_2} + \underbrace{\left[\frac{1}{m_1^f(\cdot)} \otimes \Delta_{\mathcal{G}_2^f}, i\mathcal{A}_{\mathcal{G}^f} \right]_{\circ}}_{\in \mathcal{K}(\ell^2(\mathcal{V}^f, m^f))} + K,$$

$$(11) \quad = w(\Delta_{\mathcal{G}^f}) + K'$$

where K, K' are compact operators,

$$w(x) := \frac{m_2}{2} \left(x - \frac{c_1}{m_2} \right) \left(\frac{c_2}{m_2} - x \right),$$

and

$$c_1 := e^{1/2} + e^{-1/2} - 2 \quad \text{and} \quad c_2 := e^{1/2} + e^{-1/2} + 2.$$

Note that \circ denotes the closure of the commutator which is originally defined on $\mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A})$.

Proof. We prove that $[\Delta_{\mathcal{G}^f}, \mathfrak{i}\mathcal{A}_{\mathcal{G}^f}]_{\circ} \in \mathcal{B}(\mathcal{H})$. We set $\tilde{c} := e^{-1/2} - 1$. We have that

$$\begin{aligned}
w(\Delta_{\mathcal{G}^f}) &= \frac{m_2}{2} \left(\Delta_{\mathcal{G}^f} - \frac{c_1}{m_2} \right) \left(\frac{c_2}{m_2} - \Delta_{\mathcal{G}^f} \right) \\
&= \frac{m_2}{2} \left(\Delta_{\mathcal{G}_1^f} \otimes \frac{1}{m_2} + \frac{1}{m_1} \otimes \Delta_{\mathcal{G}_2^f} - \frac{c_1}{m_2} \right) \left(\frac{c_2}{m_2} - \Delta_{\mathcal{G}_1^f} \otimes \frac{1}{m_2} - \frac{1}{m_1} \otimes \Delta_{\mathcal{G}_2^f} \right) \\
&= \frac{m_2}{2} \left(\Delta_{\mathcal{G}_1^f} \otimes \frac{1}{m_2} - \frac{c_1}{m_2} \right) \left(\frac{c_2}{m_2} - \Delta_{\mathcal{G}_1^f} \otimes \frac{1}{m_2} - \frac{1}{m_1} \otimes \Delta_{\mathcal{G}_2^f} \right) \\
&\quad + \frac{m_2}{2} \left(\frac{1}{m_1} \otimes \Delta_{\mathcal{G}_2^f} \right) \left(\frac{c_2}{m_2} - \Delta_{\mathcal{G}_1^f} \otimes \frac{1}{m_2} - \frac{1}{m_1} \otimes \Delta_{\mathcal{G}_2^f} \right) \\
&= \frac{m_2}{2} \left(\Delta_{\mathcal{G}_1^f} \otimes \frac{1}{m_2} - \frac{c_1}{m_2} \right) \left(\frac{c_2}{m_2} - \Delta_{\mathcal{G}_1^f} \otimes \frac{1}{m_2} \right) \\
&\quad + \frac{m_2}{2} \left(\frac{1}{m_1} \otimes \Delta_{\mathcal{G}_2^f} \right) \left(\frac{c_2}{m_2} - \Delta_{\mathcal{G}_1^f} \otimes \frac{1}{m_2} - \frac{1}{m_1} \otimes \Delta_{\mathcal{G}_2^f} \right) \\
&\quad - \frac{m_2}{2} \left(\Delta_{\mathcal{G}_1^f} \otimes \frac{1}{m_2} - \frac{c_1}{m_2} \right) \left(\frac{1}{m_1} \otimes \Delta_{\mathcal{G}_2^f} \right).
\end{aligned}$$

A straightforward computation leads to

$$\begin{aligned}
[\Delta_{\mathcal{G}^f}, \mathfrak{i}\mathcal{A}_{\mathcal{G}^f}]_{\circ} &= \frac{1}{2}(\Delta_{\mathcal{G}_1^f} - c_1)(c_2 - \Delta_{\mathcal{G}_1^f}) \otimes \frac{1}{m_2} \\
&\quad + \frac{1}{2} \left(\tilde{c}\delta_0(c_2 - \Delta_{\mathcal{G}_1^f}) - (\Delta_{\mathcal{G}_1^f} - c_1)\tilde{c}\delta_0 - \tilde{c}^2\delta_0 \right) \otimes \frac{1}{m_2} \\
&\quad - \tilde{c} \left[\delta_0 \otimes \frac{1}{m_2}, \mathfrak{i}\mathcal{A}_{\mathcal{G}^f} \right]_{\circ}, \\
&\quad + \left[\frac{1}{m_1^f(\cdot)} \otimes \Delta_{\mathcal{G}_2^f}, \mathfrak{i}\mathcal{A}_{\mathcal{G}^f} \right]_{\circ}.
\end{aligned}$$

in the form sense on $\ell_c^2(\mathcal{V}_1^f) \otimes \ell^2(\mathcal{V}_2^f)$. Since $\Delta_{\mathcal{G}_1^f}$ is bounded, m_2 is a constant, and δ_0 is a finite rank operator then the second line of the r.h.s. is compact. Moreover, since $\delta_0 = |\delta_0\rangle\langle\delta_0|$ and since $\delta_0 \in \mathcal{D}(T^{-1}\mathcal{A}_{\mathbb{N}}T)$, Lemma 3.12 ensures that the third line is also compact operator. We turn to the last line. We have:

$$\left[\frac{1}{m_1^f(\cdot)}, \mathfrak{i}\mathcal{A}_{\mathbb{N}} \right]_{\circ} = \frac{1}{2}(e-1)e^{-Q} \left(Q - \frac{1}{2} \right) U,$$

in the form sense on $\ell_c^2(\mathcal{V}_1^f) \otimes \ell^2(\mathcal{V}_2^f)$. The operator is a compact in $\ell^2(\mathbb{N}, 1)$ since U is bounded and $\lim_{n \rightarrow \infty} e^{-n}(n-1/2) = 0$. By conjugating back with T and T^{-1} , we infer that the fourth line is a compact operator. This implies that $[\Delta_{\mathcal{G}^f}, \mathfrak{i}\mathcal{A}_{\mathcal{G}^f}]_{\circ} \in \mathcal{B}(\ell^2(\mathcal{V}^f, m^f))$ and (10).

Note that (11) follows from (10) and the fact that $\Delta_{\mathcal{G}_1^f}$ is bounded and Lemma 3.4.

Next, as in the proof of Lemma 3.2, as $\ell_c^2(\mathcal{V}_1^f) \otimes \ell^2(\mathcal{V}_2^f)$ is a core for \mathcal{A}^f , we deduce that $\Delta_{\mathcal{G}^f} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}^f})$. \square

Lemma 3.7. *We have $\Delta_{\mathcal{G}^f} \in \mathcal{C}^2(\mathcal{A})$.*

Proof. We prove that $[[\Delta_{\mathcal{G}^f}, \mathbf{i}\mathcal{A}_{\mathcal{G}^f}]_{\circ}, \mathbf{i}\mathcal{A}_{\mathcal{G}^f}]_{\circ} \in \mathcal{B}(\mathcal{H})$. A straightforward computation leads to

$$\begin{aligned} [[\Delta_{\mathcal{G}^f}, \mathbf{i}\mathcal{A}_{\mathcal{G}^f}], \mathbf{i}\mathcal{A}_{\mathcal{G}^f}] &= \left[\frac{1}{2}(\Delta_{\mathcal{G}_1^f} - c_1)(c_2 - \Delta_{\mathcal{G}_1^f}) \otimes \frac{1}{m_2}, \mathbf{i}\mathcal{A}_{\mathcal{G}^f} \right] \\ &\quad + \left[\frac{1}{2} \left(\tilde{c}\delta_0(c_2 - \Delta_{\mathcal{G}_1^f}) - (\Delta_{\mathcal{G}_1^f} - c_1)\tilde{c}\delta_0 - \tilde{c}^2\delta_0 \right) \otimes \frac{1}{m_2}, \mathbf{i}\mathcal{A}_{\mathcal{G}^f} \right] \quad (10) \\ &\quad - \tilde{c}[[\delta_0 \otimes \frac{1}{m_2}, \mathbf{i}\mathcal{A}_{\mathcal{G}^f}], \mathbf{i}\mathcal{A}_{\mathcal{G}^f}] \\ &\quad + \left[\left[\frac{1}{m_1^f(\cdot)} \otimes \Delta_{\mathcal{G}_2^f}, \mathbf{i}\mathcal{A}_{\mathcal{G}^f} \right], \mathbf{i}\mathcal{A}_{\mathcal{G}^f} \right], \end{aligned}$$

in the form sense on $\ell_c^2(\mathcal{V}_1^f) \otimes \ell^2(\mathcal{V}_2^f)$. For the first two terms, we have

$$\begin{aligned} \left[\frac{1}{2}(\Delta_{\mathcal{G}_1^f} - c_1)(c_2 - \Delta_{\mathcal{G}_1^f}) \otimes \frac{1}{m_2}, \mathbf{i}\mathcal{A}_{\mathcal{G}^f} \right] &= \frac{1}{2} \left[\Delta_{\mathcal{G}_1^f}, \mathbf{i}T^{-1}\mathcal{A}_{\mathbb{N}T} \right] (c_2 - \Delta_{\mathcal{G}_1^f}) \otimes \frac{1}{m_2} \\ &\quad + \frac{1}{2}(\Delta_{\mathcal{G}_1^f} - c_1)[\Delta_{\mathcal{G}_1^f}, \mathbf{i}T^{-1}\mathcal{A}_{\mathbb{N}T}] \otimes \frac{1}{m_2} \end{aligned}$$

and

$$\begin{aligned} &\left[\frac{1}{2} \left(\tilde{c}\delta_0(c_2 - \Delta_{\mathcal{G}_1^f}) - (\Delta_{\mathcal{G}_1^f} - c_1)\tilde{c}\delta_0 - \tilde{c}^2\delta_0 \right) \otimes \frac{1}{m_2}, \mathbf{i}\mathcal{A}_{\mathcal{G}^f} \right] \\ &= -\frac{\tilde{c}}{2}\delta_0[\Delta_{\mathcal{G}_1^f}, \mathbf{i}T^{-1}\mathcal{A}_{\mathbb{N}T}] \otimes \frac{1}{m_2} + \frac{\tilde{c}}{2}[\delta_0, \mathbf{i}T^{-1}\mathcal{A}_{\mathbb{N}T}](c_2 - \Delta_{\mathcal{G}_1^f}) \otimes \frac{1}{m_2} \\ &\quad - \frac{\tilde{c}}{2}[\Delta_{\mathcal{G}_1^f}, \mathbf{i}T^{-1}\mathcal{A}_{\mathbb{N}T}] \otimes \frac{1}{m_2} - \frac{\tilde{c}^2}{2}[\delta_0, \mathbf{i}T^{-1}\mathcal{A}_{\mathbb{N}T}] \otimes \frac{1}{m_2}. \end{aligned}$$

Since $\delta_0 \in \mathcal{D}((T^{-1}\mathcal{A}_{\mathbb{N}T})^2)$, Lemma 3.12 ensures that the third line extends to a compact operator. Moreover, since $\Delta_{\mathcal{G}_1^f}$ is bounded, $\frac{1}{m_2}$ is a constant, and δ_0 is a finite rank operator then the second line of the r.h.s. is compact. We turn to the last line. We have $\left[\left[\frac{1}{m_1^f(\cdot)} \otimes \Delta_{\mathcal{G}_2^f}, \mathbf{i}\mathcal{A}_{\mathcal{G}^f} \right]_{\circ}, \mathbf{i}\mathcal{A}_{\mathcal{G}^f} \right]_{\circ} = \left[\left[\frac{1}{m_1^f(\cdot)}, \mathbf{i}T^{-1}\mathcal{A}_{\mathbb{N}T} \right]_{\circ}, \mathbf{i}T^{-1}\mathcal{A}_{\mathbb{N}T} \right]_{\circ} \otimes \Delta_{\mathcal{G}_2^f}$, where

$$\begin{aligned} &\mathbf{i} \left[\frac{1}{m_1^f(\cdot)}, \mathbf{i}T^{-1}\mathcal{A}_{\mathbb{N}T} \right] T^{-1}\mathcal{A}_{\mathbb{N}T} \\ &= \frac{1}{2} \left[\frac{1}{m_1^f(\cdot)}, \mathbf{i}T^{-1}\mathcal{A}_{\mathbb{N}T} \right] \left(e^{-1/2} \left(Q + \frac{1}{2} \right) U^* + e^{1/2} \left(\frac{1}{2} - Q \right) U \right) \\ &= \frac{e^{-1/2}}{4} (e-1)e^{-Q} \left(Q - \frac{1}{2} \right) \left(QUU^* - UU^* + \frac{1}{2}UU^* \right) \\ &\quad + \frac{e^{1/2}}{8} (e-1)e^{-Q} \left(Q - \frac{1}{2} \right) U^2 - \frac{e^{1/2}}{4} (e-1)e^{-Q} \left(Q - \frac{1}{2} \right) UQU, \end{aligned}$$

and

$$\begin{aligned} &\mathbf{i}T^{-1}\mathcal{A}_{\mathbb{N}T} \left[\frac{1}{m_1^f(\cdot)}, \mathbf{i}T^{-1}\mathcal{A}_{\mathbb{N}T} \right] \\ &= \frac{1}{2} \left(e^{-1/2} \left(Q + \frac{1}{2} \right) U^* + e^{1/2} \left(\frac{1}{2} - Q \right) U \right) \left[\frac{1}{m_1^f(\cdot)}, \mathbf{i}T^{-1}\mathcal{A}_{\mathbb{N}T} \right] \\ &= \frac{e^{-1/2}}{4} (e-1)U^*Qe^{-Q} \left(Q - \frac{1}{2} \right) U - \frac{e^{-1/2}}{8} (e-1)U^*e^{-Q} \left(Q - \frac{1}{2} \right) U \\ &\quad + \frac{e^{1/2}}{4} (e-1)Ue^{-Q} \left(Q - \frac{1}{2} \right) U - \frac{e^{1/2}}{4} (e-1)UQe^{-Q} \left(Q - \frac{1}{2} \right) U. \end{aligned}$$

Using the proof of the Lemma (3.6) and the dictionary (6), then the fourth line is a compact operator.

Next, as in the proof of Lemma 3.2, as $\ell^2(\mathcal{V}_1^f) \otimes \ell^2(\mathcal{V}_2^f)$ is a core for \mathcal{A}^f , we infer that $\Delta_{\mathcal{G}^f} \in \mathcal{C}^2(\mathcal{A}_{\mathcal{G}^f})$. \square

we pass to Mourre estimate.

Proposition 3.8. *We have $\Delta_{\mathcal{G}^f} \in \mathcal{C}^2(\mathcal{A}_{\mathcal{G}^f})$. Moreover, there are $c > 0$, a compact interval $\mathcal{I} \subset (c_1, c_2)$, a compact operator K such that*

$$(12) \quad E_{\mathcal{I}}(\Delta_{\mathcal{G}^f})[\Delta_{\mathcal{G}^f}, i\mathcal{A}_{\mathcal{G}^f}]_{\circ} E_{\mathcal{I}}(\Delta_{\mathcal{G}^f}) \geq cE_{\mathcal{I}}(\Delta_{\mathcal{G}^f}) + K,$$

in the form sense.

Proof. The Lemma 3.7 gives that $\Delta_{\mathcal{G}^f} \in \mathcal{C}^2(\mathcal{A}_{\mathcal{G}^f})$. By (11), we obtain

$$\begin{aligned} E_{\mathcal{I}}(\Delta_{\mathcal{G}^f})[\Delta_{\mathcal{G}^f}, i\mathcal{A}_{\mathcal{G}^f}]_{\circ} E_{\mathcal{I}}(\Delta_{\mathcal{G}^f}) &= E_{\mathcal{I}}(\Delta_{\mathcal{G}^f})w(\Delta_{\mathcal{G}^f})E_{\mathcal{I}}(\Delta_{\mathcal{G}^f}) + K \\ &\geq cE_{\mathcal{I}}(\Delta_{\mathcal{G}^f}) + K, \end{aligned}$$

where K is a compact operator and

$$c := \frac{m_2}{2} \inf_{x \in \mathcal{I}} \left(x - \frac{c_1}{m_2} \right) \left(\frac{c_2}{m_2} - x \right) > 0.$$

\square

3.4. The cusps side.

3.4.1. *The model and the low/high energy decomposition.* In the spirit of [GoMo, GoTr], we denote by P^{le} the projection on $\ker(\Delta_{\mathcal{G}_2})$ and by P^{he} is the projection on $\ker(\Delta_{\mathcal{G}_2})^{\perp}$. Here, *le* stands for *low energy* and *he* for *high energy*.

We Now, take advantage of

$$(13) \quad \begin{aligned} \ell^2(\mathcal{V}^c, m) &:= \mathcal{H}^{\text{le}} \oplus \mathcal{H}^{\text{he}} \\ &:= \ell^2(\mathbb{N}, m_1) \otimes \ker(\Delta_2) \oplus \ell^2(\mathbb{N}, m_1) \otimes \ker(\Delta_2)^{\perp}. \end{aligned}$$

We have that $\Delta_{\mathcal{G}^c} := \Delta_{\mathcal{G}^c}^{\text{le}} \oplus \Delta_{\mathcal{G}^c}^{\text{he}}$, where

$$(14) \quad \Delta_{\mathcal{G}^c}^{\text{le}} := \Delta_{\mathcal{G}_1^c} \otimes \frac{1}{m_2} P^{\text{le}},$$

on $(1 \otimes P^{\text{le}})\ell^2(\mathcal{V}^c, m)$, and

$$(15) \quad \Delta_{\mathcal{G}^c}^{\text{he}} := \Delta_{\mathcal{G}_1^c} \otimes \frac{1}{m_2} P^{\text{he}} + \frac{1}{m_1(\cdot)} \otimes P^{\text{he}} \Delta_{\mathcal{G}_2^c},$$

on $(1 \otimes P^{\text{he}})\ell^2(\mathcal{V}^c, m)$.

Let

$$\begin{aligned} T : \ell^2(\mathbb{N}, m_1^c) &\rightarrow \ell^2(\mathbb{N}, 1) \\ f &\mapsto \left(n \mapsto \sqrt{m_1^c(n)} f(n) \right). \end{aligned}$$

It is a unitary map. We see that:

$$T \Delta_{\mathcal{G}_1^c} T^{-1} = \Delta_{\mathbb{N}} - (e^{-1/2} - 1)\delta_0 + e^{1/2} + e^{-1/2} - 2 \text{ in } \ell^2(\mathbb{N}).$$

By using for instance some Jacobi matrices techniques, it is well-known that the essential spectrum of $\Delta_{\mathcal{G}^c}^{\text{le}}$ is purely absolutely continuous and equal to

$$\sigma_{\text{ac}}(\Delta_{\mathcal{G}^c}^{\text{le}}) = [e^{1/2} + e^{-1/2} - 2, e^{1/2} + e^{-1/2} + 2],$$

with multiplicity one, e.g., [We]. It has a unique eigenvalue.

We turn to the high energy part. Denote by $\{\lambda_i\}_{i=1, \dots, n}$, with $\lambda_i \leq \lambda_{i+1}$, the eigenvalues of $\Delta_{\mathcal{G}_2^c}$. Recall that $\lambda_1 = 0$. Using [GoTr, Equation (10)],

$$\frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2^c} P^{\text{he}} \leq \Delta_{\mathcal{G}^c} (1 \otimes P^{\text{he}}) \leq 2M + \frac{1}{m_1(\cdot)} \otimes \Delta_{\mathcal{G}_2^c} P^{\text{he}}.$$

Hence, $\Delta_{\mathcal{G}^c} (1 \otimes P^{\text{he}})$ has a compact resolvent.

3.4.2. *The conjugate operator.* We Now, go back to $\ell^2(\mathbb{N}, m_1) \otimes \ker(\Delta_2)$. We set:

$$(16) \quad \mathcal{A}_{\mathcal{G}^c}^{\text{le}} := T^{-1} \mathcal{A}_{\mathbb{N}} T \otimes P_{\ker(\Delta_2)}.$$

Straightforwardly we get

$$\mathcal{A}_{\mathcal{G}^c}^{\text{le}} = -\frac{i}{2} \left(e^{-1/2} \left(Q + \frac{1}{2} \right) U^* + e^{1/2} \left(\frac{1}{2} - Q \right) U \right) \otimes P_{\ker(\Delta_2)}$$

on $\ell_c^2(\mathbb{N}, m_1) \otimes \ker(\Delta_2)$. Hence, we obtain from (8):

$$(17) \quad [\Delta_{\mathcal{G}^c}^{\text{le}}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}]_{\circ} = \frac{1}{2} \Delta_{\mathcal{G}^c}^{\text{le}} (4 - \Delta_{\mathcal{G}^c}^{\text{le}}).$$

With respect to (13), we set

$$\mathcal{A}_{\mathcal{G}^c} := \mathcal{A}_{\mathcal{G}^c}^{\text{le}} \oplus \mathcal{A}_{\mathcal{G}^c}^{\text{he}},$$

where $\mathcal{A}_{\mathcal{G}^c}^{\text{he}} := 0$. We obtain:

Lemma 3.9. *We have $\Delta_{\mathcal{G}^c} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}^c})$ and*

$$(18) \quad [\Delta_{\mathcal{G}^c}, i\mathcal{A}_{\mathcal{G}^c}]_{\circ} = [\Delta_{\mathcal{G}^c}^{\text{le}}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}]_{\circ} \oplus 0 = w(\Delta_{\mathcal{G}^c}^{\text{le}}) \oplus 0 + K,$$

with respect to (13) and where $K \in \mathcal{K}(\ell^2(\mathcal{V}^c), m^c)$,

$$w(x) := \frac{m_2}{2} \left(x - \frac{c_1}{m_2} \right) \left(\frac{c_2}{m_2} - x \right),$$

and

$$c_1 := e^{1/2} + e^{-1/2} - 2 \quad \text{and} \quad c_2 := e^{1/2} + e^{-1/2} + 2.$$

In particular, $[\Delta_{\mathcal{G}^c}, i\mathcal{A}_{\mathcal{G}^c}]_{\circ} \in \mathcal{B}(\ell^2(\mathcal{V}, m))$.

