Unique solvability and exponential stability of differential hemivariational inequalities

Xiaojian Li, Zhenhai Liu, Mircea Sofonea

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Xiuwen Li[a], Zhenhai Liu[a,b] and Mircea Sofonea[c]

[a]School of Science, Nanjing University of Sciences and Technology, Nanjing, Poeple's Republic of China; [b]College of Sciences, Guangxi University for Nationalities, Nanning, Poeple's Republic of China; [c]Laboratoire de Mathématiques et Physique, Université de Perpignan, Perpignan, France

In this paper, we study a differential hemivariational inequality (DHVI, for short) in the framework of reflexive Banach spaces. Our aim is three fold. The first one is to investigate the existence and the uniqueness of mild solution, by applying a general fixed-point principle. The second one is to study its exponential stability, by employing the formula for the variation of parameters and inequality techniques. Finally, the third aim is to illustrate an application of our abstract results in the study of an initial and bound-ary value problem which describes the contact of an elastic rod with an obstacle.

KEYWORDS: Differential hemivariational inequality; C0-semigroup of operators; mild solution; fixed point principle; exponential stability; elastic rod; unilateral contact

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1. Introduction

Differential variational inequalities (DVIs, for short) are systems which couple differential or partial differential equations with a time-dependent variational inequality. They represent a powerful mathematical tool used in the analysis of a large number of nonlinear boundary value problems. Various mathematical models arising in the study of contact and impact problems, electrical circuits with ideal diodes, economical dynamics and traffic networks, lead to DVIs. For this reason, in the last decade, a considerable effort has been made in their analysis and numerical approximation.

Introduced in [1], DVIs were systematically discussed in [2], in the framework of Euclidean spaces. Later, a number of papers have been dedicated to the development of theory of DVIs and their applications. Among them, we refer the reader to [3–9], where results on solvability, stability, and bifurcation for finite-dimensional DVIs have been obtained. The study of DVIs in infinite-dimensional spaces is more recent. The reason is that, in contrast with the case of finite-dimensional spaces, the study of DVIs in infinite-dimensional spaces requires regularity results of the solution of the associated variational inequalities such as measurability, continuity, or condensivity, which enable us to convert them to differential inclusions. For more details on this topic, we refer to [10–14] and the references therein. Differential hemivariational inequalities (DHVIs, for short) represent an important extension of DVIs, which couple differential or partial differential equations with a hemivariational inequality or a variational-hemivariational inequality.

The stability of solutions is a topic of main interest in the control theory and other areas where ordinary differential equations and partial differential equations play an important role. The presence of time-dependent terms in control systems results in differential equations involving transcendental

CONTACT Zhenhai Liu zhhliu@hotmail.com

terms of the exponential type. The corresponding stability (necessary for the analysis of time systems) is often a significant matter. Recently, problems of exponential stability for differential systems have attracted the attention of many researchers, as it results from the references [15–19]. It is noteworthy that the results in all aforementioned works concern only the stability of solutions of differential equations. Nevertheless, many problems arising in engineering, operational research, economical dynamics, and physical sciences are described by systems of partial differential equations and/or by DVIs or DHVIs in infinite-dimensional spaces. For such problems, there are very few results concerning the stability of solutions, despite the fact that the need of such kind of results is widely recognized in the literature. To fill this gap, in this paper, we study the exponential stability problem for a class of DHVIs in reflexive Banach spaces. To introduce the problem, we need some notations that we present in what follows.

Everywhere in this work E will be a Banach space and U will be a reflexive Banach space, endowed with the norms $\|\cdot\|_E$ and $\|\cdot\|_U$, respectively. We denote by U^* the strong topological dual of U and by $\langle\cdot,\cdot\rangle$ the duality paring mapping between U^* and U. Moreover, $\mathcal{L}(U,E)$ represents the space of linear continuous operators from U to E. The norm on the space $\mathcal{L}(U,E)$ will be denoted by $\|\cdot\|_{\mathcal{L}(U,E)}$. We abbreviate this notation when $U=E$, i.e. we write $\mathcal{L}(E)$ instead of $\mathcal{L}(E,E)$ and $\|\cdot\|_{\mathcal{L}(E)}$ instead of $\|\cdot\|_{\mathcal{L}(E,E)}$. Finally, 0_U represents the zero element of U.

Below, I denotes either a bounded interval of the form $[0,b]$ with $b>0$ or the unbounded interval $\mathbb{R}_+=[0,+\infty)$. Notations $C(I;E)$ and $C(I;U)$ will represent the space of continuous functions on I with values in E and U, respectively. In the case $I=[0,b]$, these spaces are Banach spaces endowed with the canonical norm. In the case $I=\mathbb{R}_+$, they can be organized in a canonical way as a Fréchet space, i.e. a complete metric space in which the corresponding topology is induced by a countable family of seminorms. In the rest of the manuscript, we shall use the standard notation for the Lebesque and Sobolev spaces.

Let $A:D(A)\subseteq E\to E$ be the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{(t\geq 0)}$ on E and let $f:I\times E\to E$, $g:I\times E\to\mathcal{L}(U,E)$. Consider also the set $K\subset E$, the operators $B:U\to U^*$, $h:I\times E\to U^*$ and the functions $\varphi:U\times U\to\mathbb{R}$, $J:U\to\mathbb{R}$. We assume that J is locally Lipschitz and we denote by J^0 its generalized (Clarke) directional derivative. With these data, we consider the following problem.

(DHVI). *Find $x:I\to E$ and $u:I\to U$ such that*

$$x'(t)=Ax(t)+f(t,x(t))+g(t,x(t))u(t)\quad\text{for } t\in I, \tag{1}$$

$$u(t)\in SOL(K;B,h(t,x(t)),\varphi,J)\quad\text{for } t\in I, \tag{2}$$

$$x(0)=x_0. \tag{3}$$

Here $SOL(K;B,h(x(t),\cdot),\varphi,J)$ stands for the set of solution of the hemivariational inequality

$$\begin{aligned}u(t)\in K,\quad &\langle Bu(t)-h(t,x(t)),v-u(t)\rangle+\varphi(u(t),v)-\varphi(u(t),u(t))\\ &+J^0(u(t);v-u(t))\geq 0\quad\text{for all } v\in K\end{aligned} \tag{4}$$

and, therefore, inclusion (2) stands that u satisfies inequality (4) for $t\in I$. With this remark, we note that (DHVI) represents a system which couples the differential equation (1) with the hemivariational inequalities (4), associated to the initial condition (3). Therefore, following the terminology in [12–14], we refer to (DHVI) as a differential hemivariational inequality. The solution of (DHVI) is understood in the following sense.

Definition 1.1: A pair of functions (x,u), with $x\in C(I;E)$ and $u\in C(I;U)$, is said to be a mild solution of (1)–(3) if

$$x(t)=T(t)x_0+\int_0^t T(t-s)[f(s,x(s))+g(s,x(s))u(s)]\,\mathrm{d}s\quad\text{for all } t\in I, \tag{5}$$

where $u(t) \in SOL(K; B, h(t, x(t)), \varphi, J)$ for all $t \in I$. If (x, u) is a mild solution of (1)–(3), then $x(t)$ is called the mild trajectory and $u(t)$ is called the variational control trajectory.

We complete this definition with the following one.

Definition 1.2: A mild solution (x, u) of (1)–(3) is called globally exponentially stable if there exists two constants $L > 0$ and $\omega > 0$ such that

$$\|x(t)\|_E \leq Le^{-\omega t} \quad \text{for all } t \in I. \tag{6}$$

The rest of the manuscript is structured as follows. In Section 2, we recall some basic definitions and results needed throughout this paper. In Section 3, we state and prove the existence of a uniqueness solution for (DHVI). The first trait of novelty of our work consists in the fact that, in contrast with [12–14] (where the unique solvability of the DHVIs is obtained by converting the system into a differential inclusion), our proof is based on a fixed-point argument used in [20]. Next, in Section 4, we prove the exponential stability of the solution for the DHVI. The proof is based on arguments of semigroups of linear continuous operators. Finally, in Section 5, we consider a one-dimensional model of contact which lead to a differential hemivariational inequality of the form (1.1)–(1.3). Under appropriate assumptions on the data, we prove the unique solvability of the model as well as a stability result. The proofs are based on our abstract results in Sections 3 and 4.

