EXOTIC $\mathbb{G}_a$-QUOTIENTS OF $\mathrm{SL}_2 \times \mathbb{A}^1$

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ABSTRACT. Every deformed Koras-Russell threefold of the first kind $Y = \{x^n z = y^m - t^r + xh(x, y, t)\}$ in $\mathbb{A}^4$ is the algebraic quotient of proper Zariski locally trivial $\mathbb{G}_a$-action on $\mathrm{SL}_2 \times \mathbb{A}^1$.

INTRODUCTION

Deformed Koras-Russell threefolds (of the first kind) were introduced in [8] as a familly of smooth affine threefolds generalizing the famous Koras-Russell threefolds (of the first kind) [17, 21].

Definition. A *deformed Koras-Russell threefold of the first kind* over an algebraically closed field k of characteristic zero is a smooth affine threefold Y isomorphic to a hypersurface $Y(m, n, r, h)$ of $\mathbb{A}_k^4 = \mathrm{Spec}(k[x, y, z, t])$ defined by an equation of the form

$$x^n z = y^m - t^r + xh(x, y, t),$$

where $n \geq 2$, $m, r \geq 1$ are coprime integers, and where $h(x, y, t) \in k[x, y, t]$ is a polynomial such that $h(0, 0, 0) \in k^*$.

All these threefolds share the property to come equipped with a flat fibration $\mathrm{pr}_x : Y \to \mathbb{A}_k^1$ restricting to a trivial $\mathbb{A}^2$-bundle $(\mathbb{A}_k^1 \setminus \{0\}) \times \mathbb{A}_k^2 = \mathrm{Spec}(k[x^{\pm 1}][y, t])$ over $\mathbb{A}_k^1 \setminus \{0\}$ and whose fiber over $\{0\}$ is reduced, isomorphic to the product of the irreducible rational curve $C = \{y^m - t^r = 0\} \subset \mathbb{A}_k^2$ with $\mathbb{A}_k^1 = \mathrm{Spec}(k[z])$. If m or r is equal to 1, then $C \cong \mathbb{A}_k^1$ and $\mathrm{pr}_x : Y \to \mathbb{A}_k^1$ is isomorphic to the trivial $\mathbb{A}_k^2$-bundle [24], and hence Y is isomorphic to the affine space $\mathbb{A}_k^3$. Otherwise, if $m, r \geq 2$ then $\mathrm{pr}_x^{-1}(0)$ is not isomorphic to $\mathbb{A}_k^2$ so that so that Y cannot be isomorphic to $\mathbb{A}_k^3$ by [15]. In these cases, it is known more precisely that the fibration $\mathrm{pr}_x : Y \to \mathbb{A}_k^1$ is invariant under every algebraic action of the additive group $\mathbb{G}_{a,k}$. This property turns out to fails for the cylinders $Y \times \mathbb{A}_k^\ell$, $\ell \geq 1$, with the consequence that the known invariants associated to $\mathbb{G}_{a,k}$-actions do no longer suffice to distinguish cylinders $Y \times \mathbb{A}_k^\ell$, $\ell \geq 1$, over threefolds Y non isomorphic to $\mathbb{A}_k^3$ from affine spaces $\mathbb{A}_k^{\ell+3}$. The question whether any deformed Koras-Russell threefold $Y \not\cong \mathbb{A}_k^3$ has a cylinder $Y \times \mathbb{A}_k^\ell$ isomorphic to an affine space is totally open.

The projection $\mathrm{pr}_{x,y,t} : Y \to \mathbb{A}_k^3$ is a birational morphism which represents Y as the affine modification of $\mathbb{A}_k^3 = \mathrm{Spec}(k[x, y, t])$ with center as the closed subscheme Z_Y with defining ideal $J_Y = (x^n, y^m - t^r + xh(x, y, t))$ and principal divisor $D_Y = \{x^n = 0\}$ in the sense of [19]. Equivalently, the coordinate ring of Y is isomorphic to the quotient of the Rees algebra

$$\bigoplus_{s \geq 0} J_Y^s \cdot v^s \subset k[x, y, t][v]$$

of the ideal $J_Y \subset k[x, y, t]$ by the ideal generated by $1 - x^n v$. Over the field $\mathbb{C}$ of complex numbers, the fact that the associated closed subscheme Z_Y is supported on the topologically contractible curve $\{y^m - t^r = 0\}$ inside the topologically contractible divisor $\mathrm{Supp}(D_Y) = \mathbb{A}_\mathbb{C}^2$ implies by [19, Theorem 3.1] that every deformed Koras-Russell threefolds is a topologically contractible complex threefold, which is actually even diffeomorphic to $\mathbb{R}^6$ [21]. For arbitrary algebraically closed fields k of characteristic zero, it is known by [3]

that if $h \in k[x] \setminus xk[x]$ then a deformed Koras-Russell threefold $Y(m,n,r,h) \not\cong \mathbb{A}^3_k$ is contractible in the $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ of Morel and Voevodsky [23]. The question whether every $Y = Y(m,n,r,h)$ is contractible in this category for arbitrary $h \in k[x,y,t]$ such that $h(0,0,0) \in k^*$ is open, but it was established recently in [9] by a combination of the techniques developed in [14] and [3] that every such Y becomes contractible in $\mathcal{H}(k)$ after a single suspension with the simplicial circle S^1.

One of the steps in the proofs of these contractibility results in the $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ consists in determining the $\mathbb{A}^1$-homotopy type of the complement in Y of the curve

$$\ell = \{x = y = t = 0\} \cong \mathrm{Spec}(k[z]).$$

Over the field of complex number, the inclusion $\ell \hookrightarrow Y$ defines a smooth proper embedding of the underlying differential manifold $\mathbb{R}^2$ of ℓ into the underlying differentiable manifold $\mathbb{R}^6$ of Y. Since every two smooth proper embeddings of $\mathbb{R}^2$ into $\mathbb{R}^6$ are ambiently isotopic [13, Chapter 8], it follows that $Y \setminus \ell$ is diffeomorphic to the complement of $\mathbb{R}^2$ embedded into $\mathbb{R}^6$ as a linear subspace, hence to $(\mathbb{R}^4 \setminus \{0\}) \times \mathbb{R}^2$. The $\mathbb{A}^1$-homotopic counterpart that $Y \setminus \ell$ is $\mathbb{A}^1$-weakly equivalent to the complement $\mathbb{A}^3_k \setminus \mathbb{A}^1_k \cong (\mathbb{A}^2_k \setminus \{0\}) \times \mathbb{A}^1_k$ of an affine line $\mathbb{A}^1_k$ embedded into $\mathbb{A}^3_k$ as a linear subspace was established in [3, 9] by constructing for every Y an explicit $\mathbb{A}^1$-weak equivalence between $Y \setminus \ell$ and $\mathbb{A}^3_k \setminus \mathbb{A}^1_k$ in the form of a quasi-affine fourfold W which is simultaneously the total space of a $\mathbb{G}_{a,k}$-torsor over $Y \setminus \ell$ and $\mathbb{A}^3_k \setminus \mathbb{A}^1_k$.

Since it was not important in the next steps of the constructions in *loc. cit.*, the precise structure of this variety W was not elucidated. But since then, it has become a kind of folklore fact that it should be affine, and actually isomorphic to the product of $\mathrm{SL}_2 = \{xv - yu = 1\} \subset \mathbb{A}^4_k$ with the affine line $\mathbb{A}^1_k$, independently of the given deformed Koras-Russell threefold Y. The purpose of this article is to give a complete and detailed proof of this folklore fact, in the form of the the following theorem:

Theorem 1. *Let k be an algebraically closed field of characteristic zero and let*

$$Y = Y(m,n,r,h) = \{x^n z = y^m - t^r + xh(x,y,t)\} \subset \mathbb{A}^4_k$$

be a deformed Koras-Russell threefold. Then there exists a proper Zariski locally trivial $\mathbb{G}_{a,k}$-action on $\mathrm{SL}_2 \times \mathbb{A}^1_k$ whose algebraic quotient $(\mathrm{SL}_2 \times \mathbb{A}^1_k)//\mathbb{G}_{a,k} = \mathrm{Spec}(\Gamma(\mathrm{SL}_2 \times \mathbb{A}^1_k, \mathcal{O}_{\mathrm{SL}_2 \times \mathbb{A}^1_k})^{\mathbb{G}_{a,k}})$ is isomorphic to Y. Furthermore, the quotient morphism $\mathrm{SL}_2 \times \mathbb{A}^1_k \to Y$ restricts to a $\mathbb{G}_{a,k}$-torsor over $Y \setminus \ell$.

