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FROM NON-COMMUTATIVE DIAGRAMS TO ANTI-ELEMENTARY CLASSES

FRIEDRICH WEHRUNG

ABSTRACT. *Anti-elementarity* is a strong way of ensuring that a class of structures, in a given first-order language, is not closed under elementary equivalence with respect to any infinitary language of the form $\mathcal{L}_{\infty\lambda}$. We prove that many naturally defined classes are anti-elementary, including the following:

- the class of all lattices of finitely generated convex ℓ -subgroups of members of any class of ℓ -groups containing all Archimedean ℓ -groups;
- the class of all semilattices of finitely generated ℓ -ideals of members of any nontrivial quasivariety of ℓ -groups;
- the class of all Stone duals of spectra of MV-algebras — this yields a negative solution for the *MV-spectrum Problem*;
- the class of all semilattices of finitely generated two-sided ideals of rings;
- the class of all semilattices of finitely generated submodules of modules;
- the class of all monoids encoding the nonstable K_0 -theory of von Neumann regular rings, respectively C^* -algebras of real rank zero;
- (assuming arbitrarily large Erdős cardinals) the class of all coordinatizable sectionally complemented modular lattices with a large 4-frame.

The main underlying principle is that under quite general conditions, for a functor $\Phi: \mathcal{A} \rightarrow \mathcal{B}$, if there exists a non-commutative diagram \vec{D} of \mathcal{A} , indexed by a common sort of poset called an *almost join-semilattice*, such that

- $\Phi\vec{D}^I$ is a commutative diagram for every set I ,
- $\Phi\vec{D} \not\cong \Phi\vec{X}$ for any commutative diagram \vec{X} in \mathcal{A} ,

then the range of Φ is anti-elementary.

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1. INTRODUCTION

The present paper is an extension of a negative solution of Problem 2 in Mundici's monograph [28], sometimes called the *MV-spectrum Problem*, stated as “which topological spaces are homeomorphic to the spectrum of some MV-algebra?”. Due to the generality of the methods involved, much more came out than expected.

Formalizing the MV-spectrum problem in terms of the Stone duals of the topological spaces in question, and involving the categorical equivalence between MV-algebras and Abelian ℓ -groups with order-unit established in Mundici [27], the MV-spectrum Problem can be recast in terms of the classical problem, dating back to the seventies, of characterizing the lattices of principal ℓ -ideals of Abelian ℓ -groups. Although the author characterized the *countable* such lattices by a first-order statement called *complete normality* (cf. Wehrung [38]), it was not even known, in the uncountable case, whether those lattices could be characterized by any class of $\mathcal{L}_{\infty\lambda}$ sentences of lattice theory, for some large enough cardinal number λ .

The present paper's results imply that it cannot be so (cf. Corollary 12.10).

For any set Ω , denote by $\mathfrak{P}_{\text{inj}}(\Omega)$ the category whose objects are all subsets of Ω and whose morphisms are all one-to-one maps $f: X \rightarrow Y$ where $X, Y \subseteq \Omega$, with $g \circ f$ defined iff the domain of g is equal to the codomain of f .

Definition 1.1. Let \mathcal{C}_0 and \mathcal{C}_1 be classes of objects in a category \mathcal{S} such that $\mathcal{C}_0 \subseteq \mathcal{C}_1$. The pair $(\mathcal{C}_0, \mathcal{C}_1)$ is *anti-elementary* if for any cardinal θ there are infinite cardinals λ and κ , with λ regular and $\theta \leq \lambda < \kappa$, together with a functor $\Gamma: \mathfrak{P}_{\text{inj}}(\kappa) \rightarrow \mathcal{S}$ preserving all λ -directed colimits, such that $\Gamma(\lambda) \in \mathcal{C}_0$ and $\Gamma(\kappa) \notin \mathcal{C}_1$. If $\mathcal{C}_0 = \mathcal{C}_1$ we then say that \mathcal{C}_0 is anti-elementary.

Obviously, if $(\mathcal{C}_0, \mathcal{C}_1)$ is anti-elementary, then every class \mathcal{C} such that $\mathcal{C}_0 \subseteq \mathcal{C} \subseteq \mathcal{C}_1$ is anti-elementary. Throughout the paper we will often meet situations where the \mathcal{C}_i are subcategories (as opposed to mere classes of objects) of \mathcal{S} , in which case anti-elementarity will of course be stated on their respective classes of objects.

Let us now relate that concept to elementary equivalence with respect to infinitary languages. For any first-order language Σ , denote by $\mathbf{Str} \Sigma$ the category of all models for Σ with Σ -homomorphisms. By an extension of Feferman's [11, Theorem 6], stated in Proposition 4.1, an anti-elementary class in $\mathbf{Str} \Sigma$ cannot be closed under $\mathcal{L}_{\infty\lambda}$ -elementary equivalence, for any infinite regular cardinal λ . In particular, it is not the class of models of any class of $\mathcal{L}_{\infty\lambda}$ sentences. However, anti-elementarity also implies further forms of failure of elementarity. For example, if $\kappa = \lambda^{+2}$ witnesses the anti-elementarity of \mathcal{C} (this will be the case throughout Sections 12–14), then the least cardinal μ such that $\Gamma(\mu) \notin \mathcal{C}$ is either λ^+ or λ^{+2} , thus (using Proposition 4.1) $(\Gamma(\xi) \mid \lambda \leq \xi < \mu)$ is an $\mathcal{L}_{\infty\lambda}$ -elementary chain, of length the regular cardinal $\mu > \lambda$, in \mathcal{C} , whose union $\Gamma(\mu)$ does not belong to \mathcal{C} .

The bulk of the present paper is devoted to developing techniques enabling us to prove anti-elementarity for many classes:

- The part “ $\Gamma(\lambda) \in \mathcal{C}_0$ ” is taken care of by two new results, which we call the Uniformization Lemma (Lemma 7.2) and the Boosting Lemma (Lemma 7.3).
- The part “ $\Gamma(\kappa) \notin \mathcal{C}_1$ ” is taken care of by Lemmas 11.1 and 11.2, which are extensions, to the case of non-commutative diagrams, of the original Armature Lemma and Condensate Lifting Lemma (CLL) established in Gillibert and Wehrung [14].
- Lemmas 11.1 and 11.2 entail a collection of results of which the underlying principle can, loosely speaking, be paraphrased as follows.

Under quite general conditions, for a functor $\Phi: \mathcal{A} \rightarrow \mathcal{B}$, if there exists a poset-indexed (necessarily non-commutative) diagram \vec{D} of \mathcal{A} such that

- $\Phi \vec{D}^I$ is a commutative diagram for every set I ,
- $\Phi \vec{D} \not\cong \Phi \vec{X}$ for any commutative diagram \vec{X} in \mathcal{A} ,

then the range of Φ is anti-elementary.

The class of posets indexing the diagrams \vec{D} above needs to be restricted to the *almost join-semilattices* introduced in Gillibert and Wehrung [14] (cf. Section 9). Every join-semilattice is an almost join-semilattice.

Corollary 12.10 states that the class of all Stone duals of MV-spectra is anti-elementary; in particular, it cannot be the class of all models of any class of $\mathcal{L}_{\infty\lambda}$ sentences.

The main problem consists of determining the functor Γ witnessing anti-elementarity. This is quite a difficult task, whose framework is mostly categorical and which will take up most of Sections 3–11. The functor Γ will arise from constructions called Φ -condensates, denoted in the form $\mathbf{A} \otimes_{\Phi}^{\lambda} \vec{S}$. In that notation,

- \mathbf{A} is a *P-scaled Boolean algebra* (cf. Section 5), that is, a Boolean algebra, together with a collection of ideals (subjected to certain conditions) indexed by a poset P ;
- λ is an infinite regular cardinal;
- Φ is a functor from a category \mathcal{S} to a category \mathcal{T} ;

- \vec{S} is a P -indexed diagram in \mathcal{S} . In contrast to the general situation in Gillibert and Wehrung [14], the diagram \vec{S} will only satisfy a weak form of commutativity called Φ -commutativity (cf. Definition 6.6).

The object $\mathbf{A} \otimes_{\Phi}^{\lambda} \vec{S}$ is first defined over λ -presented P -scaled Boolean algebras \mathbf{A} (under additional assumptions on P if $\lambda > \omega$); in that case, $\mathbf{A} \otimes_{\Phi}^{\lambda} \vec{S} = \Phi(\mathbf{A} \otimes^{\lambda} \vec{S})$, where $\mathbf{A} \otimes^{\lambda} \vec{S}$ is a new construct (cf. Definition 6.4), agreeing with the $\mathbf{A} \otimes \vec{S}$ from Gillibert and Wehrung [14] if $\lambda = \omega$ and \vec{S} is a commutative diagram.

An important difference between $_ \otimes^{\lambda} \vec{S}$ and $_ \otimes \vec{S}$ is that, due to the non-commutativity of the diagram \vec{S} , the former no longer sends morphisms to morphisms: for a morphism φ , $\varphi \otimes^{\lambda} \vec{S}$ is now a nonempty set of morphisms (as opposed to a single morphism). This is where the abovementioned Φ -commutativity of \vec{S} comes into play: if that assumption holds, then the value taken by the functor Φ on all morphisms in $\varphi \otimes^{\lambda} \vec{S}$ is constant, and then naturally defined as $\varphi \otimes_{\Phi}^{\lambda} \vec{S}$.

In the general case, $\mathbf{A} \otimes_{\Phi}^{\lambda} \vec{S}$ is then defined as the λ -directed colimit of the $\mathbf{U} \otimes_{\Phi}^{\lambda} \vec{S}$, formed over λ -presented substructures \mathbf{U} of \mathbf{A} (cf. Definition 6.9).

The combinatorial background for our results mostly rests on the concept of *lifter* introduced in Gillibert and Wehrung [14], which itself rests on infinitary Ramsey-type statements based on the classical relation $(\kappa, r, \lambda) \rightarrow \rho$ (cf. Erdős *et al.* [10]). The lifters X in question are involved in the construction of certain P -scaled Boolean algebras, introduced in [14] and denoted as $\mathbf{F}(X)$ (cf. Section 9). As in [14], those lifters will be crucial in both formulations of the extended Armature Lemma and CLL (cf. Lemmas 11.1 and 11.2).

The last layer of our construction, of the functor Γ , will consist of making the λ -lifter X sufficiently functorial in a cardinal $\kappa > \lambda$ defined by a certain infinite combinatorial property. This is where *standard lifters* will come into play (cf. Definition 10.4), enabling us to write the desired lifters in the form $P(\kappa)$.

The outline above can thus be condensed into the single formula

$$\Gamma(U) = \mathbf{F}(P(U)) \otimes_{\Phi}^{\lambda} \vec{S}, \quad \text{for every } U \subseteq \kappa.$$

Due to the generality of the underlying framework, our results will enable us to establish the anti-elementarity of many further classes of algebraic structures. A sample of such classes runs as follows:

- (1) (Corollary 12.4) *The class of all lattices of finitely generated convex subgroups of the members of any class of ℓ -groups containing all Archimedean ℓ -groups.*
- (2) (Corollary 13.10) *The class of all semilattices of finitely generated ℓ -ideals of the objects of any full subcategory of the category of all ℓ -groups and ℓ -homomorphisms, closed under products and colimits indexed by all large enough regular cardinals, containing all Archimedean ℓ -groups.*
- (3) (Corollary 13.16) *The class of all semilattices of finitely generated two-sided ideals of the members of any class of unital rings containing all unital locally matricial algebras over a given field.*
- (4) (Corollary 13.16) *The class of all semilattices of finitely generated submodules of right modules.*
- (5) (Corollaries 14.7 and 14.8) *The class of all monoids encoding the nonstable K_0 -theory of unital von Neumann regular rings (resp., unital C^* -algebras of real rank zero).*

- (6) (*Corollary 15.10, assuming the existence of arbitrarily large Erdős cardinals*)
The class of all coordinatizable sectionally complemented modular lattices with a large 4-frame.

Define a *projective class* (or *PC*) within $\mathcal{L}_{\kappa\lambda}$, on a first-order language Σ , as the class of Σ -reducts of the class of all models of some $\mathcal{L}_{\kappa\lambda}$ sentence in a language containing Σ . It is worth observing that the classes of algebras considered in (1)–(6) above, which we prove to be anti-elementary, are usually PCs within $\mathcal{L}_{\infty\omega}$, thus lending some optimality to our anti-elementarity results. For example, the class \mathcal{C} , of all lattices of the form $\text{Id}_c G \stackrel{\text{def}}{=} \text{lattice of all principal } \ell\text{-ideals of } G$, for some Abelian ℓ -group G , is a PC within $\mathcal{L}_{\omega_1\omega}$. To see this, observe that if an infinite lattice D belongs to \mathcal{C} , then it is isomorphic to $\text{Id}_c G$ for some Abelian ℓ -group G with the same cardinality as D ; in other words, there are an Abelian ℓ -group structure G on D and a surjective map $f: G^+ \twoheadrightarrow D$ such that

$$f(x) \leq_D f(y) \iff \left(x \leq_G \underbrace{y +_G \cdots +_G y}_{n \text{ times}} \text{ for some positive integer } n \right)$$

whenever $x, y \in G^+$. This is PC within $\mathcal{L}_{\omega_1\omega}$. Now the remaining collection of finite lattices is easily taken care of by an $\mathcal{L}_{\omega_1\omega}$ sentence.

All our anti-elementarity results, established in Sections 12–15, are obtained by combining our new techniques (Φ -condensates) with earlier non-representability results on the functors in question.

2. NOTATION AND TERMINOLOGY

2.1. Category theory. The categorical context of our paper is much related to the one of Gabriel and Ulmer [12], Adámek and Rosický [1].

For any class \mathcal{X} of objects in a category \mathcal{A} and any functor $\Phi: \mathcal{A} \rightarrow \mathcal{B}$, we denote by $\Phi\mathcal{X}$ (or $\Phi(\mathcal{X})$) the class of all objects B of \mathcal{B} such that $B \cong \Phi(X)$ for some $X \in \mathcal{X}$. If \mathcal{X} is the class of all objects of \mathcal{A} , $\Phi\mathcal{X}$ will be called the *range of Φ* and denoted by $\text{rng } \Phi$.

We denote by id_A the identity morphism on any object A in a given category.

- Monomorphisms (in any category) and one-to-one maps (between sets) will be denoted in the form $f: A \rightarrow B$.
- A category may come along with a special class of monomorphisms called *embeddings* (e.g., order-embeddings between posets), which will then be denoted in the form $f: A \hookrightarrow B$.
- Epimorphisms (in any category) and surjective maps (between sets) will be denoted in the form $f: A \twoheadrightarrow B$.

To any category \mathcal{C} , any object C of \mathcal{C} , and any full subcategory \mathcal{S} of \mathcal{C} , we assign

- the *slice category* \mathcal{S}/C , whose objects are all the arrows with domain in \mathcal{S} and codomain C , and where a morphism from an arrow $x: X \rightarrow C$ to an arrow $y: Y \rightarrow C$ is defined as an arrow $f: X \rightarrow Y$ such that $x = y \circ f$;
- the *subobject category* $\mathcal{S} \downarrow C$, whose objects are all the monomorphisms with domain in \mathcal{S} and codomain C , and where a morphism from an arrow $x: X \hookrightarrow C$ to an arrow $y: Y \hookrightarrow C$ is defined as the unique arrow (necessarily monic), if it exists, $f: X \hookrightarrow Y$ such that $x = y \circ f$; in that case we write $x \leq y$.

Natural transformations from a functor Φ to a functor Ψ will be denoted in the form $\eta: \Phi \rightarrow \Psi$.

2.2. Set theory. We denote by $\text{card } X$ the cardinality of a set X , and we say that X is λ -small (for a cardinal λ) if $\text{card } X < \lambda$. We denote by $\mathfrak{P}(X)$ the powerset of X , by $[X]^{<\lambda}$ the set of all λ -small subsets of X , and by $[X]^\lambda$ the set of all subsets of X with cardinality λ .

For sets X and Y with $X \subseteq Y$, we denote by id_X^Y the inclusion map from X into Y , and we set $\text{id}_X \stackrel{\text{def}}{=} \text{id}_X^X$.

We denote by $\text{dom}(f)$ (resp., $\text{rng}(f)$) the domain (resp., range) of a function f . For a subset X of the domain (resp., range) of f , we denote by $f[X]$ (resp., $f^{-1}[X]$, or $f^{-1}X$) the image (resp., inverse image) of X under f .

We denote by $\sup X$ the supremum of a set X of cardinal numbers.

For any cardinal number λ , we denote by λ^+ the successor cardinal of λ . Further, we set $\lambda^{+0} \stackrel{\text{def}}{=} \lambda$, $\lambda^{+(n+1)} \stackrel{\text{def}}{=} (\lambda^{+n})^+$ for every nonnegative integer n . We also set $\text{exp}_0(\lambda) \stackrel{\text{def}}{=} \lambda$, $\text{exp}_{n+1}(\lambda) \stackrel{\text{def}}{=} 2^{\text{exp}_n(\lambda)}$, $\text{exp}_n(\lambda) \stackrel{\text{def}}{=} \sup\{\text{exp}_n(\alpha) \mid \alpha < \lambda\}$.

2.3. Partially ordered sets (posets). Let P be a poset. We set $P^\infty \stackrel{\text{def}}{=} P \cup \{\infty\}$ for a new top element ∞ .

Let λ be an infinite regular cardinal. We say that P is

- λ -directed if every nonempty λ -small subset of P is bounded above in P ;
- directed if it is ω -directed;
- λ -join-complete if every nonempty λ -small subset of P has a join (i.e., supremum).

For a subset X and an element a in P , we set

$$X \downarrow a \stackrel{\text{def}}{=} \{x \in X \mid x \leq a\} \quad \text{and} \quad X \uparrow a \stackrel{\text{def}}{=} \{x \in X \mid x \geq a\}.$$

We say that P is a *forest* if $P \downarrow p$ is a chain whenever $p \in P$.

We say that a subset X of P is

- a *lower subset* (resp., an *upper subset*) of P if for every $x \in X$, $P \downarrow x \subseteq X$ (resp., $P \uparrow x \subseteq X$);
- an *ideal* of P if it is a nonempty directed lower subset of P ; we denote by $\text{Id } P$ the set of all ideals of P , partially ordered under set inclusion;
- *cofinal in* P if every element of P is bounded above by some element of X ;
- λ -closed in P , where λ is an infinite regular cardinal and P is λ -join-complete, if the join of any λ -small subset of X belongs to X .

It is well known that whenever $\lambda > \omega$ and P is λ -join-complete, *any intersection of a λ -small set of λ -closed cofinal subsets of P is λ -closed cofinal*. This observation will be mostly applied to finite products of sets of the form $[\Omega]^{<\lambda}$.

Many of our “counterexample diagrams” will be indexed by the powerset of a three-element set, denoted as $\mathfrak{P}[3] \stackrel{\text{def}}{=} \{\emptyset, 1, 2, 3, 12, 13, 23, 123\}$ (with $12 = 21$, $123 = 312$, and so on), partially ordered by inclusion.

A poset P is *lower λ -small* if $\text{card}(P \downarrow x) < \lambda$ whenever $x \in P$. We will just say *lower finite* instead of lower ω -small.

We denote by $\text{Min } P$ (resp., $\text{Max } P$) the set of all minimal (resp. maximal) elements of P , and we set

$$P^- \stackrel{\text{def}}{=} P \setminus \text{Min } P \quad \text{and} \quad P^= \stackrel{\text{def}}{=} P \setminus \text{Max } P.$$

For posets P and Q , a map $f: P \rightarrow Q$ is *isotone* (resp., *antitone*) if for all $x, y \in P$, $x \leq y$ implies that $f(x) \leq f(y)$ (resp., $f(y) \leq f(x)$). If $x \leq y$ is equivalent to $f(x) \leq f(y)$, we say that f is an *order-embedding*.

We refer the reader to Grätzer [17] for all undefined lattice-theoretical concepts.

2.4. Lattice-ordered groups. An ℓ -group is a group endowed with a translation-invariant lattice order (cf. Bigard *et al.* [6], Anderson and Feil [2]). Although our ℓ -groups will not be assumed to be Abelian, we will denote them additively. The *positive cone* of an ℓ -group G is $G^+ \stackrel{\text{def}}{=} \{x \in G \mid x \geq 0\}$. An ℓ -subgroup of G is a subgroup closed under the lattice operations, and an ℓ -ideal is an order-convex normal ℓ -subgroup. For any element x in an ℓ -group G , we denote by $\langle x \rangle$ (resp., $\langle x \rangle^\ell$) the convex ℓ -subgroup (resp., ℓ -ideal) of G generated by x .

The Stone dual of the spectrum of any Abelian ℓ -group G is isomorphic to the (distributive) lattice $\text{Id}_c G$ of all principal (equivalently, finitely generated) ℓ -ideals of G (cf. Proposition 1.19, together with Theorem 1.10 and Lemma 1.20, in Keimel [22]).

Whenever G is a totally ordered group and H is an ℓ -group, the *lexicographical product* $G \times_{\text{lex}} H$ is the ℓ -group structure on the cartesian product $G \times H$ with the positive cone

$$(G \times_{\text{lex}} H)^+ \stackrel{\text{def}}{=} \{(x, y) \in G \times H \mid x \geq 0 \text{ and } (x = 0 \Rightarrow y \geq 0)\}.$$

2.5. Rings. Our rings will not necessarily be unital. A ring R is

- *von Neumann regular* (cf. Goodearl [16] for the unital case) if for all $x \in R$ there exists $y \in R$ such that $x = xyx$;
- *semiprimitive* if its Jacobson radical (cf. Jacobson [20, § 2], Herstein [18, page 16]) is trivial;
- an *exchange ring* if for all $x \in R$ there are $r, s \in R$ together with an idempotent $e \in R$ such that $e = rx = x + s - sx$ (cf. Warfield [31], and Ara [3] for the non-unital case).

Also recall that a C^* -algebra is an exchange ring iff it has real rank zero (cf. Ara [3, Theorem 3.8]). For more information and references, see Wehrung [35].

2.6. Model theory for infinitary languages. Let Σ be a first-order language and let λ be an infinite regular cardinal. The formulas of $\mathcal{L}_{\infty\lambda}$ are constructed inductively from atomic formulas of Σ , by allowing quantifications over less than λ variables together with arbitrary conjunctions and disjunctions over sets of formulas with less than λ free variables altogether. The satisfaction of an $\mathcal{L}_{\infty\lambda}$ formula, in a Σ -structure M , is defined the standard way. Assignments of free variables will be written in the form $\vec{a} \subseteq M$; we should keep in mind that \vec{a} is in fact a function with λ -small domain and with values in M .

We refer to Dickmann [9] for more information on model theory for infinitary languages.

3. SETTING UP THE CATEGORICAL BACKGROUND

3.1. Non-commutative diagrams. The whole paper will be articulated around the concept of a (non-commutative) diagram introduced in Wehrung [39]. Let us recall the underlying definitions.

A (*commutative*) *diagram*, in a category \mathcal{S} , is often defined as a functor D from a category \mathcal{P} (the “indexing category” of the diagram) to \mathcal{S} . Allowing any morphism in \mathcal{P} to be sent to more than one morphism in \mathcal{S} , we get a more general definition of diagram that makes D a kind of “non-deterministic functor”.

Definition 3.1. Let \mathcal{P} and \mathcal{S} be categories. A \mathcal{P} -*indexed diagram* in \mathcal{S} is an assignment D , sending each element p of \mathcal{P} to an object $D(p)$ (or D_p) of \mathcal{S} and each morphism $x: p \rightarrow q$ in \mathcal{P} to a *nonempty set* $D(x)$ of morphisms from $D(p)$ to $D(q)$, in such a way that the following statements hold:

- (1) $\text{id}_{D(p)} \in D(\text{id}_p)$ for every object p of \mathcal{P} ;
- (2) whenever $x: p \rightarrow q$ and $y: q \rightarrow r$ are morphisms in \mathcal{P} , $u \in D(x)$, and $v \in D(y)$, the composite $v \circ u$ belongs to $D(y \circ x)$.

We say that D is a *commutative diagram* if $D(x)$ is a singleton whenever x is a morphism in \mathcal{P} . A *uniformization* of a diagram D is a \mathcal{P} -indexed *commutative diagram* D^* such that $D^*(x) \subseteq D(x)$ for every morphism x in \mathcal{P} .

