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APPROXIMATE MATRIX AND TENSOR DIAGONALIZATION BY UNITARY TRANSFORMATIONS: CONVERGENCE OF JACOBI-TYPE ALGORITHMS*

KONSTANTIN USEVICH[†], JIANZE LI[‡], AND PIERRE COMON[§]

Abstract. We propose a gradient-based Jacobi algorithm for a class of maximization problems on the unitary group, with a focus on approximate diagonalization of complex matrices and tensors by unitary transformations. We provide weak convergence results, and prove local linear convergence of this algorithm. The convergence results also apply to the case of real-valued tensors.

Key words. optimization on manifolds, unitary group, Givens rotations, approximate tensor diagonalization, Lojasiewicz gradient inequality, local convergence

AMS subject classifications. 90C30,53B21,53B20,15A69,65K10,65Y20

1. Introduction. In this paper, we consider the following optimization problem

$$(1.1) \quad \mathbf{U}_* = \arg \max_{\mathbf{U} \in \mathcal{U}_n} f(\mathbf{U}),$$

where $\mathcal{U}_n \subseteq \mathbb{C}^{n \times n}$ is the unitary group and

$$(1.2) \quad f : \mathcal{U}_n \rightarrow \mathbb{R}^+$$

is a real differentiable function. An important class of such optimization problems stems from approximate matrix and tensor diagonalization problems in numerical linear and multilinear algebra, that are relevant in signal processing [17] and machine learning [5].

Jacobi-type algorithms are widely used for maximization of these cost functions. Inspired by the classic Jacobi algorithm [22] for the symmetric eigenvalue problem, these algorithms proceed by successive Givens transformations that update only a pair of columns of \mathbf{U} . The popularity of these approaches is explained by simplicity of the updates, which in many cases are very cheap. Although the Jacobi-type algorithms are similar in spirit to block-coordinate descent, they enjoy quadratic convergence for the classic matrix case [22] and the case of a pair of commuting matrices [12].

Despite their popularity, the convergence of these algorithms has not yet been studied thoroughly, except the cases listed above. For the real-valued case (orthogonal group), a gradient-based Jacobi-type algorithm (which we call *Jacobi-G*) was proposed in [26] and its weak convergence was proved. Global (single-point) convergence of this algorithm was proved for simultaneous real third-order tensor or matrix diagonalization in [30]; the proof in [30] based on the Lojasiewicz gradient inequality, that became a very popular tool for studying convergence properties of nonlinear optimization algorithms [2, 29, 36, 6], including various tensor approximation problems [37, 25].

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In this paper, we address the complex-valued (the unitary group), and focus on a number of tensor and matrix approximate diagonalization problems [17] (although the results are applicable to other instances of problem (1.1)). Unlike the real case, where the Givens transformations are univariate (“line-search” type), in the complex case the updates correspond to maximization on a sphere (similar in spirit to subspace methods). The main contributions of the paper are: (i) we generalize the Jacobi-G algorithm to the complex case, namely to the unitary group, and prove its weak convergence (to stationary points) and global rates of convergence based on the results of [9]; (ii) for the case of matrix and tensor diagonalization, we show that the local convergence can be studied by combining the tools of Lojasiewicz gradient inequality, geodesic convexity and recent results on Lojasiewicz exponent for Morse-Bott functions. In particular, local linear convergence takes place for local maxima satisfying second order regularity conditions. One of the motivations for this work was that the case of the unitary group is not common in the optimization literature, unlike the case of orthogonal group and other matrix manifolds [3].

The structure of the paper is as follows. In section 2, we recall the cost functions of main interest, the principle of Jacobi-type algorithms, and expressions for the updates in these algorithms. In section 3, we recall all the necessary facts for differentiation on the unitary group, formulate the abstract Jacobi-G algorithm and prove its correctness. Section 4 contains the expressions of the first- and second-order derivatives of various cost functions listed in section 2. In section 5, we present the result on weak convergence (to stationary points) and global convergence rates for the Jacobi-G algorithm. The general results of [9] are summarized in the same section. In section 6, we recall general results on convergence of descent algorithms on manifolds that are based on the Lojasiewicz gradient inequality, we also recall the notions of geodesic convexity and Morse-Bott functions, that will be used later on. Section 7 contains main results. While subsection 7.1 is devoted to checking the decrease conditions, subsection 7.2 contains the results on local linear convergence of Jacobi-G algorithm to local maxima satisfying the Morse-Bott property. We eventually provide in subsection 7.3 some examples of tensor and matrix diagonalization problems where these properties are satisfied.

2. Background and problem statement.

2.1. Main notation. For a matrix $\mathbf{X} \in \mathbb{C}^{m \times n}$, we denote by \mathbf{X}^T its transpose, by \mathbf{X}^* its elementwise conjugate, and by \mathbf{X}^H the Hermitian transpose, respectively. We will also frequently use the notation $\mathbf{X} = \mathbf{X}^{\Re} + i\mathbf{X}^{\Im}$ for the real and imaginary parts of \mathbf{X} , and $\Re(a)$, $\Im(a)$ for the real and imaginary part of $a \in \mathbb{C}$. Moreover, \mathcal{U}_n and $S\mathcal{U}_n$ denote the unitary and special unitary groups in $\mathbb{C}^{n \times n}$, whereas \mathcal{O}_n and $S\mathcal{O}_n$ denote the orthogonal and special orthogonal groups in $\mathbb{R}^{n \times n}$, respectively.

In this paper, we make no distinction between tensors and multi-way arrays; for simplicity, we consider only fully contravariant tensors [32]. For a tensor or a matrix $\mathcal{A} \in \mathbb{C}^{n \times \dots \times n}$, we denote by $\text{diag}\{\mathcal{A}\} \in \mathbb{C}^n$ the vector containing all the diagonal elements $\mathcal{A}_{ii\dots i}$. We denote by $\|\cdot\|$ the Frobenius norm of a tensor or a matrix, or the Euclidean norm of a vector. By $\mathcal{A} \bullet_k \mathbf{v}$ we denote the contraction on the k th index of \mathcal{A} with vector \mathbf{v} . By writing multiple contractions $\mathcal{A} \bullet_{k_1} \mathbf{v}_1 \dots \bullet_{k_\ell} \mathbf{v}_\ell$ we assume that they are performed simultaneously, *i.e.* the indexing of the tensor does not change before contractions are complete.

For a pair $i \neq j$ we introduce the projection operator $\mathcal{P}_{i,j} : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{2 \times 2}$ that

extracts the submatrix of $\mathbf{X} \in \mathbb{C}^{n \times n}$ as follows:

$$(2.1) \quad \mathcal{P}_{i,j}(\mathbf{X}) = \begin{bmatrix} X_{ii} & X_{ij} \\ X_{ji} & X_{jj} \end{bmatrix}.$$

Its adjoint operator is $\mathcal{P}_{i,j}^\top : \mathbb{C}^{2 \times 2} \rightarrow \mathbb{C}^{n \times n}$, i.e.

$$(2.2) \quad \mathcal{P}_{i,j}^\top \left(\begin{bmatrix} a & c \\ b & d \end{bmatrix} \right) = \begin{matrix} & & & i & & j & & \\ & & & \vdots & & \vdots & & \mathbf{0} \\ & & & \cdots & a & c & & \\ & & & \vdots & & \vdots & & \\ & & & \cdots & b & d & & \\ & & & \mathbf{0} & & & & \mathbf{0} \end{matrix}.$$

2.2. Cost functions under consideration. In this paper, we mainly consider the following three cost functions from \mathcal{U}_n to \mathbb{R}^+ . But, whenever possible, we formulate our results in full generality

- (i) *Approximate diagonalization of a set of matrices.* Let $\mathbf{A}^{(\ell)} = \mathbf{A}^{(\ell, \mathbb{R})} + i\mathbf{A}^{(\ell, \mathbb{S})} \in \mathbb{C}^{n \times n}$, $1 \leq \ell \leq L$, be Hermitian matrices. The cost function is defined as

$$(2.3) \quad f(\mathbf{U}) = \sum_{\ell=1}^L \|\text{diag}\{\mathbf{W}^{(\ell)}\}\|^2,$$

where $\mathbf{W}^{(\ell)} = \mathbf{U}^H \mathbf{A}^{(\ell)} \mathbf{U}$.

- (ii) *Approximate diagonalization of a partially symmetric 3rd order tensor.* Let $\mathcal{A} \in \mathbb{C}^{n \times n \times n}$ be a tensor satisfying the partial symmetry condition:

$$(2.4) \quad \mathcal{A}_{ijk} = \mathcal{A}_{ikj}$$

for any $1 \leq i, j, k \leq n$. The cost function is defined as:

$$(2.5) \quad f(\mathbf{U}) = \|\text{diag}\{\mathcal{W}\}\|^2,$$

where $\mathcal{W}_{ijk} = \sum_{p,q,r} \mathcal{A}_{pqr} U_{pi}^* U_{qj} U_{rk}$.

- (iii) *Approximate diagonalization of a 4th order tensor.* Let $\mathcal{B} \in \mathbb{C}^{n \times n \times n \times n}$ be a tensor satisfying the following partial symmetry conditions

$$(2.6) \quad \mathcal{B}_{ijkl} = \mathcal{B}_{jikl} \quad \text{and} \quad \mathcal{B}_{ijkl} = \mathcal{B}_{klij}^*$$

for any $1 \leq i, j, k, l \leq n$. The cost function is defined as

$$(2.7) \quad f(\mathbf{U}) = \sum_{p=1}^n \mathcal{V}_{pppp},$$

where $\mathcal{V}_{ijkl} = \sum_{p,q,r,s} \mathcal{B}_{pqrs} U_{pi}^* U_{qj}^* U_{rk} U_{sl}$.

Remark 2.1. As in (2.3), the simultaneous diagonalization problem of several 3rd and 4th order tensors can be considered. In this paper however, we prefer to consider the single tensor case in (2.5) and (2.7) for simplicity of presentation.

The motivation behind these cost functions comes from blind source separation.

- (i) The cost function (2.3) is used for diagonalization of covariance matrices [13, 14].

maximize the restriction $h_{(i_k, j_k), \mathbf{U}_{k-1}}$ of (1.2) defined as

$$(2.8) \quad \begin{aligned} h_{(i,j), \mathbf{U}} : \mathcal{U}_2 &\longrightarrow \mathbb{R}^+ \\ \Psi &\longmapsto f(\mathbf{U}\mathbf{G}^{(i,j, \Psi)}). \end{aligned}$$

An advantage of the Jacobi-type methods is that in many cases the maximizers of $h_{(i_k, j_k), \mathbf{U}_{k-1}}$ can be found in a closed form, and the updates are very cheap.

A typical choice of pairs (i_k, j_k) is cyclic, e.g.,

$$\begin{aligned} (1, 2) &\rightarrow (1, 3) \rightarrow \cdots \rightarrow (1, n) \rightarrow (2, 3) \rightarrow \cdots \rightarrow (2, n) \rightarrow \cdots \rightarrow (n-1, n) \rightarrow \\ (1, 2) &\rightarrow (1, 3) \rightarrow \cdots, \end{aligned}$$

which appears in the classic Jacobi algorithm for the symmetric eigenvalue problem, and is used for maximizing the cost functions in Subsection 2.2 .

The convergence of the iterations for cyclic algorithms is unknown, except in the single matrix case [22]. Recently, a gradient-based Jacobi algorithm (Jacobi-G) [26] was proposed in a context of optimization on orthogonal group for low multilinear rank approximation. Weak convergence was shown in [26] and global convergence for real matrix and 3rd order tensor case was proved in [30]. In this paper, we extend Jacobi-G to the case of the unitary group, but we postpone its formulation to Section 3.

2.4. Jacobi rotations for scale-invariant functions. First of all, consider the cost functions in (1.1) that are invariant under permutations of columns of \mathbf{U} and multiplications of columns of \mathbf{U} by complex scalars of modulus 1, i.e.

$$(2.9) \quad f(\mathbf{U}) = f(\mathbf{U}\mathbf{S})$$

for any matrix of the form

$$(2.10) \quad \mathbf{S} = \begin{bmatrix} e^{i\alpha_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\alpha_n} \end{bmatrix}.$$

In this case, we see that the restriction (2.8) satisfies

$$(2.11) \quad h_{(i,j), \mathbf{U}}(\Psi) = h_{(i,j), \mathbf{U}} \left(\Psi \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} \right)$$

for any $|z_1| = |z_2| = 1$. Hence, to maximize $h_{(i,j), \mathbf{U}}(\Psi)$, we can set

$$(2.12) \quad \begin{aligned} \Psi = \Psi(c, s_1, s_2) &= \begin{bmatrix} c & -s \\ s^* & c \end{bmatrix} = \begin{bmatrix} c & -(s_1 + is_2) \\ s_1 - is_2 & c \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta e^{i\phi} \\ \sin \theta e^{-i\phi} & \cos \theta \end{bmatrix} = \frac{1}{\sqrt{1 + |z|^2}} \begin{bmatrix} 1 & -z \\ z^* & 1 \end{bmatrix} \in S\mathcal{U}_2, \end{aligned}$$

where $c \in \mathbb{R}^+$, $s = s_1 + is_2 = \sin \theta e^{i\phi} \in \mathbb{C}$ satisfy $c^2 + |s|^2 = 1$. We also denote

$$h(c, s_1, s_2) = h_{(i,j), \mathbf{U}}(c, s_1, s_2) = h_{(i,j), \mathbf{U}} \left(\begin{bmatrix} c & -s \\ s^* & c \end{bmatrix} \right).$$

Similarly to the single matrix case [22], we will refer to the maximizers of $h_{(i,j), \mathbf{U}}$ as Jacobi rotations.