This implies in particular that the coordinate rings of all Koras-Russell threefolds of the first kind can be realized as rings of invariants of $\mathbb{G}_{a,k}$-actions on the single affine fourfold $\mathrm{SL}_2 \times \mathbb{A}^1_k$. In contrast, it is an open question raised by Freudenburg [12] whether these can be realized as rings of invariants of $\mathbb{G}_{a,k}$-actions on the affine space $\mathbb{A}^4_k$. It is also an open question whether all proper Zariski locally trivial $\mathbb{G}_{a,k}$-actions on $\mathbb{A}^4_k$ are conjugate (see e.g. [6, 16] and the references therein for known partial results on this problem). Since by [8, Theorem 1], for every fixed pair (m,r) with $m, r \geq 2$ and for every fixed big enough $n \geq 2$, there exists uncountably many pairwise non isomorphic deformed Koras-Russell threefolds $Y(m,n,r,h)$, the above theorem implies that this property fails very badly for $\mathrm{SL}_2 \times \mathbb{A}^1$:

Corollary 2. *There exists uncountably many pairwise non-conjugate proper Zariski locally trivial $\mathbb{G}_{a,k}$-actions on $\mathrm{SL}_2 \times \mathbb{A}^1$.*

The proof of Theorem 1 draws as in [3, 9] on the study of categorical quotients of certain $\mathbb{G}_{a,k}$-actions on deformed Koras-Russell threefolds in the category of algebraic spaces. The algebraic spaces which come into play are a particular class of "non-separated surfaces with an m-fold curve" which already appeared in the context of the study of proper $\mathbb{G}_{a,k}$-actions on $\mathbb{A}^4_k$ in [5, 6] and, for some special cases, in [2] and [7] in relation to the Zariski Cancellation problem for threefolds. In many respects, these spaces tend to be natural and necessary replacements in higher dimension of the non-separated curves first considered by Danielewski [1] in its famous counter-example to the Cancellation problem in dimension two, and which became ubiquitous in the study $\mathbb{A}^1$-fibered affine surfaces after the work of Fieseler [10]. With the hope to make the use of these techniques accessible to a larger community, we collect various complementary descriptions of these spaces which can be found disseminated in the literature.

1. PRELIMINARIES: SOME NON PROPER $\mathbb{G}_a$-ACTIONS ON EXOTIC AFFINE 3-SPHERES

Let $m \geq 1$ be an integer and let $X_m \subset \mathbb{A}^4_k = \mathrm{Spec}(k[x,y,u,v])$ be the smooth closed sub-variety of dimension 3 defined by the equation $x^m v - yu = 1$. The projection

$$\mathrm{pr}_{x,y} : X_m \to \mathbb{A}^2_* = \mathbb{A}^2_k \setminus \{(0,0)\}$$

is a Zariski locally trivial $\mathbb{A}^1$-bundle which is the structure morphism of a Zariski locally trivial $\mathbb{G}_{a,k}$-torsor for the $\mathbb{G}_{a,k}$-action on X_m defined by $t \cdot (x,y,u,v) = (x,y,u+tx^m, v+ty)$. For every $m, m' \geq 1$, the fiber product $W = X_m \times_{\mathbb{A}^2_*} X_{m'}$ is thus simultaneously the total space of a $\mathbb{G}_{a,k}$-torsor over X_m and $X_{m'}$ via the first and second projection respectively. Since X_m and $X_{m'}$ are affine, the vanishing of $H^1(X_m, \mathcal{O}_{X_m})$ and $H^1(X_{m'}, \mathcal{O}_{X_{m'}})$ implies that these two $\mathbb{G}_{a,k}$-torsors are trivial so that we get isomorphisms

$$(1.1) \qquad\qquad X_m \times \mathbb{A}^1_k \cong W \cong X_{m'} \times \mathbb{A}^1_k.$$

In particular, for every $m \geq 1$, $X_m \times \mathbb{A}^1_k$ is isomorphic to $X_1 \times \mathbb{A}^1_k \cong \mathrm{SL}_2 \times \mathbb{A}^1_k$.

Remark 3. Over the field of complex numbers $\mathbb{C}$, the underlying 6-dimensional real smooth manifold X_m^{diff} of X_m is diffeomorphic to that of X_1 for every $m \geq 1$. Moreover, X_1^{diff} is homotopically equivalent to the sphere $S^3 \subset \mathbb{R}^4$. It was established in [4] that for every $m > 1$, X_m is not isomorphic to X_1 as an algebraic variety. The threefolds X_m, $m > 1$, were consequently named exotic affine 3-spheres.

Each threefold X_m carries another fixed point free $\mathbb{G}_{a,k}$-action $\nu_m : \mathbb{G}_{a,k} \times X_m \to X_m$ defined by the locally nilpotent $k[y,v]$-derivation

$$\partial_m = y \frac{\partial}{\partial x} + m x^{m-1} v \frac{\partial}{\partial u}$$

of its coordinate ring $A_m = k[x,y,u,v]/(x^m v - yu - 1)$. The projection

$$q_m = \mathrm{pr}_{y,v} : X_m \to \mathbb{A}^2_* = \mathrm{Spec}(k[y,v]) \setminus \{(0,0)\}$$

is a smooth $\mathbb{G}_{a,k}$-invariant morphism which restricts to the trivial $\mathbb{G}_{a,k}$-torsor over the principal affine open subset $\mathbb{A}^2_{k,y} = \mathrm{Spec}(k[y^{\pm 1}, v])$ of $\mathbb{A}^2_*$.

1.1. Categorical quotients in the category of schemes.
If $m = 1$ then $q_1 : X_1 \to \mathbb{A}^2_*$ is again a Zariski locally trivial $\mathbb{G}_{a,k}$-torsor. This is no longer the case when $m > 1$ since then the restriction of q_m over the curve $C \cong \mathrm{Spec}(k[v^{\pm 1}])$ in $\mathbb{A}^2_*$ with equation $y = 0$ factors as the composition of the trivial $\mathbb{G}_{a,k}$-torsor

$$\mathrm{pr}_x : X_m|_C \cong \mathrm{Spec}(k[x,u,v]/(x^m v - 1)) \to \tilde{C} = \mathrm{Spec}(k[x^{\pm 1}])$$

with the cyclic étale cover $f : \tilde{C} \to C$, $x \mapsto v = x^{-m}$ of order m.

Lemma 4. *For every $m \geq 1$, the morphism $q_m : X_m \to \mathbb{A}^2_*$ is the categorical quotient of X_m by the $\mathbb{G}_{a,k}$-action ν_m in the category of schemes.*

Proof. Since the ring of $\mathbb{G}_{a,k}$-invariant functions on X_m is equal to sub-algebra $k[y,v] \subset A_m$, it follows that the composition of $q_m : X_m \to \mathbb{A}^2_*$ with the open inclusion $\mathbb{A}^2_* \hookrightarrow \mathrm{Spec}(k[y,v])$ is the categorical quotient $X_m \to X_m /\!/ \mathbb{G}_{a,k}$ of X_m in the category of affine schemes. Furthermore, for every principal affine open subset V of $\mathbb{A}^2_k$, the composition

$$q_m : q_m^{-1}(V_0) \to V_0 = V \cap \mathbb{A}^2_* \hookrightarrow V$$

is the categorical quotient in the category of affine schemes of $q_m^{-1}(V_0)$ by the $\mathbb{G}_{a,k}$-action induced by ν_m.

Now let Z be an arbitrary scheme and let $f : X_m \to Z$ be a $\mathbb{G}_{a,k}$-invariant morphism. Since X_m is irreducible, to show that $f : X_m \to Z$ factorizes as $\tilde{f} \circ q_m$ for a unique morphism $\tilde{f} : \mathbb{A}^2_* \to Z$, we may assume without loss of generally that Z is irreducible. Since $q_m : X_m \to \mathbb{A}^2_*$ is a surjective smooth morphism, hence in particular a faithfully flat morphism, it follows from faithfully flat descent that f descends to a morphism $\tilde{f} : \mathbb{A}^2_* \to Z$ if and only if it is constant on the fibers of q_m. This is clear for $m = 1$ as $q_1 : X_1 \to \mathbb{A}^2_*$ is a $\mathbb{G}_{a,k}$-torsor. We now consider the case $m \geq 2$. Since the restriction of $q_m : X_m \to \mathbb{A}^2_*$

over the principal affine open subset $\mathbb{A}^2_{k,y} = \mathbb{A}^2 \setminus C$ of $\mathbb{A}^2_*$ is a $\mathbb{G}_{a,k}$-torsor, f is constant on the fibers of $q_m|_{X_m \setminus q_m^{-1}(C)}$, and it remains to check that f is constant on the fibers of $q_m|_{q_m^{-1}(C)} : q_m^{-1}(C) \to C$. Since $\mathbb{G}_{a,k}$ acts on $q_m^{-1}(C) \cong \tilde{C} \times \mathbb{A}^1_k$ by translations on the second factor and f is $\mathbb{G}_{a,k}$-invariant, the image of $q_m^{-1}(C)$ by f is either a point and we are done, or its closure in Z is a curve D dominated by $\tilde{C}$.