We denote by $\mathcal{S}^{\mathcal{P}}$ the category of all functors from \mathcal{P} to \mathcal{S} with natural transformations as morphisms.

Specializing to the case where \mathcal{P} is the category naturally assigned to a poset P (i.e., there is an arrow from p to q iff $p \leq q$, and then the arrow is unique), we get the following definition from Wehrung [39].

Definition 3.2. Let P be a poset and let \mathcal{S} be a category. A P -*indexed diagram* in \mathcal{S} is an assignment D , sending each element p of P to an object $D(p)$ (or D_p) of \mathcal{S} and each pair (p, q) of elements of P , with $p \leq q$, to a *nonempty set* $D(p, q)$ of morphisms from $D(p)$ to $D(q)$, such that

- (1) $\text{id}_{D(p)} \in D(p, p)$ for every $p \in P$,
- (2) Whenever $p \leq q \leq r$ in P , $u \in D(p, q)$, and $v \in D(q, r)$, $v \circ u$ belongs to $D(p, r)$.

Hence D is a commutative diagram iff each $D(p, q)$, for $p \leq q$ in P , is a singleton.

We will often write poset-indexed commutative diagrams in the form

$$\vec{D} = (D_p, \delta_p^q \mid p \leq q \text{ in } P),$$

where all D_p are objects and all $\delta_p^q: D_p \rightarrow D_q$ are morphisms subjected to the usual commutation relations (i.e., $\delta_p^p = \text{id}_{D_p}$, $\delta_p^r = \delta_q^r \circ \delta_p^q$ whenever $p \leq q \leq r$); hence $\vec{D}(p, q) = \{\delta_p^q\}$. If P is a directed (resp., λ -directed, with λ an infinite regular cardinal) poset we will say that \vec{D} is a *direct system* (resp., *λ -direct system*).

Definition 3.3. Let I be a set, let \mathcal{S} be a category, let P be a poset, and let D be a P -indexed diagram in \mathcal{S} . We suppose that the product $D^I(p) \stackrel{\text{def}}{=} \prod_{i \in I} D(p_i)$ exists in \mathcal{S} for every $p = (p_i \mid i \in I) \in P^I$. Whenever $p = (p_i \mid i \in I)$ and $q = (q_i \mid i \in I)$ in P^I with $p \leq q$, let $D^I(p, q)$ consist of all morphisms in \mathcal{S} of the form $\prod_{i \in I} f_i: D^I(p) \rightarrow D^I(q)$ where each $f_i \in D(p_i, q_i)$.

It is straightforward to verify that the structure D^I introduced in Definition 3.3 is a P^I -indexed diagram (in the sense of Definition 3.2).

3.2. From finitely presented to λ -presented. Let us recall the following definition from Gabriel and Ulmer [12, Definition 6.1], see also Definitions 1.1 and 1.13 in Adámek and Rosický [1] or Gillibert and Wehrung [14, Definition 1.3.1].

Definition 3.4. Let λ be an infinite regular cardinal. An object A in a category \mathcal{A} is λ -presented if

- it is *weakly λ -presented*¹, that is, for every λ -directed colimit co-cone

$$(B, \beta_i \mid i \in I) = \varinjlim (B_i, \beta_i^j \mid i \leq j \text{ in } I) \quad \text{within } \mathcal{A},$$
 every morphism $\varphi: A \rightarrow B$ factors through some B_i , that is, $\varphi = \beta_i \circ \psi$ for some $\psi: A \rightarrow B_i$;
- for all $i \in I$ and all $\xi, \eta: A \rightarrow B_i$ such that $\beta_i \circ \xi = \beta_i \circ \eta$ there exists $j \in I$ such that $i \leq j$ and $\beta_i^j \circ \xi = \beta_i^j \circ \eta$.

Recall the following definition from Gabriel and Ulmer [12], Adámek and Rosický [1].

Definition 3.5. A category \mathcal{C} is λ -filtered if every subcategory of \mathcal{C} with less than λ morphisms has a compatible co-cone. If $\lambda = \omega$ we will just say *filtered* instead of ω -filtered.

Adámek and Rosický [1, Theorem 1.5] prove that every small filtered category \mathcal{C} admits a cofinal functor from a directed poset. They also observe in [1, Remark 1.21] that this result extends, with a similar proof, to λ -filtered categories:

Lemma 3.6. *For every small λ -filtered category \mathcal{C} , there are a λ -directed poset P and a cofinal functor from P to \mathcal{C} .*

Corollary 3.7. *A functor preserves λ -filtered colimits (indexed by small λ -filtered categories) iff it preserves λ -directed colimits (indexed by λ -directed posets).*

Next, we need to show how to extend the results of Gillibert and Wehrung [14, § 1.4], enabling us to extend a functor from λ -presented objects to all objects, from the finitely presented case to the λ -presented case. The technical result [14, Lemma 1.4.1] extends modulo the following changes:

- The category \mathcal{S} , instead of having all directed colimits, is required to have all λ -directed colimits.
- The A_i are all λ -presented.

The relevant analogue of Gillibert and Wehrung [14, Proposition 1.4.2] is then the following.

Lemma 3.8. *Let \mathcal{A}^\dagger be a full subcategory, consisting only of λ -presented objects, in a category \mathcal{A} and let \mathcal{S} be a category with all λ -directed colimits. We assume that every object of \mathcal{A} is a small λ -directed colimit of objects from \mathcal{A}^\dagger . Then every functor $\Psi: \mathcal{A}^\dagger \rightarrow \mathcal{S}$ extends to a unique (up to natural isomorphism) functor $\overline{\Psi}: \mathcal{A} \rightarrow \mathcal{S}$ which preserves λ -directed colimits from \mathcal{A}^\dagger . Furthermore, if \mathcal{A}^\dagger has small hom-sets, then $\overline{\Psi}$ preserves λ -directed colimits from \mathcal{A} .*

Necessarily, if an object A of \mathcal{A} is expressed as a λ -directed colimit $A = \varinjlim_{i \in I} A_i$, with all $A_i \in \mathcal{A}^\dagger$, then $\overline{\Psi}(A) = \varinjlim_{i \in I} \Psi(A_i)$. One of the main difficulties of the argument is to prove that $\varinjlim_{i \in I} \Psi(A_i)$ is, up to isomorphism, independent of the chosen λ -direct system based on the A_i . The proof of Lemma 3.8 is essentially the same as the one of Gillibert and Wehrung [14, Proposition 1.4.2], with the following changes:

¹Although formally different, the definition of “weakly λ -presented” introduced in Gillibert and Wehrung [14] will be equivalent to ours in all contexts occurring in the present paper.

- Change “directed” to “ λ -directed” and “directed colimits” to “ λ -directed colimits”.
- Restate the Claim on page 30 as “ \mathcal{P} is λ -filtered”. The proof remains almost identical.
- At the bottom of page 31, change the use of [1, Theorem 1.5] to [1, Remark 1.21] (cf. Lemma 3.6 above).

All along the present paper, Lemma 3.8 will be applied to the case where \mathcal{A} is the category \mathbf{Bool}_P of all P -scaled Boolean algebras (cf. Section 5), for a poset P , and \mathcal{A}^\dagger is the full subcategory of all λ -presented objects of \mathcal{A} , for an infinite regular cardinal λ (under additional conditions on P if $\lambda > \omega$).

4. A MODEL-FREE VERSION OF ELEMENTARY EMBEDDINGS

The following result is essentially a reformulation, in our context, of Feferman’s [11, Theorem 6] or Beke and Rosický’s [5, Proposition 2.14]. However, while the preservation of monomorphisms belongs to the assumptions of the two abovementioned results, it appears in our result as a conclusion (for monomorphisms with large enough domain). Also, our proof will not involve any back-and-forth argument.

Proposition 4.1. *Let λ be an infinite regular cardinal, let Σ be a first-order language, let Ω be a set, and let $\Gamma: \mathfrak{P}_{\text{inj}}(\Omega) \rightarrow \mathbf{Str} \Sigma$ be a functor such that*

$$\text{For any } Z \subseteq \Omega, \quad \Gamma(Z) = \varinjlim (\Gamma(T) \mid T \in [Z]^{<\lambda}) \quad (4.1)$$

(with the obvious transition morphisms). Then for every $f: X \rightarrow Y$ in $\mathfrak{P}_{\text{inj}}(\Omega)$ with $\text{card } X \geq \lambda$, $\Gamma(f)$ is an $\mathcal{L}_{\infty\lambda}$ -elementary embedding from $\Gamma(X)$ into $\Gamma(Y)$.

Note. It is easy to check that (4.1) is equivalent to saying that Γ preserves all λ -directed colimits.

Proof. We need to prove that $\Gamma(X) \models \varphi(\vec{a})$ iff $\Gamma(Y) \models \varphi(\Gamma(f)\vec{a})$, for every $\mathcal{L}_{\infty\lambda}$ formula φ of Σ and every assignment \vec{a} of the free variables of φ in $\Gamma(X)$. We argue by induction on the complexity of φ .

The only two nontrivial induction steps consist of proving that $\Gamma(Y) \models \varphi(\Gamma(f)\vec{a})$ implies that $\Gamma(X) \models \varphi(\vec{a})$, for any $\mathcal{L}_{\infty\lambda}$ formula φ which is either atomic or of the form $(\exists \vec{y})\psi$ for a formula ψ of smaller complexity.

Let us begin with the case where φ is atomic. From (4.1) it follows that there are $U \in [X]^{<\lambda}$ and $\vec{u} \subseteq \Gamma(U)$ such that $\vec{a} = \Gamma(\text{id}_U^X)\vec{u}$. Denoting by f_U the domain-range restriction of f from U onto $f[U]$ and setting $\vec{u}' \stackrel{\text{def}}{=} \Gamma(f_U)\vec{u}$, it follows from the equation $f \text{id}_U^X = \text{id}_{f[U]}^Y f_U$ that $\Gamma(f)\vec{a} = \Gamma(\text{id}_{f[U]}^Y)\vec{u}'$, whence $\Gamma(Y) \models \varphi(\Gamma(\text{id}_{f[U]}^Y)\vec{u}')$. By applying (4.1) to Y and by using the standard description of directed colimits within $\mathbf{Str} \Sigma$ (cf. Adámek and Rosický [1, § 5.1]), there exists $V \in [Y]^{<\lambda}$ containing $f[U]$ such that

$$\Gamma(V) \models \varphi(\Gamma(\text{id}_{f[U]}^V)\vec{u}'). \quad (4.2)$$

Since U and V are both λ -small and $\lambda \leq \text{card } X$, there exists a permutation σ of Y such that $\sigma \upharpoonright_{f[U]} = \text{id}_{f[U]}$ whereas $\sigma[V] \subseteq f[X]$. The latter implies the existence of a (unique) one-to-one map $g: V \rightarrow X$ such that $\sigma \text{id}_V^Y = fg$. Since φ is atomic and $\Gamma(g)$ is a Σ -homomorphism, (4.2) entails $\Gamma(X) \models \varphi(\Gamma(g \text{id}_{f[U]}^V)\vec{u}')$, that is,

$$\Gamma(X) \models \varphi(\Gamma(g \text{id}_{f[U]}^V f_U)\vec{u}). \quad (4.3)$$

Now $fg \text{id}_{f[U]}^V fU = \sigma \text{id}_V^Y \text{id}_{f[U]}^V fU = \sigma \text{id}_{f[U]}^Y fU = \text{id}_{f[U]}^Y fU = f \text{id}_U^X$, thus $g \text{id}_{f[U]}^V fU = \text{id}_U^X$, and thus, by (4.3), $\Gamma(X) \models \varphi(\Gamma(\text{id}_U^X)\vec{u})$, that is, $\Gamma(X) \models \varphi(\vec{a})$.

The case where $\varphi(\vec{x})$ is $(\exists \vec{y})\psi(\vec{x}, \vec{y})$, for a formula ψ with smaller complexity than φ , proceeds likewise. The assumption $\Gamma(Y) \models \varphi(\Gamma(f)\vec{a})$ means that there exists $\vec{c} \subseteq \Gamma(Y)$ such that

$$\Gamma(Y) \models \psi(\Gamma(f)\vec{a}, \vec{c}). \quad (4.4)$$

Since the assignments \vec{a} and \vec{c} are both λ -small and by (4.1), there are $U \in [X]^{<\lambda}$ and $V \in [Y]^{<\lambda}$ together with $\vec{u} \subseteq \Gamma(U)$ and $\vec{v} \subseteq \Gamma(V)$ such that $\vec{a} = \Gamma(\text{id}_U^X)\vec{u}$ and $\vec{c} = \Gamma(\text{id}_V^Y)\vec{v}$. We may further enlarge V in such a way that $f[U] \subseteq V$. Since $\text{card } V < \lambda \leq \text{card } X$, there exists a permutation σ of Y such that $\sigma \upharpoonright_{f[U]} = \text{id}_{f[U]}$ whereas $\sigma[V] \subseteq f[X]$. The former relation implies that $\sigma f \text{id}_U^X = f \text{id}_U^X$, whence

$$\Gamma(\sigma)\Gamma(f)\vec{a} = \Gamma(\sigma f \text{id}_U^X)\vec{u} = \Gamma(f \text{id}_U^X)\vec{u} = \Gamma(f)\vec{a}. \quad (4.5)$$

Again, there is $g: V \rightarrow X$ such that $\sigma \text{id}_V^Y = fg$. Setting $\vec{b} \stackrel{\text{def}}{=} \Gamma(g)\vec{v}$, it follows that

$$\Gamma(\sigma)\vec{c} = \Gamma(\sigma \text{id}_V^Y)\vec{v} = \Gamma(fg)\vec{v} = \Gamma(f)\Gamma(g)\vec{v} = \Gamma(f)\vec{b}. \quad (4.6)$$

By applying the automorphism $\Gamma(\sigma)$ to (4.4), we obtain the relation

$$\Gamma(Y) \models \psi(\Gamma(\sigma f)\vec{a}, \Gamma(\sigma)\vec{c}),$$

whence, by virtue of (4.5) and (4.6),

$$\Gamma(Y) \models \psi(\Gamma(f)\vec{a}, \Gamma(f)\vec{b}),$$

which, by the induction hypothesis, implies that

$$\Gamma(X) \models \psi(\vec{a}, \vec{b}),$$

and thus $\Gamma(X) \models \varphi(\vec{a})$, as desired. \square

Remark 4.2. The assumptions of Proposition 4.1 can be weakened, without any change in the proof, to the case where the arities of the operations and relations in Σ are only assumed to be λ -small (instead of finite). In fact it seems plausible, although we did not investigate this, that Proposition 4.1 could be extended further to the more categorical framework of admissible categories and abstract elementary classes (cf. Lieberman [25], Beke and Rosický [4, 5]).

5. P -SCALED BOOLEAN ALGEBRAS; NORMAL MORPHISMS

Let us first recall a few concepts from Gillibert and Wehrung [14]. For an arbitrary poset P , a P -scaled Boolean algebra is a structure

$$\mathbf{A} = (A, (A^{(p)} \mid p \in P)),$$

where A is a Boolean algebra, every $A^{(p)}$ is an ideal of A , $A = \bigvee (A^{(p)} \mid p \in P)$ within the ideal lattice of A , and $A^{(p)} \cap A^{(q)} = \bigvee (A^{(r)} \mid r \geq p, q)$ whenever $p, q \in P$. For P -scaled Boolean algebras \mathbf{A} and \mathbf{B} , a *morphism* from \mathbf{A} to \mathbf{B} is a homomorphism $f: A \rightarrow B$ of Boolean algebras such that $f[A^{(p)}] \subseteq B^{(p)}$ for every $p \in P$. If f is surjective and $f[A^{(p)}] = B^{(p)}$ for every p , we say that f is *normal*. The category of all P -scaled Boolean algebras is denoted by \mathbf{Bool}_P . It has all (small) directed colimits and all finite products. For a P -scaled Boolean algebra \mathbf{A} and an ideal I of A , with canonical projection $\pi_I: A \twoheadrightarrow A/I$, the *quotient algebra* \mathbf{A}/I has underlying Boolean algebra B/I and $(A/I)^{(p)} = \pi_I[A^{(p)}]$. Furthermore, π_I is a normal morphism and every normal morphism arises this way (cf. [14, § 2.5]).

For every $\mathbf{A} \in \mathbf{Bool}_P$, the ultrafilter space of A is denoted in Gillibert and Wehrung [14] by $\text{Ult } A$, and further, for every $\mathfrak{a} \in \text{Ult } A$, the subset

$$\|\mathfrak{a}\|_{\mathbf{A}} \stackrel{\text{def}}{=} \{p \in P \mid \mathfrak{a} \cap A^{(p)} \neq \emptyset\} \quad (5.1)$$

is an ideal of P (cf. Gillibert and Wehrung [14, Lemma 2.2.4]). The pair $(\text{Ult } A, \|_ \|_{\mathbf{A}})$ is a so-called *P -normed Boolean space* (cf. [14, § 2.2]).

In Gillibert and Wehrung [14] we approximated any P -scaled Boolean algebras from below by finitely presented P -scaled Boolean algebras. For uncountable cardinals our approach will be even more direct.

Definition 5.1. Let λ be an infinite regular cardinal and let P be a poset. We say that a P -scaled Boolean algebra \mathbf{A} is *λ -small* if $\text{card } P + \text{card } A < \lambda$. We denote by $\mathbf{Bool}_P^{<\lambda}$ the full subcategory of \mathbf{Bool}_P whose objects are the λ -small P -scaled Boolean algebras. (Hence $\mathbf{Bool}_P^{<\lambda}$ is nonempty iff P is λ -small.)

By Gillibert and Wehrung [14, Corollary 2.4.7], a P -scaled Boolean algebra \mathbf{A} is finitely presented within \mathbf{Bool}_P iff A is finite and for every atom a of A the ideal

$$\|a\|_{\mathbf{A}} \stackrel{\text{def}}{=} \{p \in P \mid a \in A^{(p)}\} \quad (5.2)$$

has a largest element, then denoted by $|a|_{\mathbf{A}}$. Accordingly, we shall denote by $\mathbf{Bool}_P^{\text{fin}}$ the full subcategory of \mathbf{Bool}_P consisting of all finitely presented objects. Observe that $\mathbf{Bool}_P^{<\omega}$ is contained in $\mathbf{Bool}_P^{\text{fin}}$, the equality holding iff P is finite. In this paper we will need the following sufficient condition for λ -presentability.

Lemma 5.2. *Let λ be an infinite regular cardinal, let P be a poset, and let \mathbf{A} be a P -scaled Boolean algebra. If \mathbf{A} is λ -small, then it is λ -presented within \mathbf{Bool}_P .*

Proof. A standard argument, part of which mimics the finitely presented case established in Gillibert and Wehrung [14, Lemma 2.4.3]. Let

$$(\mathbf{B}, \beta_i \mid i \in I) = \varinjlim \vec{\mathbf{B}},$$

where $\vec{\mathbf{B}} = (\mathbf{B}_i, \beta_i^j \mid i \leq j \text{ in } I)$ is a λ -direct system in \mathbf{Bool}_P . It follows from (the proof of) [14, Proposition 2.3.1] that

$$B = \bigcup (\beta_i[B_i] \mid i \in I) \quad (\lambda\text{-directed union}); \quad (5.3)$$

$$B^{(p)} = \bigcup (\beta_i[B_i^{(p)}] \mid i \in I), \quad \text{for each } p \in P \quad (\lambda\text{-directed union}); \quad (5.4)$$

$$\text{For all } i \in I \text{ and all } x, y \in B_i, \beta_i(x) \leq \beta_i(y) \Rightarrow (\exists j \geq i)(\beta_i^j(x) = \beta_i^j(y)). \quad (5.5)$$

We must prove that any morphism $\varphi: \mathbf{A} \rightarrow \mathbf{B}$ in \mathbf{Bool}_P factors through some \mathbf{B}_i in an essentially unique way. To that end, we set

$$\Gamma_i \stackrel{\text{def}}{=} \{(a, p) \in A \times P \mid \varphi(a) \in \beta_i[B_i^{(p)}]\}, \quad \text{for each } i \in I;$$

$$\Gamma \stackrel{\text{def}}{=} \{(a, p) \in A \times P \mid \varphi(a) \in B^{(p)}\}.$$

It follows from (5.4) that Γ is the directed union of the Γ_i for $i \in I$. Since Γ is a subset of the λ -small set $A \times P$ and since I is λ -directed, there exists $i \in I$ such that $\Gamma = \Gamma_i$. It follows that $\Gamma = \Gamma_k$ for any $k \geq i$.

Since $\text{card } A < \lambda$, A is a λ -presented Boolean algebra, thus, since I is λ -directed, there is $j \geq i$ such that φ factors, as a homomorphism of Boolean algebras, through B_j ; that is, there is a homomorphism $\psi: A \rightarrow B_j$ of Boolean algebras such that $\varphi = \beta_j \circ \psi$.

Now for all $p \in P$ and all $x \in A^{(p)}$, $\varphi(x)$ belongs to $B^{(p)}$, that is, $(x, p) \in \Gamma$, so $(x, p) \in \Gamma_i = \Gamma_j$, and so $(\beta_j \circ \psi)(x) = \varphi(x) \in \beta_j[B_j^{(p)}]$. By (5.5), it follows that $(\beta_j^{k_{p,x}} \circ \psi)(x) \in \beta_j^{k_{p,x}}[B_j^{(p)}]$ for some $k_{p,x} \geq j$. Since $P \times A^{(p)}$ is λ -small and I is λ -directed, the set $\{k_{p,x} \mid p \in P \text{ and } x \in A^{(p)}\}$ has an upper bound k in I . By definition, $(\beta_j^k \circ \psi)[A^{(p)}] \subseteq \beta_j^k[B_j^{(p)}]$. Therefore, φ factors, as a morphism in \mathbf{Bool}_P , through \mathbf{B}_k . This completes the proof that \mathbf{A} is weakly λ -presented in \mathbf{Bool}_P .

Since A is λ -presented as a Boolean algebra, \mathbf{A} satisfies automatically the second part of the definition of being λ -presented. \square

The information given by Lemma 5.2 will be completed in Corollary 5.5.

Definition 5.3. Let \mathbf{A} and \mathbf{B} be P -scaled Boolean algebras. We say that \mathbf{A} is an *induced subalgebra* of \mathbf{B} if A is a Boolean subalgebra of B and $A^{(p)} = A \cap B^{(p)}$ whenever $p \in P$.

In the context of Definition 5.3, \mathbf{A} is then characterized by its universe A , so we will often identify \mathbf{A} and A . The subsets of B giving rise to an induced subalgebra of \mathbf{B} are thus the Boolean subalgebras A of B such that $1 \in \bigvee(A \cap B^{(p)} \mid p \in P)$ and

$$A \cap B^{(p)} \cap B^{(q)} = \bigvee(A \cap B^{(r)} \mid r \geq p \text{ and } r \geq q) \quad \text{whenever } p, q \in P.$$

Our next lemma states that normal morphisms between P -scaled Boolean algebras can be approximated by normal morphisms between λ -small induced algebras. A similar statement, with “finitely presented” instead of “ λ -small”, requiring a less straightforward argument, is established in Gillibert and Wehrung [14, Proposition 2.5.5]. The proof of Lemma 5.4 is a straightforward Löwenheim-Skolem type argument, involving both the regularity of λ and the λ -smallness of P .

Lemma 5.4. *Let λ be an uncountable regular cardinal and let P be a λ -small poset. Then for every normal morphism $\varphi: \mathbf{A} \rightarrow \mathbf{B}$ of P -scaled Boolean algebras, the set of all pairs (U, V) where U is a λ -small induced subalgebra of \mathbf{A} , V is a λ -small induced subalgebra of \mathbf{B} , and φ restricts to a normal morphism from \mathbf{U} to \mathbf{V} (i.e., $\varphi[U] = V$ and $\varphi[U \cap A^{(p)}] = V \cap B^{(p)}$ whenever $p \in P$) is λ -closed cofinal in $[A]^{<\lambda} \times [B]^{<\lambda}$.*

Corollary 5.5. *Let λ be an infinite regular cardinal and suppose that P is λ -small. Then a P -scaled Boolean algebra is λ -presented iff it is λ -small.*

Proof. The right to left direction of our statement is provided by Lemma 5.2. For the left to right direction, we may assume that $\lambda > \omega$, and then apply Lemma 5.4 to argue that every P -scaled Boolean algebra \mathbf{A} is the λ -directed union of its λ -small induced subalgebras, thus also a λ -directed colimit of those subalgebras. If \mathbf{A} is λ -presented, it is thus a retract (i.e., split quotient) of a λ -small induced subalgebra of \mathbf{A} , and thus it is itself λ -small. \square

6. CONDENSATES AND Φ -CONDENSATES

In this section we shall introduce the crucial concepts $\mathbf{A} \otimes^\lambda \vec{S}$ (condensates) and $\mathbf{A} \otimes_\Phi^\lambda \vec{S}$ (Φ -condensates). Throughout Section 6 we shall fix a poset P .