2. Preliminaries

In this section, we review some prerequisites that are necessary in the rest of the manuscript.

A function $\phi : U \to \mathbb{R}$ is (sequentially) lower semicontinuous (l.s.c., for short) if $x_n \to x$ in U, as $n \to +\infty$ implies $\phi(x) \leq \liminf_{n\to+\infty} \phi(x_n)$. Let $J : U \to \mathbb{R}$ be a locally Lipschitz function. Following Clarke [21], the generalized directional derivative of J at $x \in U$ in the direction $v \in U$ is defined by

$$J^0(x; v) := \limsup_{\lambda\to 0^+, \xi\to x} \frac{J(\xi + \lambda v) - J(\xi)}{\lambda}.$$

The generalized gradient of $J : U \to \mathbb{R}$ at $x \in U$ is the subset of U^* given by

$$\partial J(x) := \{\xi \in U^* : J^0(x; v) \geq \langle \xi, v\rangle \ \forall\, v \in U\}.$$

The statement below collects some basic properties of the generalized directional derivative and gradient.

Lemma 2.1 ([21, Proposition 2.1.2] and [22, Proposition 5]): *If $J : U \to \mathbb{R}$ is a locally Lipschitz function, then the following statements hold.*

(i) *For all $v \in U$, one has $J^0(x; v) = \max\{\langle \xi, v\rangle : \xi \in \partial J(x)\}$.*
(ii) *For every $x \in U$, $\partial J(x)$ is a nonempty, convex, weakly* compact subset of U^* and $\|\xi\|_{U^*} \leq L_x$ for any $\xi \in \partial J(x)$, where $L_x > 0$ is the Lipschitz constant of J near x.*
(iii) *For every $x \in U$, the function $U \ni v \mapsto J^0(x; v) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e. $J^0(x; \lambda v) = \lambda J^0(x; v)$ for all $\lambda \geq 0$, $v \in U$ and $J^0(x; v_1 + v_2) \leq J^0(x; v_1) + J^0(x; v_2)$ for all $v_1, v_2 \in U$, respectively.*

Next, we proceed with the definition of some classes of operators.

Definition 2.2 ([23, Definition 3.65]): An operator $B : U \to U^*$ is said to be:

(a) monotone, if for all $u,\ v \in U$, we have $\langle Bu - Bv, u - v\rangle \geq 0$;
(b) bounded, if B maps bounded sets of U into bounded sets of U^*;
(c) pseudomonotone, if it is bounded and $u_n \to u$ weakly in U with

$$\limsup \langle Bu_n, u_n - u\rangle \leq 0$$

imply $\liminf \langle Bu_n, u_n - v\rangle \geq \langle Bu, u - v\rangle$ for all $v \in U$.

In the sequel, we denote by $C(I;E) \times C(I;U)$ the Cartesian product of the spaces $C(I;E)$ and $C(I;U)$. A typical element of $C(I;E) \times C(I;U)$ will be denoted by (x,u) or (x,η). Consider also two subsets $\mathcal{P} \subset C(I;E) \times C(I;U)$ and $\mathcal{Q} \subset C(I;E) \times C(I;U)$ defined by the equivalences below:

$$(x,u) \in \mathcal{P} \iff (x,u) \text{ satisfies (5)} \quad \text{for all } t \in I, \tag{7}$$

$$(x,u) \in \mathcal{Q} \iff (x,u) \text{ satisfies (4)} \quad \text{for all } t \in I. \tag{8}$$

It is clear that the problem (1)–(3) has at least a mild solution iff $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$ and, moreover, it has a unique solution iff $\mathcal{P} \cap \mathcal{Q} \subset C(I;E) \times C(I;U)$ reduces to a single element, i.e. is a singleton. For this reason, our aim in what follows is to provide sufficient conditions such that $\mathcal{P} \cap \mathcal{Q}$ is a singleton. To this end, we suppose that the following conditions hold.

$$\text{For each } \eta \in C(I;U) \text{ there exists a unique } x \in C(I;E) \text{ such that } (x,\eta) \in \mathcal{P}. \tag{9}$$

$$\text{For each } x \in C(I;E) \text{ there exists a unique } \eta \in C(I;U) \text{ such that } (x,\eta) \in \mathcal{Q}. \tag{10}$$

We now define the operators $\Phi\colon C(I;U) \to C(I;E)$, $\Psi\colon C(I;E) \to C(I;U)$ and $\Lambda\colon C(I;U) \to C(I;U)$ by equalities

$$x = \Phi\eta \iff (x,\eta) \in \mathcal{P}. \tag{11}$$

$$\eta = \Psi x \iff (x,\eta) \in \mathcal{Q}. \tag{12}$$

$$\Lambda\eta = \Psi\Phi\eta \quad \text{for all } \eta \in C(I;U). \tag{13}$$

We note that assumptions (9) and (10) guarantee that these operators are well-defined.

Assume that the operator Λ has a unique fixed point, i.e.

$$\text{there exists a unique element } \eta^* \in C(I;U) \text{ such that } \Lambda\eta^* = \eta^*. \tag{14}$$

Let $x^* = \Phi\eta^*$, i.e. x^* satisfies the integral equation (5) with $u = \eta^*$, and let $u^* = \Psi x^*$, i.e. u^* is the solution of the the hemivatiational inequality (4) with $x = x^*$. Then it is easy to see that the pair (x^*, u^*) has the regularity $C(I;E) \times C(I;U)$ and, moreover, it is a mild solution of the system (1)–(3). In addition, the uniqueness of the fixed point of Λ guarantees that the mild solution is unique.

We collect the properties above to deduce the following fixed point principle.

Theorem 2.3: *Assume that* (9), (10) *and* (14) *hold. Then the DHVI* (1)–(3) *has a unique mild solution with regularity* $C(I;E) \times C(I;U)$.

Theorem 2.3 represent a particular case of [24, Theorem 2], already used in a slightly different form in [20]. This theorem will play a central role in the study of the existence and uniqueness of solution for (DHVI) as it results from the next section.

3. An existence and uniqueness result

In this section, we state and prove the existence of a unique mild solution for the differential hemivariational inequality (DHVI). To this end, we consider the following assumptions on the data.

H(A)$_1$ The operator $A : D(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{(t\geq 0)} \subset \mathcal{L}(E)$.

H(f)$_1$ $f : I \times E \to E$ is such that

(1) $t \mapsto f(t,x) : I \to E$ is measurable for all $x \in E$;

(2) there is a constant $L_f > 0$ such that

$$\|f(t,x) - f(t,y)\|_E \leq L_f \|x - y\|_E \text{ for a.e. } t \in I, \text{ all } x,\ y \in E;$$

(3) there exists a function $a \in L^1(I;\mathbb{R}_+)$ such that

$$\|f(t,0_E)\|_E \leq a(t) \quad \text{for a.e. } t \in I.$$

H(g)$_1$ $g : I \times E \to \mathcal{L}(U,E)$ is such that

(1) $t \mapsto g(t,x) : I \to \mathcal{L}(U,E)$ is continuous for all $x \in E$;

(2) there is a constant $L_g > 0$ such that

$$\|g(t,x) - g(t,y)\|_{\mathcal{L}(U,E)} \leq L_g \|x - y\|_E \text{ for a.e. } t \in I, \text{ all } x, \quad y \in E;$$

(3) there exists a continuous function $d : I \to \mathbb{R}_+$ such that

$$\|g(t,0_E)\|_{\mathcal{L}(U,E)} \leq d(t) \quad \text{for all } t \in I.$$

H(K) K is a closed convex subset of U such that $0_U \in K$.