In the second case, let $U \subset Z$ be an affine open subset such that $D \cap U$ is not empty. Since X_m is affine, f is an affine morphism. It follows that $f^{-1}(U)$ is a $\mathbb{G}_{a,k}$-invariant affine open subset of X_m such that $f^{-1}(U) \cap q_m^{-1}(C)$ is not empty. Since q_m is smooth, hence open, $q_m(f^{-1}(U))$ is an open subset of $\mathbb{A}^2_*$ such that $q_m(f^{-1}(U)) \cap C$ is not empty. Let V_0 be a principal affine open subset of $\mathbb{A}^2_*$ contained in $q_m(f^{-1}(U))$ and intersecting C. Then $q_m^{-1}(V_0)$ is contained in $f^{-1}(U)$. Indeed, first note that by the choice of V_0, $q_m^{-1}(V_0 \cap C) \cap f^{-1}(U)$ is not empty. Since $q_m^{-1}(V_0)$ is affine and X_m is separated, $q_m^{-1}(V_0) \cap f^{-1}(U)$ is an affine open subset of $q_m^{-1}(V_0)$. It follows that $q_m^{-1}(V_0) \setminus (q_m^{-1}(V_0) \cap f^{-1}(U))$ is either empty or a closed subset of pure codimension one in $q_m^{-1}(V_0)$. On the other hand, since f is constant on the fibers of $q_m|_{X_m \setminus q_m^{-1}(C)}$, we have $q_m^{-1}(V_0 \setminus C) = q_m^{-1}(V_0) \setminus q_m^{-1}(C) = f^{-1}(U) \setminus q_m^{-1}(C)$. So $q_m^{-1}(V_0) \setminus (q_m^{-1}(V_0) \cap f^{-1}(U))$ is contained in $q_m^{-1}(V_0 \cap C)$. Since $q_m^{-1}(V_0 \cap C)$ is irreducible and of pure codimension one in $q_m^{-1}(V_0)$ and $q_m^{-1}(V_0 \cap C) \cap f^{-1}(U) \neq \emptyset$, it follows that $q_m^{-1}(V_0) \setminus (q_m^{-1}(V_0) \cap f^{-1}(U))$ is empty. Since U is affine and V_0 is the categorical quotient of $q_m^{-1}(V_0) \subseteq f^{-1}(U)$ in the category of affine schemes, it follows that there exists a unique morphism $\tilde{f} : V_0 \to U$ such that $f|_{q_m^{-1}(V_0)} = \tilde{f} \circ q_m|_{q_m^{-1}(V_0)}$. This implies that f is generically constant, hence constant, on the fibers of $q_m|_{q_m^{-1}(C)} : q_m^{-1}(C) \to C$ as desired. $\qquad\square$

1.2. Categorical quotients in the category of algebraic spaces.
On the other hand, since the $\mathbb{G}_{a,k}$-action ν_m on X_m is fixed point free, it admits a categorical quotient in the larger category of algebraic spaces, in the form of an étale locally trivial $\mathbb{G}_{a,k}$-torsor $\rho_m : X_m \to X_m/\mathbb{G}_{a,k}$ over a certain algebraic space $X_m/\mathbb{G}_{a,k}$ of finite type and dimension 2 (see e.g. [22, 10.4]), which is smooth as X_m is smooth. If $m = 1$ then $X_1/\mathbb{G}_{a,k} = \mathbb{A}^2_*$. But if $m \geq 2$, it follows from Lemma 4 that $X_m/\mathbb{G}_{a,k}$ cannot be a scheme. Furthermore, $X_m/\mathbb{G}_{a,k}$ is not separated for otherwise, being smooth of dimension 2 and of finite type over k, it would be a quasi-projective k-variety by Chow Lemma. Since $\rho_m : X_m \to X_m/\mathbb{G}_{a,k}$ is a $\mathbb{G}_{a,k}$-torsor, this implies that for every $m \geq 2$ the injective morphism

$$\nu_m \times \mathrm{pr}_2 : \mathbb{G}_{a,k} \times X_m \cong X_m \times_{X_m/\mathbb{G}_{a,k}} X_m \to X_m \times X_m$$

is not a closed immersion, hence that the action ν_m is not proper.

Since $\rho_m : X_m \to X_m/\mathbb{G}_{a,k}$ is a categorical quotient in the category of algebraic spaces, the surjective morphism $q_m : X_m \to \mathbb{A}^2_*$ factors as $q_m = \tilde{q}_m \circ \rho_m$ for a unique surjective morphism $\tilde{q}_m : X_m/\mathbb{G}_{a,k} \to \mathbb{A}^2_*$. Since the restriction of q_m over $\mathbb{A}^2_{k,y} = \mathbb{A}^2_* \setminus C$ is already a $\mathbb{G}_{a,k}$-torsor, $\tilde{q}_m$ restricts to an isomorphism over $\mathbb{A}^2_* \setminus C$. On the other hand, since $q_m^{-1}(C) \cong \tilde{C} \times \mathbb{A}^1_k$ on which $\mathbb{G}_{a,k}$ acts by translations on the second factor, it follows that $\tilde{q}_m^{-1}(C) \cong (\tilde{C} \times \mathbb{A}^1_k)/\mathbb{G}_{a,k} \cong \tilde{C}$. So $X_m/\mathbb{G}_{a,k}$ is somehow obtained from $\mathbb{A}^2_*$ by replacing the closed curve $C = \{y = 0\}$ by the total space of the cyclic étale cover $f : \tilde{C} \to C$, $x \mapsto v = x^{-m}$ of order m.

Let us recall from [5, § 1.1] an explicit construction of an algebraic space $\mathfrak{S}_m$ with this property. Let $U = \mathbb{A}^1_k \times C = \mathrm{Spec}(k[y, v^{\pm 1}]) \subset \mathbb{A}^2_*$, $U_* = U \setminus (\{0\} \times C) = \mathrm{Spec}(k[y^{\pm 1}, v^{\pm 1}]$, $\tilde{U} = \mathbb{A}^1_k \times \tilde{C} = \mathrm{Spec}(k[y, x^{\pm 1}])$ and let

$$\varphi = \mathrm{id} \times f : \tilde{U} = \mathbb{A}^1_k \times \tilde{C} \to \mathbb{A}^1_k \times C = U$$

be the étale morphism deduced from $f : \tilde{C} \to C$. Let $\mathrm{diag} : \tilde{U} \hookrightarrow \tilde{U} \times \tilde{U}$ be the diagonal embedding and let $j : (\tilde{U} \times_{U_*} \tilde{U}) \setminus \mathrm{Diag} \hookrightarrow \tilde{U} \times \tilde{U}$ be the natural immersion. Then the pair of morphisms

$$(1.2) \qquad (\mathrm{pr}_1 \circ (\mathrm{diag} \sqcup j), \mathrm{pr}_2 \circ (\mathrm{diag} \sqcup j)) : R = \tilde{U} \sqcup (\tilde{U} \times_{U_*} \tilde{U}) \setminus \mathrm{Diag} \rightrightarrows \tilde{U}$$

is an étale equivalence relation on $\tilde{U}$. Letting $\tilde{U}/R$ be the algebraic space defined by this étale equivalence relation, it follows that $\varphi : \tilde{U} \to U$ descends to a morphism $\beta : \tilde{U}/R \to U$. By construction, the restriction R_* of R to $\varphi^{-1}(U_*) = \tilde{U} \setminus (\{0\} \times \tilde{C})$ is equal to the equivalence relation defined by the diagonal embedding $\varphi^{-1}(U_*) \hookrightarrow \varphi^{-1}(U_*) \times_{U_*} \varphi^{-1}(U_*)$, whose quotient $\varphi^{-1}(U_*)/R_*$ is isomorphic to U_*. It follows that

$\beta : \tilde{U}/R \to U$ restricts to an isomorphism over U_*. On the other hand, since R restricts to the trivial equivalence relation on the closed subset $\{0\} \times \tilde{C} \subset \tilde{U}$, it follows that $\beta^{-1}(\{0\} \times C) \cong \{0\} \times \tilde{C}$.

Now we let $\mathfrak{S}_m$ be the algebraic space obtained by gluing $\mathbb{A}^2_{k,y} = \operatorname{Spec}(k[y^{\pm 1}, v]$ and $\tilde{U}/R$ along the open subsets $U_* = \operatorname{Spec}(k[y^{\pm 1}, v^{\pm 1}]$ and $\beta^{-1}(U_*)$ by the isomorphism $\beta^{-1}(U_*) \cong U_*$ induced by β. Then there exists a unique morphism $\delta_m : \mathfrak{S}_m \to \mathbb{A}^2_*$ whose restrictions to the corresponding open subsets $\mathbb{A}^2_{k,y}$ and $\tilde{U}/R$ of $\mathfrak{S}_m$ are equal to the open inclusion $\mathbb{A}^2_{k,y} \hookrightarrow \mathbb{A}^2_*$ and the composition of β with the open inclusion $U = \mathbb{A}^2_{k,v} \hookrightarrow \mathbb{A}^2_*$ respectively.