6.1. Condensates.

Notation 6.1. For any P -scaled Boolean algebra \mathbf{A} , we set

$$\text{Ult}^b \mathbf{A} \stackrel{\text{def}}{=} \{ \mathfrak{a} \in \text{Ult } A \mid \|\mathfrak{a}\|_{\mathbf{A}} \text{ is bounded above in } P \}.$$

Definition 6.2. We say that P is a *conditional DCPO*² if every nonempty, bounded above, directed subset of P has a join (i.e., least upper bound). In that case, for every P -scaled Boolean algebra \mathbf{A} and every $\mathfrak{a} \in \text{Ult}^b \mathbf{A}$ (cf. Notation 6.1), we set

$$|\mathfrak{a}|_{\mathbf{A}} \stackrel{\text{def}}{=} \bigvee \|\mathfrak{a}\|_{\mathbf{A}}, \quad \text{for every } \mathfrak{a} \in \text{Ult}^b \mathbf{A}.$$

For the remainder of Section 6, we shall fix an infinite regular cardinal λ such that if $\lambda > \omega$, then P is a λ -small conditional DCPO. Moreover, we shall fix a category \mathcal{S} with all binary products such that if $\lambda > \omega$, then \mathcal{S} has all products of at most 2^α objects whenever $\alpha < \lambda$ — we will express this statement by saying that \mathcal{S} has all $2^{<\lambda}$ -products³.

We shall also fix a (not necessarily commutative) P -indexed diagram $\vec{S} = (S_p, \vec{S}(p, q) \mid p \leq q \text{ in } P)$ in \mathcal{S} .

Definition 6.3. Let \mathbf{A} be a λ -presented (cf. Corollary 5.5) P -scaled Boolean algebra. We define an object $\mathbf{A} \otimes^\lambda \vec{S}$ as follows:

- If $\lambda = \omega$, then $\mathbf{A} \otimes^\lambda \vec{S} \stackrel{\text{def}}{=} \prod (S_{|a|_{\mathbf{A}}} \mid a \in \text{At } A)$ (cf. (5.2) and the comment following).
- If $\lambda > \omega$, then $\mathbf{A} \otimes^\lambda \vec{S} \stackrel{\text{def}}{=} \prod (S_{|\mathfrak{a}|_{\mathbf{A}}} \mid \mathfrak{a} \in \text{Ult}^b \mathbf{A})$.

We will say that $\mathbf{A} \otimes^\lambda \vec{S}$ is a *condensate* of \vec{S} . While this terminology extends, to the non-commutative case, the condensate construction $\mathbf{A} \otimes \vec{S}$ introduced in Gillibert and Wehrung [14], the case where $\lambda > \omega$ yields a new class of objects.

Note. The object $\mathbf{A} \otimes^\lambda \vec{S}$ is defined only in case \mathbf{A} is λ -presented.

For a homomorphism $\varphi: A \rightarrow B$ of finite Boolean algebras, denote by $\varphi^*: \text{At } B \rightarrow \text{At } A$ the dual map of φ : that is, $\varphi^*(b)$ is the unique atom a of A such that $b \leq \varphi(a)$, whenever b is an atom of B . The symbol \otimes^λ can be extended to morphisms, as follows.

Definition 6.4. For all λ -presented members \mathbf{A} and \mathbf{B} of \mathbf{Bool}_P and every morphism $\varphi: \mathbf{A} \rightarrow \mathbf{B}$ in \mathbf{Bool}_P , we define $\varphi \otimes^\lambda \vec{S}$ as the set of all morphisms from $\mathbf{A} \otimes^\lambda \vec{S}$ to $\mathbf{B} \otimes^\lambda \vec{S}$, in \mathcal{S} , of the form

$$\begin{aligned} \prod (f_b \mid b \in \text{At } B), & \quad \text{where each } f_b \in \vec{S}(|\varphi^*(b)|_{\mathbf{A}}, |b|_{\mathbf{B}}) & \quad (\text{if } \lambda = \omega) \\ \prod (f_{\mathfrak{b}} \mid \mathfrak{b} \in \text{Ult}^b \mathbf{B}), & \quad \text{where each } f_{\mathfrak{b}} \in \vec{S}(|\varphi^{-1}[\mathfrak{b}]|_{\mathbf{A}}, |\mathfrak{b}|_{\mathbf{B}}) & \quad (\text{if } \lambda > \omega). \end{aligned}$$

Observe that Definition 6.4 relies on the following facts:

- (if $\lambda = \omega$) $|\varphi^*(b)|_{\mathbf{A}} \leq |b|_{\mathbf{B}}$ whenever $b \in \text{At } B$. This trivially follows from the containment $\|\varphi^*(b)\|_{\mathbf{A}} \subseteq \|b\|_{\mathbf{B}}$, which in turns follows from φ being a morphism in \mathbf{Bool}_P .

²Recall that *DCPO* usually stands for “directed-complete partial order”.

³Taking the empty product into account, the latter (conditional to $\lambda > \omega$) implies that \mathcal{S} has a terminal object. No terminal object is assumed if $\lambda = \omega$.

- (if $\lambda > \omega$) For every $\mathfrak{b} \in \text{Ult}^b \mathbf{B}$, $\|\varphi^{-1}[\mathfrak{b}]\|_{\mathbf{A}} \subseteq \|\mathfrak{b}\|_{\mathbf{B}}$ (because φ is a morphism in \mathbf{Bool}_P), thus, since P is a conditional DCPO and $\|\mathfrak{b}\|_{\mathbf{B}}$ is bounded above (by $|\mathfrak{b}|_{\mathbf{B}}$), $|\varphi^{-1}[\mathfrak{b}]|_{\mathbf{A}}$ is defined and bounded above by $|\mathfrak{b}|_{\mathbf{B}}$.

This is to be put in contrast with Gillibert and Wehrung [14], where $\varphi \otimes \vec{S}$ denotes a single morphism from $\mathbf{A} \otimes \vec{S}$ to $\mathbf{B} \otimes \vec{S}$.

Lemma 6.5. $_{-} \otimes^{\lambda} \vec{S}$ is a diagram (in the sense of Definition 3.1), indexed by $\mathbf{Bool}_P^{\text{fin}}$ if $\lambda = \omega$ and by $\mathbf{Bool}_P^{<\lambda}$ if $\lambda > \omega$.

Proof. It is trivial that $\text{id}_{\mathbf{A}} \otimes^{\lambda} \vec{S}$ contains, as an element, the identity on $\mathbf{A} \otimes^{\lambda} \vec{S}$. Now let $\varphi: \mathbf{A} \rightarrow \mathbf{B}$ and $\psi: \mathbf{B} \rightarrow \mathbf{C}$ be morphisms of λ -presented P -scaled Boolean algebras, let $f \in \varphi \otimes^{\lambda} \vec{S}$, and let $g \in \psi \otimes^{\lambda} \vec{S}$. We need to verify that $g \circ f$ belongs to $(\psi \circ \varphi) \otimes^{\lambda} \vec{S}$. Let us verify this in case $\lambda > \omega$; the proof for $\lambda = \omega$ is similar. We can write $f = \prod (f_{\mathfrak{b}} \mid \mathfrak{b} \in \text{Ult}^b \mathbf{B})$, where each $f_{\mathfrak{b}} \in \vec{S}(|\varphi^{-1}[\mathfrak{b}]|_{\mathbf{A}}, |\mathfrak{b}|_{\mathbf{B}})$, and $g = \prod (g_{\mathfrak{c}} \mid \mathfrak{c} \in \text{Ult}^b \mathbf{C})$, where each $g_{\mathfrak{c}} \in \vec{S}(|\psi^{-1}[\mathfrak{c}]|_{\mathbf{B}}, |\mathfrak{c}|_{\mathbf{C}})$. Now $g \circ f = \prod (g_{\mathfrak{c}} \circ f_{\psi^{-1}[\mathfrak{c}]} \mid \mathfrak{c} \in \text{Ult}^b \mathbf{C})$ where each $g_{\mathfrak{c}} \circ f_{\psi^{-1}[\mathfrak{c}]}$ belongs to $\vec{S}(|\varphi^{-1}\psi^{-1}[\mathfrak{c}]|_{\mathbf{A}}, |\mathfrak{c}|_{\mathbf{C}}) = \vec{S}(|(\psi \circ \varphi)^{-1}[\mathfrak{c}]|_{\mathbf{A}}, |\mathfrak{c}|_{\mathbf{C}})$, as required. \square

6.2. Φ -condensates. Let \mathcal{T} be a category and let $\Phi: \mathcal{S} \rightarrow \mathcal{T}$ be a functor. We want to argue that under certain conditions, the composition of the diagram $_{-} \otimes^{\lambda} \vec{S}$ with Φ yields a *commutative* diagram. This condition arises from the construction \vec{S}^I introduced in Definition 3.3.

Definition 6.6. We say that a P -indexed diagram \vec{S} in \mathcal{S} is Φ -*commutative* if the composition $\Phi \vec{S}^I$ is a commutative diagram for any set I such that the diagram \vec{S}^I is defined.

This means that for any set I such that \vec{S}^I is defined, any $p = (p_i \mid i \in I)$ and $q = (q_i \mid i \in I)$ in P^I such that $p \leq q$, and any $f, g \in \prod (\vec{S}(p_i, q_i) \mid i \in I)$, the equality $\Phi(f) = \Phi(g)$ holds.

In particular, if the diagram \vec{S} is Φ -commutative, then the composite $\Phi \vec{S}$ is a commutative diagram.

The diagram of Abelian ℓ -groups denoted by \vec{A} in Wehrung [39] (cf. Section 12) is proven in that paper to be $\text{Id}_{\mathbf{c}}$ -commutative, where $\text{Id}_{\mathbf{c}}$ is the functor sending every Abelian ℓ -group to its lattice of principal ℓ -ideals. However, \vec{A} is not commutative.

Lemma 6.7. Let \vec{S} be a Φ -commutative⁴ P -indexed diagram in \mathcal{S} . Then the composite $\Phi(_{-} \otimes^{\lambda} \vec{S})$ is a commutative diagram; that is, it is a functor from the λ -presented members of \mathbf{Bool}_P to \mathcal{T} .

Proof. Suppose first that $\lambda = \omega$. We need to prove that for any finitely presented P -scaled Boolean algebras \mathbf{A} and \mathbf{B} , any morphism $\varphi: \mathbf{A} \rightarrow \mathbf{B}$, and any $f, g \in \prod (\vec{S}(|\varphi^*(b)|_{\mathbf{A}}, |b|_{\mathbf{B}}) \mid b \in \text{At } \mathbf{B})$, the equality $\Phi(f) = \Phi(g)$ holds. This follows trivially from the commutativity of the diagram $\Phi \vec{S}^{\text{At } \mathbf{B}}$.

The proof for $\lambda > \omega$, now following from the commutativity of the diagram $\Phi \vec{S}^{\text{Ult}^b \mathbf{B}}$, is similar. \square

⁴Actually we need only the weaker assumption that $\Phi \vec{S}^I$ is a commutative diagram for every nonempty finite set I (if $\lambda = \omega$) or any set I of cardinality bounded above by 2^{α} for some $\alpha < \lambda$ (if $\lambda > \omega$).

Lemma 6.8. *Suppose that \mathcal{T} has all λ -directed colimits and that \vec{S} is Φ -commutative. Then the functor $\Phi(- \otimes^\lambda \vec{S})$ extends uniquely, up to natural isomorphism, to a functor $\overline{\Psi}: \mathbf{Bool}_P^{\text{fin}} \rightarrow \mathcal{T}$ (if $\lambda = \omega$) or $\overline{\Psi}: \mathbf{Bool}_P^{<\lambda} \rightarrow \mathcal{T}$ (if $\lambda > \omega$) that preserves λ -directed colimits.*

Proof. If $\lambda = \omega$, we apply Gillibert and Wehrung [14, Proposition 1.4.2] (i.e., the case where $\lambda = \omega$ of Lemma 3.8) to $\mathcal{A} := \mathbf{Bool}_P$, $\mathcal{A}^\dagger := \mathbf{Bool}_P^{\text{fin}}$, and $\Psi := \Phi(- \otimes^\omega \vec{S})$. It follows from Proposition 2.4.6 and Corollary 2.4.7 in [14] that the assumptions of Lemma 3.8 are indeed satisfied.

If $\lambda > \omega$, we apply Lemma 3.8 to $\mathcal{A} := \mathbf{Bool}_P$, $\mathcal{A}^\dagger := \mathbf{Bool}_P^{<\lambda}$, and $\Psi := \Phi(- \otimes^\lambda \vec{S})$. It follows from Lemmas 5.2 and 5.4 that the assumptions of Lemma 3.8 are indeed satisfied. \square

Definition 6.9. We shall denote by $- \otimes_\Phi^\lambda \vec{S}$ the functor $\overline{\Psi}$ whose existence is ensured by Lemma 6.8. Objects of \mathcal{S} of the form $\mathbf{A} \otimes_\Phi^\lambda \vec{S}$, for $\mathbf{A} \in \mathbf{Bool}_P$, will be called Φ -condensates of \vec{S} .

We end this section by recalling a useful example of a P -scaled Boolean algebra, introduced in Gillibert and Wehrung [14, Definition 2.6.1]; namely,

$$\mathbf{2}[p] \stackrel{\text{def}}{=} (\mathbf{2}, (\mathbf{2}[p]^{(q)} \mid q \in P))$$

where we set $\mathbf{2} \stackrel{\text{def}}{=} \{0, 1\}$ and define $\mathbf{2}[p]^{(q)}$ as $\{0, 1\}$ if $q \leq p$, $\{0\}$ otherwise. Whenever $p \leq q$ in P , the identity map on $\mathbf{2}$ induces a monomorphism $\varepsilon_p^q: \mathbf{2}[p] \rightarrow \mathbf{2}[q]$ in $\mathbf{Bool}_P^{\text{fin}}$. The following straightforward analogue of [14, Lemma 3.1.3] remains valid.

Lemma 6.10. *Let \vec{S} be a (not necessarily commutative) P -indexed diagram in \mathcal{S} . Then the following statements hold:*

- (1) $\mathbf{2}[p] \otimes^\lambda \vec{S} = S_p$, for all $p \in P$.
- (2) $\varepsilon_p^q \otimes^\lambda \vec{S} = \vec{S}(p, q)$, for all $p \leq q$ in P .

In particular, in the context of Lemma 6.10, if \vec{S} is Φ -commutative, then we get

- For all $p \in P$, $\mathbf{2}[p] \otimes_\Phi^\lambda \vec{S} = \Phi(S_p)$.
- For all $p \leq q$ in P , $\varepsilon_p^q \otimes_\Phi^\lambda \vec{S}$ is the common value of all $\Phi(x)$ for $x \in \vec{S}(p, q)$.

7. THE UNIFORMIZATION LEMMA AND THE BOOSTING LEMMA

While the construction $- \otimes^\lambda \vec{S}$, originally defined on “small” P -scaled Boolean algebras, cannot be functorially extended to morphisms in full generality, we prove, in this section, that under mild conditions, such an extension is possible on diagrams of small P -scaled Boolean algebras indexed by *forests*. We begin with an easy lemma, whose straightforward proof we leave to the reader.

Lemma 7.1. *Let I and P be posets with I a chain, and let $f: I \rightarrow P$ be an isotone map. Then $f[I] \downarrow f(i) = f[I \downarrow i]$ for every $i \in I$.*

The main result of Section 7 states that certain diagrams of condensates of a given diagram have uniformizations (cf. Definition 3.1 for the latter).

Lemma 7.2 (Uniformization Lemma). *Let λ be an infinite regular cardinal, let P be a poset, let Λ be a forest, let $(\mathbf{A}_i, \alpha_i^j \mid i \leq j \text{ in } \Lambda)$ be a Λ -indexed commutative diagram in \mathbf{Bool}_P , and let \mathcal{S} be a category, subjected to the following conditions:*

- every bounded chain of P is finite;
- if $\lambda > \omega$ then P is a λ -small conditional DCPO;
- \mathcal{S} has all $2^{<\lambda}$ -products;
- each \mathbf{A}_i is λ -presented.

Let \vec{S} be a (not necessarily commutative) P -indexed diagram in \mathcal{S} . Then the diagram $(\mathbf{A}_i \otimes^\lambda \vec{S}, \alpha_i^j \otimes^\lambda \vec{S} \mid i \leq j \text{ in } I)$ has a uniformization.

Proof. We show the argument for the case where $\lambda > \omega$. The proof for $\lambda = \omega$ is, *mutatis mutandis*, identical.

We start by picking, for each pair $(p, q) \in P \times P$ such that $p < q$, a morphism $\sigma_p^q \in \vec{S}(p, q)$. Observe that the σ_p^q are not expected to form a commutative diagram in \mathcal{S} (which in general they do not).

Set $\mathcal{U} \stackrel{\text{def}}{=} \bigcup (\{k\} \times \text{Ult}^b \mathbf{A}_k \mid k \in \Lambda)$. For each $(k, \mathbf{c}) \in \mathcal{U}$, denote by $\pi_{k, \mathbf{c}} : \Lambda \downarrow k \rightarrow P$ the map defined by $\pi_{k, \mathbf{c}}(j) \stackrel{\text{def}}{=} |(\alpha_j^k)^{-1}[\mathbf{c}]|_{\mathbf{A}_j}$ whenever $j \in \Lambda \downarrow k$. Observe that $\pi_{k, \mathbf{c}}(k) = |\mathbf{c}|_{\mathbf{A}_k}$. For all $i \leq j \leq k$ in Λ ,

$$\pi_{k, \mathbf{c}}(i) = |(\alpha_i^k)^{-1}[\mathbf{c}]|_{\mathbf{A}_i} = |(\alpha_i^j)^{-1}(\alpha_j^k)^{-1}[\mathbf{c}]|_{\mathbf{A}_i} \leq |(\alpha_j^k)^{-1}[\mathbf{c}]|_{\mathbf{A}_j} = \pi_{k, \mathbf{c}}(j),$$

so $\pi_{k, \mathbf{c}}$ is isotone. It follows that the range $P_{k, \mathbf{c}}$ of $\pi_{k, \mathbf{c}}$ is a chain contained in P .

Claim. $P_{k, \mathbf{c}} \downarrow \pi_{k, \mathbf{c}}(j) = P_{j, (\alpha_j^k)^{-1}[\mathbf{c}]}$.

Proof of Claim. A direct calculation:

$$\begin{aligned} P_{j, (\alpha_j^k)^{-1}[\mathbf{c}]} &= \{ |(\alpha_i^j)^{-1}(\alpha_j^k)^{-1}[\mathbf{c}]|_{\mathbf{A}_i} \mid i \in \Lambda \downarrow j \} \\ &= \{ |(\alpha_i^k)^{-1}[\mathbf{c}]|_{\mathbf{A}_i} \mid i \in \Lambda \downarrow j \} \\ &= \pi_{k, \mathbf{c}}[\Lambda \downarrow j] \\ &= \pi_{k, \mathbf{c}}[(\Lambda \downarrow k) \downarrow j] \\ &= \pi_{k, \mathbf{c}}[\Lambda \downarrow k] \downarrow \pi_{k, \mathbf{c}}(j) \quad (\text{because } \Lambda \downarrow k \text{ is a chain and by Lemma 7.1}) \\ &= P_{k, \mathbf{c}} \downarrow \pi_{k, \mathbf{c}}(j). \quad \square \text{ Claim.} \end{aligned}$$

Whenever $(k, \mathbf{c}) \in \mathcal{U}$ and $j \leq k$, it follows from the assumptions on P that the subchain $P_{k, \mathbf{c}} \uparrow \pi_{k, \mathbf{c}}(j) = P_{k, \mathbf{c}} \cap [\pi_{k, \mathbf{c}}(j), \pi_{k, \mathbf{c}}(k)]$ is finite; write it as $\{p_0, \dots, p_n\}$ with $p_0 < \dots < p_n$, and then define

$$\tau_{j, k, \mathbf{c}} \stackrel{\text{def}}{=} \sigma_{p_{n-1}}^{p_n} \circ \dots \circ \sigma_{p_0}^{p_1}.$$

In particular, $\tau_{j, k, \mathbf{c}}$ belongs to $\vec{S}(p_0, p_n) = \vec{S}(|(\alpha_j^k)^{-1}[\mathbf{c}]|_{\mathbf{A}_j}, |\mathbf{c}|_{\mathbf{A}_k})$. It follows that the morphisms $\tau_{j, k, \mathbf{c}}$, for \mathbf{c} ranging over $\text{Ult}^b \mathbf{A}_k$, are the components of a morphism

$$\varphi_j^k \stackrel{\text{def}}{=} \prod (\tau_{j, k, \mathbf{c}} \mid \mathbf{c} \in \text{Ult}^b \mathbf{A}_k) \in \alpha_j^k \otimes^\lambda \vec{S}.$$

If $j = k$, then each $\tau_{j, k, \mathbf{c}}$ is the identity, thus φ_j^k is the identity on $\mathbf{A}_j \otimes^\lambda \vec{S}$.

We shall now prove that $\varphi_i^k = \varphi_j^k \circ \varphi_i^j$ whenever $i \leq j \leq k$ in Λ . This amounts to proving that the relation $\tau_{i, k, \mathbf{c}} = \tau_{j, k, \mathbf{c}} \circ \tau_{i, j, (\alpha_j^k)^{-1}[\mathbf{c}]}$ holds for every $\mathbf{c} \in \text{Ult}^b \mathbf{A}_k$. Writing $P_{k, \mathbf{c}} \uparrow \pi_{k, \mathbf{c}}(i) = \{p_0, \dots, p_n\}$ with $p_0 < \dots < p_n$, we obtain, by definition,

$$\tau_{i, k, \mathbf{c}} = \sigma_{p_{n-1}}^{p_n} \circ \dots \circ \sigma_{p_0}^{p_1}. \quad (7.1)$$

Let $m \in \{0, \dots, n\}$ such that $\pi_{k,c}(j) = p_m$. Then $P_{k,c} \uparrow \pi_{k,c}(j) = \{p_m, \dots, p_n\}$, thus

$$\tau_{j,k,c} = \sigma_{p_{n-1}}^{p_n} \circ \dots \circ \sigma_{p_m}^{p_{m+1}}. \quad (7.2)$$

Finally, it follows from the Claim above that $P_{j,(\alpha_j^k)^{-1}[c]} = \{p_0, \dots, p_m\}$, thus

$$\tau_{i,j,(\alpha_j^k)^{-1}[c]} = \sigma_{p_{m-1}}^{p_m} \circ \dots \circ \sigma_{p_0}^{p_1}. \quad (7.3)$$

The relations (7.1), (7.2), and (7.3) together imply the required relation $\tau_{i,k,c} = \tau_{j,k,c} \circ \tau_{i,j,(\alpha_j^k)^{-1}[c]}$. This concludes the proof that $(\mathbf{A}_i \otimes^\lambda \vec{S}, \varphi_i^j \mid i \leq j \text{ in } I)$ is a uniformization of $(\mathbf{A}_i \otimes^\lambda \vec{S}, \alpha_i^j \otimes^\lambda \vec{S} \mid i \leq j \text{ in } I)$. \square

For a λ -presented P -scaled Boolean algebra \mathbf{B} , the Φ -condensate $\mathbf{B} \otimes_\Phi^\lambda \vec{S}$ is defined as $\Phi(\mathbf{B} \otimes^\lambda \vec{S})$, thus it belongs to the range of Φ . The following result enables us to extend the latter observation to the case where \mathbf{B} is the colimit of a direct system, indexed by λ , of λ -presented P -scaled Boolean algebras.