2.5. Jacobi rotations for matrix/tensor diagonalization. In this subsection, we consider the cost functions in [subsection 2.2](#), which obviously satisfy the invariance property (2.9). We recall how Jacobi rotations can be computed by finding an eigenvector of a 3×3 real matrix.

LEMMA 2.3. For all $\mathbf{U} \in \mathcal{U}_n$, the cost functions in [Subsection 2.2](#) have the form

$$(2.13) \quad h_{(i,j),\mathbf{U}}(c, s_1, s_2) = \mathbf{r}^\top \mathbf{\Gamma}^{(i,j),\mathbf{U}} \mathbf{r} + C,$$

where $\mathbf{r} = \mathbf{r}(c, s_1, s_2) \stackrel{\text{def}}{=} (2c^2 - 1, -2cs_1, -2cs_2)^\top$,

$$(2.14) \quad \stackrel{\text{def}}{=} (\cos 2\theta, -\sin 2\theta \sin \phi, -\sin 2\theta \cos \phi)^\top = (2c^2 - 1, -2cs_1, -2cs_2)^\top,$$

the constant C does not depend on (c, s_1, s_2) , and $\mathbf{\Gamma}^{(i,j),\mathbf{U}} \in \mathbb{R}^{3 \times 3}$ is a symmetric matrix defined as follows:

(i) for the cost function (2.3), we have $C = \frac{1}{2} \sum_{\ell=1}^L (W_{jj}^{(\ell)} + W_{ii}^{(\ell)})^2$ and

$$\mathbf{\Gamma}^{(i,j),\mathbf{U}} = \frac{1}{2} \sum_{\ell=1}^L \begin{bmatrix} (W_{jj}^{(\ell)} - W_{ii}^{(\ell)}) \\ 2W_{ij}^{(\ell,\Re)} \\ 2W_{ij}^{(\ell,\Im)} \end{bmatrix} \begin{bmatrix} (W_{jj}^{(\ell)} - W_{ii}^{(\ell)}) & 2W_{ij}^{(\ell,\Re)} & 2W_{ij}^{(\ell,\Im)} \end{bmatrix};$$

(ii) for the cost function (2.5), we have $C = 0$ and $\mathbf{\Gamma} = \mathbf{\Gamma}^{(i,j),\mathbf{U}}$ (see [18, (9.29)] and [17, Section 5.3.2]) is:

$$\begin{aligned} \Gamma_{11} &= a_1, & \Gamma_{22} &= v_4 - \Re(v_3), & \Gamma_{33} &= v_4 + \Re(v_3), \\ \Gamma_{12} &= \Im(v_1) + \Im(v_2), & \Gamma_{13} &= \Re(v_1) - \Re(v_2), & \Gamma_{23} &= \Im(v_3), \\ a_1 &= |\mathcal{W}_{111}|^2 + |\mathcal{W}_{222}|^2, & a_2 &= |\mathcal{W}_{112}|^2 + |\mathcal{W}_{212}|^2, & a_3 &= |\mathcal{W}_{211}|^2 + |\mathcal{W}_{122}|^2, \\ a_4 &= \mathcal{W}_{111}\mathcal{W}_{112}, & a_5 &= \mathcal{W}_{111}\mathcal{W}_{211}, & a_6 &= \mathcal{W}_{222}\mathcal{W}_{122}, & a_7 &= \mathcal{W}_{222}\mathcal{W}_{212}, \\ a_8 &= \mathcal{W}_{111}\mathcal{W}_{212} + \mathcal{W}_{222}\mathcal{W}_{112}, & a_9 &= \mathcal{W}_{111}\mathcal{W}_{122} + \mathcal{W}_{222}\mathcal{W}_{211}, \\ a_{10} &= \mathcal{W}_{211}\mathcal{W}_{112} + \mathcal{W}_{122}\mathcal{W}_{212}, \\ v_1 &= a_7 - \frac{a_5}{2}, & v_2 &= a_4 - \frac{a_6}{2}, & v_3 &= \frac{a_9}{2} + a_{10}, & v_4 &= \frac{a_1 + a_3}{4} + a_2 + \Re(a_8). \end{aligned}$$

(iii) for the cost function (2.7), we have $C = 0$ and $\mathbf{\Gamma} = \mathbf{\Gamma}^{(i,j),\mathbf{U}}$ [15] is

$$\begin{aligned} \Gamma_{11} &= \mathcal{V}_{iiii} + \mathcal{V}_{jjjj}, & \Gamma_{12} &= \mathcal{V}_{ijjj}^{\Re} - \mathcal{V}_{iiij}^{\Re}, & \Gamma_{13} &= \mathcal{V}_{ijjj}^{\Im} - \mathcal{V}_{iiij}^{\Im}, & \Gamma_{23} &= \mathcal{V}_{ijij}^{\Im}, \\ \Gamma_{22} &= \frac{1}{2}(\mathcal{V}_{iiii} + \mathcal{V}_{jjjj}) + 2\mathcal{V}_{ijij} + \mathcal{V}_{ijij}^{\Re}, & \Gamma_{33} &= \frac{1}{2}(\mathcal{V}_{iiii} + \mathcal{V}_{jjjj}) + 2\mathcal{V}_{ijij} - \mathcal{V}_{ijij}^{\Re}. \end{aligned}$$

Remark 2.4. (i) By [Lemma 2.3](#), the maximization of $h_{(i,j),\mathbf{U}}(c, s_1, s_2)$ is equivalent to maximization of the quadratic form $\mathbf{r}^\top \mathbf{\Gamma}^{(i,j),\mathbf{U}} \mathbf{r}$ on the unit sphere $\|\mathbf{r}\| = 1$. Thus, the maximizer Ψ of $h_{(i,j),\mathbf{U}}$ can be obtained from an eigenvector corresponding to the maximal eigenvalue of $\mathbf{\Gamma}^{(i,j),\mathbf{U}}$ denoted by \mathbf{w} .

(ii) Even if the maximal eigenvalue is simple, \mathbf{w} is defined up to a sign change. We choose the sign such that $w_1 = \cos 2\theta = 2c^2 - 1 \geq 0$ in (2.14). Hence, we can take $\theta \in [0, \frac{\pi}{4}]$ and $c > \frac{\sqrt{2}}{2}$ (by setting $\theta = \frac{\arccos(w_1)}{2} \in [0, \frac{\pi}{4}]$, $s_1 = -\frac{w_2}{2\cos\theta}$, $s_2 = -\frac{w_3}{2\cos\theta}$).

3. Jacobi-G algorithm for \mathcal{U}_n . Before formulating the Jacobi-G algorithm, we recall some facts on complex derivatives and functions on manifolds.

3.1. Wirtinger calculus. First, we introduce the following real-valued inner product on $\mathbb{C}^{m \times n}$. For $\mathbf{X} = \mathbf{X}^{\Re} + i\mathbf{X}^{\Im}$, $\mathbf{Y} = \mathbf{Y}^{\Re} + i\mathbf{Y}^{\Im} \in \mathbb{C}^{m \times n}$, we introduce

$$(3.1) \quad \langle \mathbf{X}, \mathbf{Y} \rangle_{\Re} \stackrel{\text{def}}{=} \langle \mathbf{X}^{\Re}, \mathbf{Y}^{\Re} \rangle + \langle \mathbf{X}^{\Im}, \mathbf{Y}^{\Im} \rangle = \Re \left(\text{trace} \{ \mathbf{X}^{\text{H}} \mathbf{Y} \} \right).$$

Since a nonconstant function $f : \mathbb{C}^{m \times n} \rightarrow \mathbb{R}$ is never holomorphic, we use a shorthand notation $\frac{\partial f}{\partial \mathbf{X}^{\Re}}, \frac{\partial f}{\partial \mathbf{X}^{\Im}} \in \mathbb{R}^{m \times n}$ for the matrix derivatives with respect to the real and imaginary parts of $\mathbf{X} \in \mathbb{C}^{m \times n}$. The Wirtinger derivatives [1, 11, 28] $\frac{\partial f}{\partial \mathbf{X}}, \frac{\partial f}{\partial \mathbf{X}^*} \in \mathbb{C}^{m \times n}$ are standardly defined as

$$\frac{\partial f}{\partial \mathbf{X}} := \frac{1}{2} \left(\frac{\partial f}{\partial \mathbf{X}^{\Re}} - i \frac{\partial f}{\partial \mathbf{X}^{\Im}} \right), \quad \frac{\partial f}{\partial \mathbf{X}^*} := \frac{1}{2} \left(\frac{\partial f}{\partial \mathbf{X}^{\Re}} + i \frac{\partial f}{\partial \mathbf{X}^{\Im}} \right).$$

The matrix Euclidean gradient of f with respect to (3.1) becomes

$$\nabla f(\mathbf{X}) = \frac{\partial f}{\partial \mathbf{X}^{\Re}} + i \frac{\partial f}{\partial \mathbf{X}^{\Im}} = 2 \frac{\partial f}{\partial \mathbf{X}^*}(\mathbf{X}).$$

3.2. Riemannian gradient. Recall that $\mathcal{U}_n \subseteq \mathbb{C}^{n \times n}$ can be viewed as an embedded real submanifold of $\mathbb{C}^{n \times n}$ with the inner product induced by (3.1). By [3, Section 3.5.7], the tangent space to \mathcal{U}_n can be associated with an n^2 -dimensional \mathbb{R} -linear subspace of $\mathbb{C}^{n \times n}$:

$$\mathbf{T}_U \mathcal{U}_n = \{ \mathbf{X} \in \mathbb{C}^{n \times n} : \mathbf{X}^{\text{H}} \mathbf{U} + \mathbf{U}^{\text{H}} \mathbf{X} = 0 \}.$$

Alternatively, it is the transformed set of skew-Hermitian matrices:

$$\mathbf{T}_U \mathcal{U}_n = \{ \mathbf{X} \in \mathbb{C}^{n \times n} : \mathbf{X} = \mathbf{U} \mathbf{Z}, \quad \mathbf{Z} + \mathbf{Z}^{\text{H}} = 0 \}.$$

Then for $f : \mathbb{C}^{n \times n} \rightarrow \mathbb{R}$ differentiable in a neighborhood of \mathcal{U}_n , the Riemannian gradient is just the orthogonal projection of the Euclidean gradient $\nabla f(\mathbf{U})$ on $\mathbf{T}_U \mathcal{U}_n$:

$$(3.2) \quad \text{grad } f(\mathbf{U}) = \mathbf{U} \mathbf{\Lambda}(\mathbf{U}) \in \mathbf{T}_U \mathcal{U}_n,$$

where

$$(3.3) \quad \mathbf{\Lambda}(\mathbf{U}) = \frac{\mathbf{U}^{\text{H}} \nabla f(\mathbf{U}) - (\nabla f(\mathbf{U}))^{\text{H}} \mathbf{U}}{2} = \mathbf{U}^{\text{H}} \frac{\partial f}{\partial \mathbf{U}^*}(\mathbf{U}) - \left(\frac{\partial f}{\partial \mathbf{U}^*}(\mathbf{U}) \right)^{\text{H}} \mathbf{U}.$$

Note that $\mathbf{\Lambda}(\mathbf{U})$ is a skew-Hermitian matrix, *i.e.*,

$$(3.4) \quad \Lambda(\mathbf{U})_{i,j} = -(\Lambda(\mathbf{U})_{j,i})^*, \quad 1 \leq i, j \leq n.$$

In what follows, we will use the exponential map [3, p.102] $\text{Exp}_U : \mathbf{T}_U \mathcal{U}_n \rightarrow \mathcal{U}_n$, which maps 1-dimensional lines in the tangent space to geodesics and is given by

$$(3.5) \quad \text{Exp}_U(\mathbf{U} \Omega) = \mathbf{U} \exp(\Omega),$$

where $\exp(\cdot)$ is the matrix exponential. We will frequently use the following relation between Exp_U and the Riemannian gradient. For any $\Delta \in \mathbf{T}_U \mathcal{U}_n$, we have

$$(3.6) \quad \langle \Delta, \text{grad } f(\mathbf{U}) \rangle_{\Re} = \left(\frac{d}{dt} f(\text{Exp}_U(t \Delta)) \right) \Big|_{t=0}.$$

3.3. Riemannian Hessian and stationary points. This subsection is not mandatory to formulate the Jacobi-G algorithm, but we keep it here for convenience. For a Riemannian manifold \mathcal{M} and a C^2 function $f : \mathcal{M} \rightarrow \mathbb{R}$, the Riemannian Hessian at $x \in \mathcal{M}$ is either defined as a linear map $\mathbf{T}_x\mathcal{M} \rightarrow \mathbf{T}_x\mathcal{M}$ or as a bilinear form on $\mathbf{T}_x\mathcal{M}$; the usual definition is based on the Riemannian connection [3, p.105].