Proposition 5. *For every $m \geq 1$, $\tilde{q}_m : X_m/\mathbb{G}_{a,k} \to \mathbb{A}^2_*$ and $\delta_m : \mathfrak{S}_m \to \mathbb{A}^2_*$ are isomorphic algebraic spaces over $\mathbb{A}^2_*$.*

Proof. By construction, $\mathfrak{S}_1$ is isomorphic to $\mathbb{A}^2_*$. If $m \geq 2$, then since $\mathbb{A}^2_*$ is covered by the principal affine open subsets $\mathbb{A}^2_{k,y}$ and $U = \mathbb{A}^2_{k,v}$, it suffices to show that there exists local isomorphism $\delta_m^{-1}(\mathbb{A}^2_{k,y}) \cong \tilde{q}_m^{-1}(\mathbb{A}^2_{k,y})$ and $\tilde{q}_m^{-1}(U) \cong \delta_m^{-1}(U)$ which coincide over $\mathbb{A}^2_{k,y} \cap U = U_*$. By construction, we already have isomorphisms $\delta_m^{-1}(\mathbb{A}^2_{k,y}) \cong \mathbb{A}^2_{k,y} \cong \tilde{q}_m^{-1}(X_m/\mathbb{G}_{a,k})$ as schemes over $\mathbb{A}^2_{k,y}$. It remains to construct a compatible isomorphism $\tilde{q}_m^{-1}(U) \cong \delta_m^{-1}(U) = \tilde{U}/R$ of algebraic spaces over $U = \mathbb{A}^2_{k,v}$. Consider the morphism

$$\Phi : \tilde{U} \times \mathbb{G}_{a,k} = \operatorname{Spec}(k[x^{\pm 1}, y]) \times \mathbb{G}_{a,k} \longrightarrow X_m|_U$$
$$((x,y),t) \mapsto \nu_m(t, (x,y,0,x^{-m})) = (x + ty, y, P(x,y,t), x^{-m})$$

where

$$P(x,y,t) = \sum_{n \geq 1} \frac{\partial_m^n u}{n!}\Big|_{v=x^{-m}} t^n = \sum_{n=1}^m \frac{m!}{(m-n)!n!} x^{-n} y^{n-1} t^n = \sum_{n=1}^m \binom{m}{n} (x^{-1}t)^n y^{n-1}.$$

By definition, Φ is $\mathbb{G}_{a,k}$-equivariant for the action by translations on the second factor on $\tilde{U} \times \mathbb{G}_{a,k}$ and the action ν_m on $X_m|_U$. Furthermore, since $\frac{\partial P}{\partial t}(x, 0, t) = mx^{-1}$ does not vanish on $\tilde{U} \times \mathbb{G}_{a,k}$, it follows that the Jacobian matrix

$$J(\Phi) = \begin{pmatrix} 1 & t & y \\ 0 & 1 & 0 \\ \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial t} \\ -mx^{-m-1} & 0 & 0 \end{pmatrix}$$

of Φ has rank 3 at every point of $\tilde{U} \times \mathbb{G}_{a,k}$. So $\Phi : \tilde{U} \times \mathbb{G}_{a,k} \to X_m|_U$ is an étale trivialization of the restriction of the $\mathbb{G}_{a,k}$-action ν_m on $X_m|_U$. The coordinate ring B of the fiber product $(\tilde{U} \times \mathbb{G}_{a,k}) \times_{X_m|_U} (\tilde{U} \times \mathbb{G}_{a,k})$ is isomorphic to the quotient of $k[x_1^{\pm 1}, x_2^{\pm 1}, y, t_1, t_2]$ by the ideal I generated by the elements

$$x_1^{-m} - x_2^{-m}, \ x_1 - x_2 + y(t_1 - t_2), \text{ and } P(x_1, y, t_1) - P(x_2, y, t_2).$$

Writing $x_1^{-m} - x_2^{-m} = (x_1^{-1} - x_2^{-1})R(x_1^{-1}, x_2^{-1})$, B decomposes as the product of the rings

$$B_0 \cong k[x_1^{\pm 1}, x_2^{\pm 1}][y, t_1, t_2]/(x_1^{-1} - x_2^{-1}, x_1 - x_2 + y(t_1 - t_2), P(x_1, y, t_1) - P(x_2, y, t_2))$$

and

$$B_1 = k[x_1^{\pm 1}, x_2^{\pm 1}][y, t_1, t_2]/(R(x_1^{-1}, x_2^{-1}), x_1 - x_2 + y(t_1 - t_2), \ P(x_1, y, t_1) - P(x_2, y, t_2)).$$

Since $P(x_1, y, t_1) - P(x_1, y, t_2) = x_1^{-1}(t_1 - t_2)(1 + yS(x_1^{-1}, y, t_1, t_2))$ it follows that the homomorphism

$$k[x^{\pm 1}, y][t] \to B_0, \ (x, y, t) \mapsto (x_1, y, t_1)$$

is an isomorphism. On the other hand, since $x_1 - x_2$ in invertible in $k[x_1^{\pm 1}, x_2^{\pm 1}]/(R(x_1^{-1}, x_2^{-1}))$, y is invertible in B_1 and we get an isomorphism

$$k[x_1^{\pm 1}, x_2^{\pm 1}]/(R(x_1^{-1}, x_2^{-1}))[y^{\pm 1}][t] \to B_1, \ (x_1, x_2, y, t) \mapsto (x_1, x_2, y, t_1).$$

Summing up, $(\tilde{U} \times \mathbb{G}_{a,k}) \times_{X_m|_U} (\tilde{U} \times \mathbb{G}_{a,k})$ is $\mathbb{G}_{a,k}$-equivariantly isomorphic to the disjoint union of

$$\mathrm{Spec}(B_0) \cong \tilde{U} \times \mathbb{G}_{a,k} \quad \text{and} \quad \mathrm{Spec}(B_1) \cong (\tilde{U} \times_{U_*} \tilde{U}) \setminus \mathrm{Diag} \times \mathbb{G}_{a,k}$$

on which $\mathbb{G}_{a,k}$ acts by translations on the second factors, and we get a cartesian square of étale equivalence relations

$$
\begin{array}{ccc}
R \times \mathbb{G}_{a,k} \cong (\tilde{U} \times \mathbb{G}_{a,k}) \times_{X_m|_U} (\tilde{U} \times \mathbb{G}_{a,k}) & \xrightarrow[\mathrm{pr}_2]{\mathrm{pr}_1} & \tilde{U} \times \mathbb{G}_{a,k} \\
\Big\downarrow{\scriptstyle \mathrm{pr}_R} & & \Big\downarrow{\scriptstyle \mathrm{pr}_{\tilde{U}}} \\
R & \xrightarrow[\mathrm{pr}_2 \circ (\mathrm{diag} \sqcup j)]{\mathrm{pr}_1 \circ (\mathrm{diag} \sqcup j)} & \tilde{U}
\end{array}
$$

where $\mathrm{diag} \sqcup j$ is the morphism defined in (4) and where the vertical morphisms are trivial $\mathbb{G}_{a,k}$-torsors. By [20, I.5.8], the right-hand side morphism descends to a $\mathbb{G}_{a,k}$-torsor

$$\pi : X_m|_U = (\tilde{U} \times \mathbb{G}_{a,k})/(R \times \mathbb{G}_{a,k}) \to \tilde{U}/R = \delta_m^{-1}(U).$$

Since $\rho_m|_U : X_m|_U \to \tilde{q}_m^{-1}(U)$ is also by definition a $\mathbb{G}_{a,k}$-torsor, it follows that there exists a unique isomorphism $\alpha : \tilde{U}/R \xrightarrow{\cong} \tilde{q}_m^{-1}(U)$ such that $\rho_m|_U = \alpha \circ \pi$. This completes the proof. $\qquad\square$

1.3. **Another description of the algebraic space quotients.**

An alternative complementary description of the algebraic space $\beta : \tilde{U}/R \to U$ constructed in subsection 1.2 was given in [5] in a more general context. Since this description is sometimes more convenient to use in practice, let us review it in detail in our particular situation. We use the notation of subsection 1.2.