Lemma 7.3 (The Boosting Lemma). *Let λ be an infinite regular cardinal and let P be a poset such that if $\lambda > \omega$ then P is a λ -small conditional DCPO. Let \mathcal{S} be a category with all binary products if $\lambda = \omega$ and all $2^{<\lambda}$ -products if $\lambda > \omega$, let \mathcal{T} be a category with all λ -directed colimits, and let $\Phi: \mathcal{S} \rightarrow \mathcal{T}$ be a functor. We also assume the following additional conditions:*

- every bounded chain of P is finite;
- \mathcal{S} has all colimits indexed by λ ;
- Φ preserves all colimits indexed by λ .

Let \vec{S} be a Φ -commutative P -indexed diagram in \mathcal{S} , and consider a directed colimit cocone

$$(\mathbf{B}, \beta_\xi \mid \xi < \lambda) = \varinjlim (\mathbf{B}_\xi, \beta_\xi^\eta \mid \xi \leq \eta < \lambda) \quad \text{in } \mathbf{Bool}_P,$$

with all \mathbf{B}_ξ λ -presented. Then $\mathbf{B} \otimes_\Phi^\lambda \vec{S}$ belongs to the range of Φ .

Proof. By Lemma 7.2, the diagram $(\mathbf{B}_\xi \otimes^\lambda \vec{S}, \beta_\xi^\eta \otimes^\lambda \vec{S} \mid \xi \leq \eta < \lambda)$ has a uniformization $(\mathbf{B}_\xi \otimes^\lambda \vec{S}, \varphi_\xi^\eta \mid \xi \leq \eta < \lambda)$. Since \mathcal{S} has all colimits indexed by λ , there exists a directed colimit cocone

$$(S, \varphi_\xi \mid \xi < \lambda) = \varinjlim (\mathbf{B}_\xi \otimes^\lambda \vec{S}, \varphi_\xi^\eta \mid \xi \leq \eta < \lambda) \quad \text{in } \mathcal{S}.$$

(Of course, S will play the role of the undefined $\mathbf{B} \otimes^\lambda \vec{S}$.) Since Φ preserves all colimits indexed by λ , it follows that

$$(\Phi(S), \Phi(\varphi_\xi) \mid \xi < \lambda) = \varinjlim (\mathbf{B}_\xi \otimes_\Phi^\lambda \vec{S}, \Phi(\varphi_\xi^\eta) \mid \xi \leq \eta < \lambda) \quad \text{in } \mathcal{T}. \quad (7.4)$$

Since the functor $- \otimes_\Phi^\lambda \vec{S}$ preserves all λ -directed colimits, it also preserves all colimits indexed by λ , whence

$$(\mathbf{B} \otimes_\Phi^\lambda \vec{S}, \beta_\xi \otimes_\Phi^\lambda \vec{S} \mid \xi < \lambda) = \varinjlim (\mathbf{B}_\xi \otimes_\Phi^\lambda \vec{S}, \beta_\xi^\eta \otimes_\Phi^\lambda \vec{S} \mid \xi \leq \eta < \lambda) \quad \text{in } \mathcal{T}. \quad (7.5)$$

Now whenever $\xi \leq \eta < \lambda$, it follows from the relation $\varphi_\xi^\eta \in \beta_\xi^\eta \otimes^\lambda \vec{S}$ that $\Phi(\varphi_\xi^\eta) = \beta_\xi^\eta \otimes_\Phi^\lambda \vec{S}$. By (7.4) and (7.5) together with the uniqueness of the directed colimit, we obtain that $\Phi(S) \cong \mathbf{B} \otimes_\Phi^\lambda \vec{S}$, as desired. \square

8. TENSORING WITH NORMAL MORPHISMS

Standing Hypothesis. λ is an infinite regular cardinal, P is a poset and also a λ -small conditional DCPO if $\lambda > \omega$, \mathcal{S} is a category with all binary products if $\lambda = \omega$ and all $2^{<\lambda}$ -products if $\lambda > \omega$, \mathcal{T} is a category with all λ -directed colimits, $\Phi: \mathcal{S} \rightarrow \mathcal{T}$ is a functor, and \vec{S} is a Φ -commutative P -indexed diagram in \mathcal{S} .

In this section we shall analyze the structure of the morphisms of the form $\varphi \otimes_{\Phi}^{\lambda} \vec{S}$ in case φ is a normal morphism of P -scaled Boolean algebras.

Definition 8.1.

- (1) A *projection* in a category is the canonical morphism from a product to one of its factors.
- (2) A Φ -*projection* in \mathcal{T} is an isomorphic copy of $\Phi(p)$ for a projection p in \mathcal{S} .
- (3) A λ -*extended Φ -projection* in \mathcal{T} is a λ -directed colimit, within the category of all arrows of \mathcal{T} , of Φ -projections in \mathcal{T} .

The following result extends Gillibert and Wehrung [14, Proposition 3.1.2] to our current context.

Proposition 8.2. *For every normal morphism $\varphi: \mathbf{A} \rightarrow \mathbf{B}$ in \mathbf{Bool}_P , $\varphi \otimes_{\Phi}^{\lambda} \vec{S}$ is a λ -extended Φ -projection. Moreover, if \mathbf{A} is λ -presented, then $\varphi \otimes_{\Phi}^{\lambda} \vec{S}$ is a Φ -projection.*

Proof. By Gillibert and Wehrung [14, § 2.5], we may assume that $\mathbf{B} = \mathbf{A}/I$ and $\varphi = \pi_I$ (cf. Section 5), for an ideal I of \mathbf{A} .

By Lemma 5.4 (if $\lambda > \omega$) of Gillibert and Wehrung [14, Proposition 2.5.5] (if $\lambda = \omega$), it suffices to consider the case where either $\lambda = \omega$ and \mathbf{A} is finitely presented or $\lambda > \omega$ and \mathbf{A} is λ -small. Suppose for example that $\lambda > \omega$ and set

$$\begin{aligned} U &\stackrel{\text{def}}{=} \text{Ult}^b \mathbf{A}, \\ U_1 &\stackrel{\text{def}}{=} \{\mathfrak{a} \in \text{Ult}^b \mathbf{A} \mid \mathfrak{a} \cap I = \emptyset\}, \\ U_0 &\stackrel{\text{def}}{=} U \setminus U_1. \end{aligned}$$

Then we can write

$$\begin{aligned} \mathbf{A} \otimes^{\lambda} \vec{S} &= \prod (S_{|\mathfrak{a}|_{\mathbf{A}}} \mid \mathfrak{a} \in U), \\ \mathbf{B} \otimes^{\lambda} \vec{S} &= \prod (S_{|\mathfrak{a}/I|_{\mathbf{A}/I}} \mid \mathfrak{a} \in U_1) \\ &= \prod (S_{|\mathfrak{a}|_{\mathbf{A}}} \mid \mathfrak{a} \in U_1). \end{aligned}$$

The canonical projection $p: X \amalg Y \rightarrow X$, where $X \stackrel{\text{def}}{=} \prod (S_{|\mathfrak{a}|_{\mathbf{A}}} \mid \mathfrak{a} \in U_1)$ and $Y \stackrel{\text{def}}{=} \prod (S_{|\mathfrak{a}|_{\mathbf{A}}} \mid \mathfrak{a} \in U_0)$, is $\prod (f_{\mathfrak{a}} \mid \mathfrak{a} \in U_1)$ where each $f_{\mathfrak{a}}$ is the identity on $S_{|\mathfrak{a}|_{\mathbf{A}}}$; thus it belongs to $\varphi \otimes^{\lambda} \vec{S}$. Since \vec{S} is Φ -commutative, it follows that $\varphi \otimes_{\Phi}^{\lambda} \vec{S} = \Phi(p)$, a Φ -projection in \mathcal{T} .

The case where $\lambda = \omega$ is handled similarly, now with $U \stackrel{\text{def}}{=}} \text{At } \mathbf{A}$, $U_1 \stackrel{\text{def}}{=} U \setminus I$, and $U_0 \stackrel{\text{def}}{=} U \cap I$. \square

9. SUBMORPHISMS OF NORM-COVERINGS

Let us first recall a few concepts from Gillibert and Wehrung [14]. Following [14, Definition 2.1.2], we say that a poset P is

- a *pseudo join-semilattice* if the set U of all upper bounds of any finite subset X of P is a finitely generated upper subset of P ; then we denote by ∇X the (finite) set of all minimal elements of U ;
- *supported* if it is a pseudo join-semilattice and every finite subset of P is contained in a finite subset Y of P which is ∇ -closed, that is, $\nabla Z \subseteq Y$ whenever Z is a finite subset of Y (this definition is equivalent to the eponymous one introduced in Gillibert [13]);
- an *almost join-semilattice* if it is a pseudo join-semilattice in which every principal ideal $P \downarrow a$ is a join-semilattice.

In Gillibert and Wehrung [14, § 2.1], the non-reversible implications

join-semilattice \Rightarrow almost join-semilattice \Rightarrow supported \Rightarrow pseudo join-semilattice

are observed. As in [14], we set $a_1 \nabla \cdots \nabla a_n \stackrel{\text{def}}{=} \nabla \{a_1, \dots, a_n\}$. Following [14, § 2.6], a *norm-covering* of a poset P is a pair (X, ∂) where X is a pseudo join-semilattice and $\partial: X \rightarrow P$ is an isotone map (sometimes denoted by ∂_X if X needs to be specified). An ideal \mathbf{x} of X is *sharp* if the image $\partial[\mathbf{x}]$ has a largest element, then denoted by $\partial\mathbf{x}$. We denote by $\text{Id}_s X$ the set of all sharp ideals of X , partially ordered under set inclusion.

We denote by $\mathbf{F}(X)$ the Boolean algebra defined by generators \tilde{u} (or \tilde{u}^X in case X needs to be specified), where $u \in X$, and relations

$$\begin{aligned} \tilde{v} &\leq \tilde{u}, & \text{whenever } u \leq v \text{ in } X; \\ \tilde{u} \wedge \tilde{v} &= \bigvee (\tilde{w} \mid x \in u \nabla v), & \text{whenever } u, v \in X; \\ 1 &= \bigvee (\tilde{w} \mid w \in \text{Min } X). \end{aligned}$$

Furthermore, for every $p \in P$, we denote by $\mathbf{F}(X)^{(p)}$ the ideal of $\mathbf{F}(X)$ generated by $\{\tilde{u} \mid u \in X, p \leq \partial u\}$. The structure $\mathbf{F}(X) \stackrel{\text{def}}{=} (\mathbf{F}(X), (\mathbf{F}(X)^{(p)} \mid p \in P))$ is a P -scaled Boolean algebra (cf. [14, Lemma 2.6.5]).

For every sharp ideal \mathbf{x} of X , there is a unique morphism $\pi_{\mathbf{x}}^X: \mathbf{F}(X) \rightarrow \mathbf{2}[\partial\mathbf{x}]$ that sends every \tilde{u} , where $u \in X$, to 1 if $u \in \mathbf{x}$ and 0 otherwise. This morphism is normal (cf. Gillibert and Wehrung [14, Lemma 2.6.7]).

As already observed in Gillibert and Wehrung [14], every ∇ -closed subset Y of X defines, by restriction of the map ∂ , a norm-covering of P and the inclusion map from Y into X induces a morphism $f_Y^X: \mathbf{F}(Y) \rightarrow \mathbf{F}(X)$ in \mathbf{Bool}_P (cf. [14, Lemma 2.6.6]). We shall now extend that observation.

Definition 9.1. For norm-coverings X and Y of a poset P , a map $f: X \rightarrow Y$ is a *submorphism* if f is isotone, $\partial_X x \leq \partial_Y f(x)$ whenever $x \in X$, and for all $n < \omega$ and all $x_1, \dots, x_n \in X$ the containment $f(x_1) \nabla_Y \cdots \nabla_Y f(x_n) \subseteq f[x_1 \nabla_X \cdots \nabla_X x_n]$ holds.

The latter condition can be reformulated as stating that for all $n < \omega$, all $x_1, \dots, x_n \in X$, and all $y \in Y$, if each $f(x_i) \leq y$, then there exists $x \in X$ such that each $x_i \leq x$ and $f(x) \leq y$. It obviously suffices to check this for $n = 0$ and $n = 2$.

Lemma 9.2. *Every submorphism $f: X \rightarrow Y$ of norm-coverings of P induces a unique morphism $\mathbf{F}(f): \mathbf{F}(X) \rightarrow \mathbf{F}(Y)$ of P -scaled Boolean algebras sending \tilde{x}^X to $\widetilde{f(x)}^Y$ whenever $x \in X$.*

Proof. In order to verify the existence of $\mathbf{F}(f)$ at Boolean algebra level, it suffices to prove that the elements $\widetilde{f(x)}^Y$, for $x \in X$, satisfy the defining relations of $\mathbf{F}(X)$. Since f is isotone, $x \mapsto \widetilde{f(x)}^Y$ is antitone. For all $x_0, x_1 \in X$,

$$\begin{aligned} \widetilde{f(x_0)}^Y \cap \widetilde{f(x_1)}^Y &= \bigvee (\tilde{y}^Y \mid y \in f(x_0) \nabla^Y f(x_1)) \\ &\leq \bigvee \left(\widetilde{f(x)}^Y \mid x \in x_0 \nabla^X x_1 \right) \quad (\text{by assumption on } f), \end{aligned}$$

the converse inequality being obvious. Similarly, $1 = \bigvee (\widetilde{f(x)}^Y \mid x \in \text{Min } X)$. This completes the proof of the existence of $\varphi \stackrel{\text{def}}{=} \mathbf{F}(f)$ at Boolean algebra level. Now let $p \in P$. For each $x \in X$, $p \leq \partial_X x$ implies $p \leq \partial_Y f(x)$, thus $\widetilde{f(x)}^Y \in \mathbf{F}(Y)^{(p)}$. By definition, it follows that $\varphi[\mathbf{F}(X)^{(p)}] \subseteq \mathbf{F}(Y)^{(p)}$, thus completing the proof that φ is a morphism in \mathbf{Bool}_P . \square

10. STANDARD LIFTERS

We are now reaching the infinite combinatorial aspects of our theory. Let us first recall the concept of *lifter* introduced in Gillibert and Wehrung [14, Definition 3.2.1].

Definition 10.1. Let λ be an infinite cardinal and let P be a poset. A λ -*lifter* of P is a triple $(X, \mathbf{X}, \partial)$, with (X, ∂) a norm-covering of P and $\mathbf{X} \subseteq \text{Id}_s X$, which satisfies the following conditions:

- (1) The set $\mathbf{X}^= \stackrel{\text{def}}{=} \{x \in X \mid \partial x \text{ is not maximal in } P\}$ is lower $\text{cf}(\lambda)$ -small.
- (2) For every map $S: \mathbf{X}^= \rightarrow [X]^{<\lambda}$ there exists an isotone section $\sigma: P \hookrightarrow \mathbf{X}$ of ∂ such that $S(\sigma(p)) \cap \sigma(q) \subseteq \sigma(p)$ whenever $p < q$ in P .
- (3) If $\lambda = \omega$, then X is supported.

If $(X, \mathbf{X}, \partial)$ is a λ -lifter of P with \mathbf{X} the set of all principal ideals of X , we will say that (X, ∂) is a *principal λ -lifter* of P .

Our next construction will provide us with all the lifters we will need. The construction $P\langle K \rangle$ was introduced in the proof of Gillibert and Wehrung [14, Lemma 3.5.5].

Definition 10.2. For any poset P and any set K , let

$$P\langle K \rangle \stackrel{\text{def}}{=} \{(a, x) \mid a \in P \text{ and } (\exists \text{ finite } X \subseteq P \downarrow a)(a \in \nabla X \text{ and } x: X \rightarrow K)\}, \quad (10.1)$$

ordered componentwise (i.e., $(a, x) \leq (b, y)$ iff $a \leq b$ and y extends x) and endowed with the P -valued “norm function” ∂ defined *via* the rule $\partial(a, x) \stackrel{\text{def}}{=} a$.

The following easy result is contained in the proof of Gillibert and Wehrung [14, Lemma 3.5.5].

Lemma 10.3. *Let P be an almost join-semilattice with zero and let K be a set. Then $P\langle K \rangle$ is a lower finite almost join-semilattice with zero and $(P\langle K \rangle, \partial)$ is a norm-covering of P .*

Definition 10.4. If $P\langle K \rangle$ (together with its canonical norm function ∂) is a principal λ -lifter of P , we will call it a *standard λ -lifter* of P .

The following observation shows that the construction $P\langle K \rangle$ yields a convenient class of submorphisms of norm-coverings (cf. Definition 9.1), and thus morphisms of P -scaled Boolean algebras (cf. Lemma 9.2).

Lemma 10.5. *Let P be an almost join-semilattice with zero and let $f: X \rightarrow Y$ be a one-to-one map from a set X into a set Y . Then the rule $(a, x) \mapsto (a, f \circ x)$ defines a one-to-one submorphism $P\langle f \rangle: P\langle X \rangle \rightarrow P\langle Y \rangle$ of norm-coverings. In particular, it induces a morphism $\mathbf{F}(P\langle f \rangle): \mathbf{F}(P\langle X \rangle) \rightarrow \mathbf{F}(P\langle Y \rangle)$ of P -scaled Boolean algebras.*

Proof. Let $n < \omega$, $(a_1, x_1), \dots, (a_n, x_n) \in P\langle X \rangle$, and set $x = \bigcup_{i=1}^n x_i$. It is straightforward to verify that

$$\nabla\{(a_i, f \circ x_i) \mid 1 \leq i \leq n\} = \begin{cases} \nabla\{a_i \mid 1 \leq i \leq n\} \times \{f \circ x\}, & \text{if } x \text{ is a function,} \\ \emptyset, & \text{otherwise,} \end{cases}$$

where the left hand side is evaluated within $P\langle Y \rangle$.

The last part of the statement of Lemma 10.5 then follows from Lemma 9.2. \square

Following Gillibert and Wehrung [14, Definition 3.5.1], for cardinals κ and λ and a poset P , $(\kappa, < \lambda) \rightsquigarrow P$ means that for every $F: \mathfrak{P}(\kappa) \rightarrow [\kappa]^{< \lambda}$ there exists a one-to-one map $f: P \rightarrow \kappa$ such that

$$F(f[P \downarrow a]) \cap f[P \downarrow b] \subseteq f[P \downarrow a] \quad \text{whenever } a < b \text{ in } P.$$

Also recall from Erdős *et al.* [10] that for cardinals κ , λ , ρ , $(\kappa, < \omega, \lambda) \rightarrow \rho$ means that for every map $F: [\kappa]^{< \omega} \rightarrow [\kappa]^{< \lambda}$ there exists $H \in [\kappa]^\rho$ such that $F(X) \cap H \subseteq X$ whenever $X \in [H]^{< \omega}$.

As observed in Gillibert and Wehrung [15, Proposition 3.4], the statements $(\kappa, < \lambda) \rightsquigarrow ([\rho]^{< \omega}, \subseteq)$ and $(\kappa, < \omega, \lambda) \rightarrow \rho$ are equivalent. Since every lower finite poset P of cardinality ρ embeds into $[P]^{< \omega}$ (via $x \mapsto P \downarrow x$), thus into $[\rho]^{< \omega}$, we obtain the following.

Lemma 10.6. *Let κ and λ be infinite cardinals and let P be a lower finite poset. Set $\rho \stackrel{\text{def}}{=} \text{card } P$. If $(\kappa, < \omega, \lambda) \rightarrow \rho$, then $(\kappa, < \lambda) \rightsquigarrow P$.*

We record the following consequence of Gillibert and Wehrung [14, Lemma 3.5.5].

Lemma 10.7. *Let P be a lower finite almost join-semilattice with zero, let λ and κ be infinite cardinals such that every element of P has less than $\text{cf}(\lambda)$ upper covers and $(\kappa, < \lambda) \rightsquigarrow P$. Then $P\langle \kappa \rangle$ is a standard λ -lifter of P .*

Using Gillibert and Wehrung [15, Proposition 4.7], Lemma 10.7 enables us to find lifters for finite posets:

Corollary 10.8. *Let P be a nontrivial finite almost join-semilattice with zero and denote by n the order-dimension of P . Then for every infinite cardinal λ and for every $\kappa \geq \lambda^{+(n-1)}$, $P\langle \kappa \rangle$ is a standard λ -lifter of P .*

11. EXTENDING THE ARMATURE LEMMA AND CLL

The main aim of this section is to establish Lemmas 11.1 and 11.2, which are extensions to Φ -commutative diagrams and \otimes_{Φ}^{μ} of the original Armature Lemma and Condensate Lifting Lemma CLL (cf. Lemmas 3.2.2 and 3.4.2, respectively, in

Gillibert and Wehrung [14]). The updated statements, although still quite technical, are somehow trimmed down in comparison to the original statements from [14], by allowing an apparently smaller level of generality: for example, λ is now assumed to be regular and $\mathbf{X}^=$ is, in Lemma 11.2, assumed to be well-founded. Other differences between the original statements and the new ones are the following:

- The original statement of the Armature Lemma involved a morphism $\chi: S \rightarrow \Phi(\mathbf{F}(X) \otimes_{\Phi} \vec{A})$. There is no loss of generality in assuming that χ is the identity (just replace each $\varphi_{\mathbf{x}}$ by $\chi \circ \varphi_{\mathbf{x}}$), which we thus do in Lemma 11.1.
- Our assumptions contain the additional statement that P is a μ -small conditional DCPO if $\mu > \omega$.
- Due to the different definition of \otimes_{Φ}^{μ} (with respect to the original \otimes of Gillibert and Wehrung [14]), the functor Φ no longer needs to preserve any kind of directed colimit.
- The cardinal λ plays the same role as in Lemmas 3.2.2 and 3.4.2 of Gillibert and Wehrung [14]. This is not the case for μ , which is the parameter indexing the operator \otimes_{Φ}^{μ} .

The proofs of Lemma 11.1 and 11.2 are similar to the ones of Lemma 3.2.2 and 3.4.2 in Gillibert and Wehrung [14], with a few subtle differences. Due to the complexity of the underlying statements, we anchor the new formulations in our discussion by showing quite detailed outlines of those proofs.

Lemma 11.1 (Extended Armature Lemma). *Let λ and μ be infinite regular cardinals with $\mu \leq \lambda$, and let P be a poset with a λ -lifter $(X, \mathbf{X}, \partial)$. We also assume that if $\mu > \omega$, then P is a μ -small conditional DCPO. Let \mathcal{A} and \mathcal{S} be categories and let $\Phi: \mathcal{A} \rightarrow \mathcal{S}$ be a functor. We assume that \mathcal{A} has all binary products if $\mu = \omega$ and all $2^{<\mu}$ -products if $\mu > \omega$, and that \mathcal{S} has all μ -directed colimits.*

Let \vec{A} be a P -indexed, Φ -commutative diagram in \mathcal{A} , set $S \stackrel{\text{def}}{=} \mathbf{F}(X) \otimes_{\Phi}^{\mu} \vec{A}$, and let $((S_{\mathbf{x}}, \varphi_{\mathbf{x}}), \varphi_{\mathbf{x}}^{\mathbf{y}} \mid \mathbf{x} \subseteq \mathbf{y} \text{ in } \mathbf{X})$ be an \mathbf{X} -indexed commutative diagram in \mathcal{S}/\mathcal{S} such that $S_{\mathbf{x}}$ is weakly λ -presented whenever $\mathbf{x} \in \mathbf{X}^=$. Then there exists an isotone section $\sigma: P \hookrightarrow \mathbf{X}$ of ∂ such that the family $\left((\pi_{\sigma(p)}^X \otimes_{\Phi}^{\mu} \vec{A}) \circ \varphi_{\sigma(p)} \mid p \in P \right)$ is a natural transformation from the commutative diagram $\vec{S}\sigma$ to $\Phi\vec{A}$.

Proof. Our assumptions ensure the existence of the functor $_{-} \otimes_{\Phi}^{\mu} \vec{A}$ from \mathbf{Bool}_P to \mathcal{S} (cf. Lemma 6.8). Moreover, since the diagram \vec{A} is Φ -commutative, the diagram $\Phi\vec{A}$ is commutative. For all $p \leq q$ in P , the constant value α_p^q of $\Phi(x)$, for x ranging over $\vec{A}(p, q)$, is a morphism from $\Phi(A_p)$ to $\Phi(A_q)$.