Proof. We prove that $[\Delta_{\mathcal{G}^c}^{\text{le}}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}]_{\circ} \in \mathcal{B}(\mathcal{H}^{\text{le}})$. We set $\tilde{c} := e^{-1/2} - 1$. A straightforward computation leads to

$$\begin{aligned} [\Delta_{\mathcal{G}^c}^{\text{le}}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}]_{\circ} &= \frac{m_2}{2} (\Delta_{\mathcal{G}^c}^{\text{le}} - \frac{c_1}{m_2}) (\frac{c_2}{m_2} - \Delta_{\mathcal{G}^c}^{\text{le}}) \\ &\quad + \frac{1}{2} (\tilde{c}\delta_0(c_2 - \Delta_{\mathcal{G}^c}^{\text{le}}) - (\Delta_{\mathcal{G}^c}^{\text{le}} - c_1)\tilde{c}\delta_0 - \tilde{c}^2\delta_0) \otimes \frac{1}{m_2} P^{\text{le}} \\ &\quad - \tilde{c}[\delta_0 \otimes \frac{1}{m_2} P^{\text{le}}, i\mathcal{A}^{\text{le}}]_{\circ}, \end{aligned}$$

in the form sense on $\ell_c^2(\mathcal{V}_1^c) \otimes P^{\text{le}} \ell^2(\mathcal{V}_2^c)$. Since $\Delta_{\mathcal{G}^c}^{\text{le}}$ is bounded, P^{le} is of finite rank, and δ_0 is a finite rank operator then the second line of the r.h.s. is compact. Moreover, since $\delta_0 = |\delta_0\rangle\langle\delta_0|$ and since $\delta_0 \in \mathcal{D}(T^{-1}\mathcal{A}_{\mathbb{N}}T)$, Lemma 3.12 ensures that the last term is also compact operator. This implies that $[\Delta_{\mathcal{G}^c}^{\text{le}}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}]_{\circ} \in \mathcal{B}(\mathcal{H}^{\text{le}})$ and (18).

Next, let $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$ be a family of functions defined as follows:

$$\mathcal{X}_n(x_1, x_2) := \left(\left(1 - \frac{-n + x_1}{n^2 + 1} \right) \vee 0 \right) \wedge 1,$$

where $\text{supp}(\mathcal{X}_n) = [n, n^2 + n + 1] \times \mathcal{V}_2$ and for all $x \in [0, n]$, $\mathcal{X}_n(x_1, x_2) = 1$. We denote by $\mathcal{X}_n(Q)$ the operator of multiplication by \mathcal{X}_n .

1) We have $\|\mathcal{X}_n\|_{\infty} = 1$ then $\|\mathcal{X}_n(Q)\|_{\mathcal{B}(\ell^2(\mathcal{V}^c, m^c))} = 1$. Moreover, $\mathcal{X}_n(Q)$ tends strongly to 1 as $n \rightarrow +\infty$. Now, we shall show that $\sup_n \|\mathcal{X}_n(Q)\|_{\mathcal{D}(\Delta)} < \infty$. For

all $f \in \mathcal{C}_c(\mathcal{V}^c)$, we have

$$\begin{aligned} \|\mathcal{X}_n(Q)f\|^2 &= \sum_{x_1=n}^{n^2+n+1} \sum_{x_2 \in \mathcal{V}_2} |\mathcal{X}_n(x_1, x_2)|^2 |f(x_1, x_2)|^2 \\ &\leq \sum_{x_1=n}^{n^2+n+1} \sum_{x_2 \in \mathcal{V}_2} m(x_1, x_2) \left| \frac{n+x_1}{n^2+1} \right|^2 |f(x_1, x_2)|^2 \\ &\leq \left| \frac{n^2+n+1}{n^2+1} \right|^2 \sum_{x_1=n}^{n^2+n+1} \sum_{x_2 \in \mathcal{V}_2} m(x_1, x_2) |f(x_1, x_2)|^2 < \infty. \end{aligned}$$

Since

$$\begin{aligned} [\Delta_{\mathcal{G}_1^c} \otimes \frac{1}{m_2} + \frac{1}{m_1} \otimes \Delta_2, \mathcal{X}_n(Q)] &= [\Delta_{\mathcal{G}_1^c} \otimes \frac{1}{m_2}, \mathcal{X}_n(Q)] + [\frac{1}{m_1} \otimes \Delta_2, \mathcal{X}_n(Q)] \\ &= [\Delta_{\mathcal{G}_1^c}, \mathcal{X}_n(Q)] \otimes \frac{1}{m_2} + \underbrace{[\frac{1}{m_1}, \mathcal{X}_n(Q)] \otimes \Delta_2}_{=0} \\ &\quad \text{uniformly bounded} \end{aligned}$$

then there is $c > 0$ such that, for all $f \in \ell_c^2(\mathcal{V}^c, m^c)$ and $n \in \mathbb{N}$,

$$\|(\Delta_{\mathcal{G}^c} + i)\mathcal{X}_n(Q)(\Delta_{\mathcal{G}^c} + i)^{-1}f\|^2 \leq c\|f\|^2.$$

Since, $\Delta_{\mathcal{G}^c}$ is essentially self-adjoint then for all $f \in \ell^2(\mathcal{V}^c, m^c)$, $\|(\Delta_{\mathcal{G}_1^c} + i)\mathcal{X}_n(Q)f\|^2 \leq c\|(\Delta_{\mathcal{G}_1^c} + i)f\|^2$. Then the result follows.

2) For all $f \in \mathcal{C}_c(\mathcal{V}^c)$, we have

$$\begin{aligned} \|[\mathcal{A}_{\mathcal{G}_1^c}, \mathcal{X}_n(Q)]f\|^2 &= \sum_{x_1=n}^{n^2+n+1} \sum_{x_2 \in \mathcal{V}_2} m(x_1, x_2) |(\mathcal{A}_{\mathcal{G}_1^c} \mathcal{X}_n(Q)f)(x_1, x_2) \\ &\quad - (\mathcal{X}_n(Q)\mathcal{A}_{\mathcal{G}_1^c}f)(x_1, x_2)|^2. \end{aligned}$$

Since $f \in \mathcal{C}_c(\mathcal{V}^c)$ then $\|[\mathcal{A}_{\mathcal{G}_1^c}, \mathcal{X}_n(Q)]f\| \rightarrow 0$, as $n \rightarrow \infty$. So $\mathcal{A}_{\mathcal{G}_1^c} \mathcal{X}_n(Q)f \rightarrow \mathcal{A}_{\mathcal{G}_1^c}f$, as $n \rightarrow \infty$.

3) We have $[\Delta_{\mathcal{G}^c}, \mathcal{X}_n(Q)] = [\Delta_{\mathcal{G}_1^c}, \mathcal{X}_n(Q)] \otimes \frac{1}{m_2}$ and for all $f \in \mathcal{C}_c(\mathcal{V}^c)$,

$$\begin{aligned} \|\mathcal{A}_{\mathcal{G}_1^c}[U \otimes 1, \mathcal{X}_n(Q)]f\|^2 &= \frac{1}{2} \sum_{x_1=n}^{n^2+n+1} \sum_{x_2 \in \mathcal{V}_2} m(x_1, x_2) |e^{-1/2}(Q + \frac{1}{2})U^* \\ &\quad - e^{1/2}(\frac{1}{2} - Q)U] \otimes P_{\ker(\Delta_{\mathcal{G}_2})} ((U \otimes 1)\mathcal{X}_n(Q) - \mathcal{X}_n(Q)(U \otimes 1))f(x_1, x_2)|^2 \\ &= \frac{1}{2} \sum_{x_1=n}^{n^2+n+1} \sum_{x_2 \in \mathcal{V}_2} m(x_1, x_2) |e^{-1/2}(x_1 + \frac{1}{2}) \otimes P_{\ker(\Delta_{\mathcal{G}_2})} \mathcal{X}_n(x_1, x_2)f(x_1, x_2) \\ &\quad - e^{1/2}(\frac{1}{2} - x_1) \otimes P_{\ker(\Delta_{\mathcal{G}_2})} \mathcal{X}_n(x_1 - 2, x_2)f(x_1 - 2, x_2) \\ &\quad - e^{-1/2}(x_1 + \frac{1}{2}) \otimes P_{\ker(\Delta_{\mathcal{G}_2})} \mathcal{X}_n(x_1 + 1, x_2)f(x_1, x_2) \\ &\quad + e^{1/2}(\frac{1}{2} - x_1) \otimes P_{\ker(\Delta_{\mathcal{G}_2})} \mathcal{X}_n(x_1 - 1, x_2)f(x_1 - 2, x_2)|^2 \\ &\leq \frac{1}{2} \sum_{x_1=n}^{n^2+n+1} \sum_{x_2 \in \mathcal{V}_2} m(x_1, x_2) (e^{-1/2}(x_1 + \frac{1}{2}) \otimes P_{\ker(\Delta_{\mathcal{G}_2})} \\ &\quad \times |\mathcal{X}_n(x_1, x_2) - \mathcal{X}_n(x_1 + 1, x_2)|f(x_1, x_2) \\ &\quad + e^{1/2}(\frac{1}{2} - x_1) \otimes P_{\ker(\Delta_{\mathcal{G}_2})} |\mathcal{X}_n(x_1 - 1, x_2) - \mathcal{X}_n(x_1 - 2, x_2)|f(x_1 - 2, x_2))^2. \end{aligned}$$

In the same way, we prove that $\|\mathcal{A}_{\mathcal{G}_1^c}[U^* \otimes 1, \mathcal{X}_n(Q)]\| < \infty$.

4) Since $(\Delta_{\mathcal{G}^c} - z)^{-1} \in \ell^2(\mathcal{V}^c)$, for all $z \notin \sigma(\Delta_{\mathcal{G}^c})$ then $(\Delta_{\mathcal{G}^c} - z)^{-1} \mathcal{C}_c(\mathcal{V}^c) \subset \ell^2(\mathcal{V}^c)$ and since \mathcal{X}_n has a compact support, we have $\mathcal{X}_n(Q)(\Delta_{\mathcal{G}^c} - z)^{-1} \mathcal{C}_c(\mathcal{V}^c) \subset \mathcal{C}_c(\mathcal{V}^c)$, and as $\ell_c^2(\mathcal{V}_1^c) \otimes P^{\text{le}} \ell^2(\mathcal{V}_2^c)$ is a core for \mathcal{A}^{le} . So, by applying [GoMo, Lemma A.2], we obtain that $\Delta_{\mathcal{G}^c}^{\text{le}} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}^c}^{\text{le}})$.

Finally, it is enough to prove that $R := (\Delta_{\mathcal{G}^c} + i)^{-1} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}^c})$. Note now, that $R = (\Delta_{\mathcal{G}^c}^{\text{le}} + i)^{-1} \oplus (\Delta_{\mathcal{G}^c}^{\text{he}} + i)^{-1}$ with respect to (13). Since $\mathcal{A}_{\mathcal{G}^c} := \mathcal{A}_{\mathcal{G}^c}^{\text{le}} \oplus 0$, $(\Delta_{\mathcal{G}^c}^{\text{le}} + i)^{-1} \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}^c}^{\text{le}})$, and trivially $(\Delta_{\mathcal{G}^c}^{\text{he}} + i)^{-1} \in \mathcal{C}^1(0)$, we obtain that $R \in \mathcal{C}^1(\mathcal{A}_{\mathcal{G}^c})$. This concludes the Lemma. \square

Lemma 3.10. *We have $\Delta_{\mathcal{G}^c} \in \mathcal{C}^2(\mathcal{A}_{\mathcal{G}^c})$ and*

$$(19) \quad [[\Delta_{\mathcal{G}^c}, i\mathcal{A}_{\mathcal{G}^c}]_{\circ}, i\mathcal{A}_{\mathcal{G}^c}]_{\circ} = [[\Delta_{\mathcal{G}^c}^{\text{le}}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}]_{\circ}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}]_{\circ} \oplus 0.$$

Proof. We shall show that $[[\Delta_{\mathcal{G}^c}^{\text{le}}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}]_{\circ}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}]_{\circ} \in \mathcal{B}(\mathcal{H}^{\text{le}})$. A straightforward computation leads to

$$\begin{aligned} [[\Delta_{\mathcal{G}^c}^{\text{le}}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}]_{\circ}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}]_{\circ} &= \left[\frac{m_2}{2} (\Delta_{\mathcal{G}^c}^{\text{le}} - \frac{c_1}{m_2}) (\frac{c_2}{m_2} - \Delta_{\mathcal{G}^c}^{\text{le}}), i\mathcal{A}_{\mathcal{G}^c}^{\text{le}} \right] \\ &+ \left[\frac{1}{2} (\tilde{c}\delta_0(c_2 - \Delta_{\mathcal{G}_1^c}) - (\Delta_{\mathcal{G}_1^c} - c_1)\tilde{c}\delta_0 - \tilde{c}^2\delta_0) \otimes \frac{1}{m_2} P^{\text{le}}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}} \right] \\ &- \tilde{c} \left[[\delta_0 \otimes \frac{1}{m_2} P^{\text{le}}, i\mathcal{A}^{\text{le}}]_{\circ}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}} \right], \end{aligned}$$

in the form sense on $\ell_c^2(\mathcal{V}_1^c) \otimes P^{\text{le}} \ell^2(\mathcal{V}_2^c)$. First we have,

$$\begin{aligned} \left[\frac{m_2}{2} (\Delta_{\mathcal{G}^c}^{\text{le}} - \frac{c_1}{m_2}) (\frac{c_2}{m_2} - \Delta_{\mathcal{G}^c}^{\text{le}}), i\mathcal{A}_{\mathcal{G}^c}^{\text{le}} \right] &= [\Delta_{\mathcal{G}^c}^{\text{le}}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}] \left(\frac{c_2}{m_2} - \Delta_{\mathcal{G}^c}^{\text{le}} \right) \\ &- \frac{m_2}{2} \left(\Delta_{\mathcal{G}^c}^{\text{le}} - \frac{c_1}{m_2} \right) [\Delta_{\mathcal{G}^c}^{\text{le}}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}], \end{aligned}$$

on the other hand,

$$\begin{aligned} &\left[\frac{1}{2} (\tilde{c}\delta_0(c_2 - \Delta_{\mathcal{G}_1^c}) - (\Delta_{\mathcal{G}_1^c} - c_1)\tilde{c}\delta_0 - \tilde{c}^2\delta_0) \otimes \frac{1}{m_2} P^{\text{le}}, i\mathcal{A}_{\mathcal{G}^c}^{\text{le}} \right] \\ &= -\frac{1}{2} \tilde{c} [\Delta_{\mathcal{G}_1^c}, iT^{-1}\mathcal{A}_{\mathbb{N}}T] \delta_0 \otimes \frac{1}{m_2} P^{\text{le}} + \frac{1}{2} \tilde{c} (c_2 - \Delta_{\mathcal{G}_1^c}) [\delta_0, iT^{-1}\mathcal{A}_{\mathbb{N}}T] \otimes \frac{1}{m_2} P^{\text{le}} \\ &- \frac{1}{2} \tilde{c} (\Delta_{\mathcal{G}_1^c} - c_1) [\delta_0, iT^{-1}\mathcal{A}_{\mathbb{N}}T] \otimes \frac{1}{m_2} P^{\text{le}} - \frac{1}{2} \tilde{c} [\Delta_{\mathcal{G}_1^c}, iT^{-1}\mathcal{A}_{\mathbb{N}}T] \delta_0 \otimes \frac{1}{m_2} P^{\text{le}} \\ &- \frac{1}{2} \tilde{c} [\delta_0, iT^{-1}\mathcal{A}_{\mathbb{N}}T] \otimes \frac{1}{m_2} P^{\text{le}}. \end{aligned}$$

Since $\delta_0 \in \mathcal{D}((T^{-1}\mathcal{A}_{\mathbb{N}}T)^2)$, Lemma 3.12 ensures that the last term extends to a compact operator. Moreover, since P^{le} is of finite rank, $[\Delta_{\mathcal{G}_1^c}, iT^{-1}\mathcal{A}_{\mathbb{N}}T]_{\circ}$ is bounded and δ_0 is a finite rank operator then the first and the second lines of the r.h.s. are bounded.

Next, as in the proof of Lemma 3.2, as $\ell_c^2(\mathcal{V}_1^c) \otimes P^{\text{le}} \ell^2(\mathcal{V}_2^c)$ is a core for \mathcal{A}^{le} , we deduce that $\Delta_{\mathcal{G}^c}^{\text{le}} \in \mathcal{C}^2(\mathcal{A}_{\mathcal{G}^c}^{\text{le}})$. \square

Concerning the Mourre estimate, we prove the follows result:

Proposition 3.11. *We have $\Delta_{\mathcal{G}^c} \in \mathcal{C}^2(\mathcal{A}_{\mathcal{G}^c})$. Moreover, there are $c > 0$, a compact interval $\mathcal{I} \subset (c_1, c_2)$, a compact operator K such that*

$$(20) \quad E_{\mathcal{I}}(\Delta_{\mathcal{G}^c})[\Delta_{\mathcal{G}^c}, i\mathcal{A}_{\mathcal{G}^c}]_{\circ} E_{\mathcal{I}}(\Delta_{\mathcal{G}^c}) \geq c E_{\mathcal{I}}(\Delta_{\mathcal{G}^c}) + K,$$

in the form sense.

Proof. The Lemma 3.10 gives that $\Delta_{\mathcal{G}^c} \in \mathcal{C}^2(\mathcal{A}_{\mathcal{G}^c})$. On \mathcal{H}^{he} , $E_{\mathcal{I}}(\Delta_{\mathcal{G}^c}^{\text{he}})$ is compact since $\Delta_{\mathcal{G}^c}^{\text{he}}$ is with compact resolvent and with respect to (13), we have

$$\begin{aligned} E_{\mathcal{I}}(\Delta_{\mathcal{G}^c})[\Delta_{\mathcal{G}^c}, \mathbf{i}\mathcal{A}_{\mathcal{G}^c}] \circ E_{\mathcal{I}}(\Delta_{\mathcal{G}^c}) &= E_{\mathcal{I}}(\Delta_{\mathcal{G}^c}^{\text{le}})[\Delta_{\mathcal{G}^c}^{\text{le}}, \mathbf{i}\mathcal{A}_{\mathcal{G}^c}^{\text{le}}] \circ E_{\mathcal{I}}(\Delta_{\mathcal{G}^c}^{\text{le}}) \oplus 0 \\ &\geq cE_{\mathcal{I}}(\Delta_{\mathcal{G}^c}^{\text{le}}) \oplus 0 \geq cE_{\mathcal{I}}(\Delta_{\mathcal{G}^c}) + K, \end{aligned}$$

where K is a compact operator and

$$c := \frac{m_2}{2} \inf_{x \in I} \left(x - \frac{c_1}{m_2} \right) \left(\frac{c_2}{m_2} - x \right) > 0.$$

□

3.5. The compact part. We define the conjugate operator on $\ell^2(\mathcal{V}) = \ell^2(\mathcal{V}^f) \oplus \ell^2(\mathcal{V}^0) \oplus \ell^2(\mathcal{V}^c)$ as

$$\mathcal{A} := \mathcal{A}_{\mathcal{G}^f} \oplus 0 \oplus \mathcal{A}_{\mathcal{G}^c}.$$

It is clearly essentially self-adjoint on $\mathcal{C}_c(\mathcal{V})$. We introduce the class of operators with kernel with compact support. Define the meaning of bra-ket notation.

Lemma 3.12. *If $\varphi, \psi \in \mathcal{D}(\mathcal{A})$ then $|\varphi\rangle\langle\psi| \in \mathcal{C}^1(\mathcal{A})$. Moreover, if $\varphi, \psi \in \mathcal{D}(\mathcal{A}^2)$ then $|\varphi\rangle\langle\psi| \in \mathcal{C}^2(\mathcal{A})$.*

Definition 3.13. *We say $S \in \mathcal{K}_0$ if there $a : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}^+$ such that $a(i, j) = a(j, i)$ for all $i, j \in \mathcal{V}$, such that $\text{supp } a(\cdot, \cdot)$ is finite, and*

$$S = \sum_{i, j \in \mathcal{V}} a(i, j) |\delta_i\rangle\langle\delta_j|.$$

Remark 3.14. *Clearly $(\Delta_{\mathcal{G}} - \Delta_{\mathcal{G}^f} \oplus 0 \oplus \Delta_{\mathcal{G}^c}) \in \mathcal{K}_0$. In particular, the difference is a finite rank operator.*

Lemma 3.15. *We have $\mathcal{K}_0 \subset \mathcal{C}^2(\mathcal{A})$.*

Proof. Apply Lemma 3.12. □

3.6. The whole graph. In this section, we give the Mourre estimate in the whole graph.

Proposition 3.16. *We have $\Delta_{\mathcal{G}} \in \mathcal{C}^2(\mathcal{A})$. Moreover, there are $c > 0$, a compact interval $\mathcal{I} \subset (c_1, c_2)$, a compact operator K such that*

$$(21) \quad E_{\mathcal{I}}(\Delta_{\mathcal{G}})[\Delta_{\mathcal{G}}, \mathbf{i}\mathcal{A}] \circ E_{\mathcal{I}}(\Delta_{\mathcal{G}}) \geq cE_{\mathcal{I}}(\Delta_{\mathcal{G}}) + K.$$

Proof. First $\Delta_{\mathcal{G}} \in \mathcal{C}^2(\mathcal{A})$ because $\Delta_{\mathcal{G}^f} \oplus 0 \oplus \Delta_{\mathcal{G}^c} \in \mathcal{C}^2(\mathcal{A})$ by the Lemma 3.10, the Lemma 3.7, the Remark 3.14, and by Lemma 3.15. In particular, we have that the two operators are in $\mathcal{C}_u^1(\mathcal{A})$, see [AmBoGe]

Then by collecting (20) and (12), we obtain

$$\begin{aligned} E_{\mathcal{I}}(\Delta_{\mathcal{G}^f} \oplus 0 \oplus \Delta_{\mathcal{G}^c})[\Delta_{\mathcal{G}^f} \oplus 0 \oplus \Delta_{\mathcal{G}^c}, \mathbf{i}\mathcal{A}] \circ E_{\mathcal{I}}(\Delta_{\mathcal{G}^f} \oplus 0 \oplus \Delta_{\mathcal{G}^c}) \\ \geq cE_{\mathcal{I}}(\Delta_{\mathcal{G}^f} \oplus 0 \oplus \Delta_{\mathcal{G}^c}) + K. \end{aligned}$$

Since the two operators are in $\mathcal{C}_u^1(\mathcal{A})$ and by Theorem [AmBoGe, Theorem 7.2.9], we obtain (21). □

4. THE PERTURBED MODEL

In this section, we perturb the metrics of the previous case which will be small to infinity. We obtain similar results however the proof is more involved because we rely on the optimal class $\mathcal{C}^{1,1}(\mathcal{A})$ of the Mourre theory.

