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Vortex patterns in the almost-bosonic anyon gas

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We study theoretically and numerically the ground state of a gas of 2D anyons in an external trapping potential. We treat anyon statistics in the magnetic gauge picture, perturbatively around the bosonic end. This leads to a mean-field energy functional, whose ground state displays vortex lattices similar to those found in rotating Bose-Einstein condensates. A crucial difference is however that the vortex density is proportional to the underlying matter density of the gas.

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The eventual unambiguous observation of the elusive exotic particles going under the name of *anyons* [14, 20, 22, 34] remains one the most exciting prospects of condensed matter physics. Apart from speculative applications [21], much excitement comes from the possibility of observing quasi-particles [2, 11, 12] who do *not* fall into the ubiquitous dichotomy between bosons and fermions.

Contrarily to what happens for bosons and fermions, the many-body problem for non-interacting anyons is in general not exactly solvable. Even the most basic putative experiment thus calls for approximating schemes for its interpretation. In particular, in view of several recent proposals to create anyons in cold quantum gases [5, 19, 31, 35–37], it seems desirable to develop numerically amenable models to describe trapped gases of many anyons. In this note we study such a mean-field-type model, obtained in an “almost bosonic limit”. In previous work we have derived this model from the many-body Hamiltonian (in a somewhat idealized situation, see the details in [18]) and studied a local density approximation thereof [6]. Now we aim at a more quantitative analysis, for which we numerically simulate the ground state of the effective functional. Our main finding (see Figures 1-2 below) is that the ground state develops triangular vortex lattices akin to those found in rotating trapped Bose-Einstein condensates [1, 4, 7, 8, 10] or type II superconductors [26–28]. The important difference is however that the density of the vortex patterns is directly related to the underlying matter density profile, that one can compute to be of Thomas-Fermi (TF) shape, within local density approximation (LDA).

We begin by recalling a few facts about the basic many-anyons Hamiltonian before discussing our average-field (or mean-field) approximation. Then we recall some explicit computations one can make using the LDA, that we shall use as benchmarks for the numerical simulations. Finally we discuss briefly our numerical method and present its results. We find an excellent agreement

between the numerics and the available exact benchmarks, which gives us confidence in the new findings, namely, the inhomogeneous vortex lattice.

Many-body Hamiltonian. In the so-called magnetic gauge picture, one trades the statistical phase $e^{i\pi\alpha}$, $\alpha \in [0, 2[$ (or better, braiding phase) that the many-anyons wave function picks after particle exchanges for a Aharonov-Bohm-like magnetic flux of intensity α attached to each particle. Thus the many-body Hamiltonian for a gas of N 2D anyons in an external potential V becomes (in units where $\hbar = c = 1$ and $m = 1/2$, and with $\mathbf{x}^\perp = (x, y)^\perp = (-y, x)$)

$$H_N^\alpha = \sum_{j=1}^N \left(-i\nabla_{\mathbf{x}_j} + \alpha \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2} \right)^2 + V(\mathbf{x}_j) \quad (1)$$

acting on bosonic wave functions $\Psi_N \in L_{\text{sym}}^2(\mathbb{R}^{2N})$ symmetric under particle exchanges. This formulation makes readily clear the reason why the problem is not exactly solvable in general: free anyons (that is, with no further interactions beyond the effect of statistics) correspond to interacting bosons. It is formally equivalent to consider the action of $H_N^{\alpha-1}$ on fermionic wave functions, but we do not follow that route for reasons that will become clear below.

Average-field approximation. We are interested in the most basic question for the many-body Hamiltonian (1): compute the ground state energy and associated ground states. Since the wave functions we act on are bosonic, a basic mean-field approximation suggests itself. For weak interactions [15–17, 24, 25], the ground state of a bosonic system is close to a Bose-Einstein condensate (BEC),

$$\Psi_N(\mathbf{x}_1, \dots, \mathbf{x}_N) \approx u(\mathbf{x}_1) \dots u(\mathbf{x}_N). \quad (2)$$

This does not depend much on the particular shape of the interactions and in fact also holds [18] for ground states