For our purposes, for simplicity, we assume that the exponential map $\text{Exp}_x : \mathbf{T}_x\mathcal{M} \rightarrow \mathcal{M}$ is given, and adopt the following definition based on [3, Proposition 5.5.4]. The Riemannian Hessian $\text{Hess}_x f$ is the linear map $\mathbf{T}_x\mathcal{M} \rightarrow \mathbf{T}_x\mathcal{M}$ defined by

$$\text{Hess}_x f = \text{Hess}_{\mathbf{0}_x}(f \circ \text{Exp}_x),$$

where $\mathbf{0}_x$ is the origin in the tangent space, and $\text{Hess}_{\mathbf{0}_x} g$ is the Euclidean Hessian of $g : \mathbf{T}_x\mathcal{M} \rightarrow \mathbb{R}$. Hence, similarly to (3.6), there is the following expression for the values of Riemannian Hessian as a quadratic form at $\Delta \in \mathbf{T}_x\mathcal{M}$:

$$(3.7) \quad \langle \text{Hess}_x f[\Delta], \Delta \rangle_{\mathbb{R}} = \left(\frac{d^2}{dt^2} f(\text{Exp}_x(t\Delta)) \right) \Big|_{t=0}.$$

The Riemannian Hessian gives well-known necessary and sufficient conditions of local extrema (see, for example, [35, Theorem 4.1]).

- If x is a local maximum of f on \mathcal{M} , then $\text{Hess}_x f \preceq 0$ (negative semidefinite);
- If $\text{grad} f(x) = 0$ and $\text{Hess}_x f \prec_{\mathbf{T}_x\mathcal{M}} 0$ (i.e., $\text{Hess}_x f \preceq 0$ and $\text{rank}\{\text{Hess}_x f\} = \dim(\mathcal{M})$), then f has a strict local maximum at x .

Finally, we distinguish stationary points with nonsingular Riemannian Hessian.

DEFINITION 3.1. *A stationary point ($x \in \mathcal{M}$, $\text{grad} f(x) = 0$) is called non-degenerate if $\text{Hess}_x f$ is nonsingular on $\mathbf{T}_x\mathcal{M}$.*

3.4. Jacobi-G algorithm. We are now in a position to formulate a general-purpose Jacobi-G algorithm, which is a generalization of the algorithm proposed in [26]. The main ideas behind the algorithm are:

- optimize the cost function by successive Givens transformations;
- choose a *gradient based order* of pairs (well-aligned with $\text{grad} f(\cdot)$).

Algorithm 3.1 General Jacobi-G algorithm

Input: A differentiable function $f : \mathcal{U}_n \rightarrow \mathbb{R}^+$ defined in a neighborhood of \mathcal{U}_n , a positive constant $0 < \delta < \sqrt{2}/n$, a starting point \mathbf{U}_0 .

Output: Sequence of iterations \mathbf{U}_k .

- **For** $k = 1, 2, \dots$ until a stopping criterion is satisfied do
- Choose an index pair (i_k, j_k) satisfying

$$(3.8) \quad \|\text{grad} h_{(i_k, j_k), \mathbf{U}_{k-1}}(\mathbf{I}_2)\| \geq \delta \|\text{grad} f(\mathbf{U}_{k-1})\|.$$

- Find Ψ_k that maximizes $h_k(\Psi) \stackrel{\text{def}}{=} h_{(i_k, j_k), \mathbf{U}_{k-1}}(\Psi)$.
 - Update $\mathbf{U}_k = \mathbf{U}_{k-1} \mathbf{G}^{(i_k, j_k, \Psi_k)}$.
 - **End for**
-

Now we show that it is always possible to choose the index pair such that the inequality (3.8) is satisfied. For this, we first show how to compute the Riemannian gradient of $h_{(i,j), \mathbf{U}}$ based on that of f .

LEMMA 3.2. *The Riemannian gradient of $h_{(i,j),\mathbf{U}}$ at the identity matrix \mathbf{I}_2 is a submatrix of the matrix $\mathbf{\Lambda}(\mathbf{U})$ defined in (3.3):*

$$(3.9) \quad \text{grad } h_{(i,j),\mathbf{U}}(\mathbf{I}_2) = \mathcal{P}_{i,j}(\mathbf{\Lambda}(\mathbf{U})) = \begin{bmatrix} \mathbf{\Lambda}(\mathbf{U})_{ii} & \mathbf{\Lambda}(\mathbf{U})_{ij} \\ \mathbf{\Lambda}(\mathbf{U})_{ji} & \mathbf{\Lambda}(\mathbf{U})_{jj} \end{bmatrix}.$$

Proof. Denote $h = h_{(i,j),\mathbf{U}}$ for simplicity. For any $\mathbf{\Delta} \in \mathbf{T}_{\mathbf{I}_2}\mathcal{U}_2$, by (3.6)

$$\begin{aligned} \langle \mathbf{\Delta}, \text{grad } h(\mathbf{I}_2) \rangle_{\mathbb{R}} &= \left(\frac{d}{dt} h(\text{Exp}_{\mathbf{I}_2}(t\mathbf{\Delta})) \right) \Big|_{t=0} = \left(\frac{d}{dt} f(\mathbf{U}\mathbf{G}^{(i,j,\text{Exp}_{\mathbf{I}_2}(t\mathbf{\Delta}))}) \right) \Big|_{t=0} \\ &= \left(\frac{d}{dt} f(\text{Exp}_{\mathbf{U}}(\mathbf{U}\mathcal{P}_{i,j}^{\top}(\mathbf{\Delta})t)) \right) \Big|_{t=0} = \langle \mathbf{U}\mathcal{P}_{i,j}^{\top}(\mathbf{\Delta}), \text{grad } f(\mathbf{U}) \rangle_{\mathbb{R}} = \langle \mathbf{\Delta}, \mathcal{P}_{i,j}(\mathbf{\Lambda}(\mathbf{U})) \rangle_{\mathbb{R}}, \end{aligned}$$

which completes the proof. \square

COROLLARY 3.3. *Let f and $h_{(i,j),\mathbf{U}}$ be as in Lemma 3.2. Then*

$$\max_{1 \leq i < j \leq n} \|\text{grad } h_{(i,j),\mathbf{U}}(\mathbf{I}_2)\| \geq \frac{\sqrt{2}}{n} \|\text{grad } f(\mathbf{U})\|.$$

Proof. By (3.2) and Lemma 3.2, we see that

$$\|\text{grad } f(\mathbf{U})\|^2 = \|\mathbf{\Lambda}(\mathbf{U})\|^2 = \sum_{i,j=1}^{n,n} |\mathbf{\Lambda}(\mathbf{U})_{i,j}|^2 \leq \frac{n^2}{2} \max_{1 \leq i < j \leq n} \|\text{grad } h_{(i,j),\mathbf{U}}(\mathbf{I}_2)\|^2. \quad \square$$

4. Finding Jacobi rotations and derivatives.

4.1. Derivatives for scale-invariant functions. In this section, we consider the functions $f : \mathcal{U}_n \rightarrow \mathbb{R}$ that are invariant with respect to scaling of columns of \mathbf{U} .

LEMMA 4.1. *For f satisfying (2.9) and all \mathbf{U} the main diagonal of $\mathbf{\Lambda}(\mathbf{U})$ is zero.*

Proof. For the matrix

$$(4.1) \quad \mathbf{\Omega}_k = k \begin{bmatrix} & & & k \\ & & & \vdots \\ & \mathbf{0} & & \mathbf{0} \\ \cdots & & i & \\ \mathbf{0} & & & \mathbf{0} \end{bmatrix}.$$

we have that

$$\langle \mathbf{\Omega}_k, \mathbf{\Lambda}(\mathbf{U}) \rangle_{\mathbb{R}} = \langle \mathbf{U}\mathbf{\Omega}_k, \text{grad } f(\mathbf{U}) \rangle_{\mathbb{R}} = \left(\frac{d}{dt} f(\mathbf{U}e^{t\mathbf{\Omega}_k}) \right) \Big|_{t=0} = 0.$$

Since $\mathbf{\Lambda}(\mathbf{U})$ is skew-Hermitian, the proof is complete. \square

Next we show that the Riemannian Hessian of functions satisfying invariance property (2.9) (with matrix \mathbf{S} given in (2.10)) is rank-deficient at stationary points.

LEMMA 4.2. *Assume that $f : \mathcal{U}_n \rightarrow \mathbb{R}$ satisfies the invariance property (2.9).*

(i) *For any \mathbf{U} , $\mathbf{\Lambda} = \mathbf{\Lambda}(\mathbf{U})$ and*

$$(4.2) \quad \mathbf{Z}_k = [\mathbf{0} \quad \cdots \quad \mathbf{0} \quad i\mathbf{u}_k \quad \mathbf{0} \quad \cdots \quad \mathbf{0}] = \mathbf{U}\mathbf{\Omega}_k \in \mathbf{T}_{\mathbf{U}}\mathcal{U}_n,$$

we have that

$$\text{Hess}_{\mathcal{U}} f[\mathbf{Z}_k] = \frac{i\mathbf{U}}{2} \begin{bmatrix} 0 & \cdots & 0 & \Lambda_{1,k} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \Lambda_{k-1,k} & 0 & \cdots & 0 \\ -\Lambda_{k,1} & \cdots & -\Lambda_{k,k-1} & 0 & -\Lambda_{k,k+1} & \cdots & \Lambda_{k,n} \\ 0 & \cdots & 0 & \Lambda_{k+1,k} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \Lambda_{n,k} & 0 & \cdots & 0 \end{bmatrix},$$

(ii) If, in addition, \mathbf{U} is a stationary point of f , then $\text{rank}\{\text{Hess}_{\mathcal{U}} f\} \leq n(n-1)$, and $\text{Hess}_{\mathcal{U}} f[\mathbf{Z}_k] = \mathbf{0}$ i.e., all the matrices \mathbf{Z}_k are in the kernel of $\text{Hess}_{\mathcal{U}} f$.

Proof. (i) Recall that the parallel transport on \mathcal{U}_n along the geodesic in the direction $\mathbf{Z} = \mathbf{U}\boldsymbol{\Omega} \in \mathbf{T}_{\mathcal{U}}\mathcal{U}_n$ is given [19, (2.18)]¹, [23, Chap. 2, Ex. A.6] by

$$\tau_{t\mathbf{Z}_k} : \mathbf{T}_{\mathcal{U}}\mathcal{U}_n \rightarrow \mathbf{T}_{\text{Exp}_{\mathcal{U}}(t\mathbf{Z}_k)}\mathcal{U}_n, \quad \tau_{t\mathbf{Z}}(\mathbf{Y}) = \mathbf{U}e^{\frac{t\boldsymbol{\Omega}}{2}}\mathbf{U}^{\text{H}}\mathbf{Y}e^{\frac{t\boldsymbol{\Omega}}{2}},$$

and its inverse is then given by

$$\tau_{t\mathbf{Z}}^{-1}(\mathbf{V}) = \mathbf{U}e^{-\frac{t\boldsymbol{\Omega}}{2}}\mathbf{U}^{\text{H}}\mathbf{V}e^{-\frac{t\boldsymbol{\Omega}}{2}}.$$

From the link between parallel transport [3, (8.1)] and Riemannian connection,

$$\begin{aligned} \text{Hess}_{\mathcal{U}} f[\mathbf{Z}_k] &= \left(\frac{d}{dt} \tau_{t\mathbf{Z}}^{-1}(\text{grad } f(\text{Exp}_{\mathcal{U}}(t\mathbf{Z}_k))) \right) \Big|_{t=0} \\ &= \left(\frac{d}{dt} \mathbf{U}e^{-\frac{t\boldsymbol{\Omega}_k}{2}}\mathbf{U}^{\text{H}} \text{grad } f(\mathbf{U}e^{t\boldsymbol{\Omega}_k})e^{-\frac{t\boldsymbol{\Omega}_k}{2}} \right) \Big|_{t=0} = \frac{\mathbf{U}}{2} (-\boldsymbol{\Omega}_k \boldsymbol{\Lambda}(\mathbf{U}) + \boldsymbol{\Lambda}(\mathbf{U}) \boldsymbol{\Omega}_k). \end{aligned}$$

(ii) Since \mathbf{U} is a stationary point $\boldsymbol{\Lambda}(\mathbf{U}) = 0$, and $\text{Hess}_{\mathcal{U}} f[\mathbf{Z}_k] = \mathbf{0}$. Finally, since $\{\mathbf{Z}_k\}_{k=1}^n$, are linearly independent, $\text{rank}\{\text{Hess}_{\mathcal{U}} f\} \leq n(n-1)$. \square

Finally, we show that the Riemannian Hessian for the Givens transformations can be computed as a submatrix of the Riemannian Hessian.

PROPOSITION 4.3. *Let $h_{(i,j),\mathcal{U}}$ be as in (2.8), the projection operator $\mathcal{P}_{i,j}$ be as in (2.1), and $\mathcal{P}_{i,j}^{\text{T}}$ be its adjoint operator. Then*

$$\text{Hess}_{\mathcal{I}_2} h_{(i,j),\mathcal{U}} = \mathcal{P}_{i,j} \circ ((\mathbf{I}_n \otimes \mathbf{U}^{\text{H}}) \text{Hess}_{\mathcal{U}} f(\mathbf{I}_n \otimes \mathbf{U})) \circ \mathcal{P}_{i,j}^{\text{T}}.$$

Proof. We denote $h = h_{(i,j),\mathcal{U}}$ for simplicity. We need to check the equation only for the elements of the form (3.7). Similarly to the proof of Lemma 3.2, we have

$$\begin{aligned} \langle \text{Hess}_{\mathcal{I}_2} h[\boldsymbol{\Delta}], \boldsymbol{\Delta} \rangle_{\mathbb{R}} &= \left(\frac{d^2}{dt^2} h(\text{Exp}_{\mathcal{I}_2}(t\boldsymbol{\Delta})) \right) \Big|_{t=0} = \left(\frac{d^2}{dt^2} f(\mathbf{U}\mathbf{G}^{(i,j),\text{Exp}_{\mathcal{I}_2}(\boldsymbol{\Delta}t)}) \right) \Big|_{t=0} \\ &= \left(\frac{d^2}{dt^2} f(\text{Exp}_{\mathcal{U}}(\mathbf{U}\mathcal{P}_{i,j}^{\text{T}}(\boldsymbol{\Delta})t)) \right) \Big|_{t=0} = \langle \mathbf{U}\mathcal{P}_{i,j}^{\text{T}}(\boldsymbol{\Delta}), \text{Hess}_{\mathcal{U}} f[\mathbf{U}\mathcal{P}_{i,j}^{\text{T}}(\boldsymbol{\Delta})] \rangle_{\mathbb{R}}, \end{aligned}$$

which completes the proof. \square

¹Note that [19, (2.18)] was given for \mathcal{O}_n , but it is straightforward to show that the same expression is valid for \mathcal{U}_n , following [23, Chap. 2, Ex. A.6].