H(B) $B : U \to U^*$ is such that

(1) it is pseudomonotone;

(2) is strongly monotone, i.e. there exists $m_B > 0$ such that

$$\langle Bv_1 - Bv_2, v_1 - v_2\rangle \geq m_B \|v_1 - v_2\|_U^2 \quad \text{for all } v_1, \quad v_2 \in U.$$

H(φ) The functional $\varphi : U \times U \to \mathbb{R}$ is such that

(1) $\varphi(\eta,\cdot) : U \to \mathbb{R}$ is convex and lower semicontinuous for all $\eta \in U$;

(2) $\varphi(u,\lambda v) = \lambda\varphi(u,v)$ for all $u, v \in U,\ \lambda > 0$;

(3) $\varphi(v,v) \geq 0$ for all $v \in U$;

(4) there exists $\alpha_\varphi > 0$ such that

$$\varphi(\eta_1,v_2) - \varphi(\eta_1,v_1) + \varphi(\eta_2,v_1) - \varphi(\eta_2,v_2) \leq \alpha_\varphi \|\eta_1 - \eta_2\|_U \|v_1 - v_2\|_U$$

for all $\eta_1,\ \eta_2,\ v_1,\ v_2 \in U$.

H(J) The locally Lipschitz functional $J : U \to \mathbb{R}$ is such that

(1) there exist $\kappa_0,\ \kappa_1 \geq 0$ such that

$$\|\partial J(v)\|_{U^*} \leq \kappa_0 + \kappa_1 \|v\|_U \quad \text{for all } v \in U;$$

(2) there exists $\alpha_J > 0$ such that

$$J^0(v_1; v_2 - v_1) + J^0(v_2; v_1 - v_2) \leq \alpha_J \|v_1 - v_2\|_U^2 \quad \text{for all } v_1, \quad v_2 \in U.$$

H(h) $h : I \times E \to U^*$ is such that

(1) $h(\cdot,x) : I \to U^*$ is continuous for all $x \in E$;

(2) there is a constant $L_h > 0$ such that

$$\|h(t,x) - h(t,y)\|_{U^*} \leq L_h \|x - y\|_E \quad \text{for all } t \in I,\ x,\ y \in E;$$

(3) there exists a constant $\ell > 0$ such that

$$\|h(t,x)\|_{U^*} \leq \ell \quad \text{for all } t \in I, \quad x \in E.$$

H(0) $x_0 \in E$.

To provide our main result in this section, we recall two preliminary lemmas which will be useful to guarantee the validity of assumptions (9) and (10).

Lemma 3.1: *Let E be a Banach space, assume that $A : D(A) \subset E \to E$ satisfies condition* $\mathrm{H}(A)_1$ *and, moreover, assume that*

$$\begin{cases} \widetilde{f}\colon I \times E \to E \text{ is such that} \\ \text{(a) the mapping } t \mapsto \widetilde{f}(t,x)\colon I \to E \text{ is measurable, for all } x \in E; \\ \text{(b) for any compact set } \Sigma \subset I \text{ there exists } L_\Sigma > 0 \text{ such that} \\ \quad \|\widetilde{f}(t,x_1) - \widetilde{f}(t,x_2)\|_E \leq L_\Sigma \|x_1 - x_2\|_E \\ \quad \text{for all } x_1, x_2 \in E,\ t \in \Sigma. \end{cases} \tag{15}$$

Then, for each $x_0 \in E$, there exists a unique function $x \in C(I;E)$ such that

$$x(t) = T(t)x_0 + \int_0^t T(t-s)\widetilde{f}(s,x(s))\,\mathrm{d}s \quad \text{for all } t \in I. \tag{16}$$

Lemma 3.1 represents a well-known result in the study of differential equation in Banach spaces. It states the existence of a unique mild solution of the Cauchy problem

$$\dot{x}(t) = Ax(t) + \widetilde{f}(t,x(t)) \quad \text{for all } t \in I, \tag{17}$$

$$x(0) = x_0. \tag{18}$$

Its proof can be found in [25, Theorem 6.1.2], for instance.

Lemma 3.2: *Let U be a reflexive Banach space, assume that* $\mathrm{H}(K)$, $\mathrm{H}(B)$, $\mathrm{H}(\varphi)(1)$, (4) *and* $\mathrm{H}(J)$ *are satisfied and, moreover, assume that*

$$\alpha_\varphi + \alpha_J < m_B. \tag{19}$$

Then, for each $z \in U^*$ there exists a unique element u such that

$$u \in K, \quad \langle Bu, v-u\rangle + \varphi(u,v) - \varphi(u,u) + J^0(u; v-u) \geq \langle z, v-u\rangle \quad \text{for all } v \in K. \tag{20}$$

The proof of Lemma 3.2 is based on a surjectivity result for pseudomonotone operators combined with the properties of the subdifferential and the Banach fixed point theorem. It can be found in [26] or in [24, Theorem 84].

Next, we proceed with the following result.

Lemma 3.3: *Let U be a reflexive Banach space and assume that* H(K), H(B), H(φ)(1), (4), H(J), H(h) *and* (19) *hold. Then, for each $x \in C(I;E)$, there exists a unique function $u \in C(I;U)$ such that*

$$\begin{aligned} u(t) \in K, \quad & \langle Bu(t), v - u(t)\rangle + \varphi(u(t), v) - \varphi(u(t), u(t)) + J^0(u(t); v - u(t)) \\ & \geq \langle h(t, x(t)), v - u(t)\rangle \quad \text{for all } v \in K,\ t \in I. \end{aligned} \tag{21}$$

Moreover, if u_1, $u_2 \in C(I;U)$ represent the solution of the inequality (21) *for the functions $x_1, x_2 \in C(I;E)$, respectively, then the following inequality holds:*

$$\|u_1(t) - u_2(t)\|_U \leq \frac{L_h}{m_B - \alpha_\varphi - \alpha_J} \|x_1(t) - x_2(t)\|_E \quad \text{for all } t \in I. \tag{22}$$

In addition, if H(φ)(2), (3) *are also satisfied, then the solution satisfies the bound*

$$\|u(t)\|_U \leq \frac{1}{m_B - \alpha_J} (\|B0_U\|_{U^*} + \ell + \kappa_0) \quad \text{for all } t \in I. \tag{23}$$

Proof: Let $x \in C(I;E)$ be given and let $t \in \mathbb{R}_+$ be fixed. It follows from Lemma 3.2 that there exists a unique element $u(t)$ which solves (21) at the given moment t.

We now show that the function $t \mapsto u(t) : I \to U$ is continuous. To this end, consider $t_1, t_2 \in I$ and, for the sake of simplicity in writing, denote $x(t_i) = x_i$, $u(t_i) = u_i$ for $i = 1,2$. Using (21) we obtain

$$\begin{aligned} u_1 \in K, \quad & \langle Bu_1, v - u_1\rangle + \varphi(u_1, v) - \varphi(u_1, u_1) + J^0(u_1, v - u_1) \\ & \geq \langle h(t_1, x_1), v - u_1\rangle \quad \text{for all } v \in K, \end{aligned} \tag{24}$$

$$\begin{aligned} u_2 \in K, \quad & \langle Bu_2, v - u_2\rangle + \varphi(u_2, v) - \varphi(u_2, u_2) + J^0(u_2, v - u_2) \\ & \geq \langle h(t_2, x_2), v - u_2\rangle \quad \text{for all } v \in K. \end{aligned} \tag{25}$$

We take $v = u_2$ in (24) and $v = u_1$ in (25), then we add the resulting inequalities to obtain that

$$\begin{aligned} \langle Bu_1 - Bu_2, u_1 - u_2\rangle \leq\ & \varphi(u_1, u_2) - \varphi(u_1, u_1) + \varphi(u_2, u_1) - \varphi(u_2, u_2) \\ & + J^0(u_1; u_2 - u_1) + J^0(u_2; u_1 - u_2) \\ & + \langle h(t_1, x_1) - h(t_2, x_2), u_1 - u_2\rangle. \end{aligned} \tag{26}$$