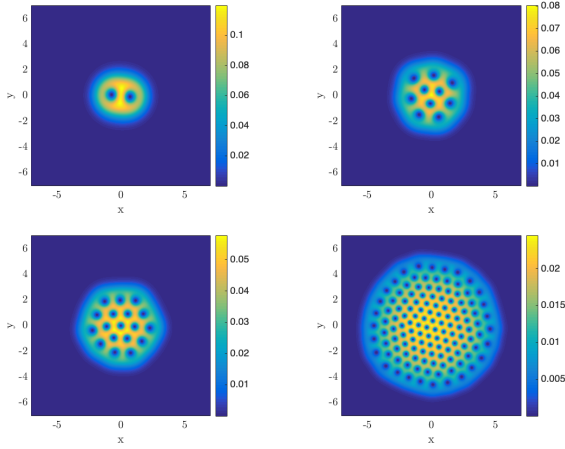


FIG. 1. Density $|u_\beta^{\text{af}}|^2$ in quadratic trap $V(\mathbf{x}) = |\mathbf{x}|^2$, at $\beta = 5, 15, 25, 140$.

of (1) in the limit $N \rightarrow \infty$ with α being scaled as

$$\alpha = \frac{\beta}{N}, \quad \beta \text{ fixed}, \quad (3)$$

a limit we refer to as ‘‘almost bosonic’’. Inserting the ansatz (2) in (1), the ground state energy and ground states of (1) are found by minimizing the ‘‘average-field’’ functional

$$\begin{aligned} \mathcal{E}_\beta^{\text{af}}[u] &= \int_{\mathbb{R}^2} |(-i\nabla + \beta \mathbf{A}[|u|^2]) u|^2 + V|u|^2 \\ \mathbf{A}[\rho](\mathbf{x}) &= \int_{\mathbb{R}^2} \frac{(\mathbf{x} - \mathbf{y})^\perp}{|\mathbf{x} - \mathbf{y}|^2} \rho(\mathbf{y}) d\mathbf{y}. \end{aligned} \quad (4)$$

From now on we shall be concerned only with the effective ground state problem

$$E_\beta^{\text{af}} = \min \left\{ \mathcal{E}_\beta^{\text{af}}[u], \quad \int_{\mathbb{R}^2} |u|^2 = 1 \right\} = \mathcal{E}_\beta^{\text{af}}[u_\beta^{\text{af}}], \quad (5)$$

which means that we expect our results to be quantitatively valid for anyons with small statistics parameter α . However, we expect the qualitative findings to be general. Note the kinship of this formalism with that of Chern-Simons theory, see [9, 13, 23] and references therein.

Local density approximation. In view of (3), β should actually be a large parameter in the $N \rightarrow \infty$ limit. This suggests looking first at the limit $\beta \rightarrow \infty$ of Problem (5). In this situation, the self-consistent magnetic interaction in (4) will make the gas expand, so that the first approximation that comes to mind is a LDA. We expect (and, in fact, proved in [6]) that in such a limit

$$\begin{aligned} E_\beta^{\text{af}} &\underset{\beta \rightarrow \infty}{\sim} E_\beta^{\text{TF}} \\ |u_\beta^{\text{af}}|^2 &\underset{\beta \rightarrow \infty}{\sim} \rho_\beta^{\text{TF}} \end{aligned} \quad (6)$$

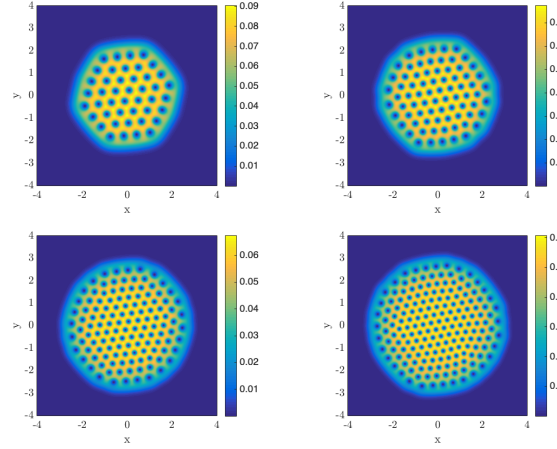


FIG. 2. Density $|u_\beta^{\text{af}}|^2$ in quartic trap $V(\mathbf{x}) = |\mathbf{x}|^4$, at $\beta = 55, 90, 140, 195$.