4.1. Perturbation of the metric. Let $\Delta_{\mathcal{G}'}$ be the closure of

$$\begin{aligned} \Delta_{\mathcal{G}'} : \mathcal{C}_c(\mathcal{V}, m') &\rightarrow \mathcal{C}_c(\mathcal{V}, m') \\ f &\mapsto \Delta_{\mathcal{G}'} f(x) := \frac{1}{m'(x)} \sum_{y \sim x} \mathcal{E}'(x, y)(f(x) - f(y)); \end{aligned}$$

with $m'(x) := (1 + \varepsilon(x))m(x)$, and $\mathcal{E}'(x, y) := (1 + \theta(x, y))\mathcal{E}(x, y)$, where $\varepsilon(x) \rightarrow 0$ if $|x| \rightarrow \infty$ and $\theta(x, y) \rightarrow 0$ if $|x|, |y| \rightarrow \infty$. The operator $\Delta_{\mathcal{G}'}$ is self-adjoint. In fact it is a compact perturbation of Δ , see Proposition 4.4 for a precise statement.

We set $m'_* := m'_{|\mathcal{V}^*}$ and $\varepsilon'_* := \varepsilon'_{|\mathcal{V}^* \times \mathcal{V}^*}$. More precisely:

$$\lim_{|x| \rightarrow 0} (\varepsilon_{|\mathcal{V}^*}(x)m^*(x)) = 0 \quad \text{and} \quad \lim_{|x|, |y| \rightarrow 0} (\theta_{|\mathcal{V}^* \times \mathcal{V}^*}(x, y)\mathcal{E}^*(x, y)) = 0,$$

with $* \in \{c, f\}$. We note that $\varepsilon_{|\mathcal{V}^*} := \varepsilon^*$ and $\theta_{|\mathcal{V}^* \times \mathcal{V}^*} := \theta^*$.

4.2. Unitary transformation. The problem here is that when we do the perturbation, we do not fall on the same Hilbert spaces. To avoid this problem, we will use the following transformation:

Proposition 4.1. *Let*

$$\begin{aligned} T_{m' \rightarrow m} : \ell^2(\mathcal{V}, m') &\rightarrow \ell^2(\mathcal{V}, m) \\ f &\mapsto T_{m' \rightarrow m} f(x, y) := \sqrt{\frac{m'(x, y)}{m(x, y)}} f(x, y). \end{aligned}$$

The transformation $T_{m' \rightarrow m}$ is unitary.

Proof. Let $f \in \ell_c^2(\mathcal{V}, m')$,

$$\begin{aligned} \|T_{m' \rightarrow m} f\|_{\ell^2(\mathcal{V}, m)}^2 &= \sum_{(x, y) \in \mathcal{V}} m(x, y) |T_{m' \rightarrow m} f(x, y)|^2 \\ &= \sum_{(x, y) \in \mathcal{V}} m(x, y) \left| \sqrt{\frac{m'(x, y)}{m(x, y)}} f(x, y) \right|^2 \\ &= \sum_{(x, y) \in \mathcal{V}} m'(x, y) |f(x, y)|^2 = \|f\|_{\ell^2(\mathcal{V}, m')}^2. \end{aligned}$$

Then, the operator T is unitary. \square

We can now, send $\Delta_{\mathcal{G}'}$ in $\ell^2(\mathcal{V}, m)$ with the help of this unitary transformation. Namely, let $\tilde{\Delta} := T_{m' \rightarrow m} \Delta_{\mathcal{G}'} T_{m' \rightarrow m}^{-1}$. We have the following result:

Proposition 4.2. *Let $f \in \ell_c^2(\mathcal{V}, m)$. Then for all $x \in \mathcal{V}$*

$$(22) \quad \tilde{\Delta} f(x) := W f(x) + \tilde{\Delta}_{m'} f(x),$$

where

$$\begin{aligned} W f(x) &:= \frac{1}{m'(x)} \sum_{z \sim x} \mathcal{E}'(x, z) f(x) - \frac{1}{m(x)} \sum_{z \sim x} \mathcal{E}'(x, z) \sqrt{\frac{m(z)m(x)}{m'(x)m'(z)}} f(x) \\ &= \frac{1}{m(x)} \sum_{z \sim x} \left(\frac{\varepsilon(z) - \varepsilon(x)}{(1 + \varepsilon(x))\sqrt{1 + \varepsilon(z)}(\sqrt{1 + \varepsilon(z)} + \sqrt{1 + \varepsilon(x)})} \right) \\ &\quad \times (1 + \theta(x, z))\mathcal{E}(x, z) f(x) \end{aligned}$$

and

$$\tilde{\Delta}_{m'} f(x) := \frac{1}{m(x)} \sum_{z \sim x} \mathcal{E}'(x, z) \sqrt{\frac{m(z)m(x)}{m'(x)m'(z)}} (f(x) - f(z)).$$

Proof. Let $f \in \ell_c^2(\mathcal{V}, m)$ and $x \in \mathcal{V}$, we have

$$\begin{aligned}
\tilde{\Delta}f(x) &= T_{m' \rightarrow m} \Delta_{\mathcal{G}'} T_{m' \rightarrow m}^{-1} f(x) \\
&= \frac{1}{\sqrt{m(x)m'(x)}} \sum_{z \sim x} \mathcal{E}'(x, z) \left(\sqrt{\frac{m(x)}{m'(x)}} f(x) - \sqrt{\frac{m(z)}{m'(z)}} f(z) \right) \\
&= \frac{1}{m'(x)} \sum_{z \sim x} \mathcal{E}'(x, z) f(x) - \frac{1}{m(x)} \sum_{z \sim x} \mathcal{E}'(x, z) \sqrt{\frac{m(z)m(x)}{m'(x)m'(z)}} f(z) \\
&= \frac{1}{m'(x)} \sum_{z \sim x} \mathcal{E}'(x, z) f(x) - \frac{1}{m(x)} \sum_{z \sim x} \mathcal{E}'(x, z) \sqrt{\frac{m(z)m(x)}{m'(x)m'(z)}} f(x) \\
&\quad + \frac{1}{m(x)} \sum_{z \sim x} \mathcal{E}'(x, z) \sqrt{\frac{m(z)m(x)}{m'(x)m'(z)}} (f(x) - f(z)) \\
&= Wf(x) + \tilde{\Delta}_{m'} f(x).
\end{aligned}$$

and

$$\begin{aligned}
Wf(x) &= \frac{1}{m'(x)} \sum_{z \sim x} \mathcal{E}'(x, z) f(x) - \frac{1}{m(x)} \sum_{z \sim x} \mathcal{E}'(x, z) \sqrt{\frac{m(z)m(x)}{m'(x)m'(z)}} f(x) \\
&= \frac{1}{m(x)} \sum_{z \sim x} \left(\frac{1}{1 + \varepsilon(x)} - \sqrt{\frac{1}{(1 + \varepsilon(x))(1 + \varepsilon(z))}} \right) \mathcal{E}'(x, z) f(x) \\
&= \frac{1}{m(x)} \sum_{z \sim x} (1 + \theta(x, z)) \left(\frac{\sqrt{1 + \varepsilon(z)} - \sqrt{1 + \varepsilon(x)}}{(1 + \varepsilon(x))\sqrt{1 + \varepsilon(z)}} \right) \mathcal{E}(x, z) f(x) \\
&= \frac{1}{m(x)} \sum_{z \sim x} (1 + \theta(x, z)) \left(\frac{\varepsilon(z) - \varepsilon(x)}{(1 + \varepsilon(x))\sqrt{1 + \varepsilon(z)}(\sqrt{1 + \varepsilon(z)} + \sqrt{1 + \varepsilon(x)})} \right) \\
&\quad \times \mathcal{E}(x, z) f(x).
\end{aligned}$$

□

After a straightforward calculus, we have the following Lemma:

Lemma 4.3. *For all $f \in \ell_c^2(\mathcal{V}, m)$, we have*

$$\begin{aligned}
(\tilde{\Delta}_{m'} - \Delta_{\mathcal{G}})f(x) &:= \frac{1}{m(x)} \sum_{z \sim x} \frac{\theta(x, z)}{\sqrt{(1 + \varepsilon(x))(1 + \varepsilon(z))}} \mathcal{E}(x, z) (f(x) - f(z)) \\
(23) \quad &+ \frac{1}{m(x)} \sum_{z \sim x} \left(\frac{1 - \sqrt{(1 + \varepsilon(x))(1 + \varepsilon(z))}}{\sqrt{(1 + \varepsilon(x))(1 + \varepsilon(z))}} \right) \mathcal{E}(x, z) (f(x) - f(z)).
\end{aligned}$$

Proposition 4.4. *Let $V : \mathcal{V} \rightarrow \mathbb{R}$ be a function, obeying $V(x) \rightarrow 0$ if $|x| \rightarrow \infty$. We assume that $\theta(x, z) \rightarrow 0$ if $|x|, |z| \rightarrow \infty$ and $\varepsilon(x) \rightarrow 0$ if $|x| \rightarrow \infty$ then $\tilde{\Delta} - \Delta_{\mathcal{G}} \in \mathcal{K}(\ell^2(\mathcal{V}), m)$, where $\tilde{\Delta}$ is the transformation of $\Delta_{\mathcal{G}'}$ in $\ell^2(\mathcal{V}, m)$. In particular*

- (1) $\mathcal{D}(\Delta_{\mathcal{G}'} + V(Q)) = \mathcal{D}(\Delta_m)$.
- (2) $\Delta_{\mathcal{G}'} + V(Q)$ is self-adjoint.
- (3) $\sigma_{\text{ess}}(\Delta_{\mathcal{G}'} + V(Q)) = \sigma_{\text{ess}}(\Delta_{\mathcal{G}})$.

Proof. First, we suppose that $\theta(x, z) \rightarrow 0$ if $|x|, |z| \rightarrow \infty$ and $\varepsilon(x) \rightarrow 0$ if $|x| \rightarrow \infty$, then by using the Propositions 4.6 and 4.14 we have that $\tilde{\Delta} - \Delta_{\mathcal{G}} \in \mathcal{K}(\ell^2(\mathcal{V}), m)$. In particular, since $\Delta_{\mathcal{G}}$ is self-adjoint and since $V(Q)$ is compact then by using Theorem [ReSi, Theorem XIII.14], we have that $\Delta_{\mathcal{G}'} + V(Q)$ is self-adjoint and $\sigma_{\text{ess}}(\Delta_{\mathcal{G}'} + V(Q)) = \sigma_{\text{ess}}(\Delta_{\mathcal{G}})$. □

4.3. Main result. The main result of this section is the following theorem:

Theorem 4.5. *Let $\mathcal{A} := \mathcal{A}_m^f \oplus 0 \oplus \mathcal{A}_m^c$, with respect to (25) and (38), be a self-adjoint operator. Let $V : \mathcal{V} \rightarrow \mathbb{R}$ be a function. We assume that*

$$\sup_{(x_1, x_2) \in \mathcal{V}} x_1^{\varepsilon+1} (\varepsilon(x_1 - 1, x_2) - \varepsilon(x_1, x_2)) < \infty,$$

$$\sup_{(x_1, x_2) \sim (z_1, z_2)} x_1^{\varepsilon+1} (\theta((x_1, x_2), (z_1, z_2)) - \theta((x_1 - 1, x_2), (z_1, z_2))) < \infty$$

and

$$\|[V, U]\langle Q \rangle^{1+\varepsilon}\| < \infty,$$

where $\theta(x, z) \rightarrow 0$ if $|x|, |z| \rightarrow \infty$, $\varepsilon(x) \rightarrow 0$ if $|x| \rightarrow \infty$ and $V(x) \rightarrow 0$ if $|x| \rightarrow \infty$. Then $\Delta_{\mathcal{G}'} + V(Q) \in \mathcal{C}^{1,1}(\mathcal{A})$, $\Delta_{\mathcal{G}'}$ is given in Subsection 4.1. Moreover, for all compact interval $\mathcal{I} \subset (c_1, c_2)$, there are $c > 0$ and a compact operator K such that

$$(24) \quad \begin{aligned} E_{\mathcal{I}}(\Delta_{\mathcal{G}'} + V(Q))[\Delta_{\mathcal{G}'} + V(Q), i\mathcal{A}]_o E_{\mathcal{I}}(\Delta_{\mathcal{G}'} + V(Q)) \\ \geq c E_{\mathcal{I}}(\Delta_{\mathcal{G}'} + V(Q)) + K. \end{aligned}$$

in the form sense, with $c_1 := e^{-1/2} + e^{1/2} - 2$ and $c_2 := e^{-1/2} + e^{1/2} + 2$.

Proof. First $\Delta_{\mathcal{G}'} + V(Q) \in \mathcal{C}^{1,1}(\mathcal{A})$ because $\Delta_{\mathcal{G}'}^f \oplus 0 \oplus \Delta_{\mathcal{G}'}^c + V^f \oplus V^c \in \mathcal{C}^{1,1}(\mathcal{A})$ by the Lemma 4.12, the Lemma 4.19, the Remark 3.14, the Lemma 4.20, the Lemma 4.13 and by Lemma 3.15. In particular, we have that the two operators are in $\mathcal{C}_u^1(\mathcal{A})$, see [AmBoGe].

Then, collecting (26) and (39), we obtain

$$\begin{aligned} E_{\mathcal{I}}(\Delta_{\mathcal{G}'}^f \oplus 0 \oplus \Delta_{\mathcal{G}'}^c + V(Q))[\Delta_{\mathcal{G}'}^f \oplus 0 \oplus \Delta_{\mathcal{G}'}^c + V(Q), i\mathcal{A}]_o E_{\mathcal{I}}(\Delta_{\mathcal{G}'}^f \oplus 0 \oplus \Delta_{\mathcal{G}'}^c + V(Q)) \\ \geq c E_{\mathcal{I}}(\Delta_{\mathcal{G}'}^f \oplus 0 \oplus \Delta_{\mathcal{G}'}^c + V(Q)) + K. \end{aligned}$$

Since the two operators are in $\mathcal{C}_u^1(\mathcal{A})$, $\tilde{\Delta} - \Delta_{\mathcal{G}} \in \mathcal{K}(\ell^2(\mathcal{V}), m)$, $\tilde{\Delta}$ is given by (22), $V(Q)$ is a compact perturbation and by [AmBoGe, Theorem 7.2.9], we obtain (24). \square

We now prove the result given in the introduction.

Proof of Theorem 1.1. First, Proposition 4.16 and Proposition 4.8 give that $\Delta_{\mathcal{G}'} \in \mathcal{C}^{1,1}(\mathcal{A})$ and that there are $c > 0$ a compact interval $\mathcal{I} \subset (c_1, c_2)$, and a compact operator K such that

$$E_{\mathcal{I}}(\Delta_{\mathcal{G}'})[\Delta_{\mathcal{G}'}, i\mathcal{A}]_o E_{\mathcal{I}}(\Delta_{\mathcal{G}'}) \geq c E_{\mathcal{I}}(\Delta_{\mathcal{G}'}) + K.$$

By Lemma 4.13 and Lemma 4.20, we prove that $V \in \mathcal{C}^{1,1}(\mathcal{A})$. And by using Proposition 4.6 and Proposition 4.14, we have that $(\tilde{\Delta} + i)^{-1} - (\Delta_{\mathcal{G}} + i)^{-1} \in \mathcal{K}(\ell^2(\mathcal{V}, m))$, $\tilde{\Delta}$ is given by (22). Finally, by applying [AmBoGe, Proposition 7.5.6] we have the Limiting Absorption Principle $\lim_{\mu \rightarrow 0} \|\langle \mathcal{A} \rangle^{-s} (H - \lambda - i\mu)^{-1} \langle \mathcal{A} \rangle^{-s}\|$, exists and finite. Now, since for all $f \in \mathcal{D}(Q)$, there is $a > 0$ such that

$$\|\langle \mathcal{A} \rangle^s f\| \leq a \|\langle Q \rangle^s f\|,$$

then $\lim_{\mu \rightarrow 0} \|\langle Q \rangle^{-s} (H - \lambda - i\mu)^{-1} \langle Q \rangle^{-s}\|$ exists and finite. \square

4.4. The funnel side. We first deal with the question of the essential spectrum.

Proposition 4.6. *Let $\tilde{\Delta}^f$ be the transformation of $\Delta_{\mathcal{G}'}^f$, on $\ell^2(\mathcal{V}^f, m^f)$, as in (22) and $V^f : \mathcal{V}^f \rightarrow \mathbb{R}$ be a function obeying $V^f(x) \rightarrow 0$ if $|x| \rightarrow \infty$. We assume $\theta^f(x, z) \rightarrow 0$ if $|x|, |z| \rightarrow \infty$ and $\varepsilon^f(x) \rightarrow 0$ if $|x| \rightarrow \infty$ then $\tilde{\Delta}^f - \Delta_{\mathcal{G}'}^f \in \mathcal{K}(\ell^2(\mathcal{V}^f), m^f)$. In particular, $\sigma_{\text{ess}}(\Delta_{\mathcal{G}'}^f + V^f(Q)) = \sigma_{\text{ess}}(\Delta_{\mathcal{G}'}^f)$.*

Proof. We recall that $\tilde{\Delta}^f := \tilde{\Delta}_{m'}^f + W^f$. First, we shall show that $\tilde{\Delta}_{m'}^f - \Delta_{\mathcal{G}^f} \in \mathcal{K}(\ell^2(\mathcal{V}^f), m^f)$, as in (23).

$$\begin{aligned}
& \left| \langle f, (\tilde{\Delta}_{m'}^f - \Delta_{\mathcal{G}^f})f \rangle_{\ell^2(\mathcal{V}^f, m^f)} \right| = \left| \sum_{x \in \mathcal{V}^f} m^f(x) (\tilde{\Delta}_{m'}^f - \Delta_{\mathcal{G}^f})f(x) \overline{f(x)} \right| \\
& \leq \left| \sum_{x \in \mathcal{V}^f} m^f(x) \frac{1}{m^f(x)} \sum_{z \sim x} \frac{\theta^f(x, z)}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \mathcal{E}^f(x, z) \right. \\
& \quad \left. \times (f(x) - f(z)) \overline{f(x)} \right| \\
& + \left| \sum_{x \in \mathcal{V}^f} m^f(x) \frac{1}{m^f(x)} \sum_{z \sim x} \frac{1 - \sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \mathcal{E}^f(x, z) \right. \\
& \quad \left. \times (f(x) - f(z)) \overline{f(x)} \right| \\
& \leq \left| \sum_{x \in \mathcal{V}^f} m^f(x) \frac{1}{m^f(x)} \sum_{z \sim x} \frac{\theta^f(x, z)}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \mathcal{E}^f(x, z) f(x) \overline{f(x)} \right| \\
& + \left| \sum_{x \in \mathcal{V}^f} m^f(x) \frac{1}{m^f(x)} \sum_{z \sim x} \frac{\theta^f(x, z)}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \mathcal{E}^f(x, z) f(z) \overline{f(x)} \right| \\
& + \left| \sum_{x \in \mathcal{V}^f} m^f(x) \frac{1}{m^f(x)} \sum_{z \sim x} \frac{1 - \sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \mathcal{E}^f(x, z) f(x) \overline{f(x)} \right| \\
& + \left| \sum_{x \in \mathcal{V}^f} m^f(x) \frac{1}{m^f(x)} \sum_{z \sim x} \frac{1 - \sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \mathcal{E}^f(x, z) f(z) \overline{f(x)} \right|
\end{aligned}$$

Then

$$\begin{aligned}
& \left| \langle f, (\tilde{\Delta}_{m'}^f - \Delta_{\mathcal{G}^f})f \rangle_{\ell^2(\mathcal{V}^f, m^f)} \right| \\
& \leq \sum_{x \in \mathcal{V}^f} m^f(x) \frac{1}{m^f(x)} \sum_{z \sim x} \frac{\theta^f(x, z)}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \mathcal{E}^f(x, z) |f(x)|^2 \\
& + 1/2 \sum_{x \in \mathcal{V}^f} m^f(x) \frac{1}{m^f(x)} \sum_{z \sim x} \frac{\theta^f(x, z)}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \mathcal{E}^f(x, z) \\
& \quad \times (|f(z)|^2 + |f(x)|^2) \\
& + \sum_{x \in \mathcal{V}^f} m^f(x) \frac{1}{m^f(x)} \sum_{z \sim x} \left| \frac{1 - \sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \right| \mathcal{E}^f(x, z) |f(x)|^2 \\
& + 1/2 \sum_{x \in \mathcal{V}^f} m^f(x) \frac{1}{m^f(x)} \sum_{z \sim x} \left| \frac{1 - \sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \right| \mathcal{E}^f(x, z) \\
& \quad \times (|f(z)|^2 + |f(x)|^2) \\
& \leq 2 \langle f, (\deg_1 + \deg_2)f \rangle,
\end{aligned}$$

where

$$\deg_1(x) := \frac{1}{m^f(x)} \sum_{z \in \mathcal{V}^f} \frac{\theta^f(x, z)}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \mathcal{E}^f(x, z)$$

and

$$\deg_2(x) := \frac{1}{m^f(x)} \sum_{z \in \mathcal{V}^f} \left| \frac{1 - \sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \right| \mathcal{E}^f(x, z), \text{ for all } x = (x_1, x_2) \in \mathcal{V}^f.$$

We have

$$\begin{aligned}
|\deg_1(x)| &= \left| \frac{1}{m^f(x)} \sum_{z \in \mathcal{V}^f} \frac{\theta^f(x, z)}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \mathcal{E}^f(x, z) \right| \\
&\leq \frac{1}{m^f(x)} \sum_{z \in \mathcal{V}^f} \left| \frac{\theta^f(x, z)}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \right| \mathcal{E}^f(x, z) \\
&\leq \sup_{z \sim x} \left| \frac{\theta^f(x, z)}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \right| \deg_{\mathcal{G}^f}(x)
\end{aligned}$$

Since \mathcal{V}_2 is a finite set and for all $x_2 \in \mathcal{V}_2$, $\frac{\theta^f((x_1, x_2), (z_1, z_2))}{\sqrt{(1 + \varepsilon^f(x_1, x_2))(1 + \varepsilon^f(z_1, z_2))}} \rightarrow 0$ if $x_1, z_1 \rightarrow \infty$ and since $\deg_{\mathcal{G}^f}(Q)$ is bounded then $\deg_1(Q)$ is compact. In the same way, using that $\forall x_2, z_2 \in \mathcal{V}_2$, $\frac{1 - \sqrt{(1 + \varepsilon^f(x_1, x_2))(1 + \varepsilon^f(z_1, z_2))}}{\sqrt{(1 + \varepsilon^f(x_1, x_2))(1 + \varepsilon^f(z_1, z_2))}} \rightarrow 0$ if $x_1, z_1 \rightarrow \infty$, we proof the compactness of $\deg_2(Q)$, then $(\tilde{\Delta}_{m'}^f - \Delta_{\mathcal{G}^f})$ is a compact operator.