We now write

$$\begin{aligned} & \langle h(t_1, x_1) - h(t_2, x_2), u_1 - u_2\rangle \\ & \quad = \langle h(t_1, x_1) - h(t_1, x_2), u_1 - u_2)\rangle + \langle h(t_1, x_2) - h(t_2, x_2), u_1 - u_2\rangle, \end{aligned}$$

then we use assumption H(h) to see that

$$\begin{aligned} & \langle h(t_1, x_1) - h(t_2, x_2), u_1 - u_2\rangle \\ & \quad \leq L_h \|x_1 - x_2\|_E \|u_1 - u_2\|_U + \|h(t_1, x_2) - h(t_2, x_2)\|_{U^*} \|u_1 - u_2\|_U. \end{aligned} \tag{27}$$

Next, inequalities (26), (27) and assumptions H(B)(2), H(φ)(4) and H(J)(2) yield

$$\begin{aligned} & (m_B - \alpha_\varphi - \alpha_J) \|u_1 - u_2\|_U \\ & \quad \leq L_h \|x_1 - x_2\|_E + \|h(t_1, x_2) - h(t_2, x_2)\|_{U^*}. \end{aligned} \tag{28}$$

Inequality (28) combined with the assumptions H(h)(1) and (19) implies that $t \mapsto u(t) : I \to U$ is a continuous function. This concludes the existence part of the lemma. The uniqueness part is a direct

consequence of the uniqueness of the solution $u(t)$ of the hemivariational inequality (21), at each $t \in I$, guaranteed by Lemma 3.3.

Next, we check that the inequality (22) hold. To this end, suppose that $x_1,\ x_2 \in C(I;E)$ and let $t \in I$ be given. Using arguments similar to those used in the proof of inequality (28) we deduce that

$$(m_B - \alpha_\varphi - \alpha_J)\,\|u_1(t) - u_2(t)\|_U \le L_h\,\|x_1(t) - x_2(t)\|_E$$

which implies (22).

Finally, we check that the solution satisfies the bound (23). To prove this, let $t \in I$ be fixed and take $v = 0_U \in K$ in (21). It follows from assumptions H(φ)(2), (3) that

$$\langle Bu(t), u(t)\rangle \le J^0(u(t); -u(t)) + \langle h(t, x(t)), u(t)\rangle.$$

Now, we write $Bu(t) = Bu(t) - B0_U + B0_U$. Then, using condition H(B)(2) and hypothesis H(h)(3), we deduce that

$$m_B\|u(t)\|_U^2 \le (\|B0_U\|_{U^*} + \ell)\|u(t)\|_U + J^0(u(t); -u(t)). \tag{29}$$

On the other hand, taking $v_1 = u(t)$ and $v_2 = 0_U$ in H(J)(2), we have

$$J^0(u(t); -u(t)) \le \alpha_J\|u(t)\|_U^2 - J^0(0_U; u(t)). \tag{30}$$

Moreover, it follows from Lemma 2.1(i) that

$$\begin{aligned} -J^0(0_U; u(t)) &\le |J^0(0_U; u(t))| \le \left|\max_{\xi\in\partial J(0)} \langle \xi, u(t)\rangle\right| \\ &\le \max_{\xi\in\partial J(0_U)} |\langle \xi, u(t)\rangle| \le \max_{\xi\in\partial J(0_U)} \|\xi\|_{U^*}\|u(t)\|_U. \end{aligned}$$

and, using condition H(J)(1) with $v = 0_U$ yields

$$-J^0(0_U; u(t)) \le \kappa_0\|u(t)\|_U. \tag{31}$$

Now, we combine inequalities (30) and (31) to see that

$$J^0(u(t); -u(t)) \le \alpha_J\|u(t)\|_U^2 + \kappa_0\|u(t)\|_U. \tag{32}$$

Finally, we substitute (32) in (29) to deduce that

$$(m_B - \alpha_J)\|u(t)\|_U \le \|B0_U\|_{U^*} + \ell + \kappa_0.$$

This shows that inequality (23) holds, which completes the proof of the lemma. ■

We proceed with the following existence and uniqueness result.

Lemma 3.4: *Let E be a Banach space and assume that* H(A)$_1$, H(f)$_1$, H(g)$_1$ *and* H(0) *hold. Then, for each $\eta \in C(I;U)$, there exists a unique function $x \in C(I;E)$ such that*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)[f(s, x(s)) + g(s, x(s))\eta(s)]\,\mathrm{d}s. \tag{33}$$

Assume now that $x_1,\ x_2 \in C(I;E)$ represent the solution of problem (33) *corresponding to the functions $\eta_1, \eta_2 \in C(I;U)$, respectively. Moreover, assume that there exists $\varrho > 0$ such that*

$$\|\eta_1(t)\|_U \le \varrho, \quad \|\eta_2(t)\|_U \le \varrho \quad \text{for all } t \in I. \tag{34}$$

Then, for each compact set $\Sigma \subset I$, there exists a constant $M_\Sigma > 0$ such that

$$\|x_1(t) - x_2(t)\|_E \le M_\Sigma \int_0^t \|\eta_1(s) - \eta_2(s)\|_U\,\mathrm{d}s \quad \text{for all } t \in \Sigma. \tag{35}$$

Proof: First, let $\eta \in C(I;U)$ be given and consider the function $\widetilde{f} : I \times E \to E$ defined by

$$\widetilde{f}(t,x) = f(t,x) + g(t,x)\eta(t) \quad \text{for all } t \in I,\ x \in E.$$

We use assumptions $\mathrm{H}(f)_1$ and $\mathrm{H}(g)_1$, to see that $\widetilde{f}$ satisfies condition (15). Therefore, the existence and uniqueness part of Lemma 3.4 is a direct consequence of Lemma 3.1.

Now, suppose that $\eta \in C(I;U)$ is such that

$$\|\eta(t)\|_U \le \varrho \quad \text{for all } t \in I \tag{36}$$

where $\varrho > 0$ is given. Consider a compact set $\Sigma \subset I$, denote

$$T_\Sigma := \sup_{s\in\Sigma} \|T(s)\|_{\mathcal{L}(E)} \tag{37}$$

and let $t \in \Sigma$. We use (33), assumptions $\mathrm{H}(f)_1$ and $\mathrm{H}(g)_1$, inequality (36) and notation (37) to obtain the following inequalities:

$$\begin{aligned}
\|x(t)\|_E &\le \|T(t)\|_{\mathcal{L}(E)}\|x_0\|_E + \int_0^t \|T(t-s)\|_{\mathcal{L}(E)}\Big[\|f(s,x(s))\|_E + \|g(s,x(s))\eta(s)\|_E\Big]\,\mathrm{d}s \\
&\le T_\Sigma\|x_0\|_E + T_\Sigma\int_0^t \Big[\|f(s,x(s)) - f(s,0_E)\|_E + \|f(s,0_E)\|_E \\
&\quad + (\|g(s,x(s)) - g(s,0_E)\|_{\mathcal{L}(U,E)} + \|g(s,0_E)\|_{\mathcal{L}(U,E)})\|\eta(s)\|_U\Big]\,\mathrm{d}s \\
&\le T_\Sigma\|x_0\|_E + T_\Sigma\int_0^t \Big(L_f\|x(s)\|_E + a(s) + [L_g\|x(s)\|_E + d(s)]\varrho\Big)\,\mathrm{d}s.
\end{aligned}$$

Denote

$$\rho_\Sigma := T_\Sigma\|x_0\|_E + T_\Sigma\big(\|a\|_{L^1(\Sigma,\mathbb{R}_+)} + \|d\|_{L^1(\Sigma,\mathbb{R}_+)}\varrho\big). \tag{38}$$

Then, it follows from the above that

$$\|x(t)\|_E \le \rho_\Sigma + T_\Sigma(L_f + L_g\varrho)\int_0^t \|x(s)\|_E\,\mathrm{d}s.$$

Then by the Gronwall inequality, we obtain that

$$\|x(t)\|_E \le \rho_\Sigma e^{T_\Sigma(L_f+L_g\varrho)t} \quad \text{for all } t \in \Sigma. \tag{39}$$

We shall use the bound (39) in the proof of inequality (35).