where E_β^{TF} is the local density approximation of the ground state energy:

$$\begin{aligned} E_\beta^{\text{TF}} &= \min \left\{ \mathcal{E}_\beta^{\text{TF}}[\rho], \quad \rho \geq 0, \int_{\mathbb{R}^2} \rho = 1 \right\} = \mathcal{E}_\beta^{\text{TF}}[\rho_\beta^{\text{TF}}] \\ \mathcal{E}_\beta^{\text{TF}}[\rho] &= \int_{\mathbb{R}^2} (e(\beta, \rho(\mathbf{x})) + V(\mathbf{x})\rho(\mathbf{x})) d\mathbf{x}. \end{aligned} \quad (7)$$

Here ρ plays the role of the matter density $|u|^2$ and $e(\beta, \rho)$ is the thermodynamic limit [6] of the ground state energy of the homogeneous analogue of (4), at average density ρ and scaled statistics parameter β . It follows from simple scaling considerations that

$$e(\beta, \rho) = \beta \rho^2 e(1, 1).$$

Thus

$$\mathcal{E}_\beta^{\text{TF}}[\rho] = \int_{\mathbb{R}^2} (\beta e(1, 1)\rho(\mathbf{x})^2 + V(\mathbf{x})\rho(\mathbf{x})) d\mathbf{x}.$$

The above is similar to the TF functional often used to describe BECs in first approximation (which in turn is, in a rather formal way, similar to the true TF functional for electrons). Note that E_β^{TF} depends on a single unknown parameter $e(1, 1)$, the thermodynamic ground state energy per unit area at density $\rho = 1$ and scaled statistics parameter $\beta = 1$.

Energy and density profile. The TF ground state problem is exactly solvable, modulo the unknown parameter $e(1, 1)$. One finds

$$\rho_\beta^{\text{TF}}(\mathbf{x}) = \frac{1}{2\beta e(1, 1)} (\lambda_\beta^{\text{TF}} - V(\mathbf{x}))_+ \quad (8)$$

where the chemical potential $\lambda_\beta^{\text{TF}}$ is set by the normalization $\int_{\mathbb{R}^2} \rho_\beta^{\text{TF}} = 1$ and $(\cdot)_+$ stands for the positive part.

More explicitly, we shall in the sequel restrict to radial power-law traps

$$V(\mathbf{x}) = |\mathbf{x}|^s, \quad s > 0. \quad (9)$$

The length scale of the cloud is then $\sim \beta^{1/(s+2)}$ in the limit $\beta \rightarrow \infty$ and, by scaling,

$$E_\beta^{\text{TF}} = \beta^{s/(s+2)} E_1^{\text{TF}} \quad (10)$$

in terms of the energy at $\beta = 1$

$$E_1^{\text{TF}} = \frac{1}{2} \frac{s}{s+1} \left(\frac{s+2}{s} \right)^{2 \frac{s+1}{s+2}} \left(\frac{2e(1,1)}{\pi} \right)^{\frac{s}{s+2}},$$

which can be combined with (10) to recover $e(1,1)$, given E_β^{TF} :

$$e(1,1) = \frac{\pi}{2} \left(2 \frac{s+1}{s} \right)^{\frac{s+2}{s}} \left(\frac{s}{s+2} \right)^{2 \frac{s+1}{s}} \frac{(E_\beta^{\text{TF}})^{\frac{s+2}{s}}}{\beta}. \quad (11)$$

Vortex density. Concerning the phase of the wave function we expect the appearance of quantized vortices with average vorticity density given by

$$\boxed{\mu_v = -2\pi\beta|u_\beta^{\text{af}}|^2 \approx -2\pi\beta\rho_\beta^{\text{TF}}}. \quad (12)$$

Actually, the very fact that a result such as (6) can hold is an indication that the system nucleates quantized vortices. Indeed, for the LDA to be acceptable, it must be that the long range forces encoded in the magnetic vector potential $\mathbf{A}[|u|^2]$ of (4) are screened. The physics is thus that a pattern of phase circulation is developed by the ground state on a ‘‘microscopic’’ length scale. This phase circulation cancels the long-range component of $\mathbf{A}[|u|^2]$ and allows for the TF profile to emerge on the macroscopic length scale $\beta^{1/(s+2)}$. We refer to [6] for details, in particular for the discussion of a trial state giving the correct energy by developing phase circulations on length scales $O(\beta^{-s/(2(s+2))})$.