Now, we will show that $W^f \in \mathcal{K}(\ell^2(\mathcal{V}^f, m^f))$. For all $x \in \mathcal{V}^f$, we have

$$\begin{aligned}
|W^f(x)| &= \left| \frac{1}{m^f(x)} \sum_{z \sim x} (1 + \theta^f(x, z)) \mathcal{E}^f(x, z) \right. \\
&\quad \times \left. \left(\frac{\varepsilon^f(z) - \varepsilon^f(x)}{(1 + \varepsilon^f(x))\sqrt{1 + \varepsilon^f(z)}(\sqrt{1 + \varepsilon^f(z)} + \sqrt{1 + \varepsilon^f(x)})} \right) \right| \\
&\leq \frac{1}{m^f(x)} \sum_{z \sim x} \left| (1 + \theta^f(x, z)) \mathcal{E}^f(x, z) \right. \\
&\quad \times \left. \left(\frac{\varepsilon^f(z) - \varepsilon^f(x)}{(1 + \varepsilon^f(x))\sqrt{1 + \varepsilon^f(z)}(\sqrt{1 + \varepsilon^f(z)} + \sqrt{1 + \varepsilon^f(x)})} \right) \right| \\
&\leq \sup_{z \sim x} \left| (1 + \theta^f(x, z)) \left(\frac{\varepsilon^f(z) - \varepsilon^f(x)}{(1 + \varepsilon^f(x))\sqrt{1 + \varepsilon^f(z)}(\sqrt{1 + \varepsilon^f(z)} + \sqrt{1 + \varepsilon^f(x)})} \right) \right| \\
&\quad \times \deg_{\mathcal{G}^f}(x).
\end{aligned}$$

Since \mathcal{V}_2 is a finite set and $(1 + \theta^f(x, z))(\varepsilon^f(z) - \varepsilon^f(x)) \rightarrow 0$ when $|x|, |z| \rightarrow \infty$, $\deg_{\mathcal{G}^f}(Q)$ is bounded and since $V^f(Q)$ is a compact perturbation then we find the result. \square

In order to go into the Mourre theory, we construct the conjugate operator in $\ell^2(\mathcal{V}^f, m^f)$.

Proposition 4.7. *Let $f \in \ell_c^2(\mathbb{N}, m_1)$, $\mathcal{A}_{m_1} f(n) = T_{1 \rightarrow m_1} \mathcal{A}_{\mathbb{N}} T_{1 \rightarrow m_1}^{-1} f(n)$, then*

$$\mathcal{A}_{m_1} = \frac{i}{2} \left(e^{1/2}(Q - 1/2)U - e^{-1/2}(Q + 1/2)U^* \right).$$

Proof. Let $f \in \ell_c^2(\mathbb{N}, m_1)$,

$$\begin{aligned}
\mathcal{A}_{m_1} f(n) &= T_{1 \rightarrow m_1} \mathcal{A}_{\mathbb{N}} T_{1 \rightarrow m_1}^{-1} f(n) \\
&= -\frac{i}{2\sqrt{m_1(n)}} \left(\frac{1}{2} (U + U^*) + Q(U^* - U) \right) T_{1 \rightarrow m_1}^{-1} f(n) \\
&= -\frac{i\sqrt{m_1(n+1)}}{4\sqrt{m_1(n)}} f(n+1) - \frac{i\sqrt{m_1(n-1)}}{4\sqrt{m_1(n)}} f(n-1) \\
&\quad - \frac{in\sqrt{m_1(n+1)}}{2\sqrt{m_1(n)}} f(n+1) + \frac{in\sqrt{m_1(n-1)}}{2\sqrt{m_1(n)}} f(n-1) \\
&= \frac{i}{2} \left(\left(n - \frac{1}{2} \right) \sqrt{\frac{m_1(n-1)}{m_1(n)}} f(n-1) - \left(n + \frac{1}{2} \right) \sqrt{\frac{m_1(n+1)}{m_1(n)}} f(n+1) \right) \\
&= \frac{i}{2} \left(e^{1/2} \left(n - \frac{1}{2} \right) U f(n) - e^{-1/2} \left(n + \frac{1}{2} \right) U^* f(n) \right).
\end{aligned}$$

□

The operator $\mathcal{A}_{\mathbb{N}}$ is essentially self-adjoint [GeGo]. On $\ell_c^2(\mathcal{V}, m)$, we set

$$(25) \quad \mathcal{A}_m^f := \mathcal{A}_{m_1} \otimes \mathbf{1}_{\mathcal{V}_2^f}.$$

The operator is essentially self-adjoint. We continue to denote the closure by the same symbol.

Now, we turn to Mourre estimate.

Proposition 4.8. *Let $V^f : \mathcal{V}^f \rightarrow \mathbb{R}$ be a function. We assume that*

$$\sup_{(x_1, x_2) \in \mathcal{V}^f} x_1^{\varepsilon+1} (\varepsilon^f(x_1 - 1, x_2) - \varepsilon^f(x_1, x_2)) < \infty,$$

$$\sup_{(x_1, x_2) \sim (z_1, z_2)} x_1^{\varepsilon+1} (\theta^f((x_1, x_2), (z_1, z_2)) - \theta^f((x_1 - 1, x_2), (z_1, z_2))) < \infty$$

and

$$\| [V^f, U] \langle Q \rangle^{1+\varepsilon} \| < \infty,$$

where $\theta^f(x, z) \rightarrow 0$ if $|x|, |z| \rightarrow \infty$, $\varepsilon^f(x) \rightarrow 0$ if $|x| \rightarrow \infty$ and $V^f(x) \rightarrow 0$ if $|x| \rightarrow \infty$. Then $\Delta_{\mathcal{G}'}^f + V^f(Q) \in \mathcal{C}^{1,1}(\mathcal{A}_{m'}^f)$. Moreover, there are $c > 0$, a compact interval $\mathcal{I} \subset (c_1, c_2)$, a compact operator K such that

$$(26) \quad E_{\mathcal{I}}(\Delta_{\mathcal{G}'}^f + V^f(Q)) [\Delta_{\mathcal{G}'}^f + V^f(Q), i\mathcal{A}_{m'}^f]_o E_{\mathcal{I}}(\Delta_{\mathcal{G}'}^f + V^f(Q)) \geq c E_{\mathcal{I}}(\Delta_{\mathcal{G}'}^f + V^f(Q)) + K,$$

in the form sense.

We shall give the proof at the end of the section. We turn to series of Lemmata.

Lemma 4.9. *Let $f \in \ell_c^2(\mathcal{V}^f, m^f)$, we have*

$$\begin{aligned}
[W^f, \mathcal{A}_m^f]f(x) &= \frac{i}{2}e^{1/2} \left((Q-1/2) \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \frac{1}{m^f(x)} \sum_{z \sim x} (1 + \theta^f(x, z)) \mathcal{E}^f(x, z) Uf(x) \\
(27) \quad & \times \left(\frac{\varepsilon^f(z) - \varepsilon^f(x)}{(1 + \varepsilon^f(x))\sqrt{1 + \varepsilon^f(z)}(\sqrt{1 + \varepsilon^f(z)} + \sqrt{1 + \varepsilon^f(x)})} \right) \\
& - \frac{i}{2}e^{1/2} \left((Q-1/2) \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \frac{1}{m^f(x_1-1, x_2)} \sum_{z \sim x} (1 + \theta^f((x_1-1, x_2), (z_1-1, z_2))) \\
& \times \left(\frac{\varepsilon^f(z_1-1, z_2) - \varepsilon^f(x_1-1, x_2)}{(1 + \varepsilon^f(x_1-1, x_2))\sqrt{1 + \varepsilon^f(z_1-1, z_2)}(\sqrt{1 + \varepsilon^f(z_1-1, z_2)} + \sqrt{1 + \varepsilon^f(x_1-1, x_2)})} \right) \\
& \quad \times \mathcal{E}^f((x_1-1, x_2), (z_1-1, z_2)) Uf(x) \\
& + \frac{i}{2}e^{-1/2} \left((Q+1/2) \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \frac{1}{m^f(x_1+1, x_2)} \sum_{z \sim x} (1 + \theta^f((x_1+1, x_2), (z_1+1, z_2))) \\
& \times \left(\frac{\varepsilon^f(z_1+1, z_2) - \varepsilon^f(x_1+1, x_2)}{(1 + \varepsilon^f(x_1+1, x_2))\sqrt{1 + \varepsilon^f(z_1+1, z_2)}(\sqrt{1 + \varepsilon^f(z_1+1, z_2)} + \sqrt{1 + \varepsilon^f(x_1+1, x_2)})} \right) \\
& \quad \times \mathcal{E}^f((x_1+1, x_2), (z_1+1, z_2)) U^*f(x) \\
& - \frac{i}{2}e^{-1/2} \left((Q+1/2) \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \frac{1}{m^f(x)} \sum_{z \sim x} (1 + \theta^f(x, z)) \mathcal{E}^f(x, z) U^*f(x) \\
& \quad \times \left(\frac{\varepsilon^f(z) - \varepsilon^f(x)}{(1 + \varepsilon^f(x))\sqrt{1 + \varepsilon^f(z)}(\sqrt{1 + \varepsilon^f(z)} + \sqrt{1 + \varepsilon^f(x)})} \right).
\end{aligned}$$

Proof. Let $f \in \ell_c^2(\mathcal{V}^f, m^f)$, we have

$$\begin{aligned}
\mathcal{A}_m^f W^f f(x) &= \frac{i}{2} \left(e^{1/2} \left((Q-1/2) U \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) - e^{-1/2} \left((Q+1/2) U^* \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \right) W^f f(x) \\
&= \frac{i}{2} e^{1/2} \left((Q-1/2) \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \frac{1}{m^f(x_1-1, x_2)} \sum_{z \sim x} (1 + \theta^f((x_1-1, x_2), (z_1-1, z_2))) \\
& \times \left(\frac{\varepsilon^f(z_1-1, z_2) - \varepsilon^f(x_1-1, x_2)}{(\sqrt{1 + \varepsilon^f(z_1-1, z_2)} + \sqrt{1 + \varepsilon^f(x_1-1, x_2)})} \right) \mathcal{E}^f((x_1-1, x_2), (z_1-1, z_2)) \\
& \times \frac{1}{(1 + \varepsilon^f(x_1-1, x_2))\sqrt{1 + \varepsilon^f(z_1-1, z_2)}} f(x_1-1, x_2) \\
& - \frac{i}{2} e^{-1/2} \left((Q+1/2) \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \frac{1}{m^f(x_1+1, x_2)} \sum_{z \sim x} (1 + \theta^f((x_1+1, x_2), (z_1+1, z_2))) \\
& \times \left(\frac{\varepsilon^f(z_1+1, z_2) - \varepsilon^f(x_1+1, x_2)}{(\sqrt{1 + \varepsilon^f(z_1+1, z_2)} + \sqrt{1 + \varepsilon^f(x_1+1, x_2)})} \right) \mathcal{E}^f((x_1+1, x_2), (z_1+1, z_2)) \\
& \times \frac{1}{(1 + \varepsilon^f(x_1+1, x_2))\sqrt{1 + \varepsilon^f(z_1+1, z_2)}} f(x_1+1, x_2).
\end{aligned}$$

and

$$\begin{aligned}
W^f \mathcal{A}_m^f f(x) &= \frac{1}{m^f(x)} \sum_{z \sim x} (1 + \theta^f(x, z)) \mathcal{E}^f(x, z) \mathcal{A}_m^f f(x) \\
&\quad \times \left(\frac{\varepsilon^f(z) - \varepsilon^f(x)}{(1 + \varepsilon^f(x)) \sqrt{1 + \varepsilon^f(z)} (\sqrt{1 + \varepsilon^f(z)} + \sqrt{1 + \varepsilon^f(x)})} \right) \\
&= \frac{i}{2} e^{1/2} \left((Q - 1/2) \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \frac{1}{m^f(x)} \sum_{z \sim x} (1 + \theta^f(x, z)) \mathcal{E}^f(x, z) U f(x) \\
&\quad \times \left(\frac{\varepsilon^f(z) - \varepsilon^f(x)}{(1 + \varepsilon^f(x)) \sqrt{1 + \varepsilon^f(z)} (\sqrt{1 + \varepsilon^f(z)} + \sqrt{1 + \varepsilon^f(x)})} \right) \\
&\quad - \frac{i}{2} e^{-1/2} \left((Q + 1/2) \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \frac{1}{m^f(x)} \sum_{z \sim x} (1 + \theta^f(x, z)) \mathcal{E}^f(x, z) U^* f(x) \\
&\quad \times \left(\frac{\varepsilon^f(z) - \varepsilon^f(x)}{(1 + \varepsilon^f(x)) \sqrt{1 + \varepsilon^f(z)} (\sqrt{1 + \varepsilon^f(z)} + \sqrt{1 + \varepsilon^f(x)})} \right).
\end{aligned}$$

Then the result follows. \square

To be able to apply the Proposition 7.5.7 in [AmBoGe], we have to check this technical points.

Proposition 4.10. *Let $\Lambda^f := (Q + 1/2) \otimes \mathbf{1}_{\mathcal{V}^f}$, then Λ^f satisfies the following assertions:*

(1) $e^{i\Lambda^f t} \mathcal{D}(\Delta_{\mathcal{G}^f}) \subset \mathcal{D}(\Delta_{\mathcal{G}^f})$ and there exists a finite constant c , such that

$$\|e^{i\Lambda^f t}\|_{\mathcal{B}(\mathcal{D}(\Delta_{\mathcal{G}^f}))} \leq c, \quad \text{for all } t \in \mathbb{R}.$$

(2) $\mathcal{D}(\Lambda^f) \subset \mathcal{D}(\mathcal{A}_m^f)$.

(3) $(\Lambda^f)^{-2} (\mathcal{A}_m^f)^2$ extends to a continuous operator in $\mathcal{D}(\Delta_{\mathcal{G}^f})$.

Note that $\Delta_{\mathcal{G}^f}$ is bounded then $\mathcal{D}(\Delta_{\mathcal{G}^f}) = \mathcal{H}^f$.

Proof. (1) Since $\Delta_{\mathcal{G}^f}$ is bounded it is verified by a functional calculus.

(2) Let $f \in \ell_c^2(\mathcal{V}^f, m^f)$,

$$\begin{aligned}
\|\mathcal{A}_m^f f\|_{\ell^2(\mathcal{V}^f, m^f)}^2 &= \sum_{x \in \mathcal{V}^f} m^f(x) |\mathcal{A}_m^f f(x)|^2 \\
&= \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e^{1/2} (Q - 1/2) U \otimes \mathbf{1}_{\mathcal{V}_2^f} - e^{-1/2} (Q + 1/2) U^* \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) f(x) \right|^2 \\
&\leq \frac{1}{2} \sum_{x \in \mathcal{V}^f} m^f(x) \left| e^{1/2} (Q - 1/2) U \otimes \mathbf{1}_{\mathcal{V}_2^f} f(x) \right|^2 + \left| e^{-1/2} (Q + 1/2) U^* \otimes \mathbf{1}_{\mathcal{V}_2^f} f(x) \right|^2 \\
&\leq c \sum_{x \in \mathcal{V}^f} m^f(x) \left| (Q + 1/2) \otimes \mathbf{1}_{\mathcal{V}_2^f} f(x) \right|^2 \\
&\leq c \|\Lambda^f f\|_{\ell^2(\mathcal{V}^f, m^f)}^2.
\end{aligned}$$

Since, Λ^f is essentially self-adjoint. On what, the result follows.

(3) For all $f \in \ell_c^2(\mathcal{V}^f, m^f)$, and by using the relations of Subsection 6, we have

$$\begin{aligned}
\mathcal{A}_{m_1}^2 f(n) &= \frac{i}{2} \left(e^{1/2}(Q-1/2)U - e^{-1/2}(Q+1/2)U^* \right) \mathcal{A}_{m_1} f(n) \\
&= -\frac{1}{4} e^{1/2}(n-1/2) \left(e^{1/2}(Q-1/2)U - e^{-1/2}(Q+1/2)U^* \right) f(n-1) \\
&\quad + \frac{1}{4} e^{-1/2}(n+1/2) \left(e^{1/2}(Q-1/2)U - e^{-1/2}(Q+1/2)U^* \right) f(n+1) \\
&= \frac{1}{4} (2n^2 + 1/2) f(n) - \frac{1}{4} e(n-1/2)(n-3/2) f(n-2) \\
&\quad - \frac{1}{4} e^{-1}(n+1/2)(n+3/2) f(n+2).
\end{aligned}$$

Then for all $f \in \ell_c^2(\mathcal{V}^f, m^f)$.

$$\begin{aligned}
\|(\Lambda^f)^{-2}(\mathcal{A}_m^f)^2 f\|^2 &= \sum_{(x_1, x_2) \in \mathcal{V}^f} m^f(x_1, x_2) |\Lambda^{-2}(\mathcal{A}_m^f)^2 f(x, y)|^2 \\
&= \sum_{(x_1, x_2) \in \mathcal{V}^f} m^f(x_1, x_2) \left| \frac{1}{4} \left(\left((Q+1/2)^{-2}(2Q^2+1/2) \right) \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) f(x_1, x_2) \right. \\
&\quad \left. - \frac{1}{4} e \left(\left((Q+1/2)^{-2}(Q-1/2)(Q-3/2) \right) \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) f(x_1-2, x_2) \right. \\
&\quad \left. - \frac{1}{4} e^{-1} \left(\left((Q+1/2)^{-2}(Q+1/2)(Q+3/2) \right) \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) f(x_1+2, x_2) \right|^2.
\end{aligned}$$

Then, there exists $C > 0$ such that for all $f \in \ell_c^2(\mathcal{V}^f, m^f)$, $\|(\Lambda^f)^{-2}(\mathcal{A}_m^f)^2 f\|^2 \leq C\|f\|^2$. By density, we find the result. \square

Proposition 4.11. *We assume that*

$$(28) \quad \sup_{(x_1, x_2), (y_1, y_2) \in \mathcal{V}^f; (x_1, x_2) \sim (y_1, y_2)} x_1^{1+\varepsilon} |\varepsilon^f(x_1, x_2) - \varepsilon^f(y_1, y_2)| < \infty$$

then $[W^f, \mathcal{A}_m^f]_\circ \in \mathcal{C}^{0,1}(\mathcal{A}_m^f)$. In particular, $W^f \in \mathcal{C}^{1,1}(\mathcal{A}_m^f)$.