4.2. Derivatives for cost functions expressed via quadratic forms. In this subsection, we find the directional derivatives of the cost functions expressed via quadratic forms (2.13), as well as the cost functions in Subsection 2.2.

LEMMA 4.4. *Let $h_{(i,j),\mathbf{U}}$ be as in (2.13) and satisfy (2.11). Then*

$$\text{grad } h_{(i,j),\mathbf{U}}(\mathbf{I}_2) = 2 \begin{bmatrix} 0 & \mathbf{\Gamma}_{12}^{(i,j,\mathbf{U})} + i\mathbf{\Gamma}_{13}^{(i,j,\mathbf{U})} \\ -\mathbf{\Gamma}_{12}^{(i,j,\mathbf{U})} + i\mathbf{\Gamma}_{13}^{(i,j,\mathbf{U})} & 0 \end{bmatrix}.$$

Proof. Denote $h = h_{(i,j),\mathbf{U}}$ and $\mathbf{\Gamma} = \mathbf{\Gamma}^{(i,j,\mathbf{U})}$. By (3.4) and Lemma 4.1, we see that $\text{grad } h(\mathbf{I}_2)$ is skew-Hermitian, i.e., it can be decomposed as

$$\text{grad } h(\mathbf{I}_2) = 2\omega_1 \mathbf{\Delta}_1 + 2\omega_2 \mathbf{\Delta}_2,$$

where $\{\mathbf{\Delta}_k\}_{k=1}^4$ is the following orthogonal basis of $\mathbf{T}_{\mathbf{I}_2}\mathcal{U}_2$:

$$(4.3) \quad \mathbf{\Delta}_1 = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \quad \mathbf{\Delta}_2 = \begin{bmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{bmatrix}, \quad \mathbf{\Delta}_3 = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{\Delta}_4 = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}.$$

Since $\|\mathbf{\Delta}_1\|^2 = \|\mathbf{\Delta}_2\|^2 = 1/2$ and $\|\mathbf{\Delta}_3\| = \|\mathbf{\Delta}_4\| = 1$, we have

$$\omega_k = \langle \mathbf{\Delta}_k, \text{grad } h(\mathbf{I}_2) \rangle_{\mathfrak{R}} = \left(\frac{d}{dt} h(e^{t\mathbf{\Delta}_k}) \right) \Big|_{t=0}$$

for $1 \leq k \leq 4$. On the other hand, we have

$$\begin{aligned} h(e^{t\mathbf{\Delta}_1}) &= h \left(\begin{bmatrix} \cos \frac{t}{2} & -\sin \frac{t}{2} \\ \sin \frac{t}{2} & \cos \frac{t}{2} \end{bmatrix} \right) = \tilde{h}(\cos t, -\sin t, 0), \\ h(e^{t\mathbf{\Delta}_2}) &= h \left(\begin{bmatrix} \cos \frac{t}{2} & -i \sin \frac{t}{2} \\ -i \sin \frac{t}{2} & \cos \frac{t}{2} \end{bmatrix} \right) = \tilde{h}(\cos t, 0, -\sin t), \end{aligned}$$

where $\tilde{h}(\mathbf{v}) = \mathbf{v}^\top \mathbf{\Gamma} \mathbf{v}$. Since $\nabla \tilde{h}(\mathbf{v}) = 2\mathbf{\Gamma} \mathbf{v}$, we have

$$\omega_1 = -\frac{\partial \tilde{h}}{\partial v_2}(1, 0, 0) = -2\Gamma_{21}, \quad \omega_2 = -\frac{\partial \tilde{h}}{\partial v_3}(1, 0, 0) = -2\Gamma_{31},$$

which completes the proof. \square

Next, we find the expressions of the Riemannian Hessians in the case of Jacobi rotations. For each pair of indices (i, j) , we define a 2×2 matrix as follows:

$$\mathfrak{D}_{\mathbf{U}}^{(i,j)} = 2 \left(\begin{bmatrix} \Gamma_{2,2} & \Gamma_{2,3} \\ \Gamma_{3,2} & \Gamma_{3,3} \end{bmatrix} - \Gamma_{1,1} \mathbf{I}_2 \right).$$

LEMMA 4.5. *Let $h_{(i,j),\mathbf{U}}$ and $\mathbf{\Gamma} = \mathbf{\Gamma}^{(i,j,\mathbf{U})}$ be as in (2.13) satisfying (2.11). Take the basis in $\mathbf{T}_{\mathbf{I}_2}\mathcal{U}_2$ as in (4.3). Then*

(i) *The 2×2 leading principal submatrix of the Riemannian Hessian of $h_{(i,j),\mathbf{U}}$ is*

$$(\text{Hess}_{\mathbf{I}_2} h_{(i,j),\mathbf{U}})_{1:2,1:2} = \mathfrak{D}_{\mathbf{U}}^{(i,j)}.$$

(ii) *If, in addition $\text{grad } h_{(i,j),\mathbf{U}}(\mathbf{I}_2) = 0$, then the Riemannian Hessian becomes*

$$\text{Hess}_{\mathbf{I}_2} h_{(i,j),\mathbf{U}} = \begin{bmatrix} \mathfrak{D}_{\mathbf{U}}^{(i,j)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Proof. (i) We denote $h = h_{(i,j),\mathbf{U}}$ for simplicity. Take $\mathbf{\Omega} = \alpha_1 \mathbf{\Delta}_1 + \alpha_2 \mathbf{\Delta}_2$, where $\alpha_1, \alpha_2 \in \mathbb{R}$ satisfy $\alpha_1^2 + \alpha_2^2 = 1$. By [3, (5.32)], we have that

$$\langle \mathbf{\Omega}, \text{Hess}_{\mathbf{I}_2} h[\mathbf{\Omega}] \rangle_{\mathbb{R}} = \left(\frac{d^2}{dt^2} h(e^{t\mathbf{\Omega}}) \right) \Big|_{t=0}.$$

Note that

$$(4.4) \quad h(e^{t\mathbf{\Omega}}) = \tilde{h}(\cos t, -\alpha_1 \sin t, -\alpha_2 \sin t).$$

It follows that

$$\frac{d}{dt} h(e^{t\mathbf{\Omega}}) = -2 \begin{bmatrix} \sin t & \alpha_1 \cos t & \alpha_2 \cos t \end{bmatrix} \mathbf{\Gamma} \begin{bmatrix} \cos t & -\alpha_1 \sin t & -\alpha_2 \sin t \end{bmatrix}^\top,$$

and thus

$$\left(\frac{d^2}{dt^2} h(e^{t\mathbf{\Omega}}) \right) \Big|_{t=0} = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \mathfrak{D}_{\mathbf{U}}^{(i,j)} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix},$$

which completes the proof.

(ii) Follows from (i) and Lemma 4.2, since $\text{Hess}_{\mathbf{I}_2} h_{(i,j),\mathbf{U}}[\mathbf{\Delta}_k] = \mathbf{0}$ for $k = 3, 4$. \square

COROLLARY 4.6. *If \mathbf{I}_2 is a local maximizer of $h_{(i,j),\mathbf{U}}$, then $\mathfrak{D}_{\mathbf{U}}^{(i,j)} \preceq 0$.*

Remark 4.7. The matrix $\mathfrak{D}_{\mathbf{U}}^{(i,j)}$ is negative definite if and only if

$$\Gamma_{11} > \lambda_{\max} \left(\begin{bmatrix} \Gamma_{22} & \Gamma_{23} \\ \Gamma_{23} & \Gamma_{33} \end{bmatrix} \right).$$

If, in addition, $\text{grad } h_{(i,j),\mathbf{U}}(\mathbf{I}_2) = \mathbf{0}$, this is equivalent to saying that $\lambda_1(\Gamma) > \lambda_2(\Gamma)$ (i.e., the first two eigenvalues are separated) and $\Gamma_{11} = \lambda_1(\Gamma)$.

4.3. Riemannian gradients for cost functions of interest. Lemma 2.3 and Lemma 3.2 allow us to find immediately the Riemannian gradients of all the cost functions in Subsection 2.2.

COROLLARY 4.8. (i) *For the cost function in (2.3), we have that*

$$(4.5) \quad \mathbf{\Lambda}(\mathbf{U}) = 2 \sum_{\ell=1}^L \begin{bmatrix} 0 & W_{12}^{(\ell)}(W_{22}^{(\ell)} - W_{11}^{(\ell)}) & \cdots & W_{1n}^{(\ell)}(W_{nn}^{(\ell)} - W_{11}^{(\ell)}) \\ -W_{21}^{(\ell)}(W_{22}^{(\ell)} - W_{11}^{(\ell)}) & 0 & \cdots & W_{2n}^{(\ell)}(W_{nn}^{(\ell)} - W_{22}^{(\ell)}) \\ \vdots & \vdots & \ddots & \vdots \\ -W_{n1}^{(\ell)}(W_{nn}^{(\ell)} - W_{11}^{(\ell)}) & -W_{n2}^{(\ell)}(W_{nn}^{(\ell)} - W_{22}^{(\ell)}) & \cdots & 0 \end{bmatrix}.$$

(ii) *For the cost function in (2.5), we have that for $1 \leq i, j \leq n$*

$$(4.6) \quad \Lambda(\mathbf{U})_{ij} = 2(\mathcal{W}_{jj}\mathcal{W}_{ji}^* + \frac{1}{2}\mathcal{W}_{jj}^*\mathcal{W}_{ij} - \mathcal{W}_{ii}^*\mathcal{W}_{ij} - \frac{1}{2}\mathcal{W}_{ii}\mathcal{W}_{ji}^*).$$

(iii) *For the cost function in (2.7), we have that*

$$(4.7) \quad \mathbf{\Lambda}(\mathbf{U}) = 2 \begin{bmatrix} 0 & \mathcal{V}_{1222} - \mathcal{V}_{2111}^* & \cdots & \mathcal{V}_{1nnn} - \mathcal{V}_{n111}^* \\ \mathcal{V}_{2111} - \mathcal{V}_{1222}^* & 0 & \cdots & \mathcal{V}_{2nnn} - \mathcal{V}_{n222}^* \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{V}_{n111} - \mathcal{V}_{1nnn}^* & \mathcal{V}_{n222} - \mathcal{V}_{2nnn}^* & \cdots & 0 \end{bmatrix}.$$

In fact, the form of the Riemannian gradient can be also derived for other cost functions, which can also be called contrast functions [15, 16], and that have the form

$$(4.8) \quad f(\mathbf{U}) = \sum_{k=1}^n \gamma(\mathbf{u}_k),$$

where $\gamma(\mathbf{u})$ are real-valued.

Remark 4.9. For contrast-like functions, if

$$(4.9) \quad \gamma(z\mathbf{u}) = \gamma(\mathbf{u}) \text{ for any } \mathbf{u} \in \mathbb{C}^n, z \in \mathbb{C}, |z| = 1,$$

then f satisfies the property (2.9) and $\Lambda(\mathbf{U})$ has zeros on its diagonal by Lemma 4.1.

Next, we show how to find Riemannian gradients for contrast-like functions. Since

$$(4.10) \quad \frac{\partial f}{\partial \mathbf{U}^*}(\mathbf{U}) = \left(\frac{\partial \gamma}{\partial \mathbf{u}^*}(\mathbf{u}_1), \dots, \frac{\partial \gamma}{\partial \mathbf{u}^*}(\mathbf{u}_n) \right),$$

we just need to compute derivatives for γ , which is easy to do for multilinear forms.