It is intuitively clear that the phase circulation responsible for the validity of the LDA must come from quantized vortices in the gas. Indeed, if one writes the magnetic kinetic energy of (4) in the manner

$$\int_{\mathbb{R}^2} |\nabla\sqrt{\rho}|^2 + \rho|\nabla\varphi + \beta\mathbf{A}[\rho]|^2$$

by separating density ρ and phase φ , minimization of the second term would suggest

$$\nabla\varphi \approx -\beta\mathbf{A}[\rho]$$

and thus, taking the curl,

$$\text{curl}\nabla\varphi \approx -2\pi\beta\rho$$

which is possible only if φ has singularities with non-trivial circulation, i.e. vortices, distributed according

to (12). Just as in a rotating Bose gas [1, 4, 10], we expect that the vortices are singly quantized, for the self-energy of a vortex of degree d should be proportional to d^2 .

Another argument is as follows. Consider the energy $E^{\text{af}}(\beta, M)$ of a homogeneous anyon gas of total mass M in a fixed container Ω . Minimization of the functional (4) leads to the variational equation

$$\begin{aligned} & (-i\nabla + \beta\mathbf{A}[|u|^2])^2 u \\ & - 2\beta \frac{\mathbf{x}^\perp}{|\mathbf{x}|^2} * (\beta\mathbf{A}[|u|^2]|u|^2 + \mathbf{J}[u]) u = \lambda u, \end{aligned} \quad (13)$$

with the current

$$\mathbf{J}[u] := \frac{i}{2} (u\nabla\bar{u} - \bar{u}\nabla u), \quad (14)$$

and the chemical potential (Lagrange multiplier)

$$\lambda = \frac{\partial E^{\text{af}}(\beta, M)}{\partial M}.$$

In the case of a fixed container, (6)-(10) yield $E^{\text{af}}(\beta, M) \propto \beta M^2$, thus we expect that $\lambda \propto \beta M$. But, multiplying (13) by \bar{u} and integrating yields (boundary terms vanish because of magnetic Neumann boundary conditions)

$$\begin{aligned} & \int_{\Omega} |(-i\nabla + \beta\mathbf{A}[|u|^2]) u|^2 - \lambda \int_{\Omega} |u|^2 \\ & = 2\beta^2 \int_{\Omega} |u|^2 \left(\frac{\mathbf{x}^\perp}{|\mathbf{x}|^2} * \mathbf{K}[u] \right), \end{aligned} \quad (15)$$

with

$$\mathbf{K}[u] := \mathbf{A}[|u|^2]|u|^2 + \frac{1}{\beta}\mathbf{J}[u].$$

The left hand side of (15) is $O(\beta)$ so we should expect, for large β

$$\begin{aligned} \int_{\Omega} |u|^2 \left(\frac{\mathbf{x}^\perp}{|\mathbf{x}|^2} * \mathbf{K}[u] \right) & = \int_{\Omega} |u|^2 (\log|\mathbf{x}| * \text{curl}\mathbf{K}[u]) \\ & = O(\beta^{-1}). \end{aligned}$$

But, on a length scale where the density $|u|^2$ is homogeneous enough we may approximate

$$\text{curl}\mathbf{K}[u] \approx 2\pi|u|^4 + \frac{|u|^2}{\beta}\text{curl}\nabla\varphi,$$

so that the above again suggests that (12) must hold on sufficiently large scales. We vindicate this expectation in the sequel, by numerical simulations of the full ground state problem.

Numerical method. To minimize (4) we approximate the evolution in imaginary time of an initial trial state.

The actual implementation is by a pre-conditioned conjugate gradient method [38, 39] using the differential of the energy functional (see (13)), with a projection step ensuring the preservation of the mass. The most costly task is to compute the non-linear non-local terms in (13). Since they have the form of convolutions, we find it more convenient to work in Fourier space. Our numerical window is thus equipped with periodic boundary conditions and we use a Fast Fourier Transform with respect to a Cartesian grid, so that the non-linearities are simply dealt with by products in the Fourier domain. Note that the kernel defining $\mathbf{A}[\rho]$ is singular and long range, which forces us to use a cut-off in the space variable and a rather large computational domain in the Fourier space to evaluate it.