Proof. We suppose that (28) is true then

$$\|(\Lambda^f)^\varepsilon [W^f, \mathcal{A}_m^f]_\circ f\|_{\ell^2(\mathcal{V}^f, m^f)} = \sum_{x \in \mathcal{V}^f} m^f(x) |(\Lambda^f)^\varepsilon [W^f, \mathcal{A}_m^f]_\circ f(x)|^2,$$

we treat the first term of $\|\langle \Lambda^f \rangle^\varepsilon [W^f, \mathcal{A}_m^f] \circ f\|_{\ell^2(\mathcal{V}^f, m^f)}$ with respect to the decomposition of (27).

$$\begin{aligned}
& \frac{e^{-1}}{4} \sum_{x \in \mathcal{V}^f} m^f(x) \left| \left((Q + 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m^f(x)} \sum_{z \sim x} (1 + \theta^f(x, z)) \right. \\
& \quad \times \left. \left(\frac{\varepsilon^f(z) - \varepsilon^f(x)}{(1 + \varepsilon^f(x))\sqrt{1 + \varepsilon^f(z)}(\sqrt{1 + \varepsilon^f(z)} + \sqrt{1 + \varepsilon^f(x)})} \right) \mathcal{E}^f(x, z) Uf(x) \right|^2 \\
& = \frac{e^{-1}}{4} \sum_{x \in \mathcal{V}^f} m^f(x) \left| \left((Q + 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m(x)} \sum_{z \sim x} (1 + \theta^f(x, z)) \right. \\
& \quad \times \left. \left(\frac{\varepsilon^f(z) - \varepsilon^f(x)}{(1 + \varepsilon^f(x))\sqrt{1 + \varepsilon^f(z)}(\sqrt{1 + \varepsilon^f(z)} + \sqrt{1 + \varepsilon^f(x)})} \right) (\mathcal{E}_1^f(x_1, z_1) \delta_{x_2=z_2} \right. \\
& \quad \left. + \mathcal{E}_2^f(x_2, z_2) \delta_{x_1=z_1}) Uf(x) \right|^2 \\
& \leq \frac{e^{-1}}{2} \sum_{x \in \mathcal{V}^f} m^f(x) \left| \left((Q + 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m^f(x)} \sum_{z_1 \sim x_1} (1 + \theta^f(x, z)) \right. \\
& \quad \times \left. \left(\frac{\varepsilon^f(z) - \varepsilon^f(x)}{(1 + \varepsilon^f(x))\sqrt{1 + \varepsilon^f(z)}(\sqrt{1 + \varepsilon^f(z)} + \sqrt{1 + \varepsilon^f(x)})} \right) \mathcal{E}_1^f(x_1, z_1) \delta_{x_2=z_2} Uf(x) \right|^2 \\
& \quad + \frac{e^{-1}}{2} \sum_{x \in \mathcal{V}^f} m^f(x) \left| \left((Q + 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m^f(x)} \sum_{z_2 \sim x_2} (1 + \theta^f(x, z)) \right. \\
& \quad \times \left. \left(\frac{\varepsilon^f(z) - \varepsilon^f(x)}{(1 + \varepsilon^f(x))\sqrt{1 + \varepsilon^f(z)}(\sqrt{1 + \varepsilon^f(z)} + \sqrt{1 + \varepsilon^f(x)})} \right) \mathcal{E}_2^f(x_2, z_2) \delta_{x_1=z_1} Uf(x) \right|^2.
\end{aligned} \tag{29}$$

First, we treat (29) and in the same way, we deduce that (30) is bounded. There exists an integer $c > 0$, such that

$$\begin{aligned}
& \frac{e^{-1}}{2} \sum_{x \in \mathcal{V}^f} m^f(x) \left| \left((Q + 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m^f(x)} \sum_{z_1 \sim x_1} (1 + \theta^f(x, z)) \mathcal{E}_1^f(x_1, z_1) \delta_{x_2=z_2} \right. \\
& \quad \times \left. \left(\frac{\varepsilon^f(z) - \varepsilon^f(x)}{(1 + \varepsilon^f(x))\sqrt{1 + \varepsilon^f(z)}(\sqrt{1 + \varepsilon^f(z)} + \sqrt{1 + \varepsilon^f(x)})} \right) Uf(x) \right|^2 \\
& \leq e^{-1} \sum_{x \in \mathcal{V}^f} m(x) \left| \left((Q + 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m^f(x)} \left(1 + \theta^f(x, (x_1 + 1, x_2)) \right) \mathcal{E}_1^f(x_1, x_1 + 1) \right. \\
& \quad \times \left. \left(\frac{\varepsilon^f(x_1 + 1, x_2) - \varepsilon^f(x)}{(1 + \varepsilon^f(x))\sqrt{1 + \varepsilon^f(x_1 + 1, x_2)}(\sqrt{1 + \varepsilon^f(x_1 + 1, x_2)} + \sqrt{1 + \varepsilon^f(x)})} \right) Uf(x) \right|^2 \\
& \quad + e^{-1} \sum_{x \in \mathcal{V}^f} m^f(x) \left| \left((Q + 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m^f(x)} \left(1 + \theta^f(x, (x_1 - 1, x_2)) \right) \mathcal{E}_1^f(x_1, x_1 - 1) \right. \\
& \quad \times \left. \left(\frac{\varepsilon^f(x_1 - 1, x_2) - \varepsilon^f(x)}{(1 + \varepsilon^f(x))\sqrt{1 + \varepsilon^f(x_1 - 1, x_2)}(\sqrt{1 + \varepsilon^f(x_1 - 1, x_2)} + \sqrt{1 + \varepsilon^f(x)})} \right) Uf(x) \right|^2.
\end{aligned}$$

Then

$$\begin{aligned}
(29) &\leq e^{-1} \sum_{x \in \mathcal{V}^f} m^f(x) \left| \left((Q + 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{e^{-x_1}}{m_2(x_2)} e^{(2x_1+1)/2} \left(1 + \theta^f(x, (x_1 + 1, x_2)) \right) \right. \\
&\quad \times \left. \left(\frac{\varepsilon^f(x_1 + 1, x_2) - \varepsilon^f(x)}{(1 + \varepsilon^f(x)) \sqrt{1 + \varepsilon^f(x_1 + 1, x_2)} (\sqrt{1 + \varepsilon^f(x_1 + 1, x_2)} + \sqrt{1 + \varepsilon^f(x)})} \right) Uf(x) \right|^2 \\
&+ e^{-1} \sum_{x \in \mathcal{V}^f} m^f(x) \left| \left((Q + 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{e^{-x_1}}{m_2(x_2)} e^{(2x_1-1)/2} \left(1 + \theta^f(x, (x_1 - 1, x_2)) \right) \right. \\
&\quad \times \left. \left(\frac{\varepsilon^f(x_1 - 1, x_2) - \varepsilon^f(x)}{(1 + \varepsilon^f(x)) \sqrt{1 + \varepsilon^f(x_1 - 1, x_2)} (\sqrt{1 + \varepsilon^f(x_1 - 1, x_2)} + \sqrt{1 + \varepsilon^f(x)})} \right) Uf(x) \right|^2 \\
&\leq e^{-1/2} \sum_{x \in \mathcal{V}^f} m^f(x) \left| \left((Q + 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m_2^f(x_2)} \left(1 + \theta^f(x, (x_1 + 1, x_2)) \right) \right. \\
&\quad \times \left. \left(\frac{\varepsilon^f(x_1 + 1, x_2) - \varepsilon^f(x)}{(1 + \varepsilon^f(x)) \sqrt{1 + \varepsilon^f(x_1 + 1, x_2)} (\sqrt{1 + \varepsilon^f(x_1 + 1, x_2)} + \sqrt{1 + \varepsilon^f(x)})} \right) Uf(x) \right|^2 \\
&+ e^{-3/2} \sum_{x \in \mathcal{V}^f} m^f(x) \left| \left((Q + 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m_2^f(x_2)} \left(1 + \theta^f(x, (x_1 - 1, x_2)) \right) \right. \\
&\quad \times \left. \left(\frac{\varepsilon^f(x_1 - 1, x_2) - \varepsilon^f(x)}{(1 + \varepsilon^f(x)) \sqrt{1 + \varepsilon^f(x_1 - 1, x_2)} (\sqrt{1 + \varepsilon^f(x_1 - 1, x_2)} + \sqrt{1 + \varepsilon^f(x)})} \right) Uf(x) \right|^2.
\end{aligned}$$

Then, there exists $c > 0$, such that

$$\| \langle \Lambda^f \rangle^{1+\varepsilon} [W^f, \mathcal{A}_m^f] \circ f \|_{\ell^2(\mathcal{V}^f, m^f)} \leq c \|f\|_{\ell^2(\mathcal{V}^f, m^f)}.$$

Using the Proposition 4.10, then by applying [AmBoGe, Proposition 7.5.7], we find the result. \square

Proposition 4.12. *We assume that*

$$(31) \quad \sup_{(x_1, x_2) \in \mathcal{V}^f} x_1^{\varepsilon+1} (\varepsilon^f(x_1 - 1, x_2) - \varepsilon^f(x_1, x_2)) < \infty$$

and

$$(32) \quad \sup_{(x_1, x_2), (z_1, z_2) \in \mathcal{V}^f; (x_1, x_2) \sim (z_1, z_2)} x_1^{\varepsilon+1} (\theta^f(x, z) - \theta^f((x_1 - 1, x_2), z)) < \infty.$$

Then $[\tilde{\Delta}_{m'}^f, \mathcal{A}_m^f] \circ \in \mathcal{C}^{0,1}(\mathcal{A}_m^f)$. In particular, $\tilde{\Delta}_{m'}^f \in \mathcal{C}^{1,1}(\mathcal{A}_m^f)$.

Proof. Let $f \in \ell_c^2(\mathcal{V}^f, m^f)$,

$$\begin{aligned}
[\tilde{\Delta}_{m'}^f, \mathcal{A}_m^f] \circ f(x) &= \frac{i}{2} \left(e^{1/2} (Q - 1/2) \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m^f(x)} \sum_{z \sim x} \frac{1 + \theta^f(x, z)}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \\
&\quad \times \mathcal{E}^f(x, z) (f(x_1 - 1, x_2) - f(z_1 - 1, z_2)) \\
- \frac{i}{2} \left(e^{-1/2} (Q + 1/2) \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m^f(x)} \sum_{z \sim x} \frac{1 + \theta^f(x, z)}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \\
&\quad \times \mathcal{E}^f(x, z) (f(x_1 + 1, x_2) - f(z_1 + 1, z_2)) \\
- \frac{i}{2} \left(e^{1/2} (Q - 1/2) \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m^f(x_1 - 1, x_2)} \sum_{z \sim x} \frac{1 + \theta^f((x_1 - 1, x_2), z)}{\sqrt{(1 + \varepsilon^f(x_1 - 1, x_2))(1 + \varepsilon^f(z))}} \\
&\quad \times \mathcal{E}^f((x_1 - 1, x_2), z) (f(x_1 - 1, x_2) - f(z_1 - 1, z_2)) \\
+ \frac{i}{2} \left(e^{-1/2} (Q + 1/2) \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m^f(x_1 + 1, x_2)} \sum_{z \sim x} \frac{1 + \theta^f((x_1 + 1, x_2), z)}{\sqrt{(1 + \varepsilon^f(x_1 + 1, x_2))(1 + \varepsilon^f(z))}} \\
&\quad \times \mathcal{E}^f((x_1 + 1, x_2), z) (f(x_1 + 1, x_2) - f(z_1 + 1, z_2)).
\end{aligned}$$

Now, we assume that (31) and (32) are true then

$$\begin{aligned}
\| \langle \Lambda^f \rangle^\varepsilon [\tilde{\Delta}_{m'}^f, \mathcal{A}_m^f] \circ f \|_{\ell^2(\mathcal{V}^f, m^f)} &= \sum_{x \in \mathcal{V}^f} m^f(x) | \langle \Lambda^f \rangle^\varepsilon [\tilde{\Delta}_{m'}^f, \mathcal{A}_m^f] \circ f(x) |^2 \\
&\leq \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e^{1/2} (Q - 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \right. \\
(33) \quad &\times \left(\frac{1}{m^f(x)} \sum_{z \sim x} \mathcal{E}^f(x, z) \frac{1 + \theta^f(x, z)}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} f(x_1 - 1, x_2) \right. \\
&\quad \left. - \frac{1}{m^f(x_1 - 1, x_2)} \sum_{z \sim x} \mathcal{E}^f((x_1 - 1, x_2), z) f(z_1 - 1, z_2) \right. \\
&\quad \left. \times \frac{1 + \theta^f((x_1 - 1, x_2), z)}{\sqrt{(1 + \varepsilon^f(x_1 - 1, x_2))(1 + \varepsilon^f(z))}} \right) \Big|^2 \\
&\quad + \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e^{1/2} (Q - 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \right. \\
&\quad \times \left(\frac{1}{m^f(x_1 - 1, x_2)} \sum_{z \sim x} \mathcal{E}^f((x_1 - 1, x_2), z) \frac{1 + \theta^f((x_1 - 1, x_2), z)}{\sqrt{(1 + \varepsilon^f(x_1 - 1, x_2))(1 + \varepsilon^f(z))}} f(z) \right. \\
&\quad \left. - \frac{1}{m^f(x)} \sum_{z \sim x} \mathcal{E}^f(x, z) \frac{1 + \theta^f(x, z)}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} f(z_1 - 1, z_2) \right) \Big|^2 \\
&\quad + \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e^{-1/2} (Q + 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \right. \\
&\quad \times \left(\frac{1}{m^f(x_1 + 1, x_2)} \sum_{z \sim x} \mathcal{E}^f((x_1 + 1, x_2), z) \frac{1 + \theta^f((x_1 + 1, x_2), z)}{\sqrt{(1 + \varepsilon^f(x_1 + 1, x_2))(1 + \varepsilon^f(z))}} \right. \\
&\quad \left. - \frac{1}{m^f(x)} \sum_{z \sim x} \mathcal{E}^f(x, z) \frac{1 + \theta^f(x, z)}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \right) f(x_1 + 1, x_2) \Big|^2
\end{aligned}$$

$$\begin{aligned}
& + \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e^{-1/2} (Q + 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \right. \\
& \times \left(\frac{1}{m^f(x)} \sum_{z \sim x} \mathcal{E}^f(x, z) \frac{1 + \theta^f(x, z)}{\sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} f(z_1 + 1, z_2) \right. \\
& \left. \left. - \frac{1}{m^f(x_1 + 1, x_2)} \sum_{z \sim x} \mathcal{E}^f((x_1 + 1, x_2), z) \frac{1 + \theta^f((x_1 + 1, x_2), z)}{\sqrt{(1 + \varepsilon^f(x_1 + 1, x_2))(1 + \varepsilon^f(z))}} f(z_1, z_2) \right) \right|^2.
\end{aligned}$$

We treat the first term of $\| \langle \Lambda^f \rangle^\varepsilon [\tilde{\Delta}_{m'}^f, \mathcal{A}_m^f] \circ f \|_{\ell^2(\mathcal{V}^f, m^f)}$ in (33).

$$\begin{aligned}
& \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e^{1/2} (Q - 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \sum_{z \sim x} \left(\frac{\mathcal{E}^f(x, z)(1 + \theta^f(x, z))}{m^f(x) \sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \right. \right. \\
& \left. \left. - \frac{\mathcal{E}^f((x_1 - 1, x_2), z)(1 + \theta^f((x_1 - 1, x_2), z))}{m(x_1 - 1, x_2) \sqrt{(1 + \varepsilon^f(x_1 - 1, x_2))(1 + \varepsilon^f(z))}} \right) f(x_1 - 1, x_2) \right|^2 \\
(34) \quad & \leq 2 \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e^{1/2} (Q - 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \right. \\
& \times \sum_{z_1 \sim x_1} \delta_{z_2 = x_2} \left(\frac{\mathcal{E}_1^f(x_1, z_1)(1 + \theta^f(x, z))}{m(x) \sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \right. \\
& \left. - \frac{\mathcal{E}_1^f(x_1 - 1, z_1)(1 + \theta^f((x_1 - 1, x_2), z))}{m^f(x_1 - 1, x_2) \sqrt{(1 + \varepsilon^f(x_1 - 1, x_2))(1 + \varepsilon^f(z))}} \right) f(x_1 - 1, x_2) \Big|^2 \\
& + 2 \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e^{1/2} (Q - 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \right. \\
(35) \quad & \times \sum_{z_2 \sim x_2} \delta_{z_1 = x_1} \left(\frac{\mathcal{E}_2^f(x_2, z_2)(1 + \theta^f(x, z))}{m(x) \sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(z))}} \right. \\
& \left. - \frac{\mathcal{E}_2^f(x_2, z_2)(1 + \theta^f((x_1 - 1, x_2), z))}{m_1^f(x_1 - 1) m_2(x_2) \sqrt{(1 + \varepsilon^f(x_1 - 1, x_2))(1 + \varepsilon^f(z))}} \right) f(x_1 - 1, x_2) \Big|^2.
\end{aligned}$$

We bound (34) as follows:

$$\begin{aligned}
(34) \leq & 2 \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e^{1/2} (Q - 1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \right. \\
& \times \left(\frac{\mathcal{E}_1^f(x_1, x_1 + 1)(1 + \theta^f(x, (x_1 + 1, x_2)))}{m_1(x_1) m_2(x_2) \sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(x_1 + 1, x_2))}} \right. \\
& - \frac{\mathcal{E}_1^f(x_1 - 1, x_1)(1 + \theta^f((x_1 - 1, x_2), (x_1, x_2)))}{m_1^f(x_1 - 1) m_2(x_2) \sqrt{(1 + \varepsilon^f(x_1 - 1, x_2))(1 + \varepsilon^f(x))}} \\
& + \frac{\mathcal{E}_1^f(x_1, x_1 - 1)(1 + \theta^f(x, (x_1 - 1, x_2)))}{m_1^f(x_1) m_2(x_2) \sqrt{(1 + \varepsilon^f(x))(1 + \varepsilon^f(x_1 - 1, x_2))}} \\
& \left. \left. - \frac{\mathcal{E}_1^f(x_1 - 1, x_1 - 2)(1 + \theta^f((x_1 - 1, x_2), (x_1 - 2, x_2)))}{m_1^f(x_1 - 1) m_2(x_2) \sqrt{(1 + \varepsilon^f(x_1 - 1, x_2))(1 + \varepsilon^f(x_1 - 2, x_2))}} \right) f(x_1 - 1, x_2) \right|^2.
\end{aligned}$$

Then

$$\begin{aligned}
(34) &\leq 2 \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e^{1/2}(Q-1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m_2(x_2)} \left(\frac{e^{1/2}(1+\theta^f(x, (x_1+1, x_2)))}{\sqrt{(1+\varepsilon^f(x))(1+\varepsilon^f(x_1+1, x_2))}} \right. \right. \\
&\quad \left. \left. - \frac{e^{1/2}(1+\theta^f((x_1-1, x_2), (x_1, x_2)))}{\sqrt{(1+\varepsilon^f(x_1-1, x_2))(1+\varepsilon^f(x))}} + \frac{e^{-1/2}(1+\theta^f(x, (x_1-1, x_2)))}{\sqrt{(1+\varepsilon^f(x))(1+\varepsilon^f(x_1-1, x_2))}} \right. \right. \\
&\quad \left. \left. - \frac{e^{-1/2}(1+\theta^f((x_1-1, x_2), (x_1-2, x_2)))}{\sqrt{(1+\varepsilon^f(x_1-1, x_2))(1+\varepsilon^f(x_1-2, x_2))}} \right) f(x_1-1, x_2) \right|^2 \\
&\leq 4 \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e(Q-1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m_2(x_2)} \right. \\
(36) &\quad \times \left(\frac{\sqrt{1+\varepsilon^f(x_1-1, x_2)}(1+\theta^f(x, (x_1+1, x_2)))}{\sqrt{(1+\varepsilon^f(x))(1+\varepsilon^f(x_1+1, x_2))(1+\varepsilon^f(x_1-1, x_2))}} \right. \\
&\quad \left. - \frac{\sqrt{1+\varepsilon^f(x_1+1, x_2)}(1+\theta^f((x_1-1, x_2), (x_1, x_2)))}{\sqrt{(1+\varepsilon^f(x_1-1, x_2))(1+\varepsilon^f(x))(1+\varepsilon^f(x_1+1, x_2))}} \right) f(x_1-1, x_2) \Big|^2 \\
&\quad + 4 \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left((Q-1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m_2(x_2)} \right. \\
(37) &\quad \times \left(\frac{\sqrt{1+\varepsilon^f(x_1-2, x_2)}(1+\theta^f(x, (x_1-1, x_2)))}{\sqrt{(1+\varepsilon^f(x))(1+\varepsilon^f(x_1-1, x_2))(1+\varepsilon^f(x_1-2, x_2))}} \right. \\
&\quad \left. - \frac{\sqrt{1+\varepsilon^f(x_1-1, x_2)}(1+\theta^f((x_1-1, x_2), (x_1-2, x_2)))}{\sqrt{(1+\varepsilon^f(x_1-1, x_2))(1+\varepsilon^f(x_1-2, x_2))(1+\varepsilon^f(x_1-1, x_2))}} \right) f(x_1-1, x_2) \Big|^2,
\end{aligned}$$

and in the same way, we deal with (35).