Next, assume that $\eta_1,\ \eta_2 \in C(I;U)$ are two functions which satisfy condition (19) and denote by $x_1,\ x_2 \in C(I;E)$ the solution of problem (33) corresponding to $\eta_1,\ \eta_2 \in C(I;U)$, respectively. Let $t \in \Sigma$. Using (33) we have

$$x_i(t) = T(t)x_0 + \int_0^t T(t-s)[f(s,x_i(s)) + g(s,x_i(s))\eta_i(s)]\,\mathrm{d}s \quad \text{for all } i = 1,2.$$

This implies that

$$x_1(t) - x_2(t) = \int_0^t T(t-s)\Big[f(s,x_1(s)) - f(s,x_2(s)) + g(s,x_1(s))\eta_1(s) - g(s,x_2(s))\eta_2(s)\Big]\,\mathrm{d}s$$

and, therefore, using (37) we deduce that

$$\|x_1(t) - x_2(t)\|_E \leq T_\Sigma \int_0^t \big[\|f(s, x_1(s)) - f(s, x_2(s))\|_E + \|g(s, x_1(s))\eta_1(s) - g(s, x_2(s))\eta_2(s)\|_E \, \mathrm{d}s \tag{40}$$

We now use assumptions H$(f)_1$ to see that

$$\|f(s, x_1(s)) - f(s, x_2(s))\|_E \leq L_f \|x_1(s) - x_2(s)\|_E \quad \text{for all } s \in [0, t]. \tag{41}$$

On the other hand, assumptions H$(g)_1$ and (34) yield

$$\begin{aligned}
&\|g(s, x_1(s))\eta_1(s) - g(s, x_2(s))\eta_2(s)\|_E \\
&\quad \leq \|g(s, x_1(s))\eta_1(s) - g(s, x_2(s))\eta_1(s)\|_E \\
&\qquad + \|g(s, x_2(s))\eta_1(s) - g(s, x_2(s))\eta_2(s)\|_E \\
&\quad \leq \|g(s, x_1(s)) - g(s, x_2(s))\|_{\mathcal{L}(U,E)} \|\eta_1(s)\|_U \\
&\qquad + \|g(s, x_2(s))\|_{\mathcal{L}(U,E)} \|\eta_1(s) - \eta_2(s)\|_U \\
&\quad \leq L_g \varrho \|x_1(s) - x_2(s)\|_E \\
&\qquad + \Big[\|g(s, x_2(s)) - g(s, 0_E)\|_{\mathcal{L}(U,E)} + \|g(s, 0_E)\|_{\mathcal{L}(U,E)}\Big] \|\eta_1(s) - \eta_2(s)\|_U \\
&\quad \leq L_g \varrho \|x_1(s) - x_2(s)\|_E + \Big[L_g \|x_2(s)\|_E + d(s)\Big] \|\eta_1(s) - \eta_2(s)\|_U
\end{aligned}$$

and, using (39) we deduce that

$$\begin{aligned}
&\|g(s, x_1(s))\eta_1(s) - g(s, x_2(s))\eta_2(s)\|_E \leq L_g \varrho \|x_1(s) - x_2(s)\|_E \\
&\qquad + \Big[L_g \rho_\Sigma e^{T_\Sigma (L_f + L_g \varrho)s} + d(s)\Big] \|\eta_1(s) - \eta_2(s)\|_U \quad \text{for all } s \in [0, t].
\end{aligned} \tag{42}$$

We now substitute inequalities (41) and (42) in (40) to find that

$$\begin{aligned}
\|x_1(t) - x_2(t)\|_E &\leq T_\Sigma (L_f + L_g \varrho) \int_0^t \|x_1(s) - x_2(s)\|_E \, \mathrm{d}s \\
&\quad + T_\Sigma \left(\sup_{s \in \Sigma} d(s) + L_g \rho_\Sigma e^{T_\Sigma (L_f + L_g \varrho)t}\right) \int_0^t \|\eta_1(s) - \eta_2(s)\|_U \, \mathrm{d}s.
\end{aligned} \tag{43}$$

Let

$$L_\Sigma := \sup_{s \in \Sigma} s, \quad Q_\Sigma := \sup_{s \in \Sigma} d(s), \tag{44}$$

$$M_\Sigma = T_\Sigma (Q_\Sigma + L_g \rho_\Sigma e^{T_\Sigma (L_f + L_g \varrho) L_\Sigma}) e^{T_\Sigma (L_f + L_g \varrho) L_\Sigma} \tag{45}$$

We now use the Gronwall argument in inequality (43) and notations (44), (45) to see that (35) holds. ■

We now recall with the following fixed-point result.

Lemma 3.5: *Let U be a Banach space and $\mathcal{S}\colon C(I; U) \to C(I; U)$ be an operator with the following property: for any compact set $\Sigma \subset I$ there exists $M_\Sigma > 0$ such that*

$$\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_U \leq M_\Sigma \int_0^t \|u_1(s) - u_2(s)\|_U \, \mathrm{d}s \quad \text{for all } u_1, u_2 \in C(I; U), \quad t \in \Sigma. \tag{46}$$

Then, $\mathcal{S}$ has a unique fixed point, i.e. there exists a unique element $\eta^* \in C(I; U)$ such that $\mathcal{S}\eta^* = \eta^*$.

Note that here and below, when no confusion arises, we use the shorthand notation $\mathcal{S}u(t)$ to represent the value of the function $\mathcal{S}u$ at the point t, i.e. $\mathcal{S}u(t) = (\mathcal{S}u)(t)$. An operator which satisfies condition (46) is called a history-dependent operator. This terminology was introduced in [27] and since it was used in many papers, see [23, 24, 28] and the references therein. Lemma 3.5 was proved in [29]. Its proof can also be found in [24, Ch.2], together with various properties of history-dependent operators.

We now are in the position to present our existence and uniqueness result for the solution of the DHVI (1)–(3).

Theorem 3.6: *Assume that E is a Banach space, U is a reflexive Banach space and, moreover, assume that* $\mathrm{H}(A)_1$, $\mathrm{H}(f)_1$, $\mathrm{H}(g)_1$, $\mathrm{H}(K)$, $\mathrm{H}(B)$, $\mathrm{H}(\varphi)$, $\mathrm{H}(J)$, $\mathrm{H}(h)$, $\mathrm{H}(0)$ *and (19) hold. Then the DHVI (1)–(3) has a unique mild solution* $(x, u) \in C(I; E) \times C(I; U)$.

Proof: We use Theorem 2.3. First, it follows from Lemma 3.4 that condition (9) holds. Next, Lemma 3.3 guarantees that condition (10) holds, too.

We now prove that the operator Λ defined by (14) is a history-dependent operator. To this end we note that, a carefully analysis of the definitions (11)–(13) reveals that the operator $\Lambda : C(I; U) \to C(I; U)$ is defined by equality

$$\Lambda\eta = u_{x_\eta} \quad \text{for all } \eta \in C(I; U). \tag{47}$$

Here, for each $\eta \in C(I; U)$, $x_\eta \in C(I; E)$ represents the solution of the integral Equation (33), guaranteed by Lemma 3.4, and u_{x_η}, represents the solution of the hemivariational inequality (21) for $x = x_\eta$, guaranteed by Lemma 3.3.