As at the beginning of the proof of Gillibert and Wehrung [14, Lemma 3.2.2], X is the λ -directed, thus also μ -directed (because $\mu \leq \lambda$) union of the set $[X]_{\nabla}^{<\lambda}$ of all its λ -small ∇ -closed subsets. For every $\mathbf{x} \in \mathbf{X}^=$, since

$$\varphi_{\mathbf{x}}: S_{\mathbf{x}} \rightarrow \mathbf{F}(X) \otimes_{\Phi}^{\mu} \vec{A} = \varinjlim (\mathbf{F}(Z) \otimes_{\Phi}^{\mu} \vec{A} \mid Z \in [X]_{\nabla}^{<\lambda}) \quad (11.1)$$

where the transition morphisms and limiting morphisms in the right hand side of (11.1) all have the form $f_{Z_0}^{Z_1} \otimes_{\Phi}^{\mu} \vec{A}$, and $S_{\mathbf{x}}$ is weakly λ -presented, there exists a λ -small ∇ -closed subset $V(\mathbf{x})$ of X such that $\varphi_{\mathbf{x}}$ factors through $\mathbf{F}(V(\mathbf{x})) \otimes_{\Phi}^{\mu} \vec{A}$. The mapping V thus goes from $\mathbf{X}^=$ to $[X]^{<\lambda}$. As in the proof of [14, Lemma 3.2.2], we may assume that the map V is isotone. By the definition of $V(\mathbf{x})$, there is a

morphism $\psi_{\mathbf{x}}: S_{\mathbf{x}} \rightarrow \mathbf{F}(V(\mathbf{x})) \otimes_{\Phi}^{\mu} \vec{A}$ such that

$$\varphi_{\mathbf{x}} = (f_{V(\mathbf{x})}^X \otimes_{\Phi}^{\mu} \vec{A}) \circ \psi_{\mathbf{x}}. \quad (11.2)$$

Since $(X, \mathbf{X}, \partial)$ is a λ -lifter of P , there is an isotone section $\sigma: P \hookrightarrow \mathbf{X}$ of ∂ such that

$$V(\sigma(p)) \cap \sigma(q) \subseteq \sigma(p) \quad \text{for all } p < q \text{ in } P. \quad (11.3)$$

The proof of the following claim is identical to the one of the Claim in the proof of [14, Lemma 3.2.2] (it is a direct translation of (11.3)) and we omit it.

Claim. *The equation $\pi_{\sigma(q)}^X \circ f_{V\sigma(p)}^X = \varepsilon_p^q \circ \pi_{\sigma(p)}^X \circ f_{V\sigma(p)}^X$ holds for all $p < q$ in P .*

By applying the functor $- \otimes_{\Phi}^{\mu} \vec{A}$ to the two sides of the Claim above and then applying Lemma 6.8, together with Lemma 6.10, we obtain the equation

$$(\pi_{\sigma(q)}^X \otimes_{\Phi}^{\mu} \vec{A}) \circ (f_{V\sigma(p)}^X \otimes_{\Phi}^{\mu} \vec{A}) = \alpha_p^q \circ (\pi_{\sigma(p)}^X \otimes_{\Phi}^{\mu} \vec{A}) \circ (f_{V\sigma(p)}^X \otimes_{\Phi}^{\mu} \vec{A}). \quad (11.4)$$

The end of the proof goes the same way as in the one of Gillibert and Wehrung [14, Lemma 3.2.2]: for all $p < q$ in P ,

$$\begin{aligned} \alpha_p^q \circ (\pi_{\sigma(p)}^X \otimes_{\Phi}^{\mu} \vec{A}) \circ \varphi_{\sigma(p)} &= \alpha_p^q \circ (\pi_{\sigma(p)}^X \otimes_{\Phi}^{\mu} \vec{A}) \circ (f_{V\sigma(p)}^X \otimes_{\Phi}^{\mu} \vec{A}) \circ \psi_{\sigma(p)} && \text{(use (11.2))} \\ &= (\pi_{\sigma(q)}^X \otimes_{\Phi}^{\mu} \vec{A}) \circ (f_{V\sigma(p)}^X \otimes_{\Phi}^{\mu} \vec{A}) \circ \psi_{\sigma(p)} && \text{(use (11.4))} \\ &= (\pi_{\sigma(q)}^X \otimes_{\Phi}^{\mu} \vec{A}) \circ \varphi_{\sigma(p)} && \text{(use (11.2))} \\ &= (\pi_{\sigma(q)}^X \otimes_{\Phi}^{\mu} \vec{A}) \circ \varphi_{\sigma(q)} \circ \varphi_{\sigma(p)}^{\sigma(q)}, \end{aligned}$$

which completes the proof of the desired naturality statement. \square

The following Lemma 11.2, extending the original CLL (viz. Gillibert and Wehrung [14, Lemma 3.4.2]) can be viewed as a more ‘‘global’’ version of Lemma 11.1. Its statement involves a subcategory $\mathcal{S}^{\Rightarrow}$ of \mathcal{S} , whose morphisms will be called the *double arrows* and denoted in the form $x: S_1 \Rightarrow S_2$. Similarly, natural transformations with all arrows in $\mathcal{S}^{\Rightarrow}$ will be denoted in the form $\vec{x}: \vec{S}_1 \Rightarrow \vec{S}_2$.

For the statement of Lemma 11.2, recall that λ -extended Φ -projections were introduced in Definition 8.1. Lemma 11.2 also involves the *projectability witnesses* introduced in Wehrung [32, Definition 3.2], see also Gillibert and Wehrung [14, Definition 1.5.1]. Heuristically, for a functor Ψ , a projectability witness for an arrow $\psi: \Psi(C) \rightarrow S$ plays the role of a ‘‘quotient’’ \overline{C} of C such that ψ induces an isomorphism $\overline{\psi}: \Psi(\overline{C}) \rightarrow S$. As a full definition of that concept is relatively technical, and as everything we need here about it has already been proved elsewhere, we refer the reader to the abovementioned references for more detail.

Lemma 11.2 (Extended CLL). *Let λ and μ be infinite regular cardinals with $\mu \leq \lambda$, and let P be a poset with a λ -lifter $(X, \mathbf{X}, \partial)$. Let \mathcal{A} , \mathcal{B} , and \mathcal{S} be categories, with functors $\Phi: \mathcal{A} \rightarrow \mathcal{S}$ and $\Psi: \mathcal{B} \rightarrow \mathcal{S}$. Let \mathcal{B}^{\dagger} be a full subcategory of \mathcal{B} and let $\mathcal{S}^{\Rightarrow}$ (the double arrows in \mathcal{S}) be a subcategory of \mathcal{S} . We are given the following data:*

- a P -indexed, Φ -commutative diagram $\vec{A} = (A_p, \vec{A}(p, q) \mid p \leq q \text{ in } P)$ in \mathcal{A} ;
- an object $B \in \mathcal{B}$ together with a double arrow $\chi: \Psi(B) \Rightarrow \mathbf{F}(X) \otimes_{\Phi}^{\mu} \vec{A}$.

We make the following assumptions:

- (WF) $\mathbf{X}^=$ is well-founded.
- (COND(μ)) If $\mu > \omega$, then P is a μ -small conditional DCPO.
- (PROD(μ)) \mathcal{A} has all binary products if $\mu = \omega$ and all $2^{<\mu}$ -products if $\mu > \omega$.

- (COLIM(μ)) \mathcal{S} has all μ -directed colimits.
 (PROJ(μ)) Every μ -extended Φ -projection belongs to $\mathcal{S}^{\Rightarrow}$.
 (PRES(λ)) For every $C \in \mathcal{B}^{\dagger}$, $\Psi(C)$ is weakly λ -presented in \mathcal{S} .
 (LS(λ)) For every $p \in P$, every $\psi: \Psi(B) \Rightarrow \Phi(A_p)$, every $\alpha < \lambda$, and every family $(\gamma_{\xi}: C_{\xi} \rightarrow B \mid \xi < \alpha)$ in the subobject category $\mathcal{B}^{\dagger} \downarrow B$, there exists a subobject γ in $\mathcal{B}^{\dagger} \downarrow B$ such that $\psi \circ \Psi(\gamma) \in \mathcal{S}^{\Rightarrow}$ and each $\gamma_{\xi} \leq \gamma$.

Then there are a commutative diagram $\vec{B} \in \mathcal{B}^P$ and a natural transformation $\vec{\chi}: \Psi \vec{B} \Rightarrow \Phi \vec{A}$ in $\mathcal{S}^{\Rightarrow}$. Furthermore, if every double arrow $\psi: \Psi(C) \Rightarrow S$, where $C \in \mathcal{B}$ and $S \in \mathcal{S}$, has a projectability witness with respect to the functor Ψ , then $\vec{\chi}$ can be taken a natural equivalence.

Proof. As at the beginning of the proof of Lemma 11.1, our assumptions ensure the commutativity of the diagram $\Phi \vec{A}$ and the existence of the functor $- \otimes_{\Phi}^{\mu} \vec{A}$ from \mathbf{Bool}_P to \mathcal{S} (cf. Lemma 6.8).

For every $\mathbf{x} \in \mathbf{X}$, the morphism $\pi_{\mathbf{x}}^X: \mathbf{F}(X) \rightarrow \mathbf{2}[\partial \mathbf{x}]$ is normal (cf. Gillibert and Wehrung [14, Lemma 2.6.7]). By Proposition 8.2, $\pi_{\mathbf{x}}^X \otimes_{\Phi}^{\mu} \vec{A}$ is a μ -extended Φ -projection from $\mathbf{F}(X) \otimes_{\Phi}^{\mu} \vec{A}$ to $\mathbf{2}[\partial \mathbf{x}] \otimes_{\Phi}^{\mu} \vec{A} = \Phi(A_{\partial \mathbf{x}})$ (cf. Lemma 6.10). By (PROJ(μ)), it follows that $\pi_{\mathbf{x}}^X \otimes_{\Phi}^{\mu} \vec{A}$ is a double arrow. Therefore, $\pi_{\mathbf{x}}^X \otimes_{\Phi}^{\mu} \vec{A}$ is a double arrow from $\mathbf{F}(X) \otimes_{\Phi}^{\mu} \vec{A}$ to $\Phi(A_{\partial \mathbf{x}})$. It follows that the composite $\rho_{\mathbf{x}} \stackrel{\text{def}}{=} (\pi_{\mathbf{x}}^X \otimes_{\Phi}^{\mu} \vec{A}) \circ \chi$ is a double arrow from $\Psi(B)$ to $\Phi(A_{\partial \mathbf{x}})$.

Due to the simplification brought by assuming (WF) from the start, all assumptions underlying the monic form of the Buttress Lemma (cf. Lemma 3.3.2 and Remark 3.3.3 in Gillibert and Wehrung [14]) are, taking $U := \mathbf{X}^=$, satisfied. This yields an $\mathbf{X}^=$ -indexed commutative diagram $(\gamma_{\mathbf{x}}, \gamma_{\mathbf{y}}^{\mathbf{y}} \mid \mathbf{x} \subseteq \mathbf{y} \text{ in } \mathbf{X}^=)$ in $\mathcal{B}^{\dagger} \downarrow B$, say $\gamma_{\mathbf{x}}: C_{\mathbf{x}} \rightarrow B$ and $\gamma_{\mathbf{x}}^{\mathbf{y}}: C_{\mathbf{x}} \rightarrow C_{\mathbf{y}}$ (all $C_{\mathbf{x}} \in \mathcal{B}^{\dagger}$), such that each $\rho_{\mathbf{x}} \circ \Phi(\gamma_{\mathbf{x}}) \in \mathcal{S}^{\Rightarrow}$. This diagram can be extended to an \mathbf{X} -indexed commutative diagram in $\mathcal{B} \downarrow B$, by setting $C_{\mathbf{x}} = B$ and $\gamma_{\mathbf{x}} = \gamma_{\mathbf{x}}^{\mathbf{y}} = \text{id}_B$ whenever $\mathbf{x} \subseteq \mathbf{y}$ in $\mathbf{X}^=$, whereas $\gamma_{\mathbf{x}}^{\mathbf{y}} = \gamma_{\mathbf{x}}$ whenever $\mathbf{x} \subseteq \mathbf{y}$, $\mathbf{x} \in \mathbf{X}^=$, and $\mathbf{y} \in \mathbf{x} \setminus \mathbf{X}^=$. The relation $\rho_{\mathbf{x}} \circ \Phi(\gamma_{\mathbf{x}}) \in \mathcal{S}^{\Rightarrow}$ now holds for all $\mathbf{x} \in \mathbf{X}$: if $\mathbf{x} \in \mathbf{X} \setminus \mathbf{X}^=$, then $\rho_{\mathbf{x}} \circ \Phi(\gamma_{\mathbf{x}}) = \rho_{\mathbf{x}} \in \mathcal{S}^{\Rightarrow}$. Hence,

$$\rho_{\mathbf{x}} \circ \Phi(\gamma_{\mathbf{x}}): \Psi(B) \Rightarrow \Phi(A_{\partial \mathbf{x}}), \quad \text{for any } \mathbf{x} \in \mathbf{X}. \quad (11.5)$$

It follows from the assumption (PRES(λ)) that $\Psi(C_{\mathbf{x}})$ is weakly λ -presented whenever $\mathbf{x} \in \mathbf{X}^=$. Setting $\varphi_{\mathbf{x}} \stackrel{\text{def}}{=} \chi \circ \Psi(\gamma_{\mathbf{x}})$ and $\varphi_{\mathbf{x}}^{\mathbf{y}} \stackrel{\text{def}}{=} \Psi(\gamma_{\mathbf{x}}^{\mathbf{y}})$, all the assumptions of Lemma 11.1 are satisfied. This yields an isotone section σ of ∂ such that the family $\vec{\chi} = (\chi_p \mid p \in P)$, where each $\chi_p \stackrel{\text{def}}{=} \rho_{\sigma(p)} \circ \Psi(\gamma_{\sigma(p)})$, is a natural transformation from the commutative diagram $\Psi \vec{B}$, where $\vec{B} \stackrel{\text{def}}{=} (C_{\sigma(p)}, \gamma_{\sigma(p)}^{\sigma(q)} \mid p \leq q \text{ in } P)$, to $\Phi \vec{A}$. By (11.5), each χ_p is a double arrow.

The last statement of Lemma 11.2, that existence of enough projectability witnesses implies that $\vec{\chi}$ can be taken a natural equivalence, is proved the same way as at the end of Gillibert and Wehrung [14, Lemma 3.4.2]. \square

In Sections 12–15 we shall explore various occurrences of anti-elementarity following from Lemmas 11.1 and 11.2. The former (intervening in Section 12) offers the advantage of providing less restrictive cardinality assumptions, at the expense of requiring a deeper understanding of the class of structures under consideration. By contrast, Lemma 11.2 (intervening in Sections 13–15) yields more streamlined proofs, enabling us to apply known non-representability results as black boxes, at the expense of more restrictive cardinality assumptions.

12. CONRAD FRAMES AND CEVIAN LATTICES

Recall from Wehrung [39] that a binary operation \searrow on a distributive lattice D with zero is *Cevian* if there exists a binary operation $(x, y) \mapsto x \searrow y$ on D such that all inequalities $x \leq y \vee (x \searrow y)$, $(x \searrow y) \wedge (y \searrow x) = 0$, and $(x \searrow y) \wedge (y \searrow z) \leq x \searrow z \leq (x \searrow y) \vee (y \searrow z)$ hold whenever $x, y, z \in D$. We also say that the lattice D is Cevian if it carries a Cevian operation.

Recall from Iberkleid *et al.* [19] that a *Conrad frame* is a lattice isomorphic to the lattice $\text{Cs}G$ of all convex ℓ -subgroups of an ℓ -group (not necessarily Abelian) G . Since $\text{Cs}G$ is an algebraic frame, it is determined by the $(\vee, 0)$ -semilattice $\text{Cs}_c G$ of all finitely generated convex ℓ -subgroups of G , which turns out to be a distributive lattice with zero.

Let us record a few properties of Cevian lattices and (lattices of compact members of) Conrad frames, established in Wehrung [39]:

Proposition 12.1.

- (1) *Every Cevian lattice is completely normal, that is, for all $a, b \in D$ there are $x, y \in D$ such that $a \vee b = a \vee y = x \vee b$ whereas $x \wedge y = 0$.*
- (2) *There exists a non-Cevian completely normal bounded distributive lattice, of cardinality \aleph_2 .*
- (3) *For every ℓ -group G , the lattice $\text{Cs}_c G$ is Cevian.*
- (4) *For every representable⁵ ℓ -group G , the $(\vee, 0)$ -semilattice $\text{Id}_c G$ of all finitely generated ℓ -ideals of G is a lattice, and also a homomorphic image of $\text{Cs}_c G$; thus it is a Cevian lattice.*

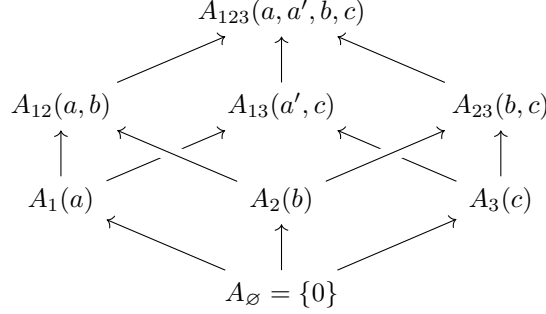
Let us recall the construction of the Id_c -commutative diagram \vec{A} , represented in Figure 12.1, introduced in Wehrung [39]. The indexing poset of our counterexample diagrams will be $P \stackrel{\text{def}}{=} \mathfrak{P}[3]$ (cf. Section 2). Denote by A_{123} the Abelian ℓ -group defined by the generators a, a', b, c subjected to the relations $0 \leq a \leq a' \leq 2a$, $0 \leq b$, and $0 \leq c$. For each $p \in P$, A_p denotes the ℓ -subgroup of A_{123} generated by $\nu(p)$ where $\nu(12) \stackrel{\text{def}}{=} \{a, b\}$, $\nu(13) \stackrel{\text{def}}{=} \{a', c\}$, $\nu(23) \stackrel{\text{def}}{=} \{b, c\}$, $\nu(1) \stackrel{\text{def}}{=} \{a\}$, $\nu(2) \stackrel{\text{def}}{=} \{b\}$, $\nu(3) \stackrel{\text{def}}{=} \{c\}$, $\nu(\emptyset) \stackrel{\text{def}}{=} \emptyset$. For $p \subseteq q$ in P , $\vec{A}(p, q)$ consists of the inclusion map, unless $p = 1$ and $q = 13$, in which case $\vec{A}(p, q)$ consists of the map sending a to a' , or $p = 1$ and $q = 123$, in which case $\vec{A}(p, q)$ consists of the two maps sending a to either a or a' . On the diagram, we emphasize each A_p with its canonical set of generators, for example $A_{12}(a, b)$, $A_{123}(a, a', b, c)$, and so on.

Notation 12.2. For any infinite regular cardinal θ and any diagram \vec{G} in the category $\ell\mathbf{Grp}$ of all ℓ -groups with ℓ -homomorphisms, we denote by $\mathcal{A}(\theta, \vec{G})$ the smallest subcategory of $\ell\mathbf{Grp}$, containing all objects and arrows of \vec{G} , and closed under products and under colimits indexed by λ , within $\ell\mathbf{Grp}$, whenever $\lambda \geq \theta$. Also, we denote by \mathbf{Cev} the class of all Cevian distributive lattices with zero.

Of course, every member of $\mathcal{A}(\theta, \vec{A})$ is an Abelian ℓ -group. Moreover, if $\theta > \omega$, then every member of $\mathcal{A}(\theta, \vec{A})$ is Archimedean (because every object in \vec{A} is Archimedean and the class of all Archimedean ℓ -groups is closed both under products and under all colimits indexed by uncountable regular cardinals).

We are now reaching this section's main result.

⁵Recall that an ℓ -group is *representable* if it is a subdirect product of totally ordered groups.

FIGURE 12.1. The non-commutative diagram \vec{A}

Theorem 12.3. *For all infinite regular cardinals θ and λ with $\theta \leq \lambda$, there exists a functor Δ , from $\mathfrak{P}_{\text{inj}}(\lambda^{+2})$ to the category of all distributive lattices with zero with $\mathcal{L}_{\infty\lambda}$ -elementary embeddings, satisfying the following statements:*

- (1) Δ preserves all λ -directed colimits;
- (2) For every λ^+ -small subset X of λ^{+2} , $\Delta(X)$ belongs to $\text{Id}_c(\mathcal{A}(\theta, \vec{A}))$;
- (3) $\Delta(\lambda^{+2})$ is not Cevian;
- (4) $\text{card } \Delta(X) \leq \exp_2(\lambda) + \text{card } X$ whenever $X \subseteq \lambda^{+2}$.

In particular, the pair $(\text{Id}_c(\mathcal{A}(\theta, \vec{A})), \mathbf{Cev})$ is anti-elementary.

Proof. Consider again the poset $P \stackrel{\text{def}}{=} \mathfrak{P}[3]$ and the P -indexed diagram \vec{A} introduced above. We shall apply the Extended Armature Lemma (i.e., Lemma 11.1), with $\lambda = \mu$, to the category $\mathcal{A} \stackrel{\text{def}}{=} \mathcal{A}(\theta, \vec{A})$, the category $\mathcal{S} := \mathbf{DLat}_0$ of all distributive lattices with zero and 0-lattice homomorphisms, the restriction Φ of the functor Id_c to $\mathcal{A}(\theta, \vec{A})$. Since $\mathcal{A}(\theta, \vec{A})$ is closed under products and since the diagram \vec{A} is Id_c -commutative, \vec{A} is also Φ -commutative.

Set $\kappa \stackrel{\text{def}}{=} \lambda^{+2}$. It follows from Corollary 10.8 that $K \stackrel{\text{def}}{=} P(\kappa)$ is a standard λ -lifter of P . By Lemma 10.5, the assignment $U \mapsto \mathbf{F}(P(U))$ extends naturally to a functor from $\mathfrak{P}_{\text{inj}}(\kappa)$ to \mathbf{Bool}_P . This functor sends every morphism $f: U \rightarrow V$ in $\mathfrak{P}_{\text{inj}}(\kappa)$ to $\mathbf{F}(P(f))$ (cf. Lemma 10.5). By composing that functor with $-\otimes_{\Phi}^{\lambda} \vec{A}$, we obtain a functor $\Gamma: \mathfrak{P}_{\text{inj}}(\kappa) \rightarrow \mathcal{S}$, $U \mapsto \mathbf{F}(P(U)) \otimes_{\Phi}^{\lambda} \vec{A}$. Any directed colimit $U = \varinjlim_{i \in I} U_i$ in $\mathfrak{P}_{\text{inj}}(\kappa)$ (essentially a directed union) gives rise to a directed colimit

$$\mathbf{F}(P(U)) = \varinjlim (\mathbf{F}(P(U_i)) \mid i \in I) \quad \text{within } \mathbf{Bool}_P.$$

Hence, the functor $U \mapsto \mathbf{F}(P(U))$ preserves all directed colimits. Since the functor $-\otimes_{\Phi}^{\lambda} \vec{A}$ preserves all λ -directed colimits, it follows that the composite $\Gamma \stackrel{\text{def}}{=} \mathbf{F}(P(-)) \otimes_{\Phi}^{\lambda} \vec{A}$ preserves all λ -directed colimits; that is, Condition (4.1) holds for that functor.

Claim 1. *For every λ^+ -small subset X of κ , the lattice $\Gamma(X)$ belongs to the range of Φ (thus, by Proposition 12.1, it is Cevian).*

Proof of Claim. If $\text{card } X < \lambda$ then $\Gamma(X) = \Phi(\mathbf{F}(P(X)) \otimes_{\Phi}^{\lambda} \vec{A})$ belongs to the range of Φ . Since Γ is a functor, the case where $\text{card } X = \lambda$ can be reduced to the case where $X = \lambda$. Since $\mathcal{A}(\theta, \vec{A})$ is closed under directed colimits indexed

by λ , those directed colimits are preserved by Φ , so we can apply the Boosting Lemma (viz. Lemma 7.3) to the λ -small P -scaled Boolean algebras $\mathbf{B}_\xi \stackrel{\text{def}}{=} \mathbf{F}(P\langle\xi\rangle)$ (for $\xi < \lambda$), $\mathbf{B} \stackrel{\text{def}}{=} \mathbf{F}(P\langle\lambda\rangle)$, each β_ξ^η is the canonical morphism $f_{P\langle\xi\rangle}^{P\langle\eta\rangle} : \mathbf{F}(P\langle\xi\rangle) \rightarrow \mathbf{F}(P\langle\eta\rangle)$, each β_ξ is the canonical morphism $f_{P\langle\xi\rangle}^{P\langle\lambda\rangle} : \mathbf{F}(P\langle\xi\rangle) \rightarrow \mathbf{F}(P\langle\lambda\rangle)$. Since $\mathbf{B} = \varinjlim_{\xi < \lambda} \mathbf{B}_\xi$, it follows that $\Gamma(\lambda) = \mathbf{B} \otimes_{\Phi}^{\lambda} \vec{A}$ belongs to the range of Φ . \square Claim 1.