Now, we concentrate on (36), since the assertions (31) and (32) hold true then there exists an integer c , such that

$$\begin{aligned}
&4 \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e(Q-1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \right. \\
&\quad \times \frac{1}{m_2(x_2)} \left(\frac{\sqrt{1+\varepsilon^f(x_1-1, x_2)}(1+\theta^f(x, (x_1+1, x_2)))}{\sqrt{(1+\varepsilon^f(x))(1+\varepsilon^f(x_1+1, x_2))(1+\varepsilon^f(x_1-1, x_2))}} \right. \\
&\quad \left. - \frac{\sqrt{1+\varepsilon^f(x_1+1, x_2)}(1+\theta^f((x_1-1, x_2), (x_1, x_2)))}{\sqrt{(1+\varepsilon^f(x_1-1, x_2))(1+\varepsilon^f(x))(1+\varepsilon^f(x_1+1, x_2))}} \right) f(x_1-1, x_2) \Big|^2 \\
&= 4 \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e(Q-1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2}^f \right) \frac{1}{m_2(x_2)} \right. \\
&\quad \times \left(\frac{(\sqrt{1+\varepsilon^f(x_1-1, x_2)} - \sqrt{1+\varepsilon^f(x_1+1, x_2)})\theta^f(x, (x_1+1, x_2))}{\sqrt{(1+\varepsilon^f(x))(1+\varepsilon^f(x_1+1, x_2))(1+\varepsilon^f(x_1-1, x_2))}} \right. \\
&\quad + \frac{\sqrt{1+\varepsilon^f(x_1+1, x_2)}(\theta^f(x, (x_1+1, x_2)) - \theta^f((x_1-1, x_2), x))}{\sqrt{(1+\varepsilon^f(x))(1+\varepsilon^f(x_1+1, x_2))(1+\varepsilon^f(x_1-1, x_2))}} \\
&\quad \left. + \frac{\sqrt{1+\varepsilon^f(x_1-1, x_2)} - \sqrt{1+\varepsilon^f(x_1+1, x_2)}}{\sqrt{(1+\varepsilon^f(x))(1+\varepsilon^f(x_1+1, x_2))(1+\varepsilon^f(x_1-1, x_2))}} \right) f(x_1-1, x_2) \Big|^2
\end{aligned}$$

$$\begin{aligned}
&= 4 \sum_{x \in \mathcal{V}^f} m^f(x) \left| \frac{i}{2} \left(e(Q-1/2)^{1+\varepsilon} \otimes \mathbf{1}_{\mathcal{V}_2^f} \right) \frac{1}{m_2(x_2)} \right. \\
&\quad \times \left(\frac{(\varepsilon^f(x_1-1, x_2) - \varepsilon^f(x_1+1, x_2)) \theta^f(x, (x_1+1, x_2))}{(\sqrt{1+\varepsilon^f(x_1-1, x_2)} + \sqrt{1+\varepsilon^f(x_1+1, x_2)})} \right. \\
&\quad \times \frac{1}{\sqrt{(1+\varepsilon^f(x))(1+\varepsilon^f(x_1+1, x_2))(1+\varepsilon^f(x_1-1, x_2))}} \\
&\quad + \frac{\sqrt{1+\varepsilon^f(x_1+1, x_2)} (\theta^f(x, (x_1+1, x_2)) - \theta^f((x_1-1, x_2), x))}{\sqrt{(1+\varepsilon^f(x))(1+\varepsilon^f(x_1+1, x_2))(1+\varepsilon^f(x_1-1, x_2))}} \\
&\quad + \frac{\varepsilon^f(x_1-1, x_2) - \varepsilon^f(x_1+1, x_2)}{(\sqrt{1+\varepsilon^f(x_1-1, x_2)} + \sqrt{1+\varepsilon^f(x_1+1, x_2)})} \\
&\quad \left. \left. \times \frac{1}{\sqrt{(1+\varepsilon^f(x))(1+\varepsilon^f(x_1+1, x_2))(1+\varepsilon^f(x_1-1, x_2))}} \right) f(x_1-1, x_2) \right|^2 \\
&\leq c \|f\|_{\ell^2(\mathcal{V}^f, m^f)}^2,
\end{aligned}$$

and in the same way, we treat (37). By density, we have proven that there exists $c > 0$ such that $\|\langle \Lambda^f \rangle^\varepsilon [\tilde{\Delta}_{m'}^f, \mathcal{A}_m^f] \circ f\|_{\ell^2(\mathcal{V}^f, m^f)}^2 \leq c \|f\|_{\ell^2(\mathcal{V}^f, m^f)}^2$. Finally, by applying [AmBoGe, Proposition 7.5.7] where the hypotheses are verified in Proposition 4.10, we find the result. \square

We turn to the regularity of $V^f(Q)$.

Lemma 4.13. *Let $V^f : \mathcal{V}^f \rightarrow \mathbb{R}$ be a function, obeying $\|[V^f, U]\langle Q+1/2 \rangle^{1+\varepsilon}\| < \infty$, then $[V^f(Q), \mathcal{A}_{G^f}] \in \mathcal{C}^{0,1}(\mathcal{A}^f)$. In particular, $V^f(Q) \in \mathcal{C}^{1,1}(\mathcal{A}^f)$.*

Proof. First, we have

$$\begin{aligned}
\|[V^f, U]\langle Q+1/2 \rangle^{1+\varepsilon}\| &= \|\langle Q+1/2 \rangle^{1+\varepsilon} [V^f, U^*]\| \\
&= \|\langle Q+1/2 \rangle^{1+\varepsilon} (V^f U^* - U^* V^f)\| \\
&= \|\langle Q+1/2 \rangle^{1+\varepsilon} V^f U^* - \langle Q+1/2 \rangle^{1+\varepsilon} U^* V^f\| \\
&= \|V^f U^* \langle Q-1 \rangle^{1+\varepsilon} - U^* V^f \langle Q-1 \rangle^{1+\varepsilon}\| \\
&= \|[V^f, U^*]\langle Q-1 \rangle^{1+\varepsilon}\|.
\end{aligned}$$

Now, we shall show that

$$\|\langle Q+1/2 \rangle^\varepsilon \otimes \mathbf{1}_{\mathcal{V}_2^f} [V^c(Q), i\mathcal{A}_{G^f}]\| < \infty.$$

We recall that

$$[V^f(Q), i\mathcal{A}_{G^f}] = \frac{e^{-1/2}}{2} \left(Q + \frac{1}{2} \right) [V^f, U^*] \otimes \mathbf{1}_{\mathcal{V}_2^f} + \frac{e^{1/2}}{2} \left(\frac{1}{2} - Q \right) [V^f, U] \otimes \mathbf{1}_{\mathcal{V}_2^f}.$$

For all $f \in \ell^2(\mathcal{V}^f, m^f)$, we obtain

$$\begin{aligned}
& \|\langle Q + 1/2 \rangle^\varepsilon \otimes \mathbf{1}_{\mathcal{V}^f} [V^f(Q), i\mathcal{A}_{\mathcal{G}^f}] f\| = \|(\langle Q + 1/2 \rangle^\varepsilon \frac{e^{-1/2}}{2} \left(Q + \frac{1}{2}\right) [V^f, U^*] \\
& \quad + \langle Q + 1/2 \rangle^\varepsilon \frac{e^{1/2}}{2} \left(\frac{1}{2} - Q\right) [V^f, U]) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\| \\
& \leq \|(\langle Q + 1/2 \rangle^\varepsilon \frac{e^{-1/2}}{2} \left(Q + \frac{1}{2}\right) [V^f, U^*]) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\| \\
& \quad + \|(\langle Q + 1/2 \rangle^\varepsilon \frac{e^{1/2}}{2} \left(\frac{1}{2} - Q\right) [V^f, U]) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\| \\
& \leq \underbrace{\frac{e^{-1/2}}{2} \|(\langle Q + 1/2 \rangle^\varepsilon Q [V^f, U^*]) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\|}_{(1)} + \underbrace{\frac{e^{-1/2}}{4} \|(\langle Q + 1/2 \rangle^\varepsilon [V^c, U^*]) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\|}_{(2)} \\
& \quad + \underbrace{\frac{e^{1/2}}{2} \|(\langle Q + 1/2 \rangle^\varepsilon Q [V^f, U]) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\|}_{(3)} + \underbrace{\frac{e^{1/2}}{4} \|(\langle Q + 1/2 \rangle^\varepsilon [V^f, U]) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\|}_{(4)}.
\end{aligned}$$

For all $f \in \ell^2(\mathcal{V}^f, m^f)$, we have

(1)

$$\begin{aligned}
& \frac{e^{-1/2}}{2} \|(\langle Q + 1/2 \rangle^\varepsilon Q [V^f, U^*]) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\| = \frac{e^{-1/2}}{2} \|([V^f, U] \langle Q + 1/2 \rangle^\varepsilon Q) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\| \\
& = \frac{e^{-1/2}}{2} \|([V^f, U] \langle Q + 1/2 \rangle^{1+\varepsilon-1} Q) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\| \\
& = \frac{e^{-1/2}}{2} \|([V^f, U] \langle Q + 1/2 \rangle^{1+\varepsilon} \underbrace{\langle Q + 1/2 \rangle^{-1} Q}_{\text{bounded}}) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\|.
\end{aligned}$$

(2)

$$\begin{aligned}
& \frac{e^{-1/2}}{4} \|(\langle Q + 1/2 \rangle^\varepsilon [V^f, U^*]) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\| = \frac{e^{-1/2}}{4} \|([V^f, U] \langle Q + 1/2 \rangle^\varepsilon) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\| \\
& = \frac{e^{-1/2}}{4} \|([V^f, U] \langle Q + 1/2 \rangle^{1+\varepsilon-1}) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\| \\
& = \frac{e^{-1/2}}{4} \|([V^f, U] \langle Q + 1/2 \rangle^{1+\varepsilon} \underbrace{\langle Q + 1/2 \rangle^{-1}}_{\text{bounded}}) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\|.
\end{aligned}$$

(3)

$$\begin{aligned}
& \frac{e^{1/2}}{2} \|(\langle Q + 1/2 \rangle^\varepsilon Q [V^f, U]) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\| = \frac{e^{1/2}}{2} \|([V^f, U^*] \langle Q + 1/2 \rangle^\varepsilon Q) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\| \\
& = \frac{e^{1/2}}{2} \|([V^f, U^*] \langle Q + 1/2 \rangle^{1+\varepsilon-1} Q) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\| \\
& = \frac{e^{1/2}}{2} \|([V^f, U^*] \langle Q + 1/2 \rangle^{1+\varepsilon} \underbrace{\langle Q + 1/2 \rangle^{-1} Q}_{\text{bounded}}) \otimes \mathbf{1}_{\mathcal{V}_2^f} f\|.
\end{aligned}$$

(4)

$$\begin{aligned}
& \frac{e^{1/2}}{4} \left\| \left((Q + 1/2)^\varepsilon [V^f, U] \right) \otimes \mathbf{1}_{\mathcal{V}_2^f} f \right\| = \frac{e^{1/2}}{4} \left\| \left([V^f, U^*] \langle Q + 1/2 \rangle^\varepsilon \right) \otimes \mathbf{1}_{\mathcal{V}_2^f} f \right\| \\
& = \frac{e^{1/2}}{4} \left\| \left([V^f, U^*] \langle Q + 1/2 \rangle^{1+\varepsilon-1} \right) \otimes \mathbf{1}_{\mathcal{V}_2^f} f \right\| \\
& = \frac{e^{1/2}}{4} \left\| \left([V^f, U^*] \langle Q + 1/2 \rangle^{1+\varepsilon} \underbrace{\langle Q + 1/2 \rangle^{-1}}_{\text{bounded}} \right) \otimes \mathbf{1}_{\mathcal{V}_2^f} f \right\|.
\end{aligned}$$

So, we can apply [AmBoGe, Proposition 7.5.7] where the hypotheses are verified in Proposition 4.10, the result follows. \square

We end this section by proving the Theorem 4.8.

Proof of Theorem 4.8. The Proposition 4.12, Proposition 4.11 and Lemma 4.13 give that $\Delta_{\mathcal{G}'}^f + V^f(Q) \in \mathcal{C}^{1,1}(\mathcal{A}_m^f)$. Since $\tilde{\Delta}_{m'}^f - \Delta_{\mathcal{G}'}^f$ is a compact operator by Proposition 4.6, thanks to (12) and by [AmBoGe, Theorem 7.2.9], we obtain (26). \square

4.5. The cusp side: Radial metric perturbation. We recall that, for all $f \in \ell_c^2(\mathcal{V}^c, m^c)$,

$$\tilde{\Delta}^c f(x) := T_{m' \rightarrow m} \Delta_{\mathcal{G}'}^c T_{m' \rightarrow m}^{-1} f(x) = W^c f(x) + \tilde{\Delta}_{m'}^c f(x),$$

where

$$\begin{aligned}
W^c f(x) := & \frac{1}{m^c(x)} \sum_{z \sim x} (1 + \theta^c(x, z)) \left(\frac{\varepsilon^c(z) - \varepsilon^c(x)}{(1 + \varepsilon^c(x)) \sqrt{1 + \varepsilon^c(z)} (\sqrt{1 + \varepsilon^c(z)} + \sqrt{1 + \varepsilon^c(x)})} \right) \\
& \times \mathcal{E}^c(x, z) f(x)
\end{aligned}$$

and

$$\tilde{\Delta}_{m'}^c f(x) := \frac{1}{m(x)} \sum_{z \sim x} \mathcal{E}'(x, z) \sqrt{\frac{m(z)m(x)}{m'(x)m'(z)}} (f(x) - f(z)).$$

We also remember that

$$\begin{aligned}
(\tilde{\Delta}_{m'}^c - \Delta_{\mathcal{G}'}^c) f(x) := & \frac{1}{m^c(x)} \sum_{z \sim x} \frac{\theta^c(x, z)}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}} \mathcal{E}^c(x, z) (f(x) - f(z)) \\
& + \frac{1}{m^c(x)} \sum_{z \sim x} \frac{1 - \sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}} \mathcal{E}^c(x, z) (f(x) - f(z)).
\end{aligned}$$

We first deal with the question of the essential spectrum.

Proposition 4.14. *Let $\tilde{\Delta}^c$ be the transformation of $\Delta_{\mathcal{G}'}^c$ on $\ell^2(\mathcal{V}^c, m^c)$, as in (22) and $V^c : \mathcal{V}^c \rightarrow \mathbb{R}$ be a function obeying $V^c(x) \rightarrow 0$ if $|x| \rightarrow \infty$. We assume that for all $x_2, z_2 \in \mathcal{V}_2$, $\theta^c((x_1, x_2), (z_1, z_2)) \rightarrow 0$ if $x_1, z_1 \rightarrow \infty$ and for all $x_2 \in \mathcal{V}_2$, $\varepsilon^c(x_1, x_2) \rightarrow 0$ if $x_1 \rightarrow \infty$ then $\tilde{\Delta}^c - \Delta_{\mathcal{G}'}^c \in \mathcal{K}(\ell^2(\mathcal{V}^c, m^c))$.*

In particular, $\sigma_{\text{ess}}(\Delta_{\mathcal{G}'}^c + V^c(Q)) = \sigma_{\text{ess}}(\Delta_{\mathcal{G}'}^c)$.

Proof. Let $f \in \ell_c^2(\mathcal{V}^c, m^c)$, we have

$$\begin{aligned}
& | \langle f, (\tilde{\Delta}_{m'}^c - \Delta_{\mathcal{G}^c}) f \rangle_{\ell_c^2(\mathcal{V}^c, m^c)} | = | \sum_{x \in \mathcal{V}^c} m^c(x) (\tilde{\Delta}_{m'}^c - \Delta_{\mathcal{G}^c}) f(x) \overline{f(x)} | \\
& \leq | \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z \sim x} \frac{\theta^c(x, z)}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}} \mathcal{E}^c(x, z) (f(x) - f(z)) \overline{f(x)} | \\
& \quad + | \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z \sim x} \frac{1 - \sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}} \\
& \quad \quad \times \mathcal{E}^c(x, z) (f(x) - f(z)) \overline{f(x)} | \\
& \leq | \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} \frac{\theta^c(x, (z_1, x_2))}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z_1, x_2))}} \mathcal{E}_1^c(x_1, z_1) f(x) \overline{f(x)} | \\
& \quad + | \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} \frac{\theta^c(x, (z_1, x_2))}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z_1, x_2))}} \mathcal{E}_1^c(x_1, z_1) f(z_1, x_2) \overline{f(x)} | \\
& \quad + | \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} \frac{\varepsilon^c(x) + \varepsilon^c(z_1, x_2) + \varepsilon^c(x)\varepsilon^c(z_1, x_2)}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z_1, x_2))}} \\
& \quad \quad \times \frac{1}{\sqrt{1 + \varepsilon^c(x)} + \sqrt{1 + \varepsilon^c(z_1, x_2)}} \mathcal{E}_1^c(x_1, z_1) f(x) \overline{f(x)} | \\
& \quad + | \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} \frac{\varepsilon^c(x) + \varepsilon^c(z_1, x_2) + \varepsilon^c(x)\varepsilon^c(z_1, x_2)}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z_1, x_2))}} \\
& \quad \quad \times \frac{1}{\sqrt{1 + \varepsilon^c(x)} + \sqrt{1 + \varepsilon^c(z_1, x_2)}} \mathcal{E}_1^c(x_1, z_1) f(z_1, x_2) \overline{f(x)} | \\
& \leq \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} \frac{\theta^c(x, (z_1, x_2))}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z_1, x_2))}} \\
& \quad \quad \times \mathcal{E}_1^c(x_1, z_1) |f(x)|^2 \\
& \quad + 1/2 \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} \frac{\theta^c(x, (z_1, x_2))}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z_1, x_2))}} \\
& \quad \quad \times \mathcal{E}^c(x, z) (|f(z)|^2 + |f(x)|^2) \\
& \quad + \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} \left| \frac{\varepsilon^c(x) + \varepsilon^c(z_1, x_2) + \varepsilon^c(x)\varepsilon^c(z_1, x_2)}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z_1, x_2))}} \right| \\
& \quad \quad \times \frac{1}{\sqrt{1 + \varepsilon^c(x)} + \sqrt{1 + \varepsilon^c(z_1, x_2)}} \mathcal{E}_1^c(x_1, z_1) |f(x)|^2 \\
& \quad + 1/2 \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} \left| \frac{\varepsilon^c(x) + \varepsilon^c(z_1, x_2) + \varepsilon^c(x)\varepsilon^c(z_1, x_2)}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z_1, x_2))}} \right| \\
& \quad \quad \times \frac{1}{\sqrt{1 + \varepsilon^c(x)} + \sqrt{1 + \varepsilon^c(z_1, x_2)}} \mathcal{E}_1^c(x_1, z_1) |f(z_1, x_2)|^2 \\
& \quad + 1/2 \sum_{x \in \mathcal{V}^c} m^c(x) \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} \left| \frac{\varepsilon^c(x) + \varepsilon^c(z_1, x_2) + \varepsilon^c(x)\varepsilon^c(z_1, x_2)}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z_1, x_2))}} \right| \\
& \quad \quad \times \frac{1}{\sqrt{1 + \varepsilon^c(x)} + \sqrt{1 + \varepsilon^c(z_1, x_2)}} \mathcal{E}_1^c(x_1, z_1) |f(x)|^2 \\
& \leq 2 \langle f, (\deg_3 + \deg_4) f \rangle,
\end{aligned}$$

where

$$\deg_3(x) := \frac{1}{m^c(x)} \sum_{z_1 \in \mathcal{V}_1^c} \frac{\theta^c(x, (z_1, x_2))}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z_1, x_2))}} \mathcal{E}_1^c(x_1, z_1)$$

and

$$\deg_4(x) := \frac{1}{m^c(x)} \sum_{z_1 \in \mathcal{V}_1^c} \left| \frac{\varepsilon^c(x) + \varepsilon^c(z_1, x_2) + \varepsilon^c(x)\varepsilon^c(z_1, x_2)}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z_1, x_2))(\sqrt{1 + \varepsilon^c(x)} + \sqrt{1 + \varepsilon^c(z_1, x_2)})}} \right| \times \mathcal{E}_1^c(x_1, z_1).$$

We have

$$\begin{aligned} |\deg_3(x)| &= \left| \frac{1}{m^c(x)} \sum_{z_1 \in \mathcal{V}_1} \frac{\theta^c(x, (z_1, x_2))}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z_1, x_2))}} \mathcal{E}_1^c(x_1, z_1) \right| \\ &\leq \frac{1}{m^c(x)} \sum_{z_1 \in \mathcal{V}_1^c} \left| \frac{\theta^c(x, (z_1, x_2))}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z_1, x_2))}} \right| \mathcal{E}_1^c(x_1, z_1) \\ &\leq \sup_{z_1 \sim x_1} \left| \frac{\theta^c(x, (z_1, x_2))}{m_2(x_2)\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z_1, x_2))}} \right| \deg_{\mathcal{G}_1^c}(x). \end{aligned}$$

Since \mathcal{V}_2 is a finite set and for all $x_2 \in \mathcal{V}_2$, $\frac{\theta^c((x_1, x_2), (z_1, x_2))}{\sqrt{(1 + \varepsilon^c(x_1, x_2))(1 + \varepsilon^c(z_1, x_2))}} \rightarrow 0$ if $x_1, z_1 \rightarrow \infty$ and since $\deg_{\mathcal{G}_1^c}(Q)$ is bounded then $\deg_3(Q)$ is compact. In the same way, using

that $\forall x_2 \in \mathcal{V}_2$, $\frac{\varepsilon^c(x) + \varepsilon^c(z_1, x_2) + \varepsilon^c(x)\varepsilon^c(z_1, x_2)}{\sqrt{(1 + \varepsilon^c(x_1, x_2))(1 + \varepsilon^c(z_1, x_2))(\sqrt{1 + \varepsilon^c(x)} + \sqrt{1 + \varepsilon^c(z_1, x_2)})}} \rightarrow 0$ if $x_1, z_1 \rightarrow \infty$,

we prove the compactness of $\deg_4(Q)$. Then, $\tilde{\Delta}_{m'}^c - \Delta_{\mathcal{G}^c}$ is a compact operator.

Now, we will show that $W^c \in \mathcal{K}(\ell^2(\mathcal{V}^c, m^c))$. For all $x \in \mathcal{V}^c$, we have

$$\begin{aligned} |W^c(x)| &= \left| \frac{1}{m^c(x)} \sum_{z \sim x} \left(\frac{\varepsilon^c(z) - \varepsilon^c(x)}{(1 + \varepsilon^c(x))\sqrt{1 + \varepsilon^c(z)}(\sqrt{1 + \varepsilon^c(z)} + \sqrt{1 + \varepsilon^c(x)})} \right) \right. \\ &\quad \left. \times (1 + \theta^c(x, z))\mathcal{E}^c(x, z) \right| \\ &\leq \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} \left| \frac{\varepsilon^c(z_1, x_2) - \varepsilon^c(x)}{(1 + \varepsilon^c(x))\sqrt{1 + \varepsilon^c(z_1, x_2)}(\sqrt{1 + \varepsilon^c(z_1, x_2)} + \sqrt{1 + \varepsilon^c(x)})} \right| \\ &\quad \times (1 + \theta^c(x, (z_1, x_2)))\mathcal{E}_1^c(x_1, z_1) \\ &\leq \sup_{z_1 \sim x_1} (1 + \theta^c(x, (z_1, x_2))) \left| \frac{\varepsilon^c(z_1, x_2) - \varepsilon^c(x)}{(1 + \varepsilon^c(x))\sqrt{1 + \varepsilon^c(z_1, x_2)}(\sqrt{1 + \varepsilon^c(z_1, x_2)} + \sqrt{1 + \varepsilon^c(x)})} \right| \\ &\quad \times \deg_{\mathcal{G}_1^c}(x). \end{aligned}$$

Since \mathcal{V}_2 is a finite set and $\forall x_2 \in \mathcal{V}_2$, $\theta^c((x_1, x_2), (z_1, x_2))(\varepsilon^c(z_1, x_2) - \varepsilon^c(x_1, x_2)) \rightarrow 0$ when $x_1, z_1 \rightarrow \infty$, and since $\deg_{\mathcal{G}_1^c}(Q)$ is bounded and since $V^c(Q)$ is a compact perturbation, then we find the result. \square

In order to go into the Mourre theory, we construct the conjugate operator. On $\ell_c^2(\mathcal{V}^c, m^c)$, we set

$$(38) \quad \mathcal{A}_m^c := \mathcal{A}_{m_1} \otimes P_{\ker(\Delta_2)}^\perp,$$

with

$$\begin{aligned} \mathcal{A}_{m_1} &:= T_{1 \rightarrow m_1} \mathcal{A}_{\mathbb{N}} T_{1 \rightarrow m_1}^{-1} \\ &:= \frac{i}{2} \left(e^{1/2}(n - 1/2)U - e^{-1/2}(n + 1/2)U^* \right). \end{aligned}$$

The operator $\mathcal{A}_{\mathbb{N}}$ is essentially self-adjoint, see [GeGo], then the operator \mathcal{A}_m^c is essentially self-adjoint. We continue to denote the closure by the same symbol.