PROPOSITION 4.10. *Let $\mathcal{A} \in \mathbb{C}^{n \times \dots \times n}$ be of order d , and consider the form*

$$(4.11) \quad g_{\mathcal{A}}(\mathbf{u}) = \mathcal{A} \bullet_1 \mathbf{u}^* \cdots \bullet_{d_1} \mathbf{u}^* \bullet_{d_1+1} \mathbf{u} \cdots \bullet_d \mathbf{u}.$$

Then it holds that

$$\begin{aligned} \frac{\partial g_{\mathcal{A}}}{\partial \mathbf{u}^*}(\mathbf{u}) &= \sum_{k=1}^{d_1} \mathcal{A} \bullet_1 \mathbf{u}^* \cdots \cancel{\bullet_k \mathbf{u}^*} \cdots \bullet_{d_1} \mathbf{u}^* \bullet_{d_1+1} \mathbf{u} \cdots \bullet_d \mathbf{u}, \\ \frac{\partial g_{\mathcal{A}}}{\partial \mathbf{u}}(\mathbf{u}) &= \sum_{k=1}^{d_2} \mathcal{A} \bullet_1 \mathbf{u}^* \cdots \bullet_{d_1} \mathbf{u}^* \bullet_{d_1+1} \mathbf{u} \cdots \cancel{\bullet_{d_1+k} \mathbf{u}} \cdots \bullet_d \mathbf{u}. \end{aligned}$$

Proof. The result follows by product differentiation and identities [24, Table IV] \square

$$\frac{\partial \mathbf{u}^H \mathbf{a}}{\partial \mathbf{u}}(\mathbf{u}) = \mathbf{0}, \quad \frac{\partial \mathbf{u}^H \mathbf{a}}{\partial \mathbf{u}^*} = \mathbf{a}, \quad \frac{\partial \mathbf{u}^T \mathbf{a}}{\partial \mathbf{u}} = \mathbf{a}, \quad \frac{\partial \mathbf{u}^T \mathbf{a}}{\partial \mathbf{u}^*} = \mathbf{0}.$$

PROPOSITION 4.11. *For $\gamma(\mathbf{u}) = |g_{\mathcal{A}}(\mathbf{u})|^2$, with $g_{\mathcal{A}}$ as in (4.11), it holds that*

$$\begin{aligned} \frac{\partial \gamma}{\partial \mathbf{u}^*} &= (g_{\mathcal{A}}(\mathbf{u}))^* \sum_{k=1}^{d_1} \mathcal{A} \bullet_1 \mathbf{u}^* \cdots \cancel{\bullet_k \mathbf{u}^*} \cdots \bullet_{d_1} \mathbf{u}^* \bullet_{d_1+1} \mathbf{u} \cdots \bullet_d \mathbf{u} \\ &+ (g_{\mathcal{A}}(\mathbf{u})) \sum_{k=1}^{d_2} \mathcal{A}^* \bullet_1 \mathbf{u} \cdots \bullet_{d_1} \mathbf{u} \bullet_{d_1+1} \mathbf{u}^* \cdots \cancel{\bullet_{d_1+k} \mathbf{u}^*} \cdots \bullet_d \mathbf{u}^*. \end{aligned}$$

Proof. The result follows² from Proposition 4.10 and the fact that \square

$$\gamma(\mathbf{u}) = (\mathcal{A} \otimes \mathcal{A}^*) \bullet_1 \mathbf{u}^* \cdots \bullet_{d_1} \mathbf{u}^* \bullet_{d_1+1} \mathbf{u} \cdots \bullet_{d+d_1} \mathbf{u} \bullet_{d+d_1+1} \mathbf{u}^* \cdots \bullet_{2d} \mathbf{u}^*.$$

²An alternative proof can be derived by combining Proposition 4.10, the rule of differentiation of composition [24, Theorem 1], and the fact that $d|z|^2 = z^* dz + z dz^*$.

Remark 4.12. **Proposition 4.11** can be used as an alternative (and simpler) way to derive the gradients in **Corollary 4.8**. For example, cost function (2.5) is a contrast-like function (4.8) for $\gamma(\mathbf{u}) = |\mathcal{A} \bullet_1 \mathbf{u}^* \bullet_2 \mathbf{u} \bullet_3 \mathbf{u}|^2$. By **Proposition 4.11**, we have that

$$\frac{\partial \gamma}{\partial \mathbf{u}^*}(\mathbf{u}) = 2(\mathcal{A} \bullet_1 \mathbf{u}^* \bullet_2 \mathbf{u} \bullet_3 \mathbf{u}) \mathcal{A}^* \bullet_1 \mathbf{u} \bullet_2 \mathbf{u}^* + (\mathcal{A} \bullet_1 \mathbf{u}^* \bullet_2 \mathbf{u} \bullet_3 \mathbf{u})^* \mathcal{A} \bullet_1 \mathbf{u}^* \bullet_2 \mathbf{u}.$$

Then by (4.10), we get

$$\begin{aligned} \left(\mathbf{U}^H \frac{\partial f}{\partial \mathbf{U}^*}(\mathbf{U}) \right)_{ij} &= \mathbf{u}_i^H \frac{\partial \gamma}{\partial \mathbf{u}^*}(\mathbf{u}_j) = 2(\mathcal{A} \bullet_1 \mathbf{u}_j^* \bullet_2 \mathbf{u}_j \bullet_3 \mathbf{u}_j)(\mathcal{A} \bullet_1 \mathbf{u}_j^* \bullet_2 \mathbf{u}_j \bullet_3 \mathbf{u}_i)^* \\ &\quad + (\mathcal{A} \bullet_1 \mathbf{u}_j^* \bullet_2 \mathbf{u}_j \bullet_3 \mathbf{u}_j)^* \mathcal{A} \bullet_1 \mathbf{u}_j^* \bullet_2 \mathbf{u}_j \bullet_3 \mathbf{u}_i. \\ &= 2\mathcal{W}_{jjj} \mathcal{W}_{jji}^* + \mathcal{W}_{jjj}^* \mathcal{W}_{jji}. \end{aligned}$$

Finally, from the definition (3.3) of $\mathbf{\Lambda}(\mathbf{U})$ we get the expression in (4.6).

The same reasoning applies to cost functions (2.3) and (2.7).

- The cost function (2.3) corresponds to $\gamma(\mathbf{u}) = \sum_{\ell=1}^L |\mathbf{u}^H \mathbf{A}^{(\ell)} \mathbf{u}|^2$, for which

$$\frac{\partial \gamma}{\partial \mathbf{u}^*}(\mathbf{u}) = \sum_{\ell=1}^L 2|\mathbf{u}^H \mathbf{A}^{(\ell)} \mathbf{u}| \mathbf{A}^{(\ell)} \mathbf{u}, \text{ and } \left(\mathbf{U}^H \frac{\partial f}{\partial \mathbf{U}^*}(\mathbf{U}) \right)_{ij} = \sum_{\ell=1}^L 2W_{ii}^{(\ell)} W_{ij}^{(\ell)},$$

by **Proposition 4.11**, which agrees with (4.5).

- The function (2.7) corresponds to $\gamma(\mathbf{u}) = \mathcal{B} \bullet_1 \mathbf{u}^* \bullet_2 \mathbf{u}^* \bullet_3 \mathbf{u} \bullet_4 \mathbf{u}$, for which

$$\frac{\partial \gamma}{\partial \mathbf{u}^*}(\mathbf{u}) = 2\mathcal{B} \bullet_2 \mathbf{u}^* \bullet_3 \mathbf{u} \bullet_4 \mathbf{u} \quad \text{and} \quad \left(\mathbf{U}^H \frac{\partial f}{\partial \mathbf{U}^*}(\mathbf{U}) \right)_{ij} = \mathcal{W}_{ijjj}$$

by **Proposition 4.10**, which agrees with **Equation (4.7)**.

5. Weak convergence results.

5.1. Global rates of convergence of descent algorithms on manifolds.

We first recall the result presented in [9] on convergence of descent algorithms. Although stated initially for retraction-based algorithms, it is valid for any descent algorithms (we provide the sketch of the proof for completeness).

THEOREM 5.1 ([9, Theorem 2.5]). *Let $f : \mathcal{M} \rightarrow \mathbb{R}$ be bounded from below by f^* . Suppose that, for a sequence of³ x_k , there exists $c > 0$ such that*

$$(5.1) \quad f(x_{k-1}) - f(x_k) \geq c \|\text{grad } f(x_k)\|^2.$$

Then

(i) $\|\text{grad } f(x_k)\| \rightarrow 0$ as $k \rightarrow \infty$;

(ii) We can find an x_k with $\|\text{grad } f(x_k)\| < \varepsilon$ and $f(x_k) \leq f(x_0)$ in at most

$$K_\varepsilon = \left\lceil \frac{f(x_0) - f^*}{c} \frac{1}{\varepsilon^2} \right\rceil$$

iterations; i.e., there exists $k \leq K_\varepsilon$ such that $\|\text{grad } f(x_k)\| < \varepsilon$.

³Note that in the original formulation of [9, Theorem 2.5] x_k were chosen as retractions of some vectors in $\mathbf{T}_{x_{k-1}}$. However, it is easy to see that this condition is not needed in the proof.

Sketch of the proof. The proof follows from an inequality for telescopic sums \square

$$f(x_0) - f^* \geq f(x_0) - f(x_K) = \sum_{k=1}^K f(x_{k-1}) - f(x_k) \geq cK \min_{1 \leq k \leq K} \|\text{grad } f(x_k)\|^2.$$

In order to check the descent condition (5.1), the following lemma about retractions is often useful (we will also use it in this paper).

DEFINITION 5.2. ([3, Definition 4.4.1]) A retraction on a manifold \mathcal{M} is a smooth mapping Retr from the tangent bundle $\mathbf{T}\mathcal{M}$ to \mathcal{M} with the following properties. Let $\text{Retr}_x : \mathbf{T}_x\mathcal{M} \rightarrow \mathcal{M}$ denote the restriction of Retr to the tangent vector space $\mathbf{T}_x\mathcal{M}$.

- (i) $\text{Retr}_x(\mathbf{0}_x) = x$, where $\mathbf{0}_x$ is the zero vector in $\mathbf{T}_x\mathcal{M}$;
- (ii) The differential of Retr_x at $\mathbf{0}_x$, $\text{DRetr}_x(\mathbf{0}_x)$, is the identity map.

LEMMA 5.3 ([9, Lemma 2.7]). Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a compact Riemannian submanifold. Let Retr be a retraction on \mathcal{M} . Suppose that $f : \mathcal{M} \rightarrow \mathbb{R}$ has Lipschitz continuous gradient in the convex hull of \mathcal{M} . Then there exists $L \geq 0$ such that for all $x \in \mathcal{M}$ and $\eta \in \mathbf{T}_x\mathcal{M}$, it holds that

$$(5.2) \quad |f(\text{Retr}_x(\eta)) - (f(x) + \langle \eta, \text{grad } f(x) \rangle)| \leq \frac{L}{2} \|\eta\|^2,$$

i.e., $f(\text{Retr}_x(\eta))$ is uniformly well approximated by its first order approximation.

5.2. Convergence of Jacobi-G algorithm to stationary points. We will show in this subsection that the iterations in Algorithm 3.1 are a special case of the iterations in Theorem 5.1, and the convergence results of Theorem 5.1 apply.

PROPOSITION 5.4. Let $f : \mathcal{U}_n \rightarrow \mathbb{R}^+$ have Lipschitz continuous gradient in the convex hull of \mathcal{U}_n . and $L \geq 0$ is such that (5.2) holds. For Algorithm 3.1, we have:

- (i) $\|\text{grad } f(\mathbf{U}_k)\| \rightarrow 0$ in Algorithm 3.1; in particular, every accumulation point in Algorithm 3.1 is a stationary point.
- (ii) For δ as in (3.8), Algorithm 3.1 needs at most

$$\left\lceil \frac{2L(f^* - f(x_0))}{\delta^2} \frac{1}{\varepsilon^2} \right\rceil$$

iterations to reach an ε -optimal solution ($\|\text{grad } f(\mathbf{U}_k)\| \leq \varepsilon$).

In order to prove Proposition 5.4, we show that the descent conditions are satisfied.

LEMMA 5.5. Let $f : \mathcal{U}_n \rightarrow \mathbb{R}^+$ have Lipschitz continuous gradient in the convex hull of \mathcal{U}_n . Then there exists $L \geq 0$ such that (5.2) holds by Lemma 5.3. Let $h_{(i,j),\mathbf{U}}$ be as in (2.8) and Ψ_{opt} be its maximizer. Then

$$h_{(i,j),\mathbf{U}}(\Psi_{\text{opt}}) - h_{(i,j),\mathbf{U}}(\mathbf{I}_2) \geq \frac{\|\text{grad } h_{(i,j),\mathbf{U}}(\mathbf{I}_2)\|^2}{2L}.$$

Proof. Denote $h = h_{(i,j),\mathbf{U}}$ for simplicity. We set

$$\Delta = \mathbf{U} \mathcal{P}_{i,j}^T \mathcal{P}_{i,j}(\Lambda(\mathbf{U})) \in \mathbf{T}_{\mathbf{U}}\mathcal{U}_n.$$

Then

$$\Delta = \mathbf{U} \cdot \begin{matrix} & i & j & \\ & \mathbf{0} & \vdots & \mathbf{0} \\ i & \cdots & \Lambda(\mathbf{U})_{ii} & \Lambda(\mathbf{U})_{ij} \\ & \mathbf{0} & \vdots & \mathbf{0} \\ j & \cdots & \Lambda(\mathbf{U})_{ji} & \Lambda(\mathbf{U})_{jj} \\ & \mathbf{0} & \vdots & \mathbf{0} \end{matrix} \Bigg],$$

which is a projection of $\text{grad } f(\mathbf{U})$ onto the tangent space to the submanifold of the matrices of type $\mathbf{UG}^{(i,j,\Psi)}$.