Claim 2. *The lattice $\Gamma(\kappa)$ is not Cevian.*

Proof of Claim. The proof of this claim is established by an argument similar to the one, in the proof of [39, Theorem 7.2], showing that the lattice denoted there by \mathbf{B} is not Cevian. We give an outline for convenience.

Suppose that $\Gamma(\kappa)$ has a Cevian operation \smile and denote by $K_{(j)}$ the set of all elements of K of height j , for $j \in \{0, 1, 2, 3\}$. For each $x \in K$, it follows from the normality of the morphism $\pi_x^K : \mathbf{F}(P\langle\kappa\rangle) \rightarrow \mathbf{2}[\partial x]$ (cf. Section 9), together with Proposition 8.2, that the morphism $\rho_x \stackrel{\text{def}}{=} \pi_x^K \otimes_{\Phi}^{\lambda} \vec{A} : \Gamma(\kappa) \rightarrow \Phi(A_{\partial x})$ is an extended Φ -projection. Since every Φ -projection is (obviously) surjective, so is every extended Φ -projection, and so ρ_x is surjective. In particular, if $x \in K_{(1)}$, then $\Phi(A_{\partial x}) = \mathbf{2}$, so we may pick $b_x \in \Gamma(\kappa)$ such that $\rho_x(b_x) = 1$; set $S_x \stackrel{\text{def}}{=} \{0, b_x\}$. Further, if $x \in K_{(2)} \cup K_{(3)}$, denote by S_x the sublattice of $\Gamma(\kappa)$ generated by

$$\{b_u \mid u \in K_{(1)} \downarrow x\} \cup \{b_u \smile b_v \mid u, v \in K_{(1)} \downarrow x\}.$$

Since K is lower finite and $\Gamma(\kappa)$ is distributive, it follows that S_x is finite. Denote by $\varphi_x : S_x \hookrightarrow \Gamma(\kappa)$ the inclusion map, and, for $x \subseteq y$ in K , denote by $\varphi_x^y : S_x \hookrightarrow S_y$ the inclusion map. By applying Lemma 11.1 with $\mu := \lambda$, we get an isotone section $\sigma : P \hookrightarrow K$ of ∂ such that the family $\vec{\chi} \stackrel{\text{def}}{=} (\chi_p \mid p \in P)$, with each $\chi_p = \rho_{\sigma(p)} \upharpoonright_{S_{\sigma(p)}}$, is a natural transformation from $\vec{S}\sigma$ to $\Phi\vec{A}$, with $\chi_p(b_{\sigma(p)}) = 1$ whenever p is an atom of P . However, the last stages of the proof of [39, Theorem 7.2] show that the existence of such a natural transformation contradicts [39, Lemma 4.3]. \square Claim 2.

Now denote by $\dot{+}$ the usual ordinal addition and set $\Delta(X) \stackrel{\text{def}}{=} \Gamma(\lambda \cup (\lambda \dot{+} X))$ (where $\lambda \dot{+} X \stackrel{\text{def}}{=} \{\lambda \dot{+} \xi \mid \xi \in X\}$) whenever $X \subseteq \kappa$. Extend Δ to a functor from $\mathfrak{P}_{\text{inj}}(\kappa)$ to the category of all distributive lattices with zero, in the natural way. Since the cardinality of $\lambda \cup (\lambda \dot{+} X)$ is always greater than or equal to λ , it follows from Proposition 4.1 that Δ sends morphisms in $\mathfrak{P}_{\text{inj}}(\kappa)$ to $\mathcal{L}_{\infty\lambda}$ -elementary embeddings. Moreover, (2) follows from Claim 1 whereas (3) follows from Claim 2. Since Γ preserves all λ -directed colimits, so does Δ ; that is, (1) holds.

Finally, each $\Gamma(Z)$, where $Z \in [\kappa]^{<\lambda}$, has cardinality bounded above by $\exp_2(\lambda)$. By elementary cardinal arithmetic, it follows that for every $X \subseteq \kappa$,

$$\text{card } \Delta(X) \leq \sum \left(\text{card } \Gamma(Z) \mid Z \in [\lambda \cup (\lambda \dot{+} X)]^{<\lambda} \right) \leq \exp_2(\lambda) + \text{card } X;$$

that is, (4) holds. \square

Corollary 12.4. *Let θ be an infinite regular cardinal and let \mathcal{G} be a class of ℓ -groups containing $\mathcal{A}(\theta, \vec{A})$. Then $\text{Cs}_c(\mathcal{G})$ is anti-elementary.*

Since every member of $\mathcal{A}(\aleph_1, \vec{A})$ is Archimedean, we obtain:

Corollary 12.5. *Let \mathcal{G} be a class of ℓ -groups containing all Archimedean ℓ -groups. Then $\text{Cs}_c(\mathcal{G})$ is anti-elementary.*

Corollary 12.6. *Let θ be an infinite regular cardinal and let \mathcal{G} be a class of representable ℓ -groups containing $\mathcal{A}(\theta, \vec{A})$. Then $\text{Id}_c(\mathcal{G})$ is anti-elementary.*

For a further extension of that result, see Theorem 13.9.

Again since every member of $\mathcal{A}(\aleph_1, \vec{A})$ is Archimedean, we obtain:

Corollary 12.7. *Let \mathcal{G} be a class of representable ℓ -groups containing all Archimedean ℓ -groups. Then $\text{Id}_c(\mathcal{G})$ is anti-elementary.*

In order to extend Corollary 12.6 to ℓ -groups with unit, we shall use the following easy observation, involving the notations P^∞ and $G \times_{\text{lex}} H$ introduced in Section 2.

Lemma 12.8. *For any ℓ -group G , $\text{Id}_c(\mathbb{Z} \times_{\text{lex}} G) \cong (\text{Id}_c G)^\infty$.*

Corollary 12.9. *Let θ be an infinite regular cardinal and let \mathcal{G} be a class of representable ℓ -groups containing $\{\mathbb{Z} \times_{\text{lex}} G \mid G \in \mathcal{A}(\theta, \vec{A})\}$. Then $\text{Id}_c(\mathcal{G})$ is anti-elementary.*

Proof. Let λ be an infinite regular cardinal with $\lambda \geq \theta$ and let Δ be the functor, defined on $\mathfrak{P}_{\text{inj}}(\lambda^{+2})$, given by Theorem 12.3. Denote by E the functor, from the category of all distributive lattices with zero to the category of all bounded distributive lattices, that sends any object D to D^∞ and any morphism f to its extension preserving ∞ . The functor $\Delta^\infty \stackrel{\text{def}}{=} E \circ \Delta$ preserves all λ -directed colimits. Since $\Delta(\lambda) \cong \text{Id}_c A$ for some $A \in \mathcal{A}(\theta, \vec{A})$, it follows from Lemma 12.8 that $\Delta^\infty(\lambda) \cong \text{Id}_c(\mathbb{Z} \times_{\text{lex}} A)$, with $\mathbb{Z} \times_{\text{lex}} A \in \mathcal{C}$ by assumption.

Since $\Delta(\lambda^{+2})$ is not Cevian, neither is $\Delta^\infty(\lambda^{+2}) = \Delta(\lambda^{+2})^\infty$. On the other hand, since every member of \mathcal{C} is representable, every member of $\text{Id}_c(\mathcal{C})$ is Cevian (cf. Proposition 12.1). \square

We do not know whether Corollary 12.7 extends to Archimedean ℓ -groups with order-unit. The problem is that the arrows in the diagram \vec{A} are not unit-preserving.

Since Mundici's well known category equivalence [27], between *Abelian ℓ -groups with unit* and *MV-algebras*, preserves the concept of ideal, we thus obtain the following solution to the *MV-spectrum problem* stated in Mundici [28, Problem 2]:

Corollary 12.10. *The class of all Stone duals of spectra of MV-algebras (or, equivalently, of all Stone duals of spectra of Abelian ℓ -groups with order-unit) is anti-elementary.*

In particular, the class of all Stone duals of spectra of MV-algebras is not closed under $\mathcal{L}_{\infty\lambda}$ -elementary equivalence for any cardinal λ (in particular, it is not definable by any class of $\mathcal{L}_{\infty\lambda}$ sentences). Mellor and Tressl [26] proved an analogue result for Stone duals of *real spectra* of commutative unital rings. We do not know whether the stronger statement, that the class of all those lattices is anti-elementary, holds.

13. STRUCTURES WITH PERMUTABLE CONGRUENCES

This section will be devoted to establishing anti-elementarity of classes of finitely generated congruences of various congruence-permutable structures, such as modules, rings, ℓ -groups. Our argument will slightly deviate from the one of Section 12,

in the sense that we will use the Extended CLL (viz. Lemma 11.2) rather than the Extended Armature Lemma (viz. Lemma 11.1), thus assuming more global, easier stated versions of the unliftability of the diagram \vec{S} of Figure 13.1, causing us to include in the statement of Theorem 13.8 the assumption that $\lambda \geq \aleph_1$.

13.1. Categorical settings for Section 13. Throughout Section 13 we shall consider the category $\mathcal{S} := \mathbf{SLat}_0$ of all $(\vee, 0)$ -semilattices with $(\vee, 0)$ -homomorphisms. Following Wehrung [37, Definition 7-3.4], we say that a $(\vee, 0)$ -homomorphism $f: S \rightarrow T$ of $(\vee, 0)$ -semilattices is *weakly distributive* if for all $s \in S$ and all $t_0, t_1 \in T$, if $f(s) \leq t_0 \vee t_1$, then there are $s_0, s_1 \in S$ such that $s \leq s_0 \vee s_1$ and each $f(s_i) \leq t_i$. We shall denote by \mathcal{S}^\Rightarrow (this section's double arrows) the subcategory of \mathcal{S} consisting of the weakly distributive $(\vee, 0)$ -homomorphisms.

The indexing poset for our crucial counterexample diagrams will again be $P \stackrel{\text{def}}{=} \mathfrak{P}[3]$ (cf. Section 2). We shall denote by \vec{S} the P -indexed commutative diagram in \mathcal{S} represented in Figure 13.1, with the maps \mathbf{e} , \mathbf{s} , \mathbf{p} defined by $\mathbf{e}(x) \stackrel{\text{def}}{=} (x, x)$, $\mathbf{s}(x, y) \stackrel{\text{def}}{=} (y, x)$, $\mathbf{p}(x, y) \stackrel{\text{def}}{=} x \vee y$ whenever $x, y \in \mathbf{2}$. Its origin can be traced back to Tuma and Wehrung [30], with a more complicated precursor of that diagram.

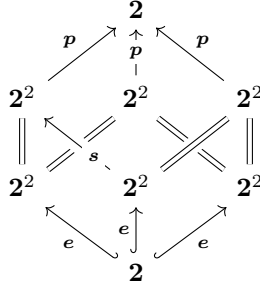


FIGURE 13.1. The commutative diagram \vec{S}

Let us recall the definition of the category \mathbf{Metr} introduced in Gillibert and Wehrung [14, § 5.1]. The objects of \mathbf{Metr} are the *semilattice-metric spaces*, that is, the triples $\mathbf{A} = (A, \delta_{\mathbf{A}}, \tilde{A})$ such that A is a set, \tilde{A} is a $(\vee, 0)$ -semilattice, and $\delta_{\mathbf{A}}: A \times A \rightarrow \tilde{A}$ is a *semilattice-valued distance*, that is,

$$\delta_{\mathbf{A}}(x, x) = 0; \quad \delta_{\mathbf{A}}(x, y) = \delta_{\mathbf{A}}(y, x); \quad \delta_{\mathbf{A}}(x, z) \leq \delta_{\mathbf{A}}(x, y) \vee \delta_{\mathbf{A}}(y, z)$$

whenever $x, y, z \in A$; we say that \mathbf{A} is of *type 1* if for all $x, y \in A$ and all $\mathbf{a}, \mathbf{b} \in \tilde{A}$, if $\delta_{\mathbf{A}}(x, y) \leq \mathbf{a} \vee \mathbf{b}$, then there exists $z \in A$ such that $\delta_{\mathbf{A}}(x, z) \leq \mathbf{a}$ and $\delta_{\mathbf{A}}(z, y) \leq \mathbf{b}$. For objects \mathbf{A} and \mathbf{B} of \mathbf{Metr} , a *morphism* from \mathbf{A} to \mathbf{B} is a pair (f, \tilde{f}) , where $f: A \rightarrow B$ is a map and $\tilde{f}: \tilde{A} \rightarrow \tilde{B}$ is a $(\vee, 0)$ -homomorphism, such that $\delta_{\mathbf{B}}(f(x), f(y)) = \tilde{f}(\delta_{\mathbf{A}}(x, y))$ whenever $x, y \in A$.

Throughout Section 13 we shall consider the full subcategory \mathcal{B} of \mathbf{Metr} whose objects are all semilattice-metric spaces \mathbf{A} of type 1 such that the range of $\delta_{\mathbf{A}}$ generates \tilde{A} as a $(\vee, 0)$ -semilattice — we will say that it *join-generates* \tilde{A} . Moreover, we shall denote by $\Psi: \mathcal{B} \rightarrow \mathcal{S}$, $\mathbf{A} \mapsto \tilde{A}$ the forgetful functor.

Recall the following observation from Růžička *et al.* [29]:

Proposition 13.1. *The $(\vee, 0)$ -semilattice of all finitely generated congruences of any congruence-permutable (universal) algebra belongs to the range of Ψ . In particular, the following $(\vee, 0)$ -semilattices all belong to the range of Ψ :*

- (1) *the $(\vee, 0)$ -semilattice $\text{Id}_c G$ of all principal ℓ -ideals of any ℓ -group G ;*
- (2) *the $(\vee, 0)$ -semilattice $\text{Id}_c R$ of all finitely generated two-sided ideals of any ring R ;*
- (3) *the $(\vee, 0)$ -semilattice $\text{Sub}_c M$ of all finitely generated submodules of any right module M .*

We omit the straightforward proof of the following lemma.

Lemma 13.2. *For any $(\vee, 0)$ -semilattice S , any object \mathbf{A} of \mathcal{B} , and any $f: \tilde{A} \Rightarrow S$, the composite $f\mathbf{A} \stackrel{\text{def}}{=} (A, f \circ \delta_{\mathbf{A}}, S)$ is an object of \mathcal{B} .*

Lemma 13.3. *There are no commutative diagram $\vec{\mathbf{B}} \in \mathcal{B}^P$ and no natural transformation $\vec{\chi}: \Psi \vec{\mathbf{B}} \Rightarrow \vec{S}$.*

Proof. Letting $\vec{\chi} = (\chi_p \mid p \in P)$, we may, thanks to Lemma 13.2, replace each \mathbf{B}_p by $\chi_p \mathbf{B}_p$ and thus assume that each $\tilde{B}_p = S_p$. But then, Wehrung [37, Theorem 9-5.1] leads to a contradiction. \square

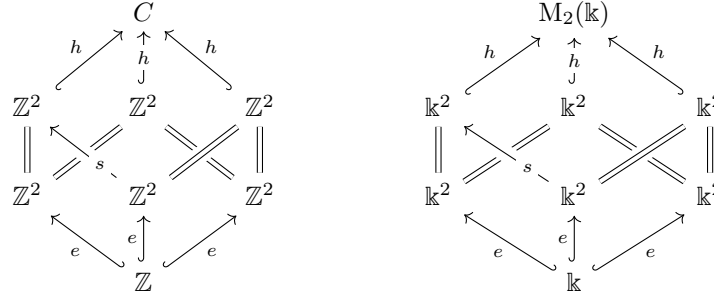


FIGURE 13.2. The non-commutative diagrams $\vec{C}_{a,g}$ and $\vec{R}_{k,g}$

The following lemma takes care of all instances of Condition (LS(λ)), from the statement of Lemma 11.2, occurring in Section 13. Its proof is a standard Löwenheim-Skolem type argument and we omit it.

Lemma 13.4. *Let λ be an uncountable regular cardinal, let \mathbf{B} be an object of \mathcal{B} , and let $f: \tilde{B} \Rightarrow S$ be a weakly distributive $(\vee, 0)$ -homomorphism with join-generating range. If S is λ -small, then the set of all $(X, Y) \in [B]^{<\lambda} \times [\tilde{B}]^{<\lambda}$, such that $(X, \delta_{\mathbf{B}}|_{X \times X}, Y)$ is an object of \mathcal{B} and $f|_Y$ is weakly distributive with join-generating range, is λ -closed cofinal in $[B]^{<\lambda} \times [\tilde{B}]^{<\lambda}$.*

13.2. Classes of non-representable ℓ -groups. It is well known that the assignment, that sends any ℓ -group G to the $(\vee, 0)$ -semilattice of its finitely generated ℓ -ideals, naturally extends to a functor $\text{Id}_c: \ell\mathbf{Grp} \rightarrow \mathcal{S}$. For any ℓ -homomorphism $f: G \rightarrow H$, $\text{Id}_c f$ sends every ℓ -ideal X of G to the ℓ -ideal of H generated by $f[X]$.

Let C be a non-representable ℓ -group. By one of the equivalent forms of representability (cf. Bigard *et al.* [6, Proposition 4.2.9]), there are $a \in C^+ \setminus \{0\}$ and $g \in C$ such that $a \wedge (g + a - g) = 0$ — we will say that (a, g) is an *NR-pair* of C .

Set $b \stackrel{\text{def}}{=} g + a - g$, then $e(x) \stackrel{\text{def}}{=} (x, x)$, $s(x, y) \stackrel{\text{def}}{=} (y, x)$, and $h(x, y) \stackrel{\text{def}}{=} xa + yb$, for all $x, y \in \mathbb{Z}$. From $a \wedge b = 0$ it follows that h is an ℓ -homomorphism. We denote by $\vec{C}_{a,g}$ the non-commutative diagram, in \mathbf{lGrp} , represented in the left hand side of Figure 13.2.

Lemma 13.5. *The diagram $\vec{C}_{a,g}$ is Id_c -commutative.*

Proof. For any set I , we need to prove that the image under the functor Id_c , of the diagram $\vec{C}_{a,g}^I$, is commutative. Let $p = (p_i \mid i \in I)$ and $q = (q_i \mid i \in I)$ in P^I such that $p \leq q$, let $(f_i \mid i \in I)$ and $(g_i \mid i \in I)$ be elements of $\prod (\vec{C}_{a,g}(p_i, q_i) \mid i \in I)$. Set $f \stackrel{\text{def}}{=} \prod (f_i \mid i \in I)$ and $g \stackrel{\text{def}}{=} \prod (g_i \mid i \in I)$. We need to prove that $f(z)$ and $g(z)$ generate the same ℓ -ideal of $\vec{C}_{a,g}^I(q)$, whenever $z = (z_i \mid i \in I)$ belongs to $\vec{C}_{a,g}^I(p)^+$. Every $i \in I$ outside $I_0 \stackrel{\text{def}}{=} \{i \in I \mid f_i = g_i\}$ satisfies $(p_i, q_i) = (1, 123)$, so, setting

$$\begin{aligned} I_1 &\stackrel{\text{def}}{=} \{i \in I \mid (p_i, q_i) = (1, 123) \text{ and } f_i = h \text{ and } g_i = h \circ s\}, \\ I_2 &\stackrel{\text{def}}{=} \{i \in I \mid (p_i, q_i) = (1, 123) \text{ and } f_i = h \circ s \text{ and } g_i = h\}, \end{aligned}$$

it follows that I is the disjoint union of I_0 , I_1 , and I_2 . For every $J \subseteq I$, denote by $z|J$ the element of $\vec{C}_{a,g}^I(p)$ agreeing with z on J and taking the constant value 0 on $I \setminus J$. Since $f(z)$ generates the same ℓ -ideal as $\{f(z|I_0), f(z|I_1), f(z|I_2)\}$, and similarly for g , it suffices to settle the case where $I = I_k$ for some $k \in \{0, 1, 2\}$. If $k = 0$ then $f = g$ and we are done. The two other cases being symmetric, it remains to settle the case where $k = 1$. For each $i \in I$, we can write $z_i = (x_i, y_i) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, thus $f_i(z_i) = x_i a + y_i b$ and $g_i(z_i) = x_i b + y_i a$. Since $nb = g + na - g$ for every integer n , we get

$$\begin{aligned} \langle (x_i a + y_i b \mid i \in I) \rangle^\ell &= \langle (x_i a \mid i \in I) \rangle^\ell \vee \langle (y_i b \mid i \in I) \rangle^\ell \\ &= \langle (x_i b \mid i \in I) \rangle^\ell \vee \langle (y_i a \mid i \in I) \rangle^\ell \\ &= \langle (x_i b + y_i a \mid i \in I) \rangle^\ell, \end{aligned}$$

that is, $\langle f(z) \rangle^\ell = \langle g(z) \rangle^\ell$, as required. \square

We set $\vec{\pi} \stackrel{\text{def}}{=} (\pi_p \mid p \in P)$, where π_{123} is the unique map from $\text{Id}_c C$ to $\mathbf{2}$ sending 0 to 0 and any nonzero element to 1, whereas, for $p \neq 123$, π_p is the canonical isomorphism from $\text{Id}_c C_{a,g}(p)$ (either $\text{Id}_c \mathbb{Z}^2$ or $\text{Id}_c \mathbb{Z}$) onto $S(p)$ (either $\mathbf{2}^2$ or $\mathbf{2}$). We omit the straightforward proof of the following lemma.

Lemma 13.6. *The family $\vec{\pi}$ is a natural transformation from $\text{Id}_c \vec{C}_{a,g}$ to \vec{S} in \mathcal{S}^\Rightarrow .*

Lemma 13.7. *There are no commutative diagram $\vec{B} \in \mathcal{B}^P$ and no natural transformation $\vec{\chi}: \Psi \vec{B} \Rightarrow \text{Id}_c \vec{C}_{a,g}$.*

Proof. If $\vec{\chi}$ were such a natural transformation, then, by Lemma 13.6, we would get $\vec{\pi} \circ \vec{\chi}: \Psi \vec{B} \Rightarrow \vec{S}$, in contradiction with Lemma 13.3. \square

We are now reaching this section's main result.

Theorem 13.8. *For all infinite regular cardinals θ and λ with $\theta + \aleph_1 \leq \lambda$, there exists a functor Δ , from $\mathfrak{P}_{\text{inj}}(\lambda^{+2})$ to the category \mathcal{S} of all $(\vee, 0)$ -semilattices with $\mathcal{L}_{\infty\lambda}$ -elementary embeddings, satisfying the following statements:*

- (1) Δ preserves all λ -directed colimits;

- (2) For every λ^+ -small subset X of λ^{+2} , $\Delta(X)$ belongs to $\text{Id}_c(\mathcal{A}(\theta, \vec{C}_{a,g}))$;
(3) There are no set A and no distance $\delta: A \times A \rightarrow \Delta(\lambda^{+2})$ of type 1 with join-generating range.

In particular, the pair $(\text{Id}_c(\mathcal{A}(\theta, \vec{C}_{a,g})), \text{rng } \Psi)$ is anti-elementary.

Proof. Set $\kappa \stackrel{\text{def}}{=} \lambda^{+2}$. It follows from Corollary 10.8 that $P(\kappa)$ is a standard λ -lifter of P . Denote by Φ the restriction of the functor Id_c to $\mathcal{A}(\theta, \vec{C}_{a,g})$. As at the beginning of the proof of Theorem 12.3, we obtain a functor $\Gamma: \mathfrak{P}_{\text{inj}}(\kappa) \rightarrow \mathfrak{S}$, $U \mapsto \mathbf{F}(P\langle U \rangle) \otimes_{\Phi}^{\lambda} \vec{C}_{a,g}$, and this functor preserves λ -directed colimits.

The proof of the following claim is, *mutatis mutandis*, identical to the one of Claim 1 of the proof of Theorem 12.3 (involving the Boosting Lemma) and we omit it.