Numerical results: energy and density. We first plot the density in color levels for two model choices of the trapping potential in Figures 1 and 2. The main virtue of these plots is to make the vortex lattice apparent. For quantitative tests we first extract from the numerical data the value of the unknown parameter $e(1, 1)$ using (11). In Figure 3 we plot the value of $e(1, 1)$ so obtained, as a function of β for two types of traps, quadratic and quartic. The convergence for large β to the same value for the two traps is a clear sign of agreement between theory and numerics.

Note that we find $e(1, 1) \approx 2\pi \times 1.18 \approx 2\pi \times (2\sqrt{\pi}/3)$, which is *different* from a frequently used [3, 13, 29, 30, 32, 33] first guess which is as follows. If one imagines the homogeneous gas as bosons in the lowest Landau level of the approximately constant magnetic field of intensity $2\pi\beta\rho$, the energy per unit area is $e(\beta, \rho) = 2\pi\beta\rho^2$ per particle, hence $e(1, 1) = 2\pi$.

A further benchmark is given by a comparison of the theoretical and numerical density profiles. Now that $e(1, 1)$ has been extracted from the numerically computed energies, we can use its value to compute the TF density profile (8). For a reliable comparison with the numerical densities, one must first average out the fast oscillations of the latter, due to the vortex lattice. Comparisons of the theoretical TF profile and rotationally-averaged numerical profile are shown in Figure 4. We find an excellent agreement, in particular as regards the essential size of the cloud (TF radius).

Numerical results: vorticity. The vortex lattice in Figures 1 and 2 is clearly visible, as expected. For large values of β , its inhomogeneity becomes apparent, see in particular the plot for $\beta = 140$ of Figure 1 where vortices are much more tightly packed at the center of the cloud than at the boundary. For a more quantitative test, we can compare the numerical data to the expected finding (12). More precisely we count the number $N_v^{\text{num}}(r)$ of vortices (spotted as zeros in the density) present on the numerical figures within a disk of radius r and compare

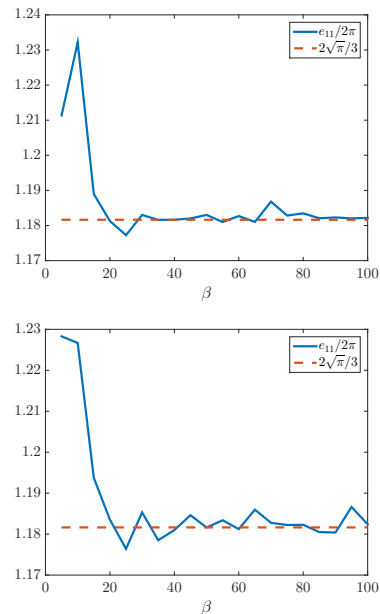


FIG. 3. Determination of the thermodynamic energy $e(1, 1)$, for quadratic (above) and quartic (below) traps.

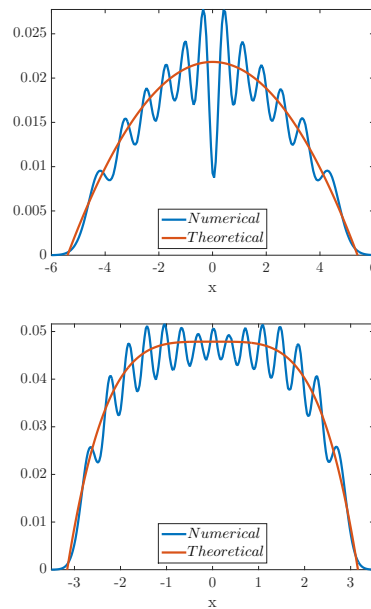


FIG. 4. Theoretical (red) and numerical (blue) density profiles, $V(\mathbf{x}) = |\mathbf{x}|^2, \beta = 90$ (above) and $V(\mathbf{x}) = |\mathbf{x}|^4, \beta = 140$ (below).