The Galois group $\mu_n = \mathrm{Spec}(k[\varepsilon]/(\varepsilon^m - 1))$ of m-th roots of unity in k^* acts on the finite étale cover

$$f : \tilde{C} = \mathrm{Spec}(k[x^{\pm 1}]) \to C = \mathrm{Spec}(k[v^{\pm 1}]), \ x \mapsto v = x^{-m}$$

by $x \mapsto \varepsilon x$. We let V_m be the scheme obtained by gluing m copies $\tilde{U}_i, i \in \mathbb{Z}/m\mathbb{Z}$, of $\tilde{U} = \mathrm{Spec}(k[y, x^{\pm 1}]) = \mathbb{A}_k^1 \times \tilde{C}$ by the identity outside the curves $\{0\} \times \tilde{C} \subset \tilde{U}_i$. The group μ_m now acts freely on V_m by

$$\tilde{U}_i \ni (y, x) \mapsto (y, \varepsilon x) \in \tilde{U}_{i+1}, \quad i \in \mathbb{Z}/m\mathbb{Z},$$

and the local isomorphisms $\mathrm{id} : \tilde{U}_i \to \tilde{U}$, glue to a global morphism $\pi_m : V_m \to \tilde{U}$ which is equivariant for the μ_m-action $(y, x) \mapsto (y, \varepsilon x)$ on $\tilde{U}$. Since the so-defined μ_m-action on V_m has trivial isotropies, a quotient $\xi : V_m \to V_m/\mu_m$ exists in the category of algebraic spaces in the form of an étale μ_m-torsor over a certain algebraic space V_m/μ_m. The μ_m-invariant morphism $\gamma_m = (\mathrm{id} \times f) \circ \pi_m : V_m \to U$ descends to a morphism $\overline{\gamma}_m : V_m/\mu_m \to U$ which restricts to an isomorphism over $U_* = U \setminus (\{0\} \times C)$. In contrast, $\overline{\gamma_m}^{-1}(\{0\} \times C)$ is isomorphic as a scheme over C to the quotient of $\mu_m \times \tilde{C}$ by the diagonal action of μ_m, hence to $\tilde{C}$.

Lemma 6. *The algebraic space $\overline{\gamma}_m : V_m/\mu_m \to U$ is U-isomorphic to $\beta : \tilde{U}/R \to U$.*

Proof. Indeed, letting $\alpha : \mu_m \times V_m \to V_m$ be the μ_m-action on V_m, the algebraic space V_m/μ_m is by definition the quotient of V_m by the étale equivalence relation $(\alpha, \mathrm{pr}_2) : \mu_m \times V_m \rightrightarrows V_m$. It is straightforward to check with the definition of α and R that the composition $\sigma : V_m \to \tilde{U}/R$ of $\pi_m : V_m \to \tilde{U}$ with the quotient morphism $\tilde{U} \to \tilde{U}/R$ is a quasi-finite μ_m-invariant morphism, which descends to a bijective quasi-finite U-morphism $\overline{\sigma} : V_m/\mu_m \to \tilde{U}/R$. To prove that $\overline{\sigma}$ is an isomorphism, it now suffices to construct a section of it. Let $i_0 : \tilde{U} \to V_m$ be the section of π_m defined by the inclusion of $\tilde{U}$ as the open subset $\tilde{U}_0 \subset V_m$. Since the restriction of $\mathrm{id} \times f : \tilde{U} \to U$ over U_* is a finite étale μ_m-cover, there exists an isomorphism

$$\tilde{U} \times_{U_*} \tilde{U} \xrightarrow{\cong} (\tilde{U} \setminus (\{0\} \times \tilde{C})) \times \mu_m$$

which maps the diagonal Diag onto $\tilde{U} \setminus (\{0\} \times \tilde{C}) \times \{1\}$. This yields an isomorphism

$$(\tilde{U} \times_{U_*} \tilde{U}) \setminus \mathrm{Diag} \xrightarrow{\cong} (\tilde{U} \setminus (\{0\} \times \tilde{C})) \times (\mu_m \setminus \{1\})$$
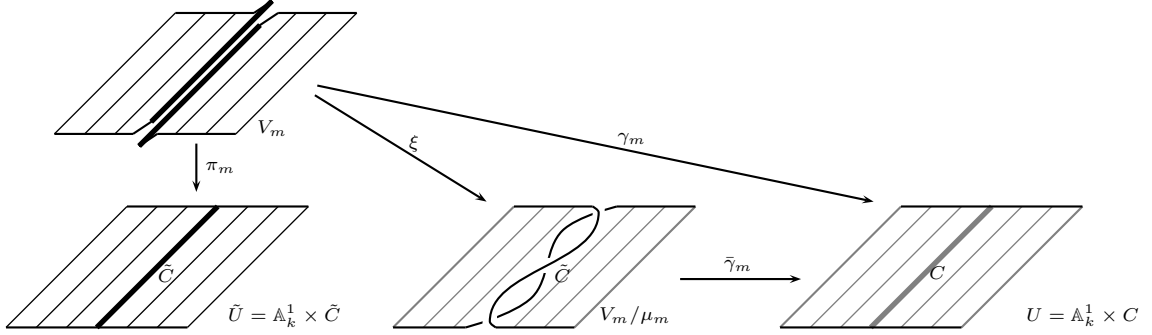
FIGURE 1.1. Construction of $\tilde{U}/R$ as a quotient of V_m by a free μ_m-action

hence an open embedding

$$\zeta_0 : R = \tilde{U} \sqcup (\tilde{U} \times_{U_*} \tilde{U}) \setminus \mathrm{Diag} \hookrightarrow V_m \times \{1\} \sqcup V_m \times (\mu_m \setminus \{1\})$$

whose image is the union of the open subsets $\tilde{U}_0 \times \{1\}$ of $V_m \times \{1\}$ and $(\tilde{U}_0 \setminus (\{0\} \times \tilde{C})) \times (\mu_m \setminus \{1\})$ of $V \times (\mu_m \setminus \{1\})$. By construction of ζ_0, the diagram

$$
\begin{array}{ccc}
R = \tilde{U} \sqcup (\tilde{U} \times_{U_*} \tilde{U}) \setminus \mathrm{Diag} & \xrightarrow{\mathrm{diag} \sqcup j} & \tilde{U} \times \tilde{U} \\
\Big\downarrow{\zeta_0} & & \Big\downarrow{i_0 \times i_0} \\
V \times \mu_m & \xrightarrow{\alpha \times \mathrm{pr}_2} & V \times V
\end{array}
$$

is cartesian. It follows that the section $i_0 : \tilde{U} \to V_m$ of $\pi_m : V_m \to \tilde{U}$ descends to a U-morphism $\tilde{U}/R \to V_m/\mu_m$ which is a section of $\overline{\sigma}$. $\qquad\square$

A practical consequence of Lemma 6 is the following:

Criterion 7. *A morphism $\tau : Y \to U$ from a scheme Y factors through a morphism $\tilde{\tau} : Y \to \tilde{U}/R$ if and only if the μ_m-equivariant morphism $\mathrm{pr}_2 : Y \times_U \tilde{U} \to \tilde{U}$ lifts to a μ_m-equivariant morphism $\tilde{\mathrm{pr}}_2 : Y \times_U \tilde{U} \to V_m$ such that $\mathrm{pr}_2 = \pi_m \circ \tilde{\mathrm{pr}}_2$.*

1.4. **An application.** To finish this section, let us give a first concrete application of Criterion 7. Given integers $m, n, r \geq 1$ such that $\gcd(m, r) = 1$, we let $X(m, n, r)$ be the smooth threefold in $\mathbb{A}_k^4 = \mathrm{Spec}(k[x, y, u, v])$ defined by the equation

$$x^m v^r - y^n u = 1.$$

We thus have $X(m, 1, 1) = X_m$. The locally nilpotent $k[y, v]$-derivation

$$\partial = y^n \frac{\partial}{\partial x} + m x^{m-1} v^r \frac{\partial}{\partial u}$$

of the coordinate ring of $X(m, n, r)$ defines a fixed point free $\mathbb{G}_{a,k}$-action on $X(m, n, r)$. As for X_m, the projection

$$q_{(m,n,r)} = \mathrm{pr}_{y,v} : X(m, n, r) \to \mathbb{A}_*^2 = \mathrm{Spec}(k[y, v]) \setminus \{(0, 0)\}$$

is a smooth $\mathbb{G}_{a,k}$-invariant morphism which restricts to the trivial $\mathbb{G}_{a,k}$-torsor over the principal affine open subset $\mathbb{A}_{k,y}^2 = \mathrm{Spec}(k[y^{\pm 1}, v])$ of $\mathbb{A}_*^2$.