Next, note that the exponential map (3.5) is a retraction (see [3, Proposition 5.4.1]). Denote $\Psi_1 = \text{Exp}_{\mathbf{I}_2}(\frac{1}{L} \text{grad } h(\mathbf{I}_2))$. Then, by Lemma 5.3, we have that

$$\begin{aligned} h(\Psi_1) - h(\mathbf{I}_2) &= f\left(\text{Exp}_{\mathbf{U}}\left(\frac{\Delta}{L}\right)\right) - f(\mathbf{U}) \\ &\geq \left\langle \frac{\Delta}{L}, \text{grad } f(\mathbf{U}) \right\rangle_{\mathbb{R}} - \frac{L}{2} \left\| \frac{\Delta}{L} \right\|^2 = \frac{\|\text{grad } h(\mathbf{I}_2)\|^2}{2L}. \end{aligned}$$

Since $h(\Psi_{\text{opt}}) - h(\mathbf{I}_2) \geq h(\Psi_1) - h(\mathbf{I}_2)$, the proof is complete. \square

Proof of Proposition 5.4. We apply Theorem 5.1 to the function $-f(\mathbf{U})$ (since we are interested in the maximization of $f(\mathbf{U})$). By Lemma 5.5, we have that

$$f(\mathbf{U}_k) - f(\mathbf{U}_{k-1}) = h_k(\Psi_k) - h_k(\mathbf{I}_2) \geq \frac{1}{2L} \|\text{grad } h_k(\mathbf{I}_2)\|^2 \geq \frac{\delta^2}{2L} \|\text{grad } f(\mathbf{U}_{k-1})\|^2,$$

and thus the descent condition (5.1) holds with the constant $\frac{\delta^2}{2L}$. \square

COROLLARY 5.6. *Proposition 5.4 applies to cost functions from Subsection 2.2.*

6. Łojasiewicz inequality and geodesic convexity. In this section, we recall known results and preliminaries that are needed for the main results in Section 7.

6.1. Łojasiewicz gradient inequality and speed of convergence. Here we recall the results on convergence of descent algorithms on analytic submanifolds that use Łojasiewicz gradient inequality [31], as presented in [37]. These results were used in [30] to prove the global convergence of Jacobi-G on the orthogonal group.

DEFINITION 6.1 (Łojasiewicz gradient inequality, [36, Definition 2.1]). *Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a Riemannian submanifold of \mathbb{R}^n . The function $f : \mathcal{M} \rightarrow \mathbb{R}$ satisfies a Łojasiewicz gradient inequality at a point $x \in \mathcal{M}$, if there exist $\delta > 0$, $\sigma > 0$ and $\zeta \in (0, \frac{1}{2}]$ such that for all $y \in \mathcal{M}$ with $\|y - x\| < \delta$, it holds that*

$$(6.1) \quad |f(x) - f(y)|^{1-\zeta} \leq \sigma \|\text{grad } f(x)\|.$$

The following lemma guarantees that (6.1) is satisfied for the real analytic functions defined on an analytic manifold.

LEMMA 6.2 ([36, Proposition 2.2 and Remark 1]). *Let $\mathcal{M} \subseteq \mathbb{R}^n$ be an analytic submanifold⁴ and $f : \mathcal{M} \rightarrow \mathbb{R}$ be a real analytic function. Then for any $x \in \mathcal{M}$, f satisfies a Łojasiewicz gradient inequality (6.1) for some⁵ $\delta, \sigma > 0$ and $\zeta \in (0, \frac{1}{2}]$.*

⁴See [27, Definition 2.7.1] or [30, Definition 5.1] for a definition of an analytic submanifold.

⁵The values of δ, σ, ζ depend on a specific point.

Łojasiewicz gradient inequality allows for proving convergence of optimization algorithms to a single limit point.

THEOREM 6.3 ([36, Theorem 2.3]). *Let $\mathcal{M} \subseteq \mathbb{R}^n$ be an analytic submanifold and $\{x_k : k \in \mathbb{N}\} \subseteq \mathcal{M}$. Suppose that f is real analytic and, for large enough k ,*
(i) there exists $\sigma > 0$ such that

$$(6.2) \quad |f(x_{k+1}) - f(x_k)| \geq \sigma \|\text{grad } f(x_k)\| \|x_{k+1} - x_k\|;$$

(ii) $\text{grad } f(x_k) = 0$ implies that $x_{k+1} = x_k$.

Then any accumulation point x_ of $\{x_k : k \in \mathbb{N}\} \subseteq \mathcal{M}$ is the only limit point.*

If, in addition, for some $\kappa > 0$ and for large enough n it holds that

$$(6.3) \quad \|x_{k+1} - x_k\| \geq \kappa \|\text{grad } f(x_k)\|,$$

then the following convergence rates apply

$$\|x_k - x^*\| \leq C \begin{cases} e^{-ck}, & \text{if } \zeta = \frac{1}{2} \text{ (for some } c > 0), \\ k^{-\frac{\zeta}{1-2\zeta}}, & \text{if } 0 < \zeta < \frac{1}{2}, \end{cases}$$

where ζ is the parameter in (6.1).

Remark 6.4. We can relax the conditions of **Theorem 6.3** as follows. We can require just that (6.2) holds for all k such that $\|x_k - x_*\| < \varepsilon$, where x_* is an accumulation point of the sequence and $\varepsilon > 0$ is some radius. This can be verified by inspecting the proof of **Theorem 6.3** (see also the proof of [2, Theorem 3.2])

In the case $\zeta = \frac{1}{2}$, according to **Theorem 6.3**, the convergence is linear (similarly to the classic results on local convergence of the gradient descent [34, 10]). In the optimization literature, the inequality (6.1) with $\zeta = \frac{1}{2}$ is often called Polyak-Łojasiewicz⁶ inequality. In the next subsection, we recall some sufficient conditions for Polyak-Łojasiewicz inequality to hold.

6.2. Łojasiewicz inequality at stationary points. It is known, and widely used in optimization (especially in the Euclidean case), that around a strong local maximum the function satisfies the Polyak-Łojasiewicz inequality. In fact, it is also valid for non-degenerate stationary points, as shown in [25]. Here we recall the most general recent result on possibly degenerate stationary points that satisfy the so-called Morse-Bott property (see also [8, p.248]).

DEFINITION 6.5 ([21, Definition 1.5]). *Let \mathcal{M} be a C^∞ submanifold and $f : \mathcal{M} \rightarrow \mathbb{R}$ be a C^2 function. Denote the set of stationary points as*

$$\text{Crit } f = \{x \in \mathcal{M} : \text{grad } f(x) = 0\}.$$

The function f is said to be Morse-Bott at $x_0 \in \mathcal{M}$ if there exists an open neighborhood $\mathcal{U} \subseteq \mathcal{M}$ of x_0 such that

- (i) $\mathcal{C} = \mathcal{U} \cap \text{Crit } f$ is a relatively open, smooth submanifold of \mathcal{M} ;*
- (ii) $\mathbf{T}_{x_0} \mathcal{C} = \text{Ker Hess}_{x_0} f$.*

Remark 6.6. (i) If $x_0 \in \mathcal{M}$ is a non-degenerate stationary point, then f is Morse-Bott at x_0 , since $\{x_0\}$ is a zero-dimensional manifold in this case.

⁶The inequality (6.1) with $\zeta = \frac{1}{2}$ goes back to Polyak [34], who used it for proving linear convergence of the gradient descent.

(ii) If $x_0 \in \mathcal{M}$ is a degenerate stationary point, then condition (ii) in [Definition 6.5](#) can be rephrased⁷ as

$$(6.4) \quad \text{rank}\{\text{Hess}_{x_0}f\} = \dim \mathcal{M} - \dim \mathcal{C}.$$

For the functions that satisfy the Morse-Bott property, it was recently shown that the Polyak-Lojasiewicz inequality holds true.

THEOREM 6.7 ([\[21, Theorem 3, Corollary 5\]](#)). *If $\mathcal{U} \subseteq \mathbb{R}^n$ is an open subset and $f : \mathcal{U} \rightarrow \mathbb{R}$ is Morse-Bott at a stationary point x , then there exist $\delta, \sigma > 0$ such that*

$$|f(y) - f(x)| \leq \sigma \|\nabla f(y)\|^2,$$

for any $y \in \mathcal{U}$ satisfying $\|y - x\| \leq \delta$.

We can also easily deduce the same result on a smooth manifold \mathcal{M} .

PROPOSITION 6.8. *If $\mathcal{U} \subseteq \mathcal{M}$ is an open subset and a C^2 function $f : \mathcal{U} \rightarrow \mathbb{R}$ is Morse-Bott at a stationary point x , then there exist an open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of x and $\sigma > 0$ such that for all $y \in \mathcal{V}$ it holds that*

$$|f(y) - f(x)| \leq \sigma \|\text{grad } f(y)\|^2.$$

Proof. Consider the exponential map $\text{Exp}_x : \mathbf{T}_x\mathcal{M} \rightarrow \mathcal{M}$, which is a local diffeomorphism. Let $\mathcal{W} \subseteq \mathbf{T}_x\mathcal{M}$ be an open subset such that $\text{Exp}_x(\mathcal{W}) = \mathcal{U}$. Let $\widehat{f} = f \circ \text{Exp}_x$ be the composite map from \mathcal{W} to \mathbb{R} . Then

$$(6.5) \quad \nabla \widehat{f}(y') = \mathbf{J}_{\text{Exp}_x}^\top(y') \text{grad } f(y),$$

where $y' \in \mathcal{W}$ and $y = \text{Exp}_x(y')$. It follows that Exp_x gives a diffeomorphism between $\text{Crit } f$ and $\text{Crit } \widehat{f}$. Since $\text{Hess}_x f = \text{H}_{\widehat{f}}(0)$ by [\[3, Proposition 5.5.5\]](#), we have that \widehat{f} is Morse-Bott at 0. Therefore, by [Theorem 6.7](#), there exist $\sigma' > 0$, $\sigma > 0$ and an open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of x such that

$$|f(y) - f(x)| = |\widehat{f}(y') - \widehat{f}(0)| \leq \sigma' \|\nabla \widehat{f}(y')\|^2 \leq \sigma \|\text{grad } f(y)\|^2,$$

for any $y \in \mathcal{V}$, where the last inequality holds because $\mathbf{J}_{\text{Exp}_x}$ is nonsingular in a neighborhood of x . \square

Remark 6.9. For the case of non-degenerate stationary points and C^∞ functions, [Proposition 6.8](#) is proved in [\[25, Lemma 4.1\]](#), which is a simple corollary of Morse Lemma [\[33, Lemma 2.2\]](#). For C^∞ functions and Morse-Bott functions, [Proposition 6.8](#) (as noted in [\[21\]](#)) is also a simple corollary of Morse-Bott Lemma [\[7\]](#).

6.3. Geodesic convexity. We recall the notion of a geodesic convexity [\[35\]](#), which is a generalization of the notion of convexity of sets and functions. In particular, we relate the definiteness of Riemannian Hessians with local convexity/concavity. In this subsection, we assume that $\mathcal{M} \subseteq \mathbb{R}^n$ is a connected Riemannian manifold.

DEFINITION 6.10 ([\[35, Definition 2.1\]](#)). *A set $\mathcal{A} \subseteq \mathcal{M}$ is called geodesically convex, if any two points $x, y \in \mathcal{A}$ are joined by a geodesic lying in \mathcal{A} .*

⁷due to the fact that $\mathbf{T}_{x_0}\mathcal{C} \subseteq \text{Ker Hess}_{x_0}f$.

DEFINITION 6.11 ([35, Definition 2.2]). Let $\mathcal{A} \subseteq \mathcal{M}$ be geodesically convex. A function $f : \mathcal{A} \rightarrow \mathbb{R}$ is called geodesically convex (resp. concave) on \mathcal{A} , if it is convex⁸ (resp. concave) when restricted to geodesics.

We will later on use a lemma that ensures the geodesic convexity of sublevel sets.

LEMMA 6.12 ([35, Lemma 2.1]). Let $\mathcal{A} \subseteq \mathcal{M}$ and $f : \mathcal{A} \rightarrow \mathbb{R}$ be geodesically convex. Then for any $x_0 \in \mathcal{A}$, the level set

$$\{x | f(x) \leq f(x_0), x \in \mathcal{A}\}$$

is geodesically convex.

The following two characterizations of geodesic convexity/concavity will be useful for analysis of the convergence properties in Section 7.

PROPOSITION 6.13 ([35, Corollary 4.1]). For any C^2 function $f : \mathcal{M} \rightarrow \mathbb{R}$, if $\text{grad } f(x) = 0$ and $\text{Hess}_x f \succ_{\mathbf{T}_x \mathcal{M}} 0$ (resp. $\prec_{\mathbf{T}_x \mathcal{M}} 0$), then there exists a neighborhood \mathcal{U} of x , such that f is geodesically convex (resp. concave) on \mathcal{U} .

7. Convergence results based on Łojasiewicz inequality.

7.1. Preliminary lemmas: checking the decrease conditions. In this subsection, we are going to find some sufficient conditions for (6.2) and (6.3) to hold in Algorithm 3.1, which will allow us to use Theorem 6.3.

Let $\mathbf{U}_k = \mathbf{U}_{k-1} \mathbf{G}^{(i_k, j_k, \Psi_k)}$ be the iterations in Algorithm 3.1. Obviously,

$$\|\mathbf{U}_k - \mathbf{U}_{k-1}\| = \|\Psi_k - \mathbf{I}_2\|.$$

Assume that Ψ_k is obtained as in Subsection 2.5, i.e., by taking \mathbf{w} as the leading eigenvector of $\mathbf{\Gamma}^{(i_k, j_k, \mathbf{U}_{k-1})}$ (normalized so that $w_1 = \cos 2\theta = 2c^2 - 1 > 0$ in (2.14)) as in Remark 2.4, and retrieving Ψ_k from \mathbf{w} according to (2.12) and (2.14). We first express $\|\Psi_k - \mathbf{I}_2\|$ through w_1 .