this to the expected number

$$N_v^{\text{theo}}(r) = 4\pi^2\beta \int_0^r \rho_\beta^{\text{TF}}(s) s ds.$$

Plots of the theoretical and numerical quantities are pro-

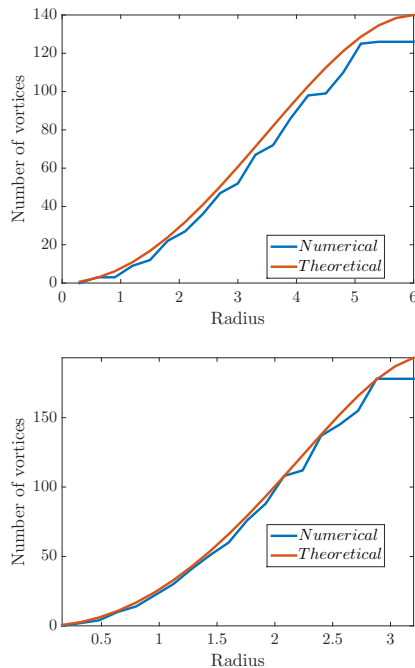


FIG. 5. Vortex density, theoretical (red) versus numerical (blue) $V(\mathbf{x}) = |\mathbf{x}|^2$, $\beta = 140$ (above) and $V(\mathbf{x}) = |\mathbf{x}|^4$, $\beta = 195$ (below)

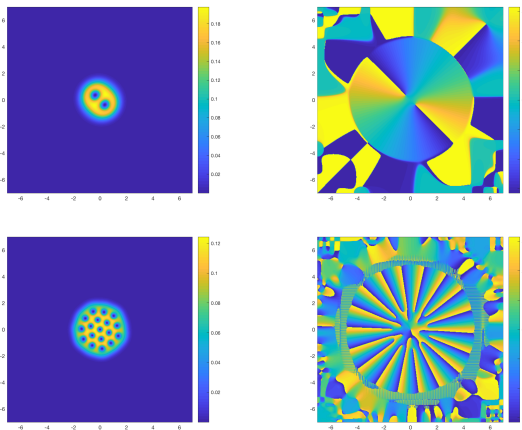


FIG. 6. Density and phase of minimizer u_{β}^{af} , quartic trap $V(\mathbf{x}) = |\mathbf{x}|^4$, at $\beta = 5, 25$.

vided in Figure 5.

Also note that the expected single quantization of vortices may readily be checked by plotting the phase of the wave function, see Figure 6.

Conclusions. We have studied a mean-field approximation of the ground state of the many-anyons gas, valid for “almost bosonic anyons”. A local density approximation suggests a Thomas-Fermi type density profile, whose

emergence is due to the screening of long-range effective magnetic interactions. This and further theoretical considerations suggest the nucleation of vortex patterns in the gas. Our numerical simulations confirm this picture and are found to be in very good agreement with the local density approximation. In particular, the vortex density is directly linked to the matter density, and thus inhomogeneous.

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- [1] A. AFTALION, *Vortices in Bose–Einstein Condensates*, vol. 67 of Progress in nonlinear differential equations and their applications, Birkhäuser, 2006.
- [2] S. AROVAS, J. SCHRIEFFER, AND F. WILCZEK, *Fractional statistics and the quantum Hall effect*, Phys. Rev. Lett., 53 (1984), pp. 722–723.
- [3] R. CHITRA AND D. SEN, *Ground state of many anyons in a harmonic potential*, Phys. Rev. B, 46 (1992), pp. 10923–10930.
- [4] N. R. COOPER, *Rapidly rotating atomic gases*, Advances in Physics, 57 (2008), pp. 539–616.
- [5] N. R. COOPER AND S. H. SIMON, *Signatures of Fractional Exclusion Statistics in the Spectroscopy of Quantum Hall Droplets*, Phys. Rev. Lett., 114 (2015), p. 106802.
- [6] M. CORREGGI, D. LUNDHOLM, AND N. ROUGERIE, *Local density approximation for the almost-bosonic anyon gas*, Analysis and PDEs, 10 (2017), pp. 1169–1200.
- [7] M. CORREGGI, F. PINSKER, N. ROUGERIE, AND J. YNGVASON, *Rotating superfluids in anharmonic traps: From vortex lattices to giant vortices*, Phys. Rev. A, 84 (2011), p. 053614.
- [8] M. CORREGGI AND N. ROUGERIE, *Inhomogeneous vortex patterns in rotating Bose-Einstein condensates*, Communications in Mathematical Physics, 321 (2013), pp. 817–860.
- [9] H. ENGER, *Vortices in Chern-Simons-Ginzburg-Landau theory and the fractional quantum Hall effect*. PhD thesis, University of Oslo, February 1998.
- [10] A. FETTER, *Rotating trapped Bose-Einstein condensates*, Rev. Mod. Phys., 81 (2009), p. 647.
- [11] F. D. M. HALDANE, *Fractional quantization of the Hall effect: A hierarchy of incompressible quantum fluid states*, Phys. Rev. Lett., 51 (1983), pp. 605–608.
- [12] B. I. HALPERIN, *Statistics of quasiparticles and the hierarchy of fractional quantized Hall states*, Phys. Rev. Lett., 52 (1984), pp. 1583–1586.
- [13] R. IENGO AND K. LECHNER, *Anyon quantum mechan-*