Let $\eta_1, \eta_2 \in C(I; U)$ and, for simplicity, denote $x_{\eta_1} = x_1$, $x_{\eta_2} = x_2$, $u_{x_{\eta_1}} = u_1$, $u_{x_{\eta_2}} = u_2$. Then, it follows from (47) that

$$\Lambda\eta_1 = u_1, \quad \Lambda\eta_2 = u_2. \tag{48}$$

Moreover, inequalities (22) and (23) imply that

$$\|u_1(t) - u_2(t)\|_U \le \frac{L_h}{m_B - \alpha_\varphi - \alpha_J}\|x_1(t) - x_2(t)\|_E \quad \text{for all } t \in I. \tag{49}$$

$$\|u_1(t)\|_U \le \varrho, \quad \|u_2(t)\|_U \le \varrho \quad \text{for all } t \in I \tag{50}$$

where, here and below, $\varrho := (1/(m_B - \alpha_J))(\|B0_U\|_{U^*} + \ell + \kappa_0)$.

Let Σ be a compact subset of I. Then, inequalities (50) allows us to use Lemma 3.4 in order to obtain that

$$\|x_1(t) - x_2(t)\|_E \le M_\Sigma \int_0^t \|\eta_1(s) - \eta_2(s)\|_U \, ds \quad \text{for all } t \in \Sigma. \tag{51}$$

We now combine definition (48) with inequalities (49), (51) to see that

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_U \le \frac{L_h M_\Sigma}{m_B - \alpha_\varphi - \alpha_J} \int_0^t \|\eta_1(s) - \eta_2(s)\|_U \, ds \quad \text{for all } t \in \Sigma.$$

This inequality shows that Λ is a history-dependent operator and, therefore, using Lemma 3.5 we deduce that it has a unique fixed point.

It follows from above that conditions (9), (10), (14) in Section 2 are verified. Therefore, we are in a position to use Theorem 2.3 in order to conclude the proof. ■

4. Exponential stability of (DHVIs)

In this section, we analyze the exponential stability of the (DHVI). To this end, we need to reinforce part of the assumption in Theorem 3.6. More precisely, we consider the following assumptions.

$\underline{\mathrm{H}(A)_2}$ $A : D(A) \subset E \to E$ satisfies condition $\mathrm{H}(A_1)$ and, moreover, there are two positive numbers w, M such that $\|T(t)\|_{\mathcal{L}(E)} \leq Me^{-wt}$ for every $t \geq 0$.
$\underline{\mathrm{H}(f)_2}$ $f : I \times E \to E$ satisfies condition $\mathrm{H}(f_1)$ with $a(t) = 0$ for all $t \in I$.
$\underline{\mathrm{H}(g)_2}$ $g : I \times E \to \mathcal{L}(U, E)$ satisfies condition $\mathrm{H}(g_1)$ with $d(t) = 0$ for all $t \in I$.

Note that condition $\mathrm{H(A)}_2$ show that the semigroup generated by A is exponentially stable. Our second main result concerning the exponential stability of (DHVI) is formulated as follows.

Theorem 4.1: *Assume that E is a Banach space, U is a reflexive Banach space,* $\mathrm{H}(A)_2$, $\mathrm{H}(f)_2$, $\mathrm{H}(g)_2$, $\mathrm{H}(K)$, $\mathrm{H}(B)$, $\mathrm{H}(\varphi)$, $\mathrm{H}(J)$, $\mathrm{H}(h)$, $\mathrm{H}(0)$ *and* (19) *hold and, moreover, assume that*

$$M[L_f(m_B - \alpha_J) + L_g(\|B0_U\|_{U^*} + \ell + \kappa_0)] < w(m_B - \alpha_J). \tag{52}$$

Then the mild solution of (1)–(3) *is globally exponentially stable.*

Proof: Denote by $(x, u) \in C(I; E) \times C(I; U)$ the mild solution of problem (1)–(3). Recall that its existence and uniqueness is guaranteed by Theorem 3.6. Moreover, using Definition 1.1 it follows that

$$x(t) = T(t)x_0 + \int_0^t T(t-s)[f(s, x(s)) + g(s, x(s))u(s)]\,\mathrm{d}s \quad \text{for all } t \in I, \tag{53}$$

and, in addition, Lemma 3.3 yields

$$\|u(t)\|_U \leq \frac{1}{m_B - \alpha_J}(\|B0_U\|_{U^*} + \ell + \kappa_0) \quad \text{for all } t \in I. \tag{54}$$

Let $t \in I$. We now use (53), assumptions $\mathrm{H}(A)_2$, $\mathrm{H}(f)_2$ and $\mathrm{H}(g)_2$ and the bound (54) to derive the estimate

$$\begin{aligned}\|x(t)\|_E &\leq \|T(t)\|_{\mathcal{L}(E)}\|x_0\|_E + \int_0^t \|T(t-s)\|_{\mathcal{L}(E)}[\|f(s, x(s))\|_E + \|g(s, x(s))u(s)\|_E]\,\mathrm{d}s \\ &\leq Me^{-wt}\|x_0\|_E + \int_0^t Me^{-w(t-s)}\Big[L_f + \frac{L_g}{m_B - \alpha_J}(\|B0_U\|_{U^*} + \ell + \kappa_0)\Big]\|x(s)\|_E\,\mathrm{d}s.\end{aligned}$$

This inequality implies that

$$e^{wt}\|x(t)\|_E \leq M\|x_0\|_E + \frac{M[L_f(m_B - \alpha_J) + L_g(\|B0_U\|_{U^*} + \ell + \kappa_0)]}{(m_B - \alpha_J)}\int_0^t e^{ws}\|x(s)\|_E\,\mathrm{d}s.$$

and, therefore, the Gronwall inequality yields

$$e^{wt}\|x(t)\|_E \leq M\|x_0\|_E e^{M[L_f(m_B-\alpha_J)+L_g(\|B0_U\|_{U^*}+\ell+\kappa_0)]/(m_B-\alpha_J)t},$$

which implies that

$$\|x(t)\|_E \leq M\|x_0\|_E e^{\left(M[L_f(m_B-\alpha_J)+L_g(\|B0_U\|_{U^*}+\ell+\kappa_0)]/(m_B-\alpha_J)-w\right)t}. \tag{55}$$

Denote

$$L = w - M\|x_0\|_E, \quad \omega = w - \frac{M[L_f(m_B - \alpha_J) + L_g(\|B0_U\|_{U^*} + \ell + \kappa_0)]}{(m_B - \alpha_J)} \tag{56}$$

and note that assumptions H$(A)_2$ and (52) guarantee that $M > 0$ and $\omega > 0$, respectively. We combine inequality (55) with notation (56) to conclude that

$$\|x(t)\|_E \le Le^{-\omega t}.$$

We now use this inequality and Definition 1.2 to conclude the proof. ■

5. A one-dimensional example

The abstract results in Sections 3–4 are useful in the study of various initial and boundary value problems. To provide an example, we consider in this section the following one-dimensional problem.

Problem $\mathcal{P}$. *Find the functions* $x : I \times (0,1) \to \mathbb{R}$ *and* $u : I \times (0,1) \to \mathbb{R}$ *such that*

$$x'(t,z) = \delta x_{zz}(t,z) - cx(t,z) + f^*(t,x(t,z)) + g^*(t,x(t,z))u(t,z)) \quad \text{for all } t \in I, \text{ a.e. } z \in (0,1), \tag{57}$$

$$x_z(t,0) = x(t,1) = 0, \quad \text{for all } t \in I, \tag{58}$$

$$\mathcal{B}u_{zz}(t,z) + h^*(t,x(t,z)) = 0, \quad \text{for all } t \in I,\ z \in (0,1), \tag{59}$$

$$u(t,1) \le k, \quad \left.\begin{array}{ll} -\mathcal{B}u_z(t,1) = \lambda + j(u(t,1)), & \text{if } u(t,1) < k \\ -\mathcal{B}u_z(t,1) \ge \lambda + j(u(t,1)), & \text{if } u(t,1) = k \end{array}\right\}, \tag{60}$$

$$u(t,0) = 0, \quad \text{for all } t \in I, \tag{61}$$

$$x(0,z) = x_0(z) \quad \text{a.e. } z \in (0,1). \tag{62}$$

Here, and below in this section, the prime denotes the derivative with respect to the time variable t and the subscripts represent the derivative with respect to the spatial variable z, i.e. $x' = \partial x/\partial t$, $u_z = \partial u/\partial z$ and $x_{zz} = \partial^2 x/\partial z^2$. Equations (57) and (58) represents the state equation, (59)–(61) is the control inequality and (62) is the initial condition.