Because of the projection in (38), we restrict to ourself radial perturbations.

Definition 4.15. *The perturbations V^c , ε and θ are called **radial** if they do not depend on the second variable, i.e. For all $(x_1, x_2), (z_1, z_2) \in \mathcal{V}$, we have $V^c(x_1, x_2) = V^c(x_1, z_2)$, $\varepsilon(x_1, x_2) = \varepsilon(x_1, z_2)$ and $\theta((x_1, x_2), (z_1, z_2)) = \theta((x_1, x_2), (z_1, x_2))$.*

Now, we turn to Mourre estimate.

Proposition 4.16. *Let $V^c : \mathcal{V}^c \rightarrow \mathbb{R}$ be a function, which does not depend on the second variable. We assume that*

$$\sup_{(x_1, x_2) \in \mathcal{V}^c} x_1^{\varepsilon+1} (\varepsilon^c(x_1 - 1, x_2) - \varepsilon^c(x_1, x_2)) < \infty,$$

$$\sup_{(x_1, x_2) \in \mathcal{V}^c, x_1 \sim z_1} x_1^{\varepsilon+1} (\theta^c((x_1, x_2), (z_1, x_2)) - \theta^c((x_1 - 1, x_2), (z_1, x_2))) < \infty$$

and

$$\|[V^c, U]\langle Q + 1/2 \rangle^{1+\varepsilon}\| < \infty$$

where $\theta^c(x, z) \rightarrow 0$ if $|x|, |z| \rightarrow \infty$, $\varepsilon^c(x) \rightarrow 0$ if $|x| \rightarrow \infty$ and $V^c(x) \rightarrow 0$ if $|x| \rightarrow \infty$. Then $\Delta_{\mathcal{G}'}^c + V^c(Q) \in \mathcal{C}^{1,1}(\mathcal{A}_{m'}^c)$. Moreover, there are $c > 0$, a compact interval $\mathcal{I} \subset (c_1, c_2)$, a compact operator K such that

$$(39) \quad E_{\mathcal{I}}(\Delta_{\mathcal{G}'}^c + V^c(Q))[\Delta_{\mathcal{G}'}^c + V^c(Q), i\mathcal{A}_{m'}^c]_0 E_{\mathcal{I}}(\Delta_{\mathcal{G}'}^c + V^c(Q)) \geq c E_{\mathcal{I}}(\Delta_{\mathcal{G}'}^c + V^c(Q)) + K,$$

in the form sense.

We shall give the proof at the end of the section. We turn to series of Lemmata. To be able to apply the [AmBoGe, Proposition 7.5.7], we have to check this point.

Proposition 4.17. *Let $\Lambda^c := (Q + 1/2) \otimes \mathbf{1}_{\mathcal{V}^c}$, then Λ^c satisfies the following assertions:*

(1) $e^{i\Lambda^c t} \mathcal{D}(\Delta_{\mathcal{G}^c}) \subset \mathcal{D}(\Delta_{\mathcal{G}^c})$ and there exists a finite constant c , such that

$$\|e^{i\Lambda^c t}\|_{\mathcal{B}(\mathcal{D}(\Delta_{\mathcal{G}^c}))} \leq c, \quad \text{for all } t \in \mathbb{R}.$$

(2) $\mathcal{D}(\Lambda^c) \subset \mathcal{D}(\mathcal{A}_m^c)$.

(3) $(\Lambda^c)^{-2} (\mathcal{A}_m^c)^2$ extends to a continuous operator in $\mathcal{D}(\Delta_{\mathcal{G}^c})$.

Proof. (1) We have

$$[\Delta_{\mathcal{G}^c}, e^{i\Lambda^c t}] = [\Delta_{\mathcal{G}_1^c}, e^{i\Lambda^c t}] \otimes \frac{1}{m_2} + [\frac{1}{m_1}, e^{i\Lambda^c t}] \otimes \Delta_2.$$

Since $\frac{1}{m_1}$ and $e^{i\Lambda^c t}$ commute and since $[\Delta_{\mathcal{G}_1^c}, e^{i\Lambda^c t}]$ is uniformly bounded, then there exists $c > 0$ such that for all $f \in \ell_c^2(\mathcal{V}^c, m^c)$

$$\|(\Delta_{\mathcal{G}^c} + i)e^{i\Lambda^c t}(\Delta_{\mathcal{G}^c} + i)^{-1}f\| \leq c\|f\|.$$

Hence, there exists $c > 0$ such that for all $f \in \ell_c^2(\mathcal{V}^c, m^c)$

$$\|(\Delta_{\mathcal{G}^c} + i)e^{i\Lambda^c t}f\| \leq c\|(\Delta_{\mathcal{G}^c} + i)f\|.$$

Since $\Delta_{\mathcal{G}^c}$ is essentially self-adjoint on $\ell_c^2(\mathcal{V}^c, m^c)$ then we find the result.

(2) Let $f \in \ell_c^2(\mathcal{V}^c, m^c)$, by using the relations of Subsection 6, we have

$$\begin{aligned}
\|\mathcal{A}_m^c f\|_{\ell^2(\mathcal{V}^c, m^c)}^2 &= \sum_{x \in \mathcal{V}^c} m^c(x) |\mathcal{A}_m^c f(x)|^2 \\
&= \sum_{x \in \mathcal{V}^c} m^c(x) \left| \frac{i}{2} \left(e^{1/2}(Q-1/2)U \otimes P_{\ker(\Delta_2)}^\perp - e^{-1/2}(Q+1/2)U^* \otimes P_{\ker(\Delta_2)}^\perp \right) f(x) \right|^2 \\
&\leq \frac{1}{2} \sum_{x \in \mathcal{V}^f} m^f(x) \left| e^{1/2}(Q-1/2)U \otimes P_{\ker(\Delta_2)}^\perp f(x) \right|^2 \\
&\quad + \left| e^{-1/2}(Q+1/2)U^* \otimes P_{\ker(\Delta_2)}^\perp f(x) \right|^2 \\
&\leq c \sum_{x \in \mathcal{V}^f} m^f(x) \left| (Q+1/2) \otimes P_{\ker(\Delta_2)}^\perp f(x) \right|^2 \\
&\leq c \|\Lambda^c f\|_{\ell^2(\mathcal{V}^f, m^f)}^2.
\end{aligned}$$

Since, Λ^c is essentially self-adjoint on $\ell_c^2(\mathcal{V}^c, m^c)$. we find the result.

(3) For all $f \in \ell_c^2(\mathcal{V}^c, m^c)$, we have

$$\begin{aligned}
\|(\Lambda^c)^{-2}(\mathcal{A}_m^c)^2 f\|^2 &= \sum_{(x_1, x_2) \in \mathcal{V}^c} m^c(x_1, x_2) |\Lambda^{-2}(\mathcal{A}_m^c)^2 f(x, y)|^2 \\
&= \sum_{(x_1, x_2) \in \mathcal{V}^c} m^c(x_1, x_2) \left| \frac{1}{4} \left(((Q+1/2)^{-2}(2Q^2+1/2)) \otimes P_{\ker(\Delta_2)}^\perp \right) f(x_1, x_2) \right. \\
&\quad \left. - \frac{1}{4} e \left(((Q+1/2)^{-2}(Q-1/2)(Q-3/2)) \otimes P_{\ker(\Delta_2)}^\perp \right) f(x_1-2, x_2) \right. \\
&\quad \left. - \frac{1}{4} e^{-1} \left(((Q+1/2)^{-2}(Q+1/2)(Q+3/2)) \otimes P_{\ker(\Delta_2)}^\perp \right) f(x_1+2, x_2) \right|^2.
\end{aligned}$$

Then there exists $C > 0$ such that for all $f \in \ell_c^2(\mathcal{V}^c, m^c)$, $\|(\Lambda^c)^{-2}(\mathcal{A}_m^c)^2 f\|^2 \leq C\|f\|^2$. By density, we find the result. \square

Proposition 4.18. *We assume that*

$$(40) \quad \sup_{(x_1, x_2), (y_1, x_2) \in \mathcal{V}^c; x_1 \sim y_1} x_1^{1+\varepsilon} |\varepsilon^c(x_1, x_2) - \varepsilon^c(y_1, x_2)| < \infty$$

then $[W^c, \mathcal{A}_m^c] \in \mathcal{C}^{0,1}(\mathcal{A}_m^c)$. In particular, $W^c \in \mathcal{C}^{1,1}(\mathcal{A}_m^c)$.

Proof. We suppose that (40) is true then

$$\|\langle \Lambda^c \rangle^\varepsilon [W^c, \mathcal{A}_m^c] f\|_{\ell^2(\mathcal{V}^c, m^c)} = \sum_{x \in \mathcal{V}^c} m^c(x) |\langle \Lambda^c \rangle^\varepsilon [W^c, \mathcal{A}_m^c] f(x)|^2,$$

we treat the first term of $\|\langle \Lambda^c \rangle^\varepsilon [W^c, \mathcal{A}_m^c] f\|_{\ell^2(\mathcal{V}^c, m^c)}$ with respect to the decomposition of (27).

$$\begin{aligned}
& \frac{e^{-1}}{4} \sum_{x \in \mathcal{V}^c} m^c(x) |((Q+1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp) \frac{1}{m^c(x)} \sum_{z \sim x} (1 + \theta^c(x, z)) \\
& \quad \left(\frac{\varepsilon^c(z) - \varepsilon^c(x)}{(1 + \varepsilon^c(x)) \sqrt{1 + \varepsilon^c(z)} (\sqrt{1 + \varepsilon^c(z)} + \sqrt{1 + \varepsilon^c(x)})} \right) \mathcal{E}^c(x, z) Uf(x) |^2 \\
& = \frac{e^{-1}}{4} \sum_{x \in \mathcal{V}^c} m^c(x) |((Q+1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp) \frac{1}{m^c(x)} \sum_{z \sim x} (1 + \theta^c(x, z)) \\
& \quad \left(\frac{\varepsilon^c(z) - \varepsilon^c(x)}{(1 + \varepsilon^c(x)) \sqrt{1 + \varepsilon^c(z)} (\sqrt{1 + \varepsilon^c(z)} + \sqrt{1 + \varepsilon^c(x)})} \right) (\mathcal{E}_1^c(x_1, z_1) \delta_{x_2=z_2} \\
& \quad + \mathcal{E}_2(x_2, z_2) \delta_{x_1=z_1}) Uf(x) |^2 \\
& \leq \frac{e^{-1}}{2} \sum_{x \in \mathcal{V}^c} m^c(x) |((Q+1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp) \frac{1}{m^c(x)} \sum_{z_1 \sim x_1} (1 + \theta^c(x, z)) \\
& \quad \left(\frac{\varepsilon^c(z) - \varepsilon^c(x)}{(1 + \varepsilon^c(x)) \sqrt{1 + \varepsilon^c(z)} (\sqrt{1 + \varepsilon^c(z)} + \sqrt{1 + \varepsilon^c(x)})} \right) \mathcal{E}_1^c(x_1, z_1) \delta_{x_2=z_2} Uf(x) |^2 \\
& \leq e^{-1} \sum_{x \in \mathcal{V}^c} m^c(x) |((Q+1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp) \frac{1}{m^c(x)} (1 + \theta^c(x, (x_1+1, x_2))) \\
& \quad \times \left(\frac{\varepsilon^c(x_1+1, x_2) - \varepsilon^c(x)}{(1 + \varepsilon^c(x)) \sqrt{1 + \varepsilon^c(x_1+1, x_2)} (\sqrt{1 + \varepsilon^c(x_1+1, x_2)} + \sqrt{1 + \varepsilon^c(x)})} \right) \\
& \quad \times \mathcal{E}_1^c(x_1, x_1+1) Uf(x) |^2 \\
& + e^{-1} \sum_{x \in \mathcal{V}^c} m^c(x) |((Q+1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp) \frac{1}{m^c(x)} (1 + \theta^c(x, (x_1-1, x_2))) \\
& \quad \times \left(\frac{\varepsilon^c(x_1-1, x_2) - \varepsilon^c(x)}{(1 + \varepsilon^c(x)) \sqrt{1 + \varepsilon^c(x_1-1, x_2)} (\sqrt{1 + \varepsilon^c(x_1-1, x_2)} + \sqrt{1 + \varepsilon^c(x)})} \right) \\
& \quad \times \mathcal{E}_1^c(x_1, x_1-1) Uf(x) |^2 \\
& \leq e^{-1} \sum_{x \in \mathcal{V}^c} m^c(x) |((Q+1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp) \frac{e^{-x_1}}{m_2(x_2)} e^{(2x_1+1)/2} (1 + \theta^c(x, (x_1+1, x_2))) \\
& \quad \times \left(\frac{\varepsilon^c(x_1+1, x_2) - \varepsilon^c(x)}{(1 + \varepsilon^c(x)) \sqrt{1 + \varepsilon^c(x_1+1, x_2)} (\sqrt{1 + \varepsilon^c(x_1+1, x_2)} + \sqrt{1 + \varepsilon^c(x)})} \right) Uf(x) |^2 \\
& + e^{-1} \sum_{x \in \mathcal{V}^c} m^c(x) |((Q+1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp) \frac{e^{-x_1}}{m_2(x_2)} e^{(2x_1-1)/2} (1 + \theta^c(x, (x_1-1, x_2))) \\
& \quad \times \left(\frac{\varepsilon^c(x_1-1, x_2) - \varepsilon^c(x)}{(1 + \varepsilon^c(x)) \sqrt{1 + \varepsilon^c(x_1-1, x_2)} (\sqrt{1 + \varepsilon^c(x_1-1, x_2)} + \sqrt{1 + \varepsilon^c(x)})} \right) Uf(x) |^2 \\
& \leq e^{-1/2} \sum_{x \in \mathcal{V}^c} m^c(x) |((Q+1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp) \frac{1}{m_2(x_2)} (1 + \theta^c(x, (x_1+1, x_2))) \\
& \quad \times \left(\frac{\varepsilon^c(x_1+1, x_2) - \varepsilon^c(x)}{(1 + \varepsilon^c(x)) \sqrt{1 + \varepsilon^c(x_1+1, x_2)} (\sqrt{1 + \varepsilon^c(x_1+1, x_2)} + \sqrt{1 + \varepsilon^c(x)})} \right) Uf(x) |^2 \\
& + e^{-3/2} \sum_{x \in \mathcal{V}^c} m^c(x) |((Q+1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp) \frac{1}{m_2(x_2)} (1 + \theta^c(x, (x_1-1, x_2))) \\
& \quad \times \left(\frac{\varepsilon^c(x_1-1, x_2) - \varepsilon^c(x)}{(1 + \varepsilon^c(x)) \sqrt{1 + \varepsilon^c(x_1-1, x_2)} (\sqrt{1 + \varepsilon^c(x_1-1, x_2)} + \sqrt{1 + \varepsilon^c(x)})} \right) Uf(x) |^2.
\end{aligned}$$

Then, there exists an integer $c > 0$, such that

$$\begin{aligned} & \frac{e^{-1}}{4} \sum_{x \in \mathcal{V}^c} m^c(x) |((Q + 1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp) \frac{1}{m^c(x)} \sum_{z \sim x} (1 + \theta^c(x, z)) \\ & \quad \left(\frac{\varepsilon^c(z) - \varepsilon^c(x)}{(1 + \varepsilon^c(x))\sqrt{1 + \varepsilon^c(z)}(\sqrt{1 + \varepsilon^c(z)} + \sqrt{1 + \varepsilon^c(x)})} \right) \mathcal{E}^c(x, z) Uf(x) \Big|^2 \\ & \leq c \|f\|_{\ell^2(\mathcal{V}^c, m^c)}^2. \end{aligned}$$

So, we can apply [AmBoGe, Proposition 7.5.7] where the hypotheses are verified in Proposition 4.17, the result follows. \square

Proposition 4.19. *We assume that*

$$(41) \quad \sup_{(x_1, x_2) \in \mathcal{V}^c} x_1^{\varepsilon+1} (\varepsilon^c(x_1 - 1, x_2) - \varepsilon^c(x_1, x_2)) < \infty$$

and

$$(42) \quad \sup_{(x_1, x_2), (z_1, z_2) \in \mathcal{V}^c; (x_1, x_2) \sim (z_1, z_2)} x_1^{\varepsilon+1} (\theta^c(x, z) - \theta^c((x_1 - 1, x_2), z)) < \infty.$$

Then $\tilde{\Delta}_{m'}^c \in \mathcal{C}^{1,1}(\mathcal{A}_m^c)$.

Proof. Let $f \in \ell_c^2(\mathcal{V}^c, m^c)$,

$$\begin{aligned} [\tilde{\Delta}_{m'}^c, \mathcal{A}_m^c]f(x) &= \frac{i}{2} \left(e^{1/2}(Q - 1/2) \otimes P_{\ker(\Delta_2)}^\perp \right) \frac{1}{m^c(x)} \sum_{z \sim x} \mathcal{E}^c(x, z) \\ & \quad \times \frac{1 + \theta^c(x, z)}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}} (f(x_1 - 1, x_2) - f(z_1 - 1, z_2)) \\ & - \frac{i}{2} \left(e^{-1/2}(Q + 1/2) \otimes P_{\ker(\Delta_2)}^\perp \right) \frac{1}{m^c(x)} \sum_{z \sim x} \mathcal{E}^c(x, z) \\ & \quad \times \frac{1 + \theta^c(x, z)}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}} (f(x_1 + 1, x_2) - f(z_1 + 1, z_2)) \\ & - \frac{i}{2} \left(e^{1/2}(Q - 1/2) \otimes P_{\ker(\Delta_2)}^\perp \right) \frac{1}{m^c(x_1 - 1, x_2)} \sum_{z \sim x} \mathcal{E}^c((x_1 - 1, x_2), z) \\ & \quad \times \frac{1 + \theta^c((x_1 - 1, x_2), z)}{\sqrt{(1 + \varepsilon^c(x_1 - 1, x_2))(1 + \varepsilon^c(z))}} (f(x_1 - 1, x_2) - f(z_1 - 1, z_2)) \\ & + \frac{i}{2} \left(e^{-1/2}(Q + 1/2) \otimes P_{\ker(\Delta_2)}^\perp \right) \frac{1}{m^c(x_1 + 1, x_2)} \sum_{z \sim x} \mathcal{E}^c((x_1 + 1, x_2), z) \\ & \quad \times \frac{1 + \theta^c((x_1 + 1, x_2), z)}{\sqrt{(1 + \varepsilon^c(x_1 + 1, x_2))(1 + \varepsilon^c(z))}} (f(x_1 + 1, x_2) - f(z_1 + 1, z_2)). \end{aligned}$$

Now, we assume that (41) and (42) are true then

$$\begin{aligned}
& \|\langle \Lambda^c \rangle^\varepsilon [\tilde{\Delta}_{m'}^c, \mathcal{A}_m^c] f\|_{\ell^2(\mathcal{V}^c, m^c)} = \sum_{x \in \mathcal{V}^c} m^c(x) |\langle \Lambda^c \rangle^\varepsilon [\tilde{\Delta}_{m'}^c, \mathcal{A}_m^c] f(x)|^2 \\
&= \sum_{x \in \mathcal{V}^c} m^c(x) \left| \frac{i}{2} (e^{1/2}(Q-1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp) \right. \\
&\quad \times \left. \left(\frac{1}{m^c(x)} \sum_{z \sim x} \mathcal{E}^c(x, z) \frac{1 + \theta^c(x, z)}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}} \right) \right. \\
(43) \quad & - \frac{1}{m^c(x_1 - 1, x_2)} \sum_{z \sim x} \mathcal{E}^c((x_1 - 1, x_2), z) \frac{1 + \theta^c((x_1 - 1, x_2), z)}{\sqrt{(1 + \varepsilon^c(x_1 - 1, x_2))(1 + \varepsilon^c(z))}} \\
&\quad \times f(x_1 - 1, x_2)|^2 \\
&+ \sum_{x \in \mathcal{V}^c} m^c(x) \left| \frac{i}{2} (e^{1/2}(Q-1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp) \right. \\
&\quad \times \left. \left(\frac{1}{m^c(x_1 - 1, x_2)} \sum_{z \sim x} \mathcal{E}^c((x_1 - 1, x_2), z) \frac{1 + \theta^c((x_1 - 1, x_2), z)}{\sqrt{(1 + \varepsilon^c(x_1 - 1, x_2))(1 + \varepsilon^c(z))}} f(z) \right) \right. \\
&\quad - \left. \frac{1}{m^c(x)} \sum_{z \sim x} \mathcal{E}^c(x, z) \frac{1 + \theta^c(x, z)}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}} f(z_1 - 1, z_2) \right|^2 \\
&+ \sum_{x \in \mathcal{V}^c} m^c(x) \left| \frac{i}{2} (e^{-1/2}(Q+1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp) \right. \\
&\quad \times \left. \left(\frac{1}{m^c(x_1 + 1, x_2)} \sum_{z \sim x} \mathcal{E}^c((x_1 + 1, x_2), z) \frac{1 + \theta^c((x_1 + 1, x_2), z)}{\sqrt{(1 + \varepsilon^c(x_1 + 1, x_2))(1 + \varepsilon^c(z))}} \right) \right. \\
&\quad - \left. \frac{1}{m^c(x)} \sum_{z \sim x} \mathcal{E}^c(x, z) \frac{1 + \theta^c(x, z)}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}} f(x_1 + 1, x_2) \right|^2 \\
&+ \sum_{x \in \mathcal{V}^c} m^c(x) \left| \frac{i}{2} (e^{-1/2}(Q+1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp) \right. \\
&\quad \times \left. \left(\frac{1}{m^c(x)} \sum_{z \sim x} \mathcal{E}^c(x, z) \frac{1 + \theta^c(x, z)}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}} f(z_1 + 1, z_2) \right) \right. \\
&\quad - \left. \frac{1}{m^c(x_1 + 1, x_2)} \sum_{z \sim x} \mathcal{E}^c((x_1 + 1, x_2), z) \frac{1 + \theta^c((x_1 + 1, x_2), z)}{\sqrt{(1 + \varepsilon^c(x_1 + 1, x_2))(1 + \varepsilon^c(z))}} f(z_1, z_2) \right|^2.
\end{aligned}$$