Claim 1. For every λ^+ -small subset X of κ , the lattice $\Gamma(X)$ belongs to the range of Φ .

Claim 2. There are no set A and no distance $\delta: A \times A \rightarrow \Gamma(\kappa)$ of type 1 with join-generating range.

Proof of Claim. Let $\chi: \tilde{A} = \Psi(\mathbf{A}) \rightarrow \Gamma(\kappa)$ be an isomorphism (or, more generally, a weakly distributive $(\vee, 0)$ -homomorphism with join-generating range) and denote by \mathcal{B}^\dagger the full subcategory of \mathcal{B} whose objects are the \mathbf{B} such that B and \tilde{B} are both λ -small. We shall verify that Lemma 11.2 applies. All assumptions are trivially satisfied except for $(\text{PROJ}(\lambda))$ and $(\text{LS}(\lambda))$. The latter follows from Lemma 13.4. For the former, for any ℓ -groups G, H and any surjective ℓ -homomorphism $f: G \rightarrow H$, the $(\vee, 0)$ -homomorphism $\varphi \stackrel{\text{def}}{=} \text{Id}_c f: \text{Id}_c G \rightarrow \text{Id}_c H$ is surjective. Moreover, specializing Gillibert and Wehrung [14, Lemma 4.5.1] to the variety of all ℓ -groups and by the natural isomorphism between $\text{Id}_c G$ and the $(\vee, 0)$ -semilattice $\text{Con}_c G$ of all finitely generated congruences of any ℓ -group G , we see that for all $\mathbf{a}, \mathbf{b} \in \text{Id}_c G$, $\varphi(\mathbf{a}) \leq \varphi(\mathbf{b})$ iff there exists $\mathbf{x} \in \varphi^{-1}\{0\}$ such that $\mathbf{a} \leq \mathbf{b} \vee \mathbf{x}$ and $\varphi(\mathbf{x}) = 0$; with the terminology of Gillibert and Wehrung [14], φ is *ideal-induced*. Since every directed colimit of ideal-induced $(\vee, 0)$ -homomorphisms is ideal-induced, and since ideal-induced implies weakly distributive, every extended Φ -projection belongs to \mathfrak{S}^\Rightarrow . Condition $(\text{PROJ}(\lambda))$ follows.

Therefore, Lemma 11.2 applies and there are a commutative diagram $\vec{\mathbf{B}} \in \mathcal{B}^P$ and a natural transformation $\vec{\chi}: \Psi \vec{\mathbf{B}} \Rightarrow \Phi \vec{C}_{a,g}$; in contradiction with Lemma 13.7. \square Claim 2.

The remainder of the proof runs as in the one of Theorem 12.3, again with $\Delta(X) \stackrel{\text{def}}{=} \Gamma(\lambda \cup (\lambda \dot{+} X))$. \square

Theorem 13.9. Let \mathcal{G} be a subcategory of $\ell\mathbf{Grp}$, closed under products and under colimits indexed by all large enough regular cardinals, and containing all objects and arrows of the diagram \vec{A} of Section 12. Then $\text{Id}_c(\mathcal{G})$ is anti-elementary.

Proof. Let θ be any infinite regular cardinal such that \mathcal{G} is closed under all λ -indexed colimits whenever λ is an infinite regular cardinal $\geq \theta$. We separate cases. Our assumptions imply that \mathcal{G} contains the diagram $\mathcal{A}(\theta, \vec{A})$ (cf. Notation 12.2). Hence, if \mathcal{G} is representable, then we can apply Corollary 12.6. If \mathcal{G} is not representable, that is, it has a non-representable member C , which in turn has an NR-pair $(a, g) \in (C^+ \setminus \{0\}) \times C$. Then $\mathcal{A}(\theta, \vec{C}_{a,g}) \subseteq \mathcal{G}$ and we can apply Theorem 13.8. \square

In particular, every nontrivial *quasivariety* \mathcal{V} of ℓ -groups contains \mathbb{Z} as a member, thus (since all objects in \vec{A} are subdirect powers of \mathbb{Z}) it contains $\mathcal{A}(\omega, \vec{A})$. Since \mathcal{V} is closed under products and under directed colimits, Theorem 13.9 applies to \mathcal{V} , so $\text{Id}_c(\mathcal{V})$ is *anti-elementary*.

Straying away from quasivarieties, we also obtain the following:

Corollary 13.10. *Let \mathcal{G} be a full subcategory of $\ell\mathbf{Grp}$, closed under products and under colimits indexed by all large enough regular cardinals, and containing all Archimedean ℓ -groups. Then $\text{Id}_c(\mathcal{G})$ is anti-elementary.*

Since the arrows in the diagram \vec{A} are not unit-preserving, we do not know whether Corollary 13.10 extends to the case of ℓ -groups with order-unit.

13.3. Ideal lattices of rings. Denote by \mathbf{Ring} the category of all unital rings with unital ring homomorphisms. The assignment, that sends any ring R to the $(\vee, 0)$ -semilattice of its finitely generated two-sided ideals, naturally extends to a functor $\text{Id}_c: \mathbf{Ring} \rightarrow \mathcal{S}$. For a ring homomorphism $f: R \rightarrow S$, $\text{Id}_c f$ sends every two-sided ideal X of R to the two-sided ideal of S generated by $f[X]$.

Now let \mathbb{k} be a field and denote by $M_2(\mathbb{k})$ the ring of all 2×2 square matrices over \mathbb{k} . For all $x, y \in \mathbb{k}$, we set $e(x) \stackrel{\text{def}}{=} (x, x)$, $s(x, y) \stackrel{\text{def}}{=} (y, x)$, and $h(x, y) \stackrel{\text{def}}{=} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. We denote by $\vec{R}_{\mathbb{k}}$ the non-commutative diagram, in \mathbf{Ring} , represented in the right hand side of Figure 13.2.

Lemma 13.11. *The diagram $\vec{R}_{\mathbb{k}}$ is Id_c -commutative.*

Proof. For any set I , we need to prove that the image under the functor Id_c , of the diagram $\vec{R}_{\mathbb{k}}^I$, is commutative. Let $p = (p_i \mid i \in I)$ and $q = (q_i \mid i \in I)$ in P^I such that $p \leq q$, let $(f_i \mid i \in I)$ and $(g_i \mid i \in I)$ be elements of $\prod (\vec{R}_{\mathbb{k}}(p_i, q_i) \mid i \in I)$. Set $f \stackrel{\text{def}}{=} \prod (f_i \mid i \in I)$ and $g \stackrel{\text{def}}{=} \prod (g_i \mid i \in I)$. In order to prove that the maps $\text{Id}_c f$ and $\text{Id}_c g$ are equal, it suffices to prove that they agree on the two-sided ideal of $\vec{R}_{\mathbb{k}}^I(p)$ generated by any $z = (z_i \mid i \in I) \in \vec{R}_{\mathbb{k}}^I(p)$; that is, it suffices to verify that the elements $f(z)$ and $g(z)$ generate the same two-sided ideal of $\vec{R}_{\mathbb{k}}^I(q)$. The complement in I of $I_0 \stackrel{\text{def}}{=} \{i \in I \mid f_i = g_i\}$ is contained in $\{i \in I \mid (p_i, q_i) = (1, 123)\}$, so, setting

$$I_1 \stackrel{\text{def}}{=} \{i \in I \mid (p_i, q_i) = (1, 123) \text{ and } f_i = h \text{ and } g_i = h \circ s\},$$

$$I_2 \stackrel{\text{def}}{=} \{i \in I \mid (p_i, q_i) = (1, 123) \text{ and } f_i = h \circ s \text{ and } g_i = h\},$$

it follows that I is the disjoint union of I_0 , I_1 , and I_2 . Denote by χ_J the characteristic function of any subset J of I . Since $f(z)$ generates the same two-sided ideal as $\{f(z\chi_{I_0}), f(z\chi_{I_1}), f(z\chi_{I_2})\}$, and similarly for g , it suffices to settle the case where $I = I_k$ for some $k \in \{0, 1, 2\}$. If $k = 0$ then $f = g$ and we are done. The two other cases being symmetric, it remains to settle the case where $k = 1$. For each $i \in I_1$, we can write $z_i = (x_i, y_i) \in \mathbb{k} \times \mathbb{k}$, thus $f_i(z_i) = \begin{pmatrix} x_i & 0 \\ 0 & y_i \end{pmatrix}$ and $g_i(z_i) = \begin{pmatrix} y_i & 0 \\ 0 & x_i \end{pmatrix}$. Defining \bar{u} as the constant I -indexed family with value $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we get $\bar{u} = \bar{u}^{-1}$ and $g(z) = \bar{u} \cdot f(z) \cdot \bar{u}^{-1}$. The desired conclusion follows. \square

We set $\vec{\pi} \stackrel{\text{def}}{=} (\pi_p \mid p \in P)$ where each $\pi_p: \text{Id}_c R_{\mathbb{k}}(p) \rightarrow S(p)$ is the canonical isomorphism. The following lemma is trivial.

Lemma 13.12. *The family $\vec{\pi}$ is a natural isomorphism from $\text{Id}_c \vec{R}_{\mathbb{k}}$ onto \vec{S} .*

The following lemma is then an immediate consequence of Lemma 13.3.

Lemma 13.13. *There are no commutative diagram $\vec{B} \in \mathcal{B}^P$ and no natural transformation $\vec{\chi}: \Psi \vec{B} \Rightarrow \text{Id}_c \vec{R}_{\mathbb{k}}$.*

Notation 13.14. For any infinite regular cardinal θ , we denote by $\mathcal{R}(\theta, \mathbb{k})$ the smallest subcategory of **Ring**, containing all objects and arrows of $\vec{R}_{\mathbb{k}}$, and closed under products and under colimits, within **Ring**, indexed by any $\lambda \geq \theta$.

We are now reaching this section's main result.

Theorem 13.15. *For all infinite regular cardinals θ and λ with $\theta + \aleph_1 \leq \lambda$ and every field \mathbb{k} , there exists a functor Δ , from $\mathfrak{P}_{\text{inj}}(\lambda^{+2})$ to the category \mathcal{S} of all $(\vee, 0)$ -semilattices with $\mathcal{L}_{\infty\lambda}$ -elementary embeddings, satisfying the following statements:*

- (1) Δ preserves all λ -directed colimits;
- (2) For every λ^+ -small subset X of λ^{+2} , $\Delta(X)$ belongs to $\text{Id}_c(\mathcal{R}(\theta, \mathbb{k}))$;
- (3) There are no set A and no distance $\delta: A \times A \rightarrow \Delta(\lambda^{+2})$ of type 1 with join-generating range.

In particular, the pair $(\text{Id}_c(\mathcal{R}(\theta, \mathbb{k})), \text{rng } \Psi)$ is anti-elementary.

Proof. The proof follows the lines of the one of Theorem 13.8, with Φ now defined as the restriction of the functor Id_c to $\mathcal{R}(\theta, \mathbb{k})$, Lemma 13.13 used instead of Lemma 13.7 and with $\Gamma: \mathfrak{P}_{\text{inj}}(\kappa) \rightarrow \mathcal{S}$, $U \mapsto \mathbf{F}(P\langle U \rangle) \otimes_{\Phi}^{\lambda} \vec{R}_{\mathbb{k}}$. \square

Recall that an algebra over a field \mathbb{k} is *locally matricial* if it is a directed colimit of finite products of finite-dimensional matrix rings over \mathbb{k} .

Corollary 13.16. *The following classes of $(\vee, 0)$ -semilattices are all anti-elementary:*

- (1) $\text{Id}_c(\mathcal{R})$, whenever \mathcal{R} is a class of unital rings containing all unital locally matricial \mathbb{k} -algebras for some field \mathbb{k} ;
- (2) The class of all semilattices of finitely generated submodules of right modules.

Proof. Ad (1). Since all the vertices of $\vec{R}_{\mathbb{k}}$ are locally matricial \mathbb{k} -algebras, we get $\mathcal{R}(\omega, \mathbb{k}) \subseteq \mathcal{R} \subseteq \mathbf{Ring}$. By Proposition 13.1(2), $\text{Id}_c(\mathbf{Ring}) \subseteq \text{rng } \Psi$. Apply Theorem 13.15.

Ad (2). By a well known module-theoretical trick (see, for example, the proof of Růžička *et al.* [29, Theorem 4.2]), the semilattice of all finitely two-sided ideals of any unital von Neumann regular ring is isomorphic to $\text{Sub}_c M$ for some right module M . Moreover, by Proposition 13.1(3), $\text{Sub}_c M$ belongs to the range of Ψ for every right module M . Apply Theorem 13.15. \square

14. NONSTABLE K_0 -THEORY OF EXCHANGE RINGS

Section 14 will be devoted to proving anti-elementarity of classes of monoids arising from the nonstable K_0 -theory of rings and operator algebras. The statement of this section's main result (viz. Theorem 14.6) will deviate from those of Sections 12 and 13, in the sense that the category \mathcal{R} in question (playing the role

of the category \mathcal{A} in Lemma 11.2) will no longer be assumed to be closed under directed colimits (indexed by regular cardinals $\geq \theta$) within the category of all unital rings. This way, we will be able to cover both the case of von Neumann regular rings (where directed colimits from the category **Ring** of all unital rings are preserved; see Corollary 14.7) and C^* -algebras of real rank zero (where those colimits are not preserved — we need to take the completion; see Corollary 14.8).

Throughout Section 14 we will consider the category $\mathcal{S} := \mathbf{CMon}$ of all commutative monoids with monoid homomorphisms. Moreover, following the terminology in Wehrung [35], a monoid homomorphism $f: M \rightarrow N$ is a *pre-V-homomorphism* if whenever $a, b \in N$ and $c \in M$, $f(c) = a + b$ implies the existence of $x, y \in M$ such that $c = x + y$, $f(x) = a$, and $f(y) = b$. We shall denote by $\mathcal{S}^{\Rightarrow}$ (this section's double arrows) the subcategory of \mathcal{S} consisting of all pre-V-homomorphisms.

Still following standard terminology, we denote by $M_\infty(R)$ the directed union, over all positive integers n , of all matrix rings $M_n(R)$, where every $a \in M_n(R)$ is identified with $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+1}(R)$. Whenever n is either a positive integer or ∞ , we denote by $I_n(R)$ the set of all idempotent elements in the ring $M_n(R)$. Matrices $a, b \in I_\infty(R)$ are *Murray-von Neumann equivalent*, in notation $a \sim b$, if there are $x, y \in M_\infty(R)$ such that $a = xy$ and $b = yx$. We denote by $[a]$ the Murray-von Neumann equivalence class of an idempotent matrix a . Those equivalence classes can be added, *via* the rule $[a] + [b] \stackrel{\text{def}}{=} [a + b]$ whenever $ab = ba = 0$ (then we write $a \oplus b$ instead of $a + b$), and $V(R) \stackrel{\text{def}}{=} \{[a] \mid a \in I_\infty(R)\}$ is a commutative monoid. This monoid is *conical*, that is, $\mathbf{x} + \mathbf{y} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{y} = \mathbf{0}$, whenever $\mathbf{x}, \mathbf{y} \in V(R)$. It encodes the so-called *nonstable K_0 -theory* of R .

A ring R is *V-semiprimitive* (cf. Wehrung [35, Definition 9.1]) if for any $a, b \in I_\infty(R)$, $ab \neq 0$ implies that there are decompositions $a = u \oplus u'$ and $b = v \oplus v'$ in $I_\infty(R)$ such that $u \neq 0$ and $u \sim v$. A classical argument shows that it is equivalent to require that every matrix ring $M_n(R)$, with n a positive integer, satisfies the statement above. In particular, *V-semiprimitivity can be expressed by a collection of first-order sentences in ring theory*. Moreover, *any product of V-semiprimitive rings is V-semiprimitive*.

The assignment $R \mapsto V(R)$ naturally extends to a functor, from the category of all rings and ring homomorphisms to \mathcal{S} . Throughout Section 14 we shall denote by \mathcal{B} the category of all V-semiprimitive rings with ring homomorphisms, and by $\Psi: \mathcal{B} \rightarrow \mathcal{S}$ the restriction of V to \mathcal{B} .

We record here the following observation from Wehrung [35, Proposition 9.2].

Proposition 14.1. *Every semiprimitive exchange ring is V-semiprimitive. In particular, every ring which is either von Neumann regular or a C^* -algebra of real rank zero is V-semiprimitive.*

The following lemma takes care of all instances of Condition (LS(λ)), from the statement of Lemma 11.2, occurring in Section 14. Since, as observed above, V-semiprimitivity is first-order, the proof of Lemma 14.2 follows from a standard Löwenheim-Skolem type argument, so we omit it.

Lemma 14.2. *For any uncountable regular cardinal λ , any λ -small commutative monoid M , any V-semiprimitive ring R , and any pre-V-homomorphism $\psi: V(R) \Rightarrow M$, the collection \mathcal{C} of all λ -small elementary subrings R' of R , such*

that $\psi \circ \mathbf{V}(\text{id}_{R'}^R)$ is a pre- V -homomorphism, is a λ -closed cofinal subset of the powerset of R . Moreover, every member of \mathcal{C} is V -semiprimitive.

Set again $P \stackrel{\text{def}}{=} \mathfrak{P}[3]$ (cf. Section 2). Consider the maps $\mathbf{e}: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \times \mathbb{Z}^+$, $\mathbf{s}: \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \times \mathbb{Z}^+$, and $\mathbf{p}: \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ defined via the rules $\mathbf{e}(x) \stackrel{\text{def}}{=} (x, x)$, $\mathbf{s}(x, y) \stackrel{\text{def}}{=} (y, x)$, and $\mathbf{p}(x, y) \stackrel{\text{def}}{=} x + y$ whenever $x, y \in \mathbb{Z}^+$. Further, we denote by \vec{D} the P -indexed commutative diagram in \mathcal{S} represented in Figure 14.1.

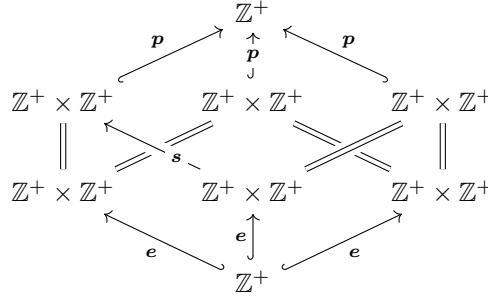


FIGURE 14.1. The commutative diagram \vec{D}

The proof of the following lemma is, *mutatis mutandis*, identical to the one of Lemma 13.11, the main change being that instead of saying that $f(z)$ generates the same two-sided ideal as $\{f(z\chi_{I_0}), f(z\chi_{I_1}), f(z\chi_{I_2})\}$, we now observe that $[f(z)] = [f(z\chi_{I_0})] + [f(z\chi_{I_1})] + [f(z\chi_{I_2})]$, within $\mathbf{V}(\vec{R}_{\mathbb{k}}^I(q))$, for every idempotent matrix z over $\vec{R}_{\mathbb{k}}^I(p)$.

Lemma 14.3. *For any field \mathbb{k} , the diagram $\vec{R}_{\mathbb{k}}$ (introduced in Subsection 13.1), of unital rings and unital ring homomorphisms, is V -commutative.*

We omit the trivial proof of our next observation.

Lemma 14.4. *For any field \mathbb{k} , the diagrams $\mathbf{V} \vec{R}_{\mathbb{k}}$ and \vec{D} are naturally isomorphic.*

The following crucial result is established in Wehrung [35, Theorem 10.1].

Lemma 14.5. *There are no commutative diagram $\vec{B} \in \mathcal{B}^P$ and no natural transformation $\vec{\chi}: \Psi \vec{B} \Rightarrow \vec{D}$.*

We are now reaching this section's main result.

Theorem 14.6. *Let \mathbb{k} be a field, let θ be an infinite regular cardinal, and let \mathcal{R} be a subcategory of the category of all unital rings and unital ring homomorphisms satisfying the following assumptions:*

- (i) *all objects and arrows of the diagram $\vec{R}_{\mathbb{k}}$ (introduced in Subsection 13.3) belong to \mathcal{R} ;*
- (ii) *every object of \mathcal{R} is V -semiprimitive;*
- (iii) *\mathcal{R} is closed under products (within the category of all unital rings and unital ring homomorphisms);*
- (iv) *\mathcal{R} has all colimits indexed by regular cardinals $\geq \theta$, and the functor \mathbf{V} preserves those colimits.*

Denote by \mathcal{C} the class of all commutative monoids that are the image of $V(R)$ by a pre- V -homomorphism, for some V -semiprimitive ring R . Then for every regular cardinal $\lambda \geq \theta + \aleph_1$, there exists a functor Δ , from $\mathfrak{P}_{\text{inj}}(\lambda^{+2})$ to the category of all commutative monoids with $\mathcal{L}_{\infty\lambda}$ -elementary embeddings, satisfying the following statements:

- (1) Δ preserves all λ -directed colimits;
- (2) for every λ^+ -small subset X of λ^{+2} , $\Delta(X)$ belongs to $V(\mathcal{R})$;
- (3) $\Delta(\lambda^{+2})$ does not belong to \mathcal{C} .

In particular, the pair $(V(\mathcal{R}), \mathcal{C})$ is anti-elementary.

Proof. Denote by Φ the restriction of the functor V from \mathcal{R} to \mathcal{S} . It follows from Assumption (iii) and Lemma 14.3 that the diagram $\vec{R}_{\mathbb{k}}$ is Φ -commutative. Moreover, it follows from Assumption (ii) that $V(\mathcal{R})$ is contained in the range of Ψ , thus in \mathcal{C} .

Using Assumption (i) and setting $\kappa \stackrel{\text{def}}{=} \lambda^{+2}$, we obtain, as at the beginning of the proof of Theorem 12.3, a functor $\Gamma: \mathfrak{P}_{\text{inj}}(\kappa) \rightarrow \mathcal{S}$, $U \mapsto \mathbf{F}(P\langle U \rangle) \otimes_{\Phi}^{\lambda} \vec{R}_{\mathbb{k}}$, and this functor preserves λ -directed colimits.

Then we need to prove the analogue of Claim 1 of Theorem 13.8, which states that for every λ^+ -small subset X of κ , the lattice $\Gamma(X)$ belongs to the range of Φ . The proof goes the same way, using the Boosting Lemma (viz. Lemma 7.3). A crucial observation is that Assumption (iv) ensures that all assumptions required for applying the Boosting Lemma are satisfied.

The remainder our proof follows the lines of the one of Theorem 13.8, with Lemmas 14.2–14.5 used instead of Lemmas 13.4–13.7 and with the functor $\Gamma: \mathfrak{P}_{\text{inj}}(\kappa) \rightarrow \mathcal{S}$, $U \mapsto \mathbf{F}(P\langle U \rangle) \otimes_{\Phi}^{\lambda} \vec{R}_{\mathbb{k}}$. The property (PROJ(λ)) follows from the argument of the part of the proof of Wehrung [35, Lemma 13.1] establishing the property denoted there by (PROJ($\Phi, \mathbf{CMon}^{\Rightarrow}$)). \square

Let us now present two applications of Theorem 14.6, obtained by specializing the category \mathcal{R} , both for $\theta := \omega$.

Defining \mathcal{R} as the category \mathbf{vNRing} of all unital von Neumann regular rings with unital ring homomorphisms, we obtain the following.

Corollary 14.7. *The class $V(\mathbf{vNRing})$ is anti-elementary.*

Now define \mathcal{R} as the category \mathbf{RR}_0 of all unital C^* -algebras of real rank zero with unital C^* -homomorphisms. We need to verify Assumptions (i)–(iv) of the statement of Theorem 14.6, for $\mathbb{k} := \mathbb{C}$. Assumption (i) follows from the fact that $\vec{R}_{\mathbb{C}}$ is a diagram in \mathbf{RR}_0 . Assumption (ii) follows from Proposition 14.1. Assumption (iii) is obvious. Finally, the colimit of any direct system in \mathbf{RR}_0 is the norm-completion of its colimit in the category of unital rings and unital ring homomorphisms; invoking Blackadar [7, § 5.1], Assumption (iv) follows.

Corollary 14.8. *The class $V(\mathbf{RR}_0)$ is anti-elementary.*

15. COORDINATIZABILITY BY VON NEUMANN REGULAR RINGS WITHOUT UNIT

15.1. Some background on coordinatization and frames. We refer the reader to Wehrung [34] for more references and detail.