Proposition 8. *The morphism $q_{(m,n,r)} : X(m,n,r) \to \mathbb{A}^2_*$ factors through an étale locally trivial $\mathbb{G}_{a,k}$-torsor $\rho_{(m,n,r)} : X(m,n,r) \to \mathfrak{S}_m$.*

Proof. By construction of $\mathfrak{S}_m$ as the gluing of $\tilde{U}/R$ and $\mathbb{A}^2_{k,y}$ described in subsection 1.2, it is enough to check as in the case of $X(m,1,1)$ that $q_{(m,n,r)}|_U : X(m,n,r)|_U \to U$ factors through a $\mathbb{G}_{a,k}$-torsor over $\tilde{U}/R \cong V_m/\mu_m$. Let $\tilde{U} = \mathrm{Spec}(k[y,\lambda^{\pm 1}]) \to U$ be the étale μ_m-cover defined by $(y,\lambda) \mapsto (y,\lambda^{-m})$. The fiber product $\tilde{Y} = Y \times_U \tilde{U}$ is isomorphic to the closed subscheme in $\tilde{U} \times \mathrm{Spec}(k[x,u])$ defined by the equation

$$y^n u = (\lambda^{-r}x)^m - 1$$

on which μ_m acts by $(y,\lambda,x,u) \mapsto (y,\varepsilon\lambda,x,u)$, where $\varepsilon \in k^*$ is a primitive m-th root of unity. The induced $\mathbb{G}_{a,k}$-action on $X(m,n,r)|_U$ lifts on $\tilde{Y}$ to the $\mathbb{G}_{a,k}$-action commuting with the action of μ_m defined by the locally nilpotent $k[y,\lambda^{\pm 1}]$-derivation

$$\tilde{\partial} = y^n \frac{\partial}{\partial x} + mx^{m-1}\lambda^r \frac{\partial}{\partial u}.$$

Since $\gcd(m,r) = 1$, the inverse image by $\mathrm{pr}_2 : \tilde{Y} \to \tilde{U}$ of the curve $\tilde{C} = \{y = 0\} \cong \mathrm{Spec}(k[\lambda^{\pm 1}])$ is the disjoint union of m irreducible surfaces $\tilde{S}_i = \tilde{C}_i \times \mathrm{Spec}(k[u])$, where

$$\tilde{C}_i = \mathrm{Spec}(k[\lambda^{\pm 1},x]/(\lambda^{-r}x - \varepsilon^{ri})) \cong \mathrm{Spec}(k[\lambda^{\pm 1}]), \ i \in \mathbb{Z}/m\mathbb{Z}.$$

Furthermore, the group μ_m acts transitively on $\mathrm{pr}_2^{-1}(\tilde{C})$ by $\tilde{S}_i \ni (\lambda,u) \mapsto (\varepsilon\lambda,u) \in \tilde{S}_{i+1}$. For every $i \in \mathbb{Z}/m\mathbb{Z}$, the $\tilde{U}$-morphism

$$\tilde{U}_i \times \mathbb{A}^1_k = \mathrm{Spec}(k[y,\lambda^{\pm 1}][v_i]) \to \tilde{Y}, \ (y,\lambda,v_i) \mapsto (y,\lambda,y^n v_i + (\varepsilon^i \lambda)^r, \lambda^{-mr}v_i \prod_{j \neq i}(y^n v_i + \lambda^r(\varepsilon^{ri} - \varepsilon^{rj})))$$

induces a $\tilde{U}$-isomorphism between $\tilde{U}_i \times \mathbb{A}^1_k \cong \tilde{U} \times \times \mathbb{A}^1_k$ and the $\mathbb{G}_{a,k}$-invariant open subset $\tilde{Y} \setminus \bigcup_{j \neq i} \tilde{S}_j$ of $\tilde{Y}$. Furthermore, using the expression $v_i = y^{-n}(x - (\varepsilon^i \lambda)^r)$ as a rational function on $\tilde{Y}$, we see that $\tilde{\partial}(v_i) = 1$ so that $\tilde{Y} \setminus \bigcup_{j \neq i} \tilde{S}_j$ is $\mathbb{G}_{a,k}$-equivarianlty isomorphic to $\tilde{U}_i \times \mathbb{A}^1_k$ on which $\mathbb{G}_{a,k}$ acts by translations on the second factor. The restriction of pr_2 to $\tilde{Y} \setminus \bigcup_{j \neq i} \tilde{S}_j$ is thus the trivial $\mathbb{G}_{a,k}$-torsor over $\tilde{U}_i$. It follows that $\mathrm{pr}_2 : \tilde{Y} \to \tilde{U}$ factors through a $\mathbb{G}_{a,k}$-torsor $\tilde{\mathrm{pr}}_2 : \tilde{Y} \to V_m$ with gluing isomorphisms defined by

$$(y,\lambda^{\pm 1},v_i) \mapsto (y,\lambda^{\pm 1},v_i + y^{-n}\lambda^r(\varepsilon^{ri} - \varepsilon^{rj})).$$

By construction $\tilde{\mathrm{pr}}_2 : \tilde{Y} \to V_m$ is equivariant for the μ_m-actions on $\tilde{Y}$ and V_m respectively. So $\tilde{\mathrm{pr}}_2 : \tilde{Y} \to V_m$ descends to an étale locally trivial $\mathbb{G}_{a,k}$-torsor $\rho_{(m,n,r),U} : \tilde{Y}/\mu_m = X(m,n,r)|_U \to V_m/\mu_m = \tilde{U}/R$ such that $q_{(m,n,r)}|_U = \beta \circ \rho_{(m,n,r),U}$ as desired. $\square$

It seems that the discrete family of threefolds $X(m,n,r)$ has not been studied yet in the literature. In particular, to the author's knowledge, the dependence of their isomorphism types in terms of the parameters m, n and r is unknown. The following result implies that some of these could provide new types of exotic affine 3-spheres:

Corollary 9. *For every triple (m,n,r) of positive integers such that $\gcd(m,r) = 1$, the affine fourfold $X(m,n,r) \times \mathbb{A}^1_k$ is isomorphic to $\mathrm{SL}_2 \times \mathbb{A}^1_k$.*

Proof. Since $\rho_{(m,n,r)} : X(m,n,r) \to \mathfrak{S}_m$ is the total space of an étale locally trivial $\mathbb{G}_{a,k}$-torsor, it follows that the fiber product $X(m,n,r) \times_{\mathfrak{S}_m} X(m,1,1)$ is simultaneously the total space of an étale $\mathbb{G}_{a,k}$-torsor over $X(m,n,r)$ and $X_m = X(m,1,1)$ via the first and second projection respectively. Since X_m and $X(m,n,r)$ are affine, these torsors are the trivial ones, which yields isomorphisms

$$X(m,n,r) \times \mathbb{A}^1_k \cong X(m,n,r) \times_{\mathfrak{S}_m} X(m,1,1) \cong X_m \times \mathbb{A}^1_k.$$

The result follows since on the other hand $X_m \times \mathbb{A}^1_k \cong X_1 \times \mathbb{A}^1_k = \mathrm{SL}_2 \times \mathbb{A}^1_k$ by (1.1). $\square$

2. FIXED POINT FREE $\mathbb{G}_{a,k}$- ACTIONS ON PUNCTURED DEFORMED KORAS-RUSSELL THREEFOLDS

Every deformed Koras-Russell threefold

$$Y = Y(m, n, r, h) = \{x^n z = y^m - t^r + xh(x, y, t)\} \subset \mathbb{A}_k^4$$

admits a $\mathbb{G}_{a,k}$-action defined by the locally nilpotent derivation $k[y, t]$-derivation

$$x^n \frac{\partial}{\partial y} + (my^{m-1} + x\frac{\partial h}{\partial y}(x, y, t))\frac{\partial}{\partial z}$$

of its coordinate ring. The fixed point locus of this action is equal to the affine line

$$\ell = \{x = y = t = 0\} \cong \mathrm{Spec}(k[z]),$$

so that the action restricts to a fixed point free $\mathbb{G}_{a,k}$-action on the quasi-affine threefold $Y_* = Y \setminus \ell$.

The proof of Theorem 1 we give in the next subsection then essentially follows from the basic observation that the categorical quotient $Y_*/\mathbb{G}_{a,k}$ taken in the category of algebraic spaces is isomorphic to the algebraic space $\mathfrak{S}_m$ described in subsection 1.2.