LEMMA 7.1. For the iterations Ψ_k obtained as in Subsection 2.5, it holds that

$$(7.1) \quad 2\|\Psi_k - \mathbf{I}_2\| \geq \sqrt{1 - w_1^2} \geq 0.65\|\Psi_k - \mathbf{I}_2\|$$

Proof. Note that

$$\|\Psi_k - \mathbf{I}_2\| = \sqrt{2(1 - c)^2 + 2|s|^2} = 2\sqrt{1 - c}.$$

By (2.14) and Remark 2.4, we see that

$$\sqrt{1 - w_1^2} = 2c\sqrt{1 - c^2} \geq 1.3\sqrt{1 - c} \text{ and } 2c\sqrt{1 - c^2} \leq 2 \cdot 2\sqrt{1 - c}. \quad \square$$

Since we are looking at Algorithm 3.1, we can replace in both inequalities of (7.1) $\text{grad } f(\mathbf{U}_{k-1})$ with $\text{grad } h_{(i,j), \mathbf{U}}(\mathbf{I}_2)$. Next, we prove a result for condition (6.3).

LEMMA 7.2. Let $f : \mathcal{U}_n \rightarrow \mathbb{R}^+$ be a C^3 function. Then there exists a universal constant $\kappa > 0$ such that

$$\|\Psi_k - \mathbf{I}_2\| \geq \kappa \|\text{grad } h_k(\mathbf{I}_2)\|.$$

⁸with respect to the arc length parameters.

Proof. We denote $\mathbf{\Gamma} = \mathbf{\Gamma}^{(i_k, j_k, \mathcal{U}_{k-1})}$ as in (2.13). By Lemma 4.4, we have that

$$\|\text{grad } h_k(\mathbf{I}_2)\| = 2\sqrt{2}\sqrt{\Gamma_{12}^2 + \Gamma_{13}^2}.$$

By Lemma 7.1, it is sufficient to prove that

$$1 - w_1^2 \geq \kappa'(\Gamma_{12}^2 + \Gamma_{13}^2)$$

for a universal constant $\kappa' > 0$. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ be the eigenvalues of $\mathbf{\Gamma}$. Without loss of generality, we set $\mathbf{\Gamma}' = \mathbf{\Gamma} - \lambda_3 \mathbf{I}_3$, $\mu_1 = \lambda_1 - \lambda_3$ and $\mu_2 = \lambda_2 - \lambda_3$. Then

$$(7.2) \quad \mathbf{\Gamma}' = \mu_1 \mathbf{w} \mathbf{w}^\top + \mu_2 \mathbf{v} \mathbf{v}^\top,$$

where \mathbf{v} is the second eigenvector of $\mathbf{\Gamma}$. It follows that

$$(7.3) \quad \begin{aligned} \Gamma_{12}^2 + \Gamma_{13}^2 &= (\Gamma'_{12})^2 + (\Gamma'_{13})^2 = (\mu_1 w_1 w_2 + \mu_2 v_1 v_2)^2 + (\mu_1 w_1 w_3 + \mu_2 v_1 v_3)^2 \\ &= \mu_1^2 w_1^2 (1 - w_1^2) - 2\mu_1 \mu_2 w_1^2 v_1^2 + \mu_2^2 v_1^2 (1 - v_1^2). \end{aligned}$$

Since $v_1^2 \leq 1 - w_1^2$ (due to orthonormality of \mathbf{v} and \mathbf{w}) and $\mu_2, \mu_1 \leq 2\|\mathbf{\Gamma}\|$, we get that

$$\Gamma_{12}^2 + \Gamma_{13}^2 \leq (1 - w_1^2) 4\|\mathbf{\Gamma}\|^2.$$

By Lemma 4.4, Lemma 4.5 and C^3 smoothness of f , $\mathbf{\Gamma}$ continuously depends on $\mathcal{U} \in \mathcal{U}_n$. Therefore, $\|\mathbf{\Gamma}\|$ is bounded from above, and thus the proof is completed. \square

We are ready to check the sufficient decrease condition (6.2).

LEMMA 7.3. *Let $\mathbf{\Gamma} = \mathbf{\Gamma}^{(i_k, j_k, \mathcal{U}_{k-1})}$ be as in (2.13). Let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ be the eigenvalues of $\mathbf{\Gamma}$, and $\eta = \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}$. Suppose that $1 - \eta \geq \varepsilon$ for some $\varepsilon > 0$. Then*

$$|h_k(\mathbf{\Psi}_k) - h_k(\mathbf{I}_2)| \geq \frac{\varepsilon}{4} \|\text{grad } h_k(\mathbf{I}_2)\| \sqrt{1 - w_1^2}.$$

Proof. Define the ratio

$$(7.4) \quad q(\mathbf{\Gamma}, \mathbf{w}) = \frac{(\mathbf{w}^\top \mathbf{\Gamma} \mathbf{w} - \Gamma_{11})^2}{(\Gamma_{12}^2 + \Gamma_{13}^2)(1 - w_1^2)}.$$

It is sufficient to prove that $q(\mathbf{\Gamma}, \mathbf{w}) \geq \varepsilon/2$. Denote

$$\rho = 1 - w_1^2, \quad \tau = \frac{v_1^2}{\rho} \in [0, 1],$$

where \mathbf{v} is as in the proof of Lemma 7.2. From (7.2) and (7.3) we immediately have

$$(7.5) \quad \mathbf{w}^\top \mathbf{\Gamma} \mathbf{w} - \Gamma_{11} = \mu_1 - (\mu_1 w_1^2 + \mu_2 v_1^2) = \mu_1 \rho (1 - \tau \eta),$$

$$(7.6) \quad \Gamma_{12}^2 + \Gamma_{13}^2 = \rho \mu_1^2 ((1 - \tau) + \tau(1 - \eta)^2 - \rho(1 - \tau \eta)^2).$$

By substituting (7.5) and (7.6) into (7.4), we get

$$\frac{1}{q(\mathbf{\Gamma}, \mathbf{w})} = \frac{\rho^2 \mu_1^2 ((1 - \tau) + \tau(1 - \eta)^2 - \rho(1 - \tau \eta)^2)}{\mu_1^2 \rho^2 (1 - \tau \eta)^2} \leq \frac{1 - 2\tau \eta + \tau \eta^2}{(1 - \tau \eta)^2} \leq 1 + \frac{\tau \eta^2}{(1 - \eta)} \leq \frac{2}{\varepsilon}.$$

The proof is complete. \square

7.2. Main results.

THEOREM 7.4. *Suppose that $f : \mathcal{U}_n \rightarrow \mathbb{R}^+$ has Lipschitz continuous gradient in the convex hull of \mathcal{U}_n , and $\bar{\mathbf{U}}$ is an accumulation point of [Algorithm 3.1](#), where the restrictions to Givens transformations are given by [\(2.13\)](#) (and $\text{grad} f(\mathbf{U}) = 0$ by [Proposition 5.4](#)). Assume that all $\mathfrak{D}_{\bar{\mathbf{U}}}^{(i,j)}$ are negative definite for all pairs (i, j) . Then*

- (i) $\bar{\mathbf{U}}$ is the only limit point and convergence rates in [Theorem 6.3](#) apply.
- (ii) If the rank of Riemannian Hessian is maximal at $\bar{\mathbf{U}}$ (i.e., $\text{rank}\{\text{Hess}_{\bar{\mathbf{U}}}f\} = n^2 - n$), then the speed of convergence is linear.

Proof. (i) Since $\mathfrak{D}_{\bar{\mathbf{U}}}^{(i,j)}$ is negative definite for any $i \neq j$, the two top eigenvalues of $\mathbf{\Gamma}^{(i,j,\bar{\mathbf{U}})}$ are separated by [Remark 4.7](#). Therefore, there exists $\varepsilon > 0$ such that

$$\frac{\lambda_2(\mathbf{\Gamma}^{(i,j,\bar{\mathbf{U}})}) - \lambda_3(\mathbf{\Gamma}^{(i,j,\bar{\mathbf{U}})})}{\lambda_1(\mathbf{\Gamma}^{(i,j,\bar{\mathbf{U}})}) - \lambda_3(\mathbf{\Gamma}^{(i,j,\bar{\mathbf{U}})})} < 1 - \varepsilon.$$

By the continuity of $\mathbf{\Gamma}^{(i,j,\mathbf{U})}$ with respect to \mathbf{U} , the conditions of [Lemma 7.3](#) are satisfied in a neighborhood of $\bar{\mathbf{U}}$. Therefore, there exists $c > 0$ such that

$$|f(\mathbf{U}_k) - f(\mathbf{U}_{k-1})| \geq c \|\text{grad} h_k(\mathbf{I}_2)\| \|\mathbf{U}_k - \mathbf{U}_{k-1}\|,$$

in a neighborhood of $\bar{\mathbf{U}}$. By [Remark 6.4](#), it is enough to use [Theorem 6.3](#), hence $\bar{\mathbf{U}}$ is the only limit point. Moreover, by [Lemma 7.2](#), the convergence rates apply.

- (ii) Due to the scaling invariance, $\bar{\mathbf{U}}$ belongs to an n -dimensional submanifold of stationary points defined by $\bar{\mathbf{U}}\mathbf{S}$, where \mathbf{S} is as in [\(2.10\)](#). Since $\text{rank}\{\text{Hess}_{\bar{\mathbf{U}}}f\} = n^2 - n$, f is Morse-Bott at $\bar{\mathbf{U}}$ by [Remark 6.6](#). Therefore, by [Proposition 6.8](#), $\zeta = 1/2$ in [\(6.1\)](#) at $\bar{\mathbf{U}}$, and thus the convergence is linear by [Theorem 6.3](#).

THEOREM 7.5. *Let \mathbf{U}_* be a semi-strict local maximum of f (i.e. $\text{rank}\{\text{Hess}_{\mathbf{U}_*}f\} = n^2 - n$). Then there exists a neighborhood \mathcal{W} of \mathbf{U}_* , such that for any $\mathbf{U}_0 \in \mathcal{W}$, [Algorithm 3.1](#) converges linearly to $\mathbf{U}_*\mathbf{S}$, where \mathbf{S} is of the form [\(2.10\)](#).*

Proof. Let \mathbb{T} be the unit circle in \mathbb{C} . Consider the action of \mathbb{T}^n on \mathcal{U}_n defined as

$$\mathbf{U} \cdot (t_1, \dots, t_n) = \mathbf{U} \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{bmatrix}.$$

Since the action of \mathbb{T}^n on \mathcal{U}_n is free and proper, the quotient manifold $\widetilde{\mathcal{U}}_n = \mathcal{U}_n/\mathbb{T}^n$ is well-defined. In order to define the gradient and Hessians on $\widetilde{\mathcal{U}}_n$, we use the standard splitting into horizontal and vertical space

$$\mathbf{T}_{\mathbf{U}}\mathcal{U}_n = \mathcal{V}_{\mathbf{U}}\mathcal{U}_n \oplus \mathcal{H}_{\mathbf{U}}\mathcal{U}_n,$$

where $\mathcal{H}_{\mathbf{U}}\mathcal{U}_n$ contains the skew-symmetric matrices with zero diagonal:

$$\mathcal{H}_{\mathbf{U}}\mathcal{U}_n = \{\mathbf{X} \in \mathbb{C}^{n \times n} : \mathbf{X} = \mathbf{U}\mathbf{Z}, \quad \mathbf{Z} + \mathbf{Z}^H = 0, \quad \text{diag}\{\mathbf{Z}\} = \mathbf{0}\}.$$

In this case, an element $\tilde{\mathbf{U}} \in \widetilde{\mathcal{U}}_n$ can be represented as its representative \mathbf{U} and the tangent space $\mathbf{T}_{\tilde{\mathbf{U}}}\widetilde{\mathcal{U}}_n$ is identified with $\mathcal{H}_{\mathbf{U}}\mathcal{U}_n$, see [[3](#), Section 3.5.8]. Moreover, the

Riemannian metric on $\widetilde{\mathcal{U}}_n$ can be defined as

$$\langle \widetilde{\xi}, \widetilde{\eta} \rangle_{\mathbf{T}_{\widetilde{\mathcal{U}}_n}} = \langle \xi, \eta \rangle_{\mathbf{T}_U \mathcal{U}_n},$$

because the inner product is invariant with respect to the choice of representative \mathbf{U} , see [3, Section 3.6.2]. This makes $\widetilde{\mathcal{U}}_n$ a Riemannian manifold; the natural projection $\pi : \mathcal{U} \mapsto \widetilde{\mathcal{U}}$ then becomes a Riemannian submersion.

Due to the invariance property (2.9), the function f is, in fact, defined on $\widetilde{\mathcal{U}}_n$ (we will denote the corresponding function $\widetilde{f} : \widetilde{\mathcal{U}}_n \rightarrow \mathbb{R}$). Obviously, $\text{grad } f(\mathbf{U}) \in \mathcal{H}_U \mathcal{U}_n$; moreover, the Riemannian Hessian is given by

$$\text{Hess}_{\widetilde{\mathcal{U}}} \widetilde{f}[\mathbf{Z}] = \mathcal{P}_{\mathcal{H}_U \mathcal{U}_n} \text{Hess}_U f[\mathbf{Z}],$$

see [3, Section 5.3.4].