Problem $\mathcal{P}$ represents the variational formulation of a mathematical model which describes the contact of an elastic rod with an obstacle. A brief description of the physical setting is the following. An elastic rod occupies, in the reference configuration, the interval $[0,1]$ on the Oz axis. The rod is fixed in $z = 0$, is acted by body forces of density h^* along Oz, and its extremity $z = 1$ is in contact with an obstacle made of a rigid body covered by a deformable layer of thickness $k > 0$. This layer behaves rigid-elastically, i.e. it allows penetration, but only when the magnitude of the stress in the contact point reaches a critical positive value, the yield limit, denoted by λ. In addition, the reaction of this layer depends on the penetration, this dependence being described by a given continuous function j. The contact process is time-dependent, the time interval of interest is I and can be bounded or unbounded. The material's behaviour is described with the elasticity operator $\mathcal{B}$, assumed to be nonlinear.

Details concerning the previous contact model (59)–(61), including the passage from the classical formulation of the problem can be found in [22] and, therefore, we skip them. Nevertheless we note that, in contrast with the model presented in [22], in this current paper we assume that the force h^* depends on an additional variable x, which satisfies a diffusion equation of the form (57) and (58), associated to the initial condition (62). Here $\delta > 0$ is the diffusion coefficient, the parameter $c > 0$ is used to regulate the convergence speed, f^* and g^* are arbitrary external forcing functions, and x_0 represents the initial data. From physical point of view x could be interpreted as being the temperature field, the adhesion field the a moisture field. For more details we refer the reader to [30], where models of contact governed by a parameter x which satisfies a diffusion equation have been considered.

We now turn to the analysis of Problem $\mathcal{P}$. To this end, we define the operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ as follows

$$\begin{aligned} &(Ax)(z) = \delta x_{zz}(z) - cx(z), \\ &D(A) = \{x \in H^2(0,1) \mid x_z(0) = 0,\ x_z(1) = 0\}. \end{aligned} \tag{63}$$

It is well know from [25] that A satisfies assumption $\mathrm{H}(A)_1$ on the space $E = L^2(0,1)$. Moreover, the semigroup $T(t) = e^{tA}$ generated by A is exponential stable, that is,

$$\|T(t)\|_{\mathcal{L}(L^2(0,1))} \leq e^{-(c+\delta\pi^2/4)t} \quad \text{for all } t \in I. \tag{64}$$

This shows that assumption $\mathrm{H}(A)_2$ holds, too.

Next, we define the space U by equality

$$U = \{v \in H^1(0,1) \mid v(0) = 0\}, \tag{65}$$

which is a real Hilbert space endowed with the inner product

$$(u,v)_U = \int_0^1 u_x\, v_x \,\mathrm{d}x \quad \text{for all } u, v \in U$$

and the associated norm $\|\cdot\|_U$. Recall that the completeness of the space $(U, \|\cdot\|_U)$ follows from the Friedrichs-Poincaré inequality. Moreover, from the Sobolev trace theorem, it follows that

$$|v(1)| \leq \|v\|_U \quad \text{for all } v \in U. \tag{66}$$

We denote by U^* and $\langle\cdot,\cdot\rangle$ the dual of U and the duality pairing between U^* and U, respectively.

We now consider the following assumptions on the data.

$\underline{\mathrm{H}(f^*)}$ $f^* : I \times \mathbb{R} \to \mathbb{R}$ is such that
(1) there exists $L_{f^*} > 0$ such that

$$|f^*(t_1,\vartheta_1) - f^*(t_2,\vartheta_2)| \leq L_{f^*}\big(|t_1 - t_2| + |\vartheta_1 - \vartheta_2| \quad \text{for all } t_1,\ t_2 \in I,\ \vartheta_1,\ \vartheta_2 \in \mathbb{R};$$

(2) $f^*(t,0) = 0$ for all $t \in I$.

$\underline{\mathrm{H}(g^*)}$ $g^* : I \times \mathbb{R} \to \mathbb{R}$ is such that
(1) there exists $L_{g^*} > 0$ such that

$$|g^*(t_1,\vartheta_1) - g^*(t_2,\vartheta_2)| \leq L_{g^*}\big(|t_1 - t_2| + |\vartheta_1 - \vartheta_2| \quad \text{for all } t_1,\ t_2 \in I,\ \vartheta_1,\ \vartheta_2 \in \mathbb{R};$$

(2) $g^*(t,0) = 0$ for all $t \in I$.

$\underline{\mathrm{H}(\mathcal{B})}$ $\mathcal{B} : (0,1) \times \mathbb{R} \to \mathbb{R}$ is such that
(1) there exists $L_{\mathcal{B}} > 0$ such that

$$|\mathcal{B}(z,\vartheta_1) - \mathcal{B}(z,\vartheta_2)| \leq L_{\mathcal{B}}|\vartheta_1 - \vartheta_2| \quad \text{for all } \vartheta_1,\ \vartheta_2 \in \mathbb{R},\ \text{a.e. } z \in (0,1);$$

(2) there exists $m_B > 0$ such that

$$(\mathcal{B}(z,\vartheta_1) - \mathcal{B}(z,\vartheta_2))(\vartheta_1 - \vartheta_2) \geq m_B|\vartheta_1 - \vartheta_2|^2 \quad \text{for all } \vartheta_1,\ \vartheta_2 \in \mathbb{R},\ \text{a.e. } z \in (0,1);$$

(3) the map $z \mapsto \mathcal{B}(z,\vartheta)$ is measurable on $(0,1)$, for each $\vartheta \in \mathbb{R}$;
(4) $\mathcal{B}(z,0) = 0$ for each $z \in (0,1)$.

$\underline{\mathrm{H}(j)}$ The functional $j : \mathbb{R} \to \mathbb{R}$ is such that
(1) j is continuous;

(2) there are $\kappa_0,\ \kappa_1 \geq 0$ such that

$$|j(\theta)| \leq \kappa_0 + \kappa_1|\theta| \text{ for all } \theta \in \mathbb{R};$$

(3) there exists $\alpha_j > 0$ such that $\theta \mapsto \alpha_j\theta + j(\theta)$ is nondecreasing;
(4) $j(\theta) \geq 0$ if $\theta > 0$ and $j(\theta) = 0$ if $\theta \leq 0$.

H(h^*) $h^* : I \times \mathbb{R} \to \mathbb{R}$ is such that
(1) there exists $L_{h^*} > 0$ such that

$$|h^*(t_1, \vartheta_1) - h^*(t_2, \vartheta_2)| \leq L_{g^*}\big(|t_1 - t_2| + |\vartheta_1 - \vartheta_2|\big) \quad \text{for all } t_1,\ t_2 \in I,\ \vartheta_1,\ \vartheta_2 \in \mathbb{R};$$

(2) there exists a constant $\ell > 0$ such that

$$\|h(t, \vartheta)\|_{L^2(0,1)} \leq \ell \text{ for a.e. } t \in I,\ \vartheta \in \mathbb{R}.$$

H(0) $x_0(z) \in L^2(0, 1)$.

Note that condition H(j)(4) is imposed from mechanical reasons. It shows that when there is a contact between the rod and the obstacle then the reaction of the foundation is toward the rod and when there is separation, it vanishes.