2.1. **Proof of Theorem 1.** Since the $\mathbb{G}_{a,k}$-action on Y_* defined above is fixed point free, the categorical quotient $Y_* \to Y_*/\mathbb{G}_{a,k}$ exists in the form of an étale locally trivial $\mathbb{G}_{a,k}$-torsor over a certain algebraic space $X_*/\mathbb{G}_{a,k}$. The $\mathbb{G}_{a,k}$-invariant projection $\mathrm{pr}_{x,t} : Y \to \mathbb{A}_k^2$ induces a surjective morphism $\pi : Y_* \to \mathbb{A}_*^2 = \mathbb{A}_k^2 \setminus \{(0, 0)\}$, which restricts further over the principal affine open subset $\mathbb{A}_{k,x}^2 = \mathrm{Spec}(k[x^{\pm 1}, t])$ to the trivial $\mathbb{G}_{a,k}$-torsor. On the other hand, the restriction of π over the curve $B \cong \mathrm{Spec}(k[t^{\pm 1}])$ in $\mathbb{A}_*^2$ with equation $x = 0$ factors as the composition of the trivial $\mathbb{G}_{a,k}$-torsor

$$\pi : Y|_B \cong \mathrm{Spec}(k[y, t^{\pm 1}, z]/(y^m - t^r)) \to \tilde{B} = \mathrm{Spec}(k[y, t^{\pm 1}]/(y^m - t^r))$$

with the projection $f : \tilde{B} \to B$, $(y, t) \mapsto t$. Since $\gcd(m, r) = 1$, the curves B and $\tilde{B}$ are both isomorphic to the punctured affine line $\mathbb{A}^1 \setminus \{0\}$, and f is a finite cyclic étale cover of order m. This strongly suggests that $Y_*/\mathbb{G}_{a,k}$ should be isomorphic to $\mathfrak{S}_m$, and this is indeed the case:

Proposition 10. *The categorical quotient $Y_*/\mathbb{G}_{a,k}$ in the category of algebraic spaces is isomorphic to $\mathfrak{S}_m$.*

Proof. The result can be extracted from the proof of Lemma 4.6 in [9] (see also [3, Lemma 3.2]). Let us nevertheless sketch the main steps which proceed along the same lines as the method employed in the proof of Proposition 8. Since the restriction of π over $\mathbb{A}_{k,x}^2$ is the trivial $\mathbb{G}_{a,k}$-torsor, it is again enough to check that the restriction of π over $U = \mathbb{A}_{k,t}^2 = \mathrm{Spec}(k[x, t^{\pm 1}])$ factors through a $\mathbb{G}_{a,k}$-torsor over $\tilde{U}/R \cong V_m/\mu_m$, where $\tilde{U} = \mathrm{Spec}(k[x, \lambda^{\pm 1}]) \to U$ is étale μ_m-cover defined by $(x, \lambda) \mapsto (x, \lambda^m)$. The fiber product $\tilde{Y} = Y \times_U \tilde{U}$ is isomorphic to the closed subscheme in $\tilde{U} \times \mathrm{Spec}(k[y, z])$ defined by the equation

$$x^n z = y^m - \lambda^{mr} + xh(x, y, \lambda^m),$$

on which μ_m acts by $(x, y, \lambda, z) \mapsto (x, y, \varepsilon\lambda, z)$, where $\varepsilon \in k^*$ is a primitive m-th root of unity. The induced $\mathbb{G}_{a,k}$-action on $Y|_U$ lifts on $\tilde{Y}$ to the $\mathbb{G}_{a,k}$-action commuting with the action of μ_m defined by the locally nilpotent $k[x, \lambda^{\pm 1}]$-derivation

$$\tilde{\partial} = x^n \frac{\partial}{\partial y} + (my^{m-1} + x\frac{\partial h}{\partial y}(x, y, \lambda^m))\frac{\partial}{\partial z}$$

Since $\gcd(m, r) = 1$, the inverse image by $\mathrm{pr}_2 : \tilde{Y} \to \tilde{U}$ of the curve $\tilde{B} = \{x = 0\} \cong \mathrm{Spec}(k[\lambda^{\pm 1}])$ is the disjoint union of the $\mathbb{G}_{a,k}$-invariant irreducible surfaces $\tilde{S}_i = \tilde{B}_i \times \mathrm{Spec}(k[z])$, where

$$\tilde{B}_i = \mathrm{Spec}(k[\lambda^{\pm 1}, y]/(y - (\varepsilon^i \lambda)^r) \cong \mathrm{Spec}(k[\lambda^{\pm 1}]), \ i \in \mathbb{Z}/m\mathbb{Z}.$$

The restriction of the $\mathbb{G}_{a,k}$-action on each $\tilde{S}_i$ is given by the locally nilpotent $k[\lambda^{\pm 1}]$-derivation $m(\varepsilon^i \lambda)^{r(m-1)} \frac{\partial}{\partial z}$, so that the projection $\mathrm{pr}_{\tilde{B}_i} : \tilde{S}_i \to \tilde{B}_i$ is a trivial $\mathbb{G}_{a,k}$-torsor. Furthermore, the group μ_m acts transitively on $\mathrm{pr}_2^{-1}(\tilde{B})$ by $\tilde{S}_i \ni (\lambda, z) \mapsto (\varepsilon\lambda, z) \in \tilde{S}_{i+1}$. For every $i \in \mathbb{Z}/m\mathbb{Z}$, pr_2 restricts on the open subset

$\tilde{Y} \setminus \bigcup_{j \neq i} \tilde{S}_j$ of $\tilde{Y}$ to a surjective $\mathbb{G}_{a,k}$-invariant smooth morphism $\mathrm{pr}_{2,i} : \tilde{Y} \setminus \bigcup_{j \neq i} \tilde{S}_j \to \tilde{U}_i$ whose fibers each consist of a unique $\mathbb{G}_{a,k}$-orbit. It follows that $\mathrm{pr}_{2,i}$ is a $\mathbb{G}_{a,k}$-torsor, hence is isomorphic to the trivial one as $\tilde{U}_i \cong \tilde{U}$ is affine. We conclude that $\mathrm{pr}_2 : \tilde{Y} \to \tilde{U}$ factors in a unique way through a $\mathbb{G}_{a,k}$-torsor $\tilde{\mathrm{pr}}_2 : \tilde{Y} \to V_m$ equivariant for the μ_m-actions on $\tilde{Y}$ and V_m respectively. So $\tilde{\mathrm{pr}}_2 : \tilde{Y} \to V_m$ descends to an étale locally trivial $\mathbb{G}_{a,k}$-torsor $\tilde{\pi} : \tilde{Y}/\mu_m = Y|_U \to V_m/\mu_m = \tilde{U}/R$ which factors the projection $\pi : Y|_U \to U$. $\qquad\qquad\square$

We can now finish the proof of Theorem 1 as follows. By Proposition 5 and Proposition 10, X_m and Y_* are étale locally trivial $\mathbb{G}_{a,k}$-torsors over the same algebraic space $\mathfrak{S}_m$. This implies that the fiber product $W_m = Y_* \times_{\mathfrak{S}_m} X_m$ is simultaneously the total space of étale $\mathbb{G}_{a,k}$-torsors over Y_* and X_m via the first and second projection respectively. Since Y_* is separated, the $\mathbb{G}_{a,k}$-action on W_m corresponding the $\mathbb{G}_{a,k}$-torsor $\mathrm{pr}_1 : W_m \to Y_*$ is proper. Furthermore, since Y_* is a scheme, $\mathrm{pr}_1 : W_m \to Y_*$ is in fact locally trivial in the Zariski topology [11]. On the other hand, since X_m is affine $\mathrm{pr}_2 : W_m \to X_m$ is the trivial $\mathbb{G}_{a,k}$-torsor. Thus $W_m \cong X_m \times \mathbb{A}^1_k$ and hence $W_m \cong \mathrm{SL}_2 \times \mathbb{A}^1_k$ by (1.1). The $\mathbb{G}_{a,k}$-action on W_m defining the $\mathbb{G}_{a,k}$-torsor $\mathrm{pr}_1 : W_m \to Y_*$ thus corresponds via these isomorphisms to a proper and Zariski locally trivial $\mathbb{G}_{a,k}$-action on $\mathrm{SL}_2 \times \mathbb{A}^1_k$, whose categorical quotient $(\mathrm{SL}_2 \times \mathbb{A}^1_k)/\mathbb{G}_{a,k}$ in the category of algebraic spaces is isomorphic to the quasi-affine variety $Y_* = Y \setminus \ell$.

Lemma 11. *The categorical quotient* $(\mathrm{SL}_2 \times \mathbb{A}^1_k)//\mathbb{G}_{a,k} = \mathrm{Spec}(\Gamma(\mathrm{SL}_2 \times \mathbb{A}^1_k, \mathcal{O}_{\mathrm{SL}_2 \times \mathbb{A}^1_k})^{\mathbb{G}_{a,k}})$ *in the category of affine schemes is isomorphic to* Y.