By Lemma 4.2 have that $\text{rank}\{\text{Hess}_{\widetilde{\mathcal{U}}_*} \widetilde{f}\} = \text{rank}\{\text{Hess}_{U_*} f\} = n(n-1)$, and therefore it is negative definite. Hence, by Proposition 6.13 there exists an open neighborhood $\widetilde{\mathcal{U}}$ of $\widetilde{\mathcal{U}}_*$ where \widetilde{f} is geodesically concave. For simplicity assume that $f(\mathbf{U}_*) = 0$. Without loss of generality, by Lemma 6.12 we can take $\widetilde{\mathcal{W}}$ such that its boundary $\delta(\widetilde{\mathcal{W}})$ is a level set for $a < 0$, i.e.

$$f(\widetilde{\mathcal{U}}) = a, \quad \forall \widetilde{\mathcal{U}} \in \delta(\widetilde{\mathcal{W}}).$$

Moreover, for any $a < b < 0$ we have that the level set

$$\widetilde{\mathcal{W}}_b = \{f(\widetilde{\mathcal{U}}) \geq b, \widetilde{\mathcal{U}} \in \widetilde{\mathcal{W}}\}$$

is a geodesically convex neighbourhood of $\widetilde{\mathcal{U}}_*$.

Next, assume that $\widetilde{\mathcal{U}}_{k-1} \in \widetilde{\mathcal{W}}$, and consider the $\mathbf{U}_k = \mathbf{U}_{k-1} \mathbf{G}^{(i,j,\Psi)}$ with Ψ given as the maximizer of (2.13). Define $b = f(\mathbf{U}_{k-1})$. In what follows, we are going to prove that $\widetilde{\mathcal{U}}_{k-1} \in \widetilde{\mathcal{W}}_b$, so that the sequence $\widetilde{\mathcal{U}}_k$ never leaves the set $\widetilde{\mathcal{W}}$.

Recall that Ψ is computed as follows (see (2.4)): take the vector \mathbf{w} as in (2.14). Take $\alpha_1 = -w_2/\sqrt{1-w_1^2}$, $\alpha_2 = -w_3/\sqrt{1-w_1^2}$ (we can assume $w_1 \neq 1$ because otherwise $\Psi = \mathbf{I}_2$ and this case is trivial), and consider the following geodesic in \mathcal{U}_n :

$$\gamma(t) = \mathbf{U}_{k-1} \mathcal{P}_{i,j}^\top \left(\begin{bmatrix} \cos \frac{t}{2} & -(\alpha_1 + i\alpha_2) \sin \frac{t}{2} \\ (\alpha_1 - i\alpha_2) \sin \frac{t}{2} & \cos \frac{t}{2} \end{bmatrix} \right),$$

which starts at $\gamma(0) = \mathbf{U}_{k-1}$. Note that at each point t_1 , $\frac{d}{dt} \gamma(t_1) \in \mathcal{V}_{\gamma(t_1)} \mathcal{U}_n$, hence the corresponding curve $\widetilde{\gamma}$ is a geodesic in the quotient manifold $\widetilde{\mathcal{U}}_n$.

Next, as in (4.4), we have that

$$f(\gamma(t)) = [\cos t \quad -\alpha_1 \sin t \quad -\alpha_2 \sin t] \mathbf{\Gamma} [\cos t \quad -\alpha_1 \sin t \quad -\alpha_2 \sin t]^\top + C,$$

hence

$$f(\gamma(t)) = A \cos(2(t - t_*)) + C,$$

where $t_* = \arccos(w_1) \in [0, \frac{\pi}{2}]$ and $\gamma(t_*) = \mathbf{U}_k$. Note that by geodesic concavity of f at \mathbf{U}_{k-1} we have $\frac{d}{dt^2} f(\gamma(0)) = -4A \cos(-2t_*) < 0$ and therefore $\cos(2t_*) > 0$ and $t_* \in [0, \frac{\pi}{4}]$. Hence we have that $\frac{d}{dt} f(\gamma(t)) = -4A \sin(2(t - t_*)) > 0$ for any

$t \in [0, t_*)$, the cost function is decreasing; note that $\frac{d}{dt}f(\gamma(t_*)) = 0$ and there are no other stationary points in $t \in [0, t_*)$.

Next, by continuity and because $\tilde{\mathcal{W}}$ is open, there exist a small $\varepsilon > 0$ such that $\tilde{\gamma}(\varepsilon)$ is in the interior of \mathcal{W}_b . By periodicity of $f(\gamma(t))$ and continuity, we have that there exists t_2 such that $\gamma(t_2) \in \partial\tilde{\mathcal{W}}_b$ and $\tilde{\gamma}(t) \in \tilde{\mathcal{W}}_b$ for all $t \in [0, t_2]$. By Rolle's theorem, there exists a local maximum of $f(\gamma(t))$ in $[0, t_2]$. Note that by construction, the closest positive local maximum to 0 is at t_* , therefore $\tilde{\mathbf{U}}_k = \tilde{\gamma}(t_*) \in \tilde{\mathcal{W}}_b$, hence we stay in the same neighbourhood $\tilde{\mathcal{W}}$.

Finally, as a neighborhood of $\mathbf{U}_* \in \mathcal{U}_n$, we can take the preimage $\mathcal{W} = \pi^{-1}(\tilde{\mathcal{W}})$; also linear convergence rate follows from [Theorem 7.4](#). \square

7.3. Examples of cost functions satisfying regularity conditions. In this subsection, that the regularity conditions are satisfied for diagonalizable tensors and matrices. Recall that $\mathcal{A} \in \mathbb{C}^{n \times \dots \times n}$ is a diagonal tensor if all the elements are zero except the ones on the diagonal ($\mathcal{A}_{i \dots i}$).

PROPOSITION 7.6. (i) For a set of jointly orthogonally diagonalisable matrices

$$\mathbf{A}^{(\ell)} = \mathbf{U}_0 \begin{bmatrix} \mu_1^{(\ell)} & & 0 \\ & \ddots & \\ 0 & & \mu_n^{(\ell)} \end{bmatrix} \mathbf{U}_0^H,$$

such that for any pair $i \neq j$

$$\sum_{\ell=1}^m (\mu_i^{(\ell)} - \mu_j^{(\ell)})^2 > 0$$

the matrix \mathbf{U}_0 is a semi-strict local maximum (as in conditions of [Theorem 7.5](#)).

(ii) For an orthogonally diagonalizable 3rd order tensor

$$\mathcal{A} = \mathcal{D} \bullet_1 \mathbf{U}_0 \bullet_2 \mathbf{U}_0^H \bullet_3 \mathbf{U}_0^H,$$

where \mathcal{D} is a diagonal tensor with at most one zero element on the diagonal, the matrix \mathbf{U}_0 is a semi-strict local maximum of f (as in [Theorem 7.5](#)).

(iii) For an orthogonally diagonalizable tensor

$$\mathcal{A} = \mathcal{D} \bullet_1 \mathbf{U}_0 \bullet_2 \mathbf{U}_0 \bullet_3 \mathbf{U}_0^H \bullet_4 \mathbf{U}_0^H,$$

where the values on the diagonals are either (a) all positive or (b) there is at most one i with $\mathcal{D}_{iiii} \leq 0$, for which $\mathcal{D}_{iiii} + \mathcal{D}_{jjjj} > 0$ for all $j \neq i$.

For proving [Proposition 7.6](#), we need a lemma about Hessians of multilinear forms.

LEMMA 7.7. Let $\gamma(\mathbf{u})$ be a real-valued function that is either $\gamma(\mathbf{u}) = g_{\mathcal{A}}(\mathbf{u})$ or $\gamma(\mathbf{u}) = |g_{\mathcal{A}}(\mathbf{u})|^2$, where \mathcal{A} is a d -th order diagonal tensor and $g_{\mathcal{A}}$ is defined as in [\(4.11\)](#). Then for any distinct indices $1 \leq i \neq j \neq k \leq n$ it holds that

$$\mathbf{e}_i^\top \left(\frac{\partial^2 \gamma}{\partial \mathbf{u}^* \partial \mathbf{u}^*}(\mathbf{e}_k) \right) \mathbf{e}_j = 0, \quad \mathbf{e}_i^\top \left(\frac{\partial^2 \gamma}{\partial \mathbf{u}^* \partial \mathbf{u}}(\mathbf{e}_k) \right) \mathbf{e}_j = 0.$$

Proof. Consider the case $\gamma(\mathbf{u}) = g_{\mathcal{A}}(\mathbf{u})$ and the derivative $\frac{\partial^2 \gamma}{\partial \mathbf{u}^* \partial \mathbf{u}^*}$. By continuing

the differentiation as in (4.10), we get that

$$e_i^\top \left(\frac{\partial^2 \gamma}{\partial \mathbf{u}^* \partial \mathbf{u}^*} (e_s) \right) e_j = \sum_{\substack{s \neq t \\ 1 \leq s, t \leq d_1}} (\mathcal{A} \bullet_s e_i \bullet_t e_j)_{k \dots k} = 0,$$

$$e_i^\top \left(\frac{\partial^2 \gamma}{\partial \mathbf{u}^* \partial \mathbf{u}} (e_s) \right) e_j = \sum_{s=1}^{d_1} \sum_{t=d_1+1}^d (\mathcal{A} \bullet_s e_i \bullet_t e_j)_{k \dots k} = 0,$$

because the offdiagonal elements are zero. Similarly, for $\gamma(\mathbf{u}) = |g\mathcal{A}(\mathbf{u})|^2$, we get

$$e_i^\top \left(\frac{\partial^2 \gamma}{\partial \mathbf{u}^* \partial \mathbf{u}^*} (e_k) \right) e_j$$

$$= (g\mathcal{A}(\mathbf{u}))^* \sum_{\substack{s \neq t \\ 1 \leq s, t \leq d_1}} (\mathcal{A} \bullet_s e_i \bullet_s e_j)_{k \dots k} + \left(\sum_{t=d_1+1}^d (\mathcal{A}^* \bullet_t e_j)_{k \dots k} \right) \sum_{s=1}^{d_1} (\mathcal{A} \bullet_s e_i)_{k \dots k},$$

$$+ (g\mathcal{A}(\mathbf{u})) \sum_{\substack{s \neq t \\ d_1+1 \leq s, t \leq d}} (\mathcal{A}^* \bullet_s e_i \bullet_t e_j)_{k \dots k} + \left(\sum_{t=1}^{d_1} (\mathcal{A} \bullet_t e_j)_{k \dots k} \right) \sum_{s=d_1+1}^d (\mathcal{A}^* \bullet_s e_i)_{k \dots k},$$

by Proposition 4.11, where each term is equal to zero, because there are at least two different indices, hence the off-diagonal elements are taken. \square

Proof of Proposition 7.6. Without loss of generality, we can consider only the case $\mathbf{U}_0 = \mathbf{I}_n$, so that all the matrices/tensors are diagonal. Due to diagonality of matrices/tensors (the off-diagonal elements are zero) from Remark 4.12 we have that Euclidean gradients vanish, i.e., $\frac{\partial f}{\partial \mathbf{U}^*}(\mathbf{I}_n) = 0$. Hence \mathbf{I}_n is a stationary point (see also Corollary 4.8), and moreover, by [4, Eq. (7)] (see also [35]) the Riemannian Hessian is just the projection of the Euclidean Hessian on the tangent space

$$(7.7) \quad \text{Hess}_{\mathbf{I}_n} f[\eta] = \Pi_{\mathbf{T}_{\mathbf{I}_n} \mathcal{U}_n} \mathbf{H}_{\tilde{f}}(\mathbf{I}_n)[\eta],$$

where $\tilde{f}(\mathbf{U})$ is the Euclidean extension of f (which has vanishing gradient at \mathbf{I}_n).

Next, we show that the Hessian does not contain off-diagonal blocks. From (7.7), we just need to look at the Euclidean Hessian. Take two pairs of indices (i, k) and (j, l) and look at the Hessian terms

$$\frac{\partial^2 f}{\partial \mathbf{U}_{i,k}^* \partial \mathbf{U}_{j,l}} \quad \text{and} \quad \frac{\partial^2 f}{\partial \mathbf{U}_{i,k}^* \partial \mathbf{U}_{j,l}^*}.$$

Since by (4.10), $(\frac{\partial f}{\partial \mathbf{U}^*})_{i,k}$ is a function of \mathbf{u}_k only, the Hessian terms can only be nonzero if $j = k$ or $l = k$. Let us choose $l = k$. In that case,

$$\frac{\partial^2 f}{\partial \mathbf{U}_{i,k}^* \partial \mathbf{U}_{j,k}}(\mathbf{I}_n) = \frac{\partial^2 \gamma}{\partial \mathbf{u}_i^* \partial \mathbf{u}_j}(e_k) = e_i^\top \left(\frac{\partial^2 \gamma}{\partial \mathbf{u}^* \partial \mathbf{u}^*} (e_k) \right) e_j = 0$$

by Lemma 7.7. Similarly, the second Hessian term is also equal to zero. Thus the Hessian is block-diagonal with the terms given in Lemma 4.5.

Finally, we apply Lemma 2.3 and get that for each cost function

- (i) $\mathfrak{D}_{\mathcal{U}}^{(i,j)} = -\mathbf{I}_2 \sum_{\ell=1}^m (\mu_i^{(\ell)} - \mu_j^{(\ell)})^2;$
 - (ii) $\mathfrak{D}_{\mathcal{U}}^{(i,j)} = -\frac{3}{2}\mathbf{I}_2(|\mathcal{D}_{iii}|^2 + |\mathcal{D}_{jjj}|^2);$
 - (iii) $\mathfrak{D}_{\mathcal{U}}^{(i,j)} = -\mathbf{I}_2(\mathcal{D}_{iii} + \mathcal{D}_{jjj});$
- which are negative definite if and only if the conditions of the proposition are satisfied. \square

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