Next, define the set K, the operator $B : U \to U^*$ and the functions $\varphi : U \times U \to \mathbb{R}$, $q : \mathbb{R} \to \mathbb{R}$, $J : U \to \mathbb{R}, f : I \times E \to E, g : I \times E \to \mathcal{L}(E, U), h : I \times U \to U^*$ by equalities

$$K = \{u \in U \mid u(1) \leq k\}, \tag{67}$$

$$\langle Bu, v\rangle = \int_0^1 \mathcal{B}u_z v_z \, \mathrm{d}z \quad \text{for all } u,\ v \in U, \tag{68}$$

$$\varphi(u, v) = \lambda v(1) \quad \text{for all } u,\ v \in U, \tag{69}$$

$$q(r) = \int_0^r j(s) \, \mathrm{d}s \quad \text{for all } r \in \mathbb{R}, \tag{70}$$

$$J(v) = q(v(1)) \quad \text{for all } v \in U, \tag{71}$$

$$f(t, x)(z) = f^*(t, x(z)) \quad \text{for all } t \in I,\ x \in E, \text{ a.e. } z \in (0, 1), \tag{72}$$

$$[g(t, x)u](z) = g^*(t, x(z))u(z) \quad \text{for all } t \in I,\ x \in E,\ u \in U, \text{ a.e. } z \in (0, 1), \tag{73}$$

$$\langle h(t, x), v\rangle = \int_0^1 h^*(t, x(z))\, v(z) \, \mathrm{d}z, \quad \text{for all } t \in I,\ x \in E,\ v \in U. \tag{74}$$

In the sequel, to derive the variational-hemivariational formulation of Problem $\mathcal{P}$, we let $v \in K$ and perform an integration by parts on Equation (59) with $v(0) = u(0) = 0$ and the condition (60) to obtain

$$\int_0^1 \mathcal{B}u_z(v_z - u_z) \, \mathrm{d}z + \lambda v(1) - \lambda u(1) + j(u(1))(v(1) - u(1)) \geq \int_0^1 h^*(t, x(t, z))(v - u) \, \mathrm{d}z. \tag{75}$$

Recall that the assumptions H(f^*)–H(0) above guarantee that these functions and operators are well defined. For instance, we precise that the definition (73) has to be understood in the following sense: for all $t \in I$ and $x \in E$, $g(t, x)$ represents the operator which associate at each function $u \in U$ the function $g(t, x)u : (0, 1) \to \mathbb{R}$ defined by $[g(t, x)u](z) = g^*(t, x(z))u(z)$ a.e. $z \in (0, 1)$. It is easy to see that this function belongs to E and the operator $u \mapsto g(t, x)u$ is linear and continuous, i.e. it belongs to $\mathcal{L}(U, E)$. Also, note that the operator B and the function h^* are defined by using Riesz's

representation theorem. The function q could be nonconvex. Nevertheless, it is a regular function in the sense of Clarke. Moreover, it satisfies the equality

$$q^0(s;\theta) = j(s)\theta, \quad \text{for all } s,\ \theta \in \mathbb{R}, \tag{76}$$

where $q^0(s;\theta)$ denotes the generalized directional derivative of J at the point s in the direction θ. On the other hand, using a standard argument (Theorem 3.47 in [23] or Lemma 8 (vi) in [24], for instance), we have

$$J^0(u;v) = q^0(u(1);v(1)) \quad \text{for all } u, v \in V, \tag{77}$$

where $J^0(u;v)$ denotes the generalized directional derivative of J at the point u in the direction v.

With these preliminaries, using notation (67)–(71) and equalities (74)–(77), it is easy to see that inequality (59)–(61) could be written, equivalently, as follows:

$$\begin{aligned} &\boldsymbol{u}(t) \in K, \quad \langle Bu(t), v - u(t)\rangle + \varphi(u(t), v) - \varphi(u(t), u(t)) \\ &\quad + J^0(u(t); v - u(t)) \geq \langle h(t, x(t)), v - u(t)\rangle \quad \text{for all } v \in K, \end{aligned} \tag{78}$$

for all $t \in I$. Moreover, in what follows we skip the dependence of the variables on z, i.e. we simply write $x(t)$ instead of $x(t,z)$ and $u(t)$ instead of $u(t,z)$. Therefore, using the definition (63) of the operator A, we deduce that Problem $\mathcal{P}$ can be formulated, equivalently, in a form of the following differential variational-hemivariational inequality.

(DHVI)$_{\mathcal{P}}$. *Find $x : I \to E$ and $u : I \to U$ such that*

$$x'(t) = Ax(t) + f(t, x(t)) + g(t, x(t))u(t), \quad \text{a.e. } t \in I, \tag{79}$$

$$u(t) \in SOL(K; B, h(t, x(t)), \varphi, J), \quad \text{for all } t \in I, \tag{80}$$

$$x(0) = x_0. \tag{81}$$

Recall that here $SOL(K; B, h(x(t), \cdot), \varphi, J)$ stands for the set of solution of the hemivariational inequality (78).

The existence of a unique solution of the system (79)–(81) is provided by the following result.

Theorem 5.1: *Assume that* H(f^*), H(g^*), H($\mathcal{B}$), H(j), H(h^*) *and* H(0) *hold. Moreover, assume that $m_{\mathcal{B}} > \alpha_j$. Then, Problem (79)–(81) has a unique mild solution $(x, u) \in C(I; E) \times C(I; U)$).*

Proof: The proof of is based on Theorem 3.6 with the choice $E = L^2(0,1)$, U being the space defined by (65). For this reason, we check in what follows the validity of the conditions of the theorem.

First, note that as already mentioned, the operator A satisfies condition H(A)$_2$. Moreover, conditions H(f^*) and H(g^*) imply that the functions f and g defined by (72) and (73) satisfy conditions H(f)$_2$ and H(g)$_2$, respectively, with $L_f = L_{f^*}$ and $L_g = L_{g^*}$. The proof is based on standard arguments and, therefore we skip it. Next, since $k > 0$ it is clear that condition H(K) holds. We now use condition H($\mathcal{B}$) of the function $\mathcal{B}$ to see that the operator B given by (68) satisfies assumption H(B)(2) with $m_B = m_{\mathcal{B}}$. Moreover, B satisfies

$$\|Bu - Bv\|_{U^*} \leq L_{\mathcal{B}}\|u - v\|_U, \quad \forall\, u, \quad v \in L^2(0,1),$$

i.e. is Lipschitz continuous. We conclude from here that B satisfies assumption H(B)(1), too.

In addition, it is easy to see that the function φ defined by (69) satisfies condition H(φ) with $\alpha_\varphi = 0$. Moreover, using the standard arguments on subdifferential calculus, we see that the function J defined

by (71) satisfies the hypothesis H(J). In fact, using (77), (76) and (66) we have

$$J^0(u; v-u) + J^0(v, u-v) = (j(u(1)) - j(v(1))(u(1) - v(1))$$
$$\leq \alpha_j |u(1) - v(1)|^2 \leq \alpha_j \|u(1) - v(1)\|_U^2,$$

Finally, we note that assumption H(h^*) on the function h^* guarantees that the function h defined by (74) satisfies condition H(h) and we recall that condition H(0) holds. We also note that inequality $m_{\mathcal{B}} > \alpha_j$ implies that the smallness condition (19) holds, too.

We are now in a position to apply Theorem 3.6 to conclude the proof. ■

Now, we also proceed with the following globally exponentially stability result.

Theorem 5.2: *Assume that* H(f^*), H(g^*), H($\mathcal{B}$), H(j), H(h^*) *and* H(0) *hold. Moreover, assume that* $m_{\mathcal{B}} > \alpha_j$ *and, in addition,*

$$L_{f^*}(m_{\mathcal{B}} - \alpha_j) + L_{g^*}(\ell + \kappa_0) < \left(c + \frac{\delta \pi^2}{4}\right)(m_{\mathcal{B}} - \alpha_j). \tag{82}$$

Then the mild solution of (79)–(81) is globally exponentially stable.

***Proof*:** Note that (82) combined with inequality (64) and assumption H($\mathcal{B}$)(4) guarantees that condition (52) holds. Therefore, Theorem 5.2 is a direct consequence of Theorem 4.1. ■

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