Proof. The universal properties of the categorical quotient $\mathrm{pr}_1 : W_m \to W_m/\mathbb{G}_{a,k} = Y_*$ and the affinization morphism $W_m/\mathbb{G}_{m,k} \to \mathrm{Spec}(\Gamma(W_m/\mathbb{G}_{m,k}, \mathcal{O}_{W_m/\mathbb{G}_{m,k}}))$ imply that

$$W_m//\mathbb{G}_{a,k} = \mathrm{Spec}(\Gamma(W_m/\mathbb{G}_{m,k}, \mathcal{O}_{W_m/\mathbb{G}_{m,k}})) = \mathrm{Spec}(\Gamma(Y_*, \mathcal{O}_{Y_*})).$$

Since Y is a smooth, hence normal, affine variety and ℓ has pure codimension 2 in Y, we have $\Gamma(Y_*, \mathcal{O}_{Y_*}) = \Gamma(Y, \mathcal{O}_Y)$. Thus

$$(\mathrm{SL}_2 \times \mathbb{A}^1_k)//\mathbb{G}_{a,k} \cong W_m//\mathbb{G}_{a,k} \cong \mathrm{Spec}(\Gamma(Y, \mathcal{O}_Y)) = Y.$$

$\qquad\qquad\square$

2.2. **Complements.** The proof of Theorem 1 provides a systematic method to construct for each given deformed Koras-Russell threefold $Y = Y(m, n, r, h)$ an explicit locally nilpotent derivation δ of the coordinate ring of $\mathrm{SL}_2 \times \mathbb{A}^1_k$ with kernel isomorphic to $\Gamma(Y, \mathcal{O}_Y)$, usually at the cost of a series of tedious calculations.

To explain the scheme of this method, let $\mathbb{A}^2_{x,t} = \mathrm{Spec}(k[x,t])$, $\mathbb{A}^2_{x,v} = \mathrm{Spec}(k[x,v])$, and choose the coordinates so that $Y = Y(m, n, r, h)$ and X_m are given by the equations

$$x^n z = y^m - t^r + xh(x, y, t) \quad \text{and} \quad v^m t - xu = 1$$

in $\mathbb{A}^2_{x,t} \times \mathrm{Spec}(k[y,z])$ and $\mathbb{A}^2_{x,t} \times \mathrm{Spec}(k[u,v]) = \mathbb{A}^2_{x,v} \times \mathrm{Spec}(k[u,t])$ respectively. Let

$$\partial = x^n \frac{\partial}{\partial y} + (my^{m-1} + x\frac{\partial h}{\partial y}(x, y, t))\frac{\partial}{\partial z} \quad \text{and} \quad \partial_m = mv^{m-1}t\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}$$

be the locally nilpotent derivations defining the $\mathbb{G}_{a,k}$-actions on Y and X_m respectively with algebraic spaces quotients $Y_*/\mathbb{G}_{a,k} \cong \mathfrak{S}_m \cong X_m/\mathbb{G}_{a,k}$. By the construction used in the proof of Theorem 1, we have a

commutative diagram

$$W_m = Y_* \times_{\mathfrak{S}_m} X_m \xrightarrow{\cong} X_m \times \mathbb{A}_k^1 \xrightarrow{\cong} X_m \times_{\mathbb{A}_{x,v}^2} X_1 \xrightarrow{\cong} \mathrm{SL}_2 \times \mathbb{A}_k^1$$

$$Y_* \qquad X_m \qquad X_1 = \mathrm{SL}_2$$

$$\mathfrak{S}_m$$

$$\mathbb{A}_{x,t}^2 \qquad\qquad \mathbb{A}_{x,v}^2$$

$$\mathbb{A}_x^1 = \mathrm{Spec}(k[x]).$$

Let $\tilde{\partial}_m$ and $\tilde{\partial}$ be the commuting locally nilpotent $k[x,t]$-derivations of the coordinate ring of W_m with kernels equal to $\Gamma(Y, \mathcal{O}_Y)$ and $\Gamma(X_m, \mathcal{O}_{X_m})$ corresponding to the $\mathbb{G}_{a,k}$-torsors $\mathrm{pr}_1 : W_m = Y_* \times_{\mathfrak{S}_m} X_m \to Y_*$ and $\mathrm{pr}_2 : W_m = Y_* \times_{\mathfrak{S}_m} X_m \to X_m$ respectively. Via the left-hand side isomorphism $\psi : W_m \to X_m \times \mathbb{A}_k^1$ of the top line of the diagram, the derivation $\tilde{\partial}$ corresponds to the locally nilpotent derivation $\frac{\partial}{\partial w}$ of the coordinate ring $\Gamma(X_m, \mathcal{O}_{X_m})[w]$ of $X_m \times \mathbb{A}_k^1$, whereas $\tilde{\partial}_m$ corresponds to the unique $k[x,t]$-derivation δ_m commuting with $\frac{\partial}{\partial w}$, whose restriction to $\mathrm{Ker}(\frac{\partial}{\partial w}) = \Gamma(X_m, \mathcal{O}_{X_m})$ is equal to ∂_m and such that $\delta_m(w) = \psi(\tilde{\partial}_m(\psi^{-1}(w)))$. In practice, the element $\psi^{-1}(w) \in \mathrm{Ker}\tilde{\partial}$ as well as its image by $\tilde{\partial}_m$ can be explicitly determined by considering an étale cover of $S \to \mathfrak{S}_m$ of $\mathfrak{S}_m$ on which the $\mathbb{G}_{a,k}^2$-torsor $W_m \to \mathfrak{S}_m$ becomes trivial.

The commutativity of the diagram then implies that the locally nilpotent derivation δ of the coordinate ring of $\mathrm{SL}_2 \times \mathbb{A}_k^1$ corresponding to δ_m through the isomorphisms

$$X_m \times \mathbb{A}_k^1 \cong X_m \times_{\mathbb{A}_{x,v}^2} X_1 \cong \mathrm{SL}_2 \times \mathbb{A}_k^1$$

of schemes over $\mathrm{Spec}(k[x,v])$ is a $k[x]$-derivation. Explicit isomorphisms $X_m \times \mathbb{A}_k^1 \cong \mathrm{SL}_2 \times \mathbb{A}_k^1$ for each $m \geq 2$ can be constructed by finding explicit trivializations of the $\mathbb{G}_{a,k}$-bundles $\mathrm{pr}_1 : X_m \times_{\mathbb{A}_{x,v}^2} X_1 \to X_m$ and $\mathrm{pr}_2 : X_m \times_{\mathbb{A}_{x,v}^2} X_1 \to X_1$. In practice, this amounts to describe these bundles in terms Čech 1-cocyles with values in $\mathcal{O}_{X_m}$ and $\mathcal{O}_{X_1}$ on suitable open cover of X_m and X_1 respectively, and find explicit expressions of each of these 1-cocyles as coboundaries.

Note in addition that letting $X(m,n,r) = \{v^m t^r - x^n u = 1\}$, where $\gcd(m,r) = 1$, endowed with the $\mathbb{G}_{a,k}$-action determined by the locally nilpotent $k[x,t]$-derivation

$$\partial_{(m,n,r)} = m v^{m-1} t^r \frac{\partial}{\partial u} + x^n \frac{\partial}{\partial v},$$

we have by Proposition 8 isomorphisms

$$X_m \times \mathbb{A}_k^1 \cong X_m \times_{\mathfrak{S}_m} X(m,n,r) \cong X(m,n,r) \times \mathbb{A}_k^1$$

of schemes over $\mathbb{A}_{x,t}^2$. The derivation $\tilde{\partial}_m$ corresponds via these isomorphisms to a unique $k[x,t]$-derivation $\delta_{(m,n,r)}$ of the coordinate ring $\Gamma(X(m,n,r), \mathcal{O}_{X(m,n,r)})[\omega]$ of $X(m,n,r) \times \mathbb{A}_k^1$ commuting with $\frac{\partial}{\partial \omega}$ and whose restriction to $\mathrm{Ker}(\frac{\partial}{\partial \omega}) = \Gamma(X(m,n,r), \mathcal{O}_{X(m,n,r)})$ is equal to $\partial_{(m,n,r)}$. Depending on the deformed

Koras-Russell threefold Y, the element $\delta_{(m,n,r)}(\omega) \in \Gamma(X(m,n,r), \mathcal{O}_{X(m,n,r)})$ can be easier to determine for suitably chosen n and r than the element $\delta_m(w) \in \Gamma(X_m, \mathcal{O}_{X_m})$. For instance, we have:

Example 12. The fixed point free $\mathbb{G}_{a,k}$-actions on

$$X_{(2,2,3)} \times \mathbb{A}_k^1 = \mathrm{Spec}(k[x,t,u,v][\omega]/(v^2 t^3 - x^2 u - 1))$$

whose algebraic quotients are the deformed Russell cubic threefolds $Y_\alpha \subset \mathbb{A}_k^4$ with equations

$$x^2 z = y^2 - t^3 + x(1 + \alpha t), \quad \alpha \in k,$$

are given by the locally nilpotent $k[x,t]$-derivations

$$\delta_{(2,2,3),\alpha} = 2vt^3 \frac{\partial}{\partial u} + x^2 \frac{\partial}{\partial v} + (\frac{1}{2}(1 + \alpha t)x - t^3)\frac{\partial}{\partial \omega}.$$

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