We treat the first term of $\|\langle \Lambda^c \rangle^\varepsilon [\tilde{\Delta}_{m'}^c, \mathcal{A}_m^c] f\|_{\ell^2(\mathcal{V}^c, m^c)}$ in (43)

$$\begin{aligned}
& \sum_{x \in \mathcal{V}^c} m^c(x) \left| \frac{i}{2} \left(e^{1/2}(Q-1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp \right) \right. \\
&\quad \times \sum_{z \sim x} \left(\frac{\mathcal{E}^c(x, z)(1 + \theta^c(x, z))}{m^c(x) \sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}} \right. \\
&\quad - \left. \frac{\mathcal{E}^c((x_1 - 1, x_2), z)(1 + \theta^c((x_1 - 1, x_2), z))}{m^c(x_1 - 1, x_2) \sqrt{(1 + \varepsilon^c(x_1 - 1, x_2))(1 + \varepsilon^c(z))}} \right) f(x_1 - 1, x_2) \Big|^2 \\
&\leq 2 \sum_{x \in \mathcal{V}^c} m^c(x) \left| \frac{i}{2} \left(e^{1/2}(Q-1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp \right) \right. \\
(44) \quad & \times \sum_{z_1 \sim x_1} \delta_{z_2=x_2} \left(\frac{\mathcal{E}_1^c(x_1, z_1)(1 + \theta^c(x, z))}{m(x) \sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}} \right. \\
&\quad - \left. \frac{\mathcal{E}_1^c(x_1 - 1, z_1)(1 + \theta^c((x_1 - 1, x_2), z))}{m^c(x_1 - 1, x_2) \sqrt{(1 + \varepsilon^c(x_1 - 1, x_2))(1 + \varepsilon^c(z))}} \right) f(x_1 - 1, x_2) \Big|^2
\end{aligned}$$

$$\begin{aligned}
& + 2 \sum_{x \in \mathcal{V}^c} m^c(x) \Big| \frac{i}{2} \left(e^{1/2} (Q - 1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp \right) \\
(45) \quad & \times \sum_{\substack{z_1=x_1 \\ z_2 \sim x_2}} \delta_{z_1=x_1} \left(\frac{\mathcal{E}_2(x_2, z_2)(1 + \theta^c(x, z))}{m(x) \sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}} \right. \\
& \left. - \frac{\mathcal{E}_2(x_2, z_2)(1 + \theta^c((x_1 - 1, x_2), z))}{m_1^c(x_1 - 1) m_2(x_2) \sqrt{(1 + \varepsilon^c(x_1 - 1, x_2))(1 + \varepsilon^c(z))}} \right) f(x_1 - 1, x_2) \Big|^2.
\end{aligned}$$

We treat (44).

$$\begin{aligned}
& 2 \sum_{x \in \mathcal{V}^c} m^c(x) \Big| \frac{i}{2} \left(e^{1/2} (Q - 1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp \right) \\
& \quad \times \sum_{\substack{z_2=x_2 \\ z_1 \sim x_1}} \delta_{z_2=x_2} \left(\frac{\mathcal{E}_1^c(x_1, z_1)(1 + \theta^c(x, z))}{m_1(x_1) m_2(x_2) \sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(z))}} \right. \\
& \quad \left. - \frac{\mathcal{E}_1^c(x_1 - 1, z_1)(1 + \theta^c((x_1 - 1, x_2), z))}{m_1^c(x_1 - 1) m_2(x_2) \sqrt{(1 + \varepsilon^c(x_1 - 1, x_2))(1 + \varepsilon^c(z))}} \right) f(x_1 - 1, x_2) \Big|^2 \\
& \leq 2 \sum_{x \in \mathcal{V}^c} m^c(x) \Big| \frac{i}{2} \left(e^{1/2} (Q - 1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp \right) \\
& \quad \times \left(\frac{\mathcal{E}_1^c(x_1, x_1 + 1)(1 + \theta^c(x, (x_1 + 1, x_2)))}{m_1^c(x_1) m_2(x_2) \sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(x_1 + 1, x_2))}} \right. \\
& \quad - \frac{\mathcal{E}_1^c(x_1 - 1, x_1)(1 + \theta^c((x_1 - 1, x_2), (x_1, x_2)))}{m_1^c(x_1 - 1) m_2(x_2) \sqrt{(1 + \varepsilon^c(x_1 - 1, x_2))(1 + \varepsilon^c(x))}} \\
& \quad + \frac{\mathcal{E}_1^c(x_1, x_1 - 1)(1 + \theta^c(x, (x_1 - 1, x_2)))}{m_1(x_1) m_2(x_2) \sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(x_1 - 1, x_2))}} \\
& \quad \left. - \frac{\mathcal{E}_1^c(x_1 - 1, x_1 - 2)(1 + \theta^c((x_1 - 1, x_2), (x_1 - 2, x_2)))}{m_1(x_1 - 1) m_2(x_2) \sqrt{(1 + \varepsilon^c(x_1 - 1, x_2))(1 + \varepsilon^c(x_1 - 2, x_2))}} \right) f(x_1 - 1, x_2) \Big|^2 \\
& \leq 2 \sum_{x \in \mathcal{V}^c} m^c(x) \Big| \frac{i}{2} \left(e^{1/2} (Q - 1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp \right) \\
& \quad \times \frac{1}{m_2(x_2)} \left(\frac{e^{1/2}(1 + \theta^c(x, (x_1 + 1, x_2)))}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(x_1 + 1, x_2))}} \right. \\
& \quad - \frac{e^{1/2}(1 + \theta^c((x_1 - 1, x_2), (x_1, x_2)))}{\sqrt{(1 + \varepsilon^c(x_1 - 1, x_2))(1 + \varepsilon^c(x))}} + \frac{e^{-1/2}(1 + \theta^c(x, (x_1 - 1, x_2)))}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(x_1 - 1, x_2))}} \\
& \quad \left. - \frac{e^{-1/2}(1 + \theta^c((x_1 - 1, x_2), (x_1 - 2, x_2)))}{\sqrt{(1 + \varepsilon^c(x_1 - 1, x_2))(1 + \varepsilon^c(x_1 - 2, x_2))}} \right) f(x_1 - 1, x_2) \Big|^2 \\
& \leq 4 \sum_{x \in \mathcal{V}^c} m^c(x) \Big| \frac{i}{2} \left(e(Q - 1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp \right) \\
(46) \quad & \quad \times \frac{1}{m_2(x_2)} \left(\frac{\sqrt{1 + \varepsilon^c(x_1 - 1, x_2)}(1 + \theta^c(x, (x_1 + 1, x_2)))}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(x_1 + 1, x_2))(1 + \varepsilon^c(x_1 - 1, x_2))}} \right. \\
& \quad \left. - \frac{\sqrt{1 + \varepsilon^c(x_1 + 1, x_2)}(1 + \theta^c((x_1 - 1, x_2), (x_1, x_2)))}{\sqrt{(1 + \varepsilon^c(x_1 - 1, x_2))(1 + \varepsilon^c(x))(1 + \varepsilon^c(x_1 + 1, x_2))}} \right) f(x_1 - 1, x_2) \Big|^2
\end{aligned}$$

$$\begin{aligned}
& + 4 \sum_{x \in \mathcal{V}^c} m^c(x) \left| \frac{i}{2} \left((Q - 1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp \right) \right. \\
& \quad \times \frac{1}{m_2(x_2)} \left(\frac{\sqrt{1 + \varepsilon^c(x_1 - 2, x_2)}(1 + \theta^c(x, (x_1 - 1, x_2)))}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(x_1 - 1, x_2))(1 + \varepsilon^c(x_1 - 2, x_2))}} \right. \\
(47) \quad & \quad \left. - \frac{\sqrt{1 + \varepsilon^c(x_1 - 1, x_2)}(1 + \theta^c((x_1 - 1, x_2), (x_1 - 2, x_2)))}{\sqrt{(1 + \varepsilon^c(x_1 - 1, x_2))(1 + \varepsilon^c(x_1 - 2, x_2))(1 + \varepsilon^c(x_1 - 1, x_2))}} \right) \\
& \quad \left. \times f(x_1 - 1, x_2) \right|^2,
\end{aligned}$$

and in the same way, we deal with (45).

Now, we concentrate on (46), since the assertions (41) and (42) hold true then there exist an integer c , such that

$$\begin{aligned}
& 4 \sum_{x \in \mathcal{V}^c} m^c(x) \left| \frac{i}{2} \left(e(Q - 1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp \right) \right. \\
& \quad \times \frac{1}{m_2(x_2)} \left(\frac{\sqrt{1 + \varepsilon^c(x_1 - 1, x_2)}(1 + \theta^c(x, (x_1 + 1, x_2)))}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(x_1 + 1, x_2))(1 + \varepsilon^c(x_1 - 1, x_2))}} \right. \\
& \quad \left. - \frac{\sqrt{1 + \varepsilon^c(x_1 + 1, x_2)}(1 + \theta^c((x_1 - 1, x_2), (x_1, x_2)))}{\sqrt{(1 + \varepsilon^c(x_1 - 1, x_2))(1 + \varepsilon^c(x))(1 + \varepsilon^c(x_1 + 1, x_2))}} \right) f(x_1 - 1, x_2) \Big|^2 \\
& = 4 \sum_{x \in \mathcal{V}^c} m^c(x) \left| \frac{i}{2} \left(e(Q - 1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp \right) \right. \\
& \quad \times \frac{1}{m_2(x_2)} \left(\frac{(\sqrt{1 + \varepsilon^c(x_1 - 1, x_2)} - \sqrt{1 + \varepsilon^c(x_1 + 1, x_2)})\theta^c(x, (x_1 + 1, x_2))}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(x_1 + 1, x_2))(1 + \varepsilon^c(x_1 - 1, x_2))}} \right. \\
& \quad + \frac{\sqrt{1 + \varepsilon^c(x_1 + 1, x_2)}(\theta^c(x, (x_1 + 1, x_2)) - \theta^c((x_1 - 1, x_2), x))}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(x_1 + 1, x_2))(1 + \varepsilon^c(x_1 - 1, x_2))}} \\
& \quad \left. + \frac{\sqrt{1 + \varepsilon^c(x_1 - 1, x_2)} - \sqrt{1 + \varepsilon^c(x_1 + 1, x_2)}}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(x_1 + 1, x_2))(1 + \varepsilon^c(x_1 - 1, x_2))}} \right) f(x_1 - 1, x_2) \Big|^2 \\
& = 4 \sum_{x \in \mathcal{V}^c} m^c(x) \left| \frac{i}{2} \left(e(Q - 1/2)^{1+\varepsilon} \otimes P_{\ker(\Delta_2)}^\perp \right) \frac{1}{m_2(x_2)} \right. \\
& \quad \times \left(\frac{(\varepsilon^c(x_1 - 1, x_2) - \varepsilon^c(x_1 + 1, x_2))\theta^c(x, (x_1 + 1, x_2))}{(\sqrt{1 + \varepsilon^c(x_1 - 1, x_2)} + \sqrt{1 + \varepsilon^c(x_1 + 1, x_2)})} \right. \\
& \quad \times \frac{1}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(x_1 + 1, x_2))(1 + \varepsilon^c(x_1 - 1, x_2))}} \\
& \quad + \frac{\sqrt{1 + \varepsilon^c(x_1 + 1, x_2)}(\theta^c(x, (x_1 + 1, x_2)) - \theta^c((x_1 - 1, x_2), x))}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(x_1 + 1, x_2))(1 + \varepsilon^c(x_1 - 1, x_2))}} \\
& \quad + \frac{\varepsilon^c(x_1 - 1, x_2) - \varepsilon^c(x_1 + 1, x_2)}{(\sqrt{1 + \varepsilon^c(x_1 - 1, x_2)} + \sqrt{1 + \varepsilon^c(x_1 + 1, x_2)})} \\
& \quad \left. \times \frac{1}{\sqrt{(1 + \varepsilon^c(x))(1 + \varepsilon^c(x_1 + 1, x_2))(1 + \varepsilon^c(x_1 - 1, x_2))}} \right) f(x_1 - 1, x_2) \Big|^2 \\
& \leq c \|f\|_{\ell^2(\mathcal{V}^c, m^c)}^2,
\end{aligned}$$

and in the same way, we treat (47). By density, we have proven that there exists $c > 0$ such that $\|(\Lambda^c)^\varepsilon[\tilde{\Delta}_{m'}^c, \mathcal{A}_m^c]f\|_{\ell^2(\mathcal{V}^c, m^c)}^2 \leq c \|f\|_{\ell^2(\mathcal{V}^c, m^c)}^2$. Finally, by applying [AmBoGe, Proposition 7.5.7] where the hypotheses are verified in Proposition 4.10, we find the result. \square

We turn to the regularity of $V^c(Q)$.

Lemma 4.20. *Let $V^c : \mathcal{V}^c \rightarrow \mathbb{R}$ be a function, obeying V^c does not depend on the second variable and $\| [V^c, U] \langle Q + 1/2 \rangle^{1+\varepsilon} \| < \infty$, then $[V^c(Q), \mathcal{A}_{\mathcal{G}^c}] \in \mathcal{C}^{0,1}(\mathcal{A}^c)$. In particular, $V^c(Q) \in \mathcal{C}^{1,1}(\mathcal{A}^c)$.*

Proof. We note that $\mathcal{V}^c := \mathcal{V}^c \otimes \mathbf{1}_{\mathcal{V}^c}$.

First, we have

$$\begin{aligned} \| [V^c, U] \langle Q + 1/2 \rangle^{1+\varepsilon} \| &= \| \langle Q + 1/2 \rangle^{1+\varepsilon} [V^c, U^*] \| \\ &= \| \langle Q + 1/2 \rangle^{1+\varepsilon} (V^c U^* - U^* V^c) \| \\ &= \| \langle Q + 1/2 \rangle^{1+\varepsilon} V^c U^* - \langle Q + 1/2 \rangle^{1+\varepsilon} U^* V^c \| \\ &= \| V^c U^* \langle Q - 1 \rangle^{1+\varepsilon} - U^* V^c \langle Q - 1 \rangle^{1+\varepsilon} \| \\ &= \| [V^c, U^*] \langle Q - 1 \rangle^{1+\varepsilon} \|. \end{aligned}$$

Now, we shall show that

$$\| \langle Q + 1/2 \rangle^\varepsilon \otimes \mathbf{1}_{\mathcal{V}^c} [V^c(Q), i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}] \| < \infty.$$

We recall that

$$[V^c(Q), i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}] = \frac{e^{-1/2}}{2} \left(Q + \frac{1}{2} \right) [V^c, U^*] \otimes P_{\ker(\Delta_2)}^\perp + \frac{e^{1/2}}{2} \left(\frac{1}{2} - Q \right) [V^c, U] \otimes P_{\ker(\Delta_2)}^\perp.$$

For all $f \in \ell^2(\mathcal{V}^c, m^c)$, we obtain

$$\begin{aligned} \| \langle Q + 1/2 \rangle^\varepsilon \otimes \mathbf{1}_{\mathcal{V}^c} [V^c(Q), i\mathcal{A}_{\mathcal{G}^c}^{\text{le}}] f \| &= \| \langle \langle Q + 1/2 \rangle^\varepsilon \frac{e^{-1/2}}{2} \left(Q + \frac{1}{2} \right) [V^c, U^*] \\ &\quad + \langle Q + 1/2 \rangle^\varepsilon \frac{e^{1/2}}{2} \left(\frac{1}{2} - Q \right) [V^c, U] \rangle \otimes P_{\ker(\Delta_2)}^\perp f \| \\ &\leq \| \langle \langle Q + 1/2 \rangle^\varepsilon \frac{e^{-1/2}}{2} \left(Q + \frac{1}{2} \right) [V^c, U^*] \rangle \otimes P_{\ker(\Delta_2)}^\perp f \| \\ &\quad + \| \langle \langle Q + 1/2 \rangle^\varepsilon \frac{e^{1/2}}{2} \left(\frac{1}{2} - Q \right) [V^c, U] \rangle \otimes P_{\ker(\Delta_2)}^\perp f \| \\ &\leq \underbrace{\frac{e^{-1/2}}{2} \| \langle \langle Q + 1/2 \rangle^\varepsilon Q [V^c, U^*] \rangle \otimes P_{\ker(\Delta_2)}^\perp f \|}_{(1)} \\ &\quad + \underbrace{\frac{e^{-1/2}}{4} \| \langle \langle Q + 1/2 \rangle^\varepsilon [V^c, U^*] \rangle \otimes P_{\ker(\Delta_2)}^\perp f \|}_{(2)} \\ &\quad + \underbrace{\frac{e^{1/2}}{2} \| \langle \langle Q + 1/2 \rangle^\varepsilon Q [V^c, U] \rangle \otimes P_{\ker(\Delta_2)}^\perp f \|}_{(3)} \\ &\quad + \underbrace{\frac{e^{1/2}}{4} \| \langle \langle Q + 1/2 \rangle^\varepsilon [V^c, U] \rangle \otimes P_{\ker(\Delta_2)}^\perp f \|}_{(4)}. \end{aligned}$$

For all $f \in \ell^2(\mathcal{V}^c, m^c)$, we have

(1)

$$\begin{aligned}
& \frac{e^{-1/2}}{2} \left\| \left(\langle Q + 1/2 \rangle^\varepsilon Q [V^c, U^*] \right) \otimes P_{\ker(\Delta_2)}^\perp f \right\| \\
&= \frac{e^{-1/2}}{2} \left\| \left([V^c, U] \langle Q + 1/2 \rangle^\varepsilon Q \right) \otimes P_{\ker(\Delta_2)}^\perp f \right\| \\
&= \frac{e^{-1/2}}{2} \left\| \left([V^c, U] \langle Q + 1/2 \rangle^{1+\varepsilon} \underbrace{\langle Q + 1/2 \rangle^{-1} Q}_{\text{bounded}} \right) \otimes P_{\ker(\Delta_2)}^\perp f \right\|.
\end{aligned}$$

(2)

$$\begin{aligned}
& \frac{e^{-1/2}}{4} \left\| \left(\langle Q + 1/2 \rangle^\varepsilon [V^c, U^*] \right) \otimes P_{\ker(\Delta_2)}^\perp f \right\| \\
&= \frac{e^{-1/2}}{4} \left\| \left([V^c, U] \langle Q + 1/2 \rangle^\varepsilon \right) \otimes P_{\ker(\Delta_2)}^\perp f \right\| \\
&= \frac{e^{-1/2}}{4} \left\| \left([V^c, U] \langle Q + 1/2 \rangle^{1+\varepsilon} \underbrace{\langle Q + 1/2 \rangle^{-1}}_{\text{bounded}} \right) \otimes P_{\ker(\Delta_2)}^\perp f \right\|.
\end{aligned}$$

(3)

$$\begin{aligned}
& \frac{e^{1/2}}{2} \left\| \left(\langle Q + 1/2 \rangle^\varepsilon Q [V^c, U] \right) \otimes P_{\ker(\Delta_2)}^\perp f \right\| \\
&= \frac{e^{1/2}}{2} \left\| \left([V^c, U^*] \langle Q + 1/2 \rangle^\varepsilon Q \right) \otimes P_{\ker(\Delta_2)}^\perp f \right\| \\
&= \frac{e^{1/2}}{2} \left\| \left([V^c, U^*] \langle Q + 1/2 \rangle^{1+\varepsilon} \underbrace{\langle Q + 1/2 \rangle^{-1} Q}_{\text{bounded}} \right) \otimes P_{\ker(\Delta_2)}^\perp f \right\|.
\end{aligned}$$

(4)

$$\begin{aligned}
& \frac{e^{1/2}}{4} \left\| \left(\langle Q + 1/2 \rangle^\varepsilon [V^c, U] \right) \otimes P_{\ker(\Delta_2)}^\perp f \right\| \\
&= \frac{e^{1/2}}{4} \left\| \left([V^c, U^*] \langle Q + 1/2 \rangle^\varepsilon \right) \otimes P_{\ker(\Delta_2)}^\perp f \right\| \\
&= \frac{e^{1/2}}{4} \left\| \left([V^c, U^*] \langle Q + 1/2 \rangle^{1+\varepsilon} \underbrace{\langle Q + 1/2 \rangle^{-1}}_{\text{bounded}} \right) \otimes P_{\ker(\Delta_2)}^\perp f \right\|.
\end{aligned}$$

So, we can apply [AmBoGe, Proposition 7.5.7] where the hypotheses are verified in Proposition 4.17, the result follows. \square

We end this section by proving the result 4.16.

Proof of Theorem 4.16. The Proposition 4.19, Proposition 4.18 and Lemma 4.20 gives that $\Delta_{\mathcal{G}'} \in \mathcal{C}^{1,1}(\mathcal{A}_m^c)$. Since $\tilde{\Delta}_{m'}^c - \Delta_{\mathcal{G}^c}$ is a compact operator by Proposition 4.14, thanks to (20) and by [AmBoGe, Theorem 7.2.9] we obtain (39). \square

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