A lattice L is

- modular if $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$ whenever $x, y, z \in L$;

- *sectionally complemented* if it has a zero (i.e., smallest) element 0 and for all $x \leq y$ in L there exists $z \in L$ such that $x \wedge z = 0$ and $x \vee z = y$ (in abbreviation, $y = x \oplus z$).

Two elements a and b of L are *perspective* if there exists $x \in L$ such that $a \oplus x = b \oplus x$. An ideal I of a sectionally complemented modular L is *neutral* if every element of L perspective to some element of I belongs to I . For a positive integer n , an $(n + 1)$ -*frame* of L is a pair $((a_i \mid 0 \leq i \leq n), (c_i \mid 1 \leq i \leq n))$ such that $a_0 \oplus c_i = a_i \oplus c_i = a_0 \oplus a_i$ whenever $1 \leq i \leq n$. The frame is *large* if L is generated by a_0 (equivalently, by any a_i) as a neutral ideal.

For any von Neumann regular ring R , the set $\mathbb{L}(R) \stackrel{\text{def}}{=} \{xR \mid x \in R\}$, partially ordered by set inclusion, is a sectionally complemented modular lattice. A lattice is *coordinatizable* if it is isomorphic to $\mathbb{L}(R)$ for some von Neumann regular ring R .

The argument of Wehrung [33, Theorem 9.4] shows that the class of all coordinatizable complemented modular lattices (with unit!) is not closed under $\mathcal{L}_{\infty\lambda}$ -elementary equivalence, for any infinite regular cardinal λ . We deal here with the more difficult case of lattices without unit, but with a large 4-frame.

Jónsson proved in [21] that every sectionally complemented modular lattice with a countable cofinal sequence and a large 4-frame is coordinatizable. The author proved in Wehrung [34], *via* a counterexample of cardinality \aleph_1 , that the countable cofinal sequence assumption cannot be dispensed with.

In this section we shall go further, by proving that the class of coordinatizable lattices with a large 4-frame is anti-elementary. In contrast to our settings in Sections 12–14, the indexing poset of our main diagram counterexample will be the transfinite chain ω_1 , with the drawback that the proof of our main result (viz. Theorem 15.9) will require a large cardinal assumption.

We define binary relations \trianglelefteq_r , \succsim_r , and \trianglelefteq on the set $\text{Idp } R$ of all idempotent elements of any ring R by

$$\begin{aligned} a \trianglelefteq_r b & \quad \text{if} \quad a = ba, \\ a \succsim_r b & \quad \text{if} \quad a = ba \text{ and } b = ab, \\ a \trianglelefteq b & \quad \text{if} \quad a = ab = ba, \end{aligned}$$

for all $a, b \in \text{Idp } R$. In particular, $a \trianglelefteq_r b$ iff $aR \subseteq bR$ and $a \succsim_r b$ iff $aR = bR$.

The proof of Jónsson [21, Lemma 10.2] remains valid for non-unital rings so we omit it.

Lemma 15.1. *For any idempotent element e in a von Neumann regular ring R , there are mutually inverse isomorphisms $\varphi_e: \mathbb{L}(R) \downarrow eR \rightarrow \mathbb{L}(eRe)$, $\mathbf{x} \mapsto \mathbf{x} \cap eRe = \mathbf{x}e$ and $\psi_e: \mathbb{L}(eRe) \rightarrow \mathbb{L}(R) \downarrow eR$, $\mathbf{y} \mapsto \mathbf{y}R$.*

Lemma 15.2. *Let n be a positive integer and let a_1, \dots, a_n be idempotent elements in a ring R with $a_1 \trianglelefteq_r \dots \trianglelefteq_r a_n$. Then $u \stackrel{\text{def}}{=} a_1 \dots a_n$ and $v \stackrel{\text{def}}{=} a_1 a_n$ are both idempotent and $u \succsim_r v$.*

Proof. Since $a_i = a_n a_i$ and $a_1 = a_i a_1$ for every i , we get $u = a_n u$ and $a_1 = u a_1$, thus $u^2 = u a_1 \dots a_n = u$ and $v^2 = v$. Further, $uv = u a_1 a_n = a_1 a_n = v$ and $vu = a_1 a_n u = a_1 u = u$. \square

Lemma 15.3. *Let R and S be von Neumann regular rings, let $f: R \rightarrow S$ be a ring homomorphism, let $a \in \text{Idp } R$ and $b \in \text{Idp } S$ such that $f(a) \trianglelefteq_r b$. Denote by φ the domain-range restriction of $\mathbb{L}(f)$ from $\mathbb{L}(R) \downarrow aR$ to $\mathbb{L}(S) \downarrow bS$. Then the assignment*

$x \mapsto f(x)b$ defines a ring homomorphism $f': aRa \rightarrow bSb$ and the following diagram commutes:

$$\begin{array}{ccc} \mathbb{L}(aRa) & \xrightarrow{\mathbb{L}(f')} & \mathbb{L}(bSb) \\ \cong \downarrow \psi_a & & \psi_b \downarrow \cong \\ \mathbb{L}(R) \downarrow aR & \xrightarrow{\varphi} & \mathbb{L}(S) \downarrow bS \end{array}$$

Proof. For any $x, y \in aRa$, $bf(y) = bf(a)f(y) = f(a)f(y) = f(y)$ thus $f'(x)f'(y) = f(x)bf(y)b = f(xy)b = f'(xy)$, so f' is a ring homomorphism. Now let $\mathbf{x} \in \mathbb{L}(aRa)$. We can write $\mathbf{x} = xRa$ for some idempotent $x \leq a$, so $(\psi_b \circ \mathbb{L}(f'))(\mathbf{x}) = \psi_b(f(x)bSb) = f(x)bSbS = f(x)bS$. Since $f(x) = f(x)f(x) = f(x)bf(x) \in f(x)bS$, we get $(\psi_b \circ \mathbb{L}(f'))(\mathbf{x}) = f(x)S = \varphi(\mathbf{x}R) = (\varphi \circ \psi_a)(\mathbf{x})$, as required. \square

Definition 15.4. Let P be a poset and let $\vec{L} = (L_p, f_p^q \mid p \leq q \text{ in } P)$ be a P -indexed commutative diagram of lattices and lattice homomorphisms. An \vec{L} -thread is a family $\vec{a} = (a_p \mid p \in P) \in \prod(L_p \mid p \in P)$ such that $f_p^q(a_p) \leq a_q$ whenever $p \leq q$ in P . Then we define the *truncation of \vec{L} at \vec{a}* as the commutative diagram $\vec{L} \upharpoonright \vec{a} \stackrel{\text{def}}{=} (L_p \downarrow a_p, g_p^q \mid p \leq q \text{ in } P)$, where for all $p \leq q$ in P , g_p^q denotes the domain-range restriction of f_p^q from $L_p \downarrow a_p$ to $L_q \downarrow a_q$.

The following lemma states that for a commutative diagram \vec{R} of von Neumann regular rings, any truncation of $\mathbb{L}\vec{R}$ can be represented as $\mathbb{L}\vec{R}'$ for some \mathbb{L} -commutative diagram \vec{R}' of von Neumann regular rings.

Lemma 15.5. Let P be a poset, let $\vec{R} = (R_p, f_p^q \mid p \leq q \text{ in } P)$ be a P -indexed commutative diagram of von Neumann regular rings and ring homomorphisms, and let $(a_p \mid p \in P) \in \prod(\text{Idp } R_p \mid p \in P)$ such that $f_p^q(a_p) \leq_r a_q$ whenever $p \leq q$ in P . Set $\vec{a} \stackrel{\text{def}}{=} (a_p R_p \mid p \in P)$. For all $p \leq q$ in P , set $R'_p \stackrel{\text{def}}{=} a_p R_p a_p$ and let $\vec{R}'(p, q)$ be the set of all maps of the form $f: R'_p \rightarrow R'_q$, $x \mapsto f_p^q(x) \prod_{i=0}^{n-1} f_{r_i}^q(a_{r_i})$ where $p = r_0 \leq \dots \leq r_n = q$ in P . Then \vec{R}' is an \mathbb{L} -commutative diagram of von Neumann regular rings and ring homomorphisms, and $\vec{\psi} \stackrel{\text{def}}{=} (\psi_p \mid p \in P)$ is a natural isomorphism from $\mathbb{L}\vec{R}'$ to $\mathbb{L}\vec{R} \downarrow \vec{a}$.

Proof. It is obvious that \vec{R}' is a diagram of von Neumann regular rings and ring homomorphisms. For any $p \leq q$ in P , any $f \in \vec{R}'(p, q)$ is defined by a rule of the form $f(x) = f_p^q(x) \prod_{i=0}^{n-1} f_{r_i}^q(a_{r_i})$ where $p = r_0 \leq \dots \leq r_n = q$. Hence, defining $g: R'_p \rightarrow R'_q$ via the rule $g(x) \stackrel{\text{def}}{=} x f_p^q(a_p) a_q$ (whenever $x \in R_p$) and setting $u \stackrel{\text{def}}{=} \prod_{i=0}^{n-1} f_{r_i}^q(a_{r_i})$, $v \stackrel{\text{def}}{=} f_p^q(a_p) a_q$, we obtain that $f(x) = f_p^q(x)u$ and $g(x) = f_p^q(x)v$ whenever $x \in R_p$. Moreover, it follows from Lemma 15.2 that u and v are idempotent with $u \asymp_r v$.

Now let I be a set and let $(p_i \mid i \in I)$ and $(q_i \mid i \in I)$ be families of elements of P such that $p_i \leq q_i$ whenever $i \in I$. Further, let $f_i, g_i \in \vec{R}'(p_i, q_i)$ for $i \in I$, let $f \stackrel{\text{def}}{=} \prod(f_i \mid i \in I)$ and $g \stackrel{\text{def}}{=} \prod(g_i \mid i \in I)$. For each $i \in I$, it follows from the paragraph above that there are idempotent elements $u_i \asymp_r v_i$ of R_{q_i} such that $f_i(x) = f_{p_i}^{q_i}(x)u_i$ and $g_i(x) = f_{p_i}^{q_i}(x)v_i$ for every $x \in R_{p_i}$. Hence, the elements $u \stackrel{\text{def}}{=} (u_i \mid i \in I)$ and $v \stackrel{\text{def}}{=} (v_i \mid i \in I)$ of $\prod(R_{q_i} \mid i \in I)$ are idempotent with $u \asymp_r v$, and further, setting

$h(x) \stackrel{\text{def}}{=} (f_{p_i}^{q_i}(x_i) \mid i \in I)$, we obtain that $f(x) = h(x)u$ and $g(x) = h(x)v$ for every $x \in \prod(R_{p_i} \mid i \in I)$. In particular, $\mathbb{L}(f) = \mathbb{L}(g)$. This completes the proof of the \mathbb{L} -commutativity of \vec{R}' . The final statement of Lemma 15.5, about $\vec{\psi}$ being a natural isomorphism, follows immediately from Lemma 15.3. \square

15.2. Categorical settings for Section 15. Let us define a category \mathcal{A} as follows. The objects of \mathcal{A} are the structures (R, μ_R) where R is a von Neumann regular ring (not necessarily unital) and $\mu_R = (e_{ij}^R \mid 0 \leq i, j \leq 3)$ where the e_{ij}^R form a system of matrix units of R (i.e., $e_{ij}^R e_{kl}^R = \delta_{jk} e_{il}^R$ whenever $i, j, k, l \in \{0, 1, 2, 3\}$, where δ denotes Kronecker's symbol), such that, setting $e^R \stackrel{\text{def}}{=} \sum_{i=0}^3 e_{ii}^R$,

$$\forall x \in R, e^R x = x e^R = 0 \Rightarrow (\exists y, z \in R)(x = y e^R z). \quad (15.1)$$

We shall often omit the superscript R in e_{ij}^R and e^R when the ring R is understood. The canonical 4-frame of $\mathbb{L}(R)$ associated with the system μ_R of matrix units is

$$\tau_R \stackrel{\text{def}}{=} ((e_{ii}^R R \mid 0 \leq i \leq 3), ((e_{ii}^R - e_{0i}^R) R \mid 0 \leq i \leq 3)).$$

For objects (R, μ_R) and (S, μ_S) of \mathcal{A} , define the morphisms $f: (R, \mu_R) \rightarrow (S, \mu_S)$ in \mathcal{A} as the ring homomorphisms $f: R \rightarrow S$ such that $f(e_{ij}^R) = e_{ij}^S$ whenever $0 \leq i, j \leq 3$.

Lemma 15.6. *Let R be an object of \mathcal{A} . Then τ_R is a large 4-frame in $\mathbb{L}(R)$.*

Proof. Since the e_{ij} form a system of matrix units in R , τ_R is a 4-frame in $\mathbb{L}(R)$. In order to prove that it is a large 4-frame, it is, by Wehrung [36, Theorem 8-3.24], sufficient to verify that the two-sided ideal J of R generated by e is equal to R . Now any $x \in R$ can be written as $y + ex + xe - exe$, where $ey = ye = 0$. Since R satisfies the axiom (15.1), it follows that $y \in J$. Moreover, $e \in J$ implies that $ex + xe - exe \in J$. Therefore, $x \in J$. \square

We shall consider the following categories and functors (represented in Figure 15.1):

- the above-defined category \mathcal{A} , of all (not necessarily unital) regular rings with “large” systems of 4×4 matrix units;
- \mathcal{S} is the category of all sectionally complemented modular lattices with 0-lattice homomorphisms;
- $\mathcal{S}^{\Rightarrow}$ (this section's double arrows) is the subcategory of \mathcal{S} consisting of all surjective homomorphisms;
- $\Phi(R, \mu_R) = \mathbb{L}(R)$ for every object (R, μ_R) of \mathcal{A} , and $\Phi(f) = \mathbb{L}(f)$ for every morphism f in \mathcal{A} ;
- \mathcal{B} is the category of all (not necessarily unital) von Neumann regular rings and ring homomorphisms;
- $\Psi \stackrel{\text{def}}{=} \mathbb{L}: \mathcal{B} \rightarrow \mathcal{S}$.

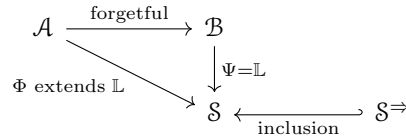


FIGURE 15.1. Categories and functors for Section 15

The following lemma collects mere routine facts, with straightforward proofs that we shall thus omit. Actually, most of it got already checked in Gillibert and Wehrung [14, Ch. 6] and Wehrung [34]; Lemma 15.7(3) is contained in Gillibert and Wehrung [14, Lemma 6.2.1].

Lemma 15.7.

- (1) *The categories \mathcal{A} , \mathcal{B} , and \mathcal{S} have all directed colimits and all products.*
- (2) *Both functors Φ and Ψ preserve all directed colimits.*
- (3) *Every double arrow $\psi: \Psi(C) \Rightarrow S$, where $C \in \mathcal{B}$ and $S \in \mathcal{S}$, has a projectability witness with respect to the functor Ψ .*

15.3. The \mathbb{L} -commutative diagram \vec{R} . We shall now consider the diagram $\vec{L} \stackrel{\text{def}}{=} (L_\xi, e_\xi^\eta \mid \xi \leq \eta < \omega_1)$ of \mathcal{S} introduced in Wehrung [34, § 7] and denoted there by \vec{A} , of sectionally complemented modular lattices and 0-lattice embeddings. Although the full construction of that diagram is quite complex, we will need here only a small part of its underlying information. The main part of the construction, which we will not need here, is the one of an increasing ω_1 -sequence $(A_\xi \mid \xi < \omega_1)$ of countable unital von Neumann regular rings, all sharing the same unit, and an \leq_r -increasing sequence $(\tilde{\xi} \mid \xi < \omega_1)$ of idempotents $\tilde{\xi} \in A_\xi$. We set $B_\xi \stackrel{\text{def}}{=} A_\xi^{5 \times 5}$ (the ring of all square matrices of order 5 over A_ξ), with canonical system of matrix units $(e_{ij} \mid 0 \leq i, j \leq 4)$ and $e \stackrel{\text{def}}{=} \sum_{i=0}^3 e_{ii}$. Further, we set $b_\xi \stackrel{\text{def}}{=} e + \tilde{\xi} \cdot e_{44}$ and $L_\xi \stackrel{\text{def}}{=} \mathbb{L}(B_\xi) \downarrow b_\xi B_\xi$, for any $\xi < \omega_1$. We need to know that $(b_\xi \mid \xi < \omega_1)$ is an \leq_r -increasing ω_1 -sequence of idempotents and that all matrix units e_{ij} , for $0 \leq i, j \leq 3$, belong to all $b_\xi B_\xi b_\xi$. For $\xi < \eta < \omega_1$, $b_\xi = b_\eta b_\xi \neq b_\xi b_\eta$.

Let \vec{R} be constructed from \vec{B} as \vec{R}' is constructed from \vec{R} in the proof of Lemma 15.5. Hence \vec{R} is an ω_1 -indexed, \mathbb{L} -commutative diagram in \mathcal{B} such that $\vec{L} = \mathbb{L}\vec{R}$. In particular, whenever $\xi \leq \eta < \omega_1$, $R_\xi = b_\xi B_\xi b_\xi$ and all morphisms in $\vec{R}(\xi, \eta)$ are finite compositions of right multiplications by elements of the form b_ζ where $\xi \leq \zeta \leq \eta$. Since all the matrix units e_{ij} belong to all $b_\zeta B_\zeta b_\zeta$, it follows that they are fixed points of every morphism in $\vec{R}(\xi, \eta)$. It follows that, expanding each R_ξ by $\mu \stackrel{\text{def}}{=} (e_{ij} \mid 0 \leq i, j \leq 3)$, we obtain a Φ -commutative diagram (\vec{R}, μ) in \mathcal{A} such that $\Phi(\vec{R}, \mu) = \mathbb{L}\vec{R} = \vec{L}$.

Now a complication arises. We would like to apply the Extended CLL (i.e., Lemma 11.2) to the Φ -commutative diagram (\vec{R}, μ) , together with the Boosting Lemma (viz. Lemma 7.3). The former step causes no difficulty, *via* Gillibert [13, Proposition 4.6] (stating that every tree has “many lifters”). However, the latter step is definitely a problem, as the poset ω_1 is not lower finite.

In order to get around that difficulty, we need to resort to the *pseudo-retracts* introduced in Gillibert and Wehrung [14, § 3.6]. By definition, a *pseudo-retraction* for a pair (P, Q) of posets is a pair of isotone maps $\varepsilon: P \rightarrow \text{Id } Q$ and $\pi: Q \rightarrow P$ such that $\pi[\varepsilon(p)]$ is cofinal in $P \downarrow p$ whenever $p \in P$. Then we say that P is a *pseudo-retract* of Q . By Gillibert and Wehrung [14, Lemma 3.6.6], every almost join-semilattice P is a pseudo-retract of some lower finite almost join-semilattice Q with the same cardinality. We may assume, in addition, that if P has a zero, then so does Q (replace Q by $Q' \stackrel{\text{def}}{=} Q \uparrow q_0$ for any $q_0 \in \varepsilon(0)$, set $\varepsilon'(p) \stackrel{\text{def}}{=} \varepsilon(p) \uparrow q_0$ and $f' \stackrel{\text{def}}{=} f \upharpoonright_{Q'}$). We apply the latter observation to $P := \omega_1$; denote by $\varepsilon: \omega_1 \rightarrow \text{Id } Q$, $\pi: Q \rightarrow \omega_1$ the implied pseudo-retraction. Since the diagram \vec{R} is \mathbb{L} -commutative,

so is the composite $\vec{R}\pi$; in particular, $\mathbb{L}\vec{R}\pi$ is a Q -indexed commutative diagram in \mathcal{S} .

Lemma 15.8. *For any Q -indexed commutative diagram $\vec{R}' \in \mathcal{B}^Q$, there is no natural transformation $\vec{\chi}: \mathbb{L}\vec{R}' \Rightarrow \mathbb{L}\vec{R}\pi$ in \mathcal{S}^\Rightarrow .*

Proof. By Gillibert and Wehrung [14, Lemma 3.7.1], any natural transformation $\vec{\chi}: \mathbb{L}\vec{R}' \Rightarrow \mathbb{L}\vec{R}\pi$ would yield a natural transformation $\mathbb{L}\vec{R}' \Rightarrow \mathbb{L}\vec{R}$. However, we established in Wehrung [34, Lemma 7.4] that the latter cannot occur. \square

Theorem 15.9. *Assume that for every cardinal λ there exists a cardinal κ such that $(\kappa, <\omega, \lambda) \rightarrow \aleph_1$. Then the class of all coordinatizable sectionally complemented modular lattices with a large 4-frame is anti-elementary.*

Proof. Let κ and λ satisfy $(\kappa, <\omega, \lambda) \rightarrow \aleph_1$. It follows from Lemma 10.6, applied to the poset Q , that $(\kappa, <\lambda) \rightsquigarrow Q$. By Lemma 10.7, it follows that for every infinite cardinal $\theta \geq \aleph_1$, Q has a standard θ^+ -lifter $Q\langle\kappa\rangle$ for some infinite cardinal κ . The assumptions (WF)–(LS(λ)) of the Extended CLL (i.e., Lemma 11.2), with $\lambda = \mu = \theta^+$ and defining \mathcal{B}^\dagger as the full subcategory of \mathcal{B} consisting of all λ -small von Neumann regular rings, are easily seen to be satisfied.

Claim 1. *The lattice $\mathbf{F}(Q\langle\kappa\rangle) \otimes_{\mathbb{F}}^\lambda \vec{R}\pi$ is not coordinatizable.*

Proof of Claim. Suppose otherwise, so there are a von Neumann regular ring B and a double arrow $\chi: \mathbb{L}(B) \Rightarrow \mathbf{F}(Q\langle\kappa\rangle) \otimes_{\mathbb{F}}^\lambda \vec{R}\pi$. By Lemma 11.2, this implies the existence of a natural transformation $\vec{\chi}: \mathbb{L}\vec{R}' \Rightarrow \mathbb{L}\vec{R}\pi$ in \mathcal{S}^\Rightarrow ; in contradiction with Lemma 15.8. \square Claim 1.

Claim 2. *The lattice $\mathbf{F}(Q\langle\lambda\rangle) \otimes_{\mathbb{F}}^\lambda \vec{R}\pi$ belongs to the range of Φ ; thus it is coordinatizable and it has a large 4-frame.*

Proof of Claim. Since $Q\langle\lambda\rangle$ is the directed union of all $Q\langle\xi\rangle$ for $\xi < \lambda$, we get $\mathbf{F}(Q\langle\lambda\rangle) = \varinjlim_{\xi < \lambda} \mathbf{F}(Q\langle\xi\rangle)$, so the first part of the statement follows from the Boosting Lemma (viz. Lemma 7.3). For every object (R, μ_R) of \mathcal{A} , τ_R is a large 4-frame of $\mathbb{L}(R)$ (cf. Lemma 15.6) and $\Phi(R, \mu_R) = \mathbb{L}(R)$ is coordinatizable. \square Claim 2.

Now the assignment $X \mapsto \mathbf{F}(Q\langle X\rangle) \otimes_{\mathbb{F}}^\lambda \vec{R}\pi$ naturally extends to a functor from $\mathfrak{P}_{\text{inj}}(\kappa)$ to \mathcal{S} preserving all λ -directed colimits. The desired conclusion then follows from Claims 1 and 2. \square

Corollary 15.10. *If there are arbitrarily large Erdős cardinals, then the class of all coordinatizable sectionally complemented modular lattices with a large 4-frame is anti-elementary.*

By Devlin and Paris [8] (see also Koepke [23, 24] for further relative consistency strength results), the existence of κ such that $(\kappa, <\omega, \aleph_1) \rightarrow \aleph_1$ entails the existence of 0^\sharp (thus it is a large cardinal axiom). In the author's opinion, the apparent reliance of Theorem 15.9 on the large cardinal assumption $\forall\lambda\exists\kappa (\kappa, <\omega, \lambda) \rightarrow \aleph_1$ is accidental: at the time the main counterexample of Wehrung [34] (of a non-coordinatizable sectionally complemented modular lattice with a large 4-frame) was constructed, the Uniformization Lemma (Lemma 7.2) was not known, so the most natural (although not easy) way to obtain the counterexample seemed to be the application of the condensate construction to an ω_1 -indexed diagram; and it seems very unlikely that no finitely indexed diagram could play a similar role.

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