- ics and Chern-Simons theory*, Phys. Rep., 213 (1992), pp. 179–269.
- [14] A. KHARE, *Fractional Statistics and Quantum Theory*, World Scientific, Singapore, 2nd ed., 2005.
- [15] M. LEWIN, *Mean-Field limit of Bose systems: rigorous results*, Preprint (2015) arXiv:1510.04407.
- [16] ———, *Gaz de bosons dans le régime de champ moyen : les théories de Hartree et Bogoliubov*, in Séminaire Laurent Schwartz – EDP et applications, IHÉS, 2012-2013. Exp. no 3.
- [17] E. H. LIEB, R. SEIRINGER, J. P. SOLOVEJ, AND J. YNGVASON, *The mathematics of the Bose gas and its condensation*, Oberwolfach Seminars, Birkhäuser, 2005.
- [18] D. LUNDHOLM AND N. ROUGERIE, *The average field approximation for almost bosonic extended anyons*, J. Stat. Phys., 161 (2015), pp. 1236–1267.
- [19] ———, *Emergence of fractional statistics for tracer particles in a Laughlin liquid*, Phys. Rev. Lett., 116 (2016), p. 170401.
- [20] J. MYRHEIM, *Anyons*, in Topological aspects of low dimensional systems, A. Comtet, T. Jolicœur, S. Ouvry, and F. David, eds., vol. 69 of Les Houches - Ecole d'Ete de Physique Theorique, 1999, pp. 265–413.
- [21] C. NAYAK, S. H. SIMON, A. STERN, M. FREEDMAN, AND S. DAS SARMA, *Non-abelian anyons and topological quantum computation*, Rev. Mod. Phys., 80 (2008), pp. 1083–1159.
- [22] S. OUVRY, *Anyons and lowest Landau level anyons*, Séminaire Poincaré, 11 (2007), pp. 77–107.
- [23] K. RAJARATNAM, *Vortex lattice solutions of the ZHK Chern-Simons equations*. PhD thesis, University of Toronto, 2019.
- [24] N. ROUGERIE, *De Finetti theorems, mean-field limits and Bose-Einstein condensation*. arXiv:1506.05263, 2014. LMU lecture notes.
- [25] ———, *Théorèmes de De Finetti, limites de champ moyen et condensation de Bose-Einstein*, Les cours Peccot, Spartacus IDH, Paris, 2016. Cours Peccot, Collège de France : février-mars 2014.
- [26] E. SANDIER AND S. SERFATY, *Vortices in the magnetic Ginzburg-Landau model*, Progress in Nonlinear Differential Equations and their Applications, 70, Birkhäuser Boston, Inc., Boston, MA, 2007.
- [27] D. R. TILLEY AND J. TILLEY, *Superfluidity and Superconductivity*, Institute of Physics publishing, Bristol and Philadelphia, 1990.
- [28] M. TINKHAM, *Introduction to Superconductivity*, McGraw-Hill, New York, 1975.
- [29] C. TRUGENBERGER, *Ground state and collective excitations of extended anyons*, Phys. Lett. B, 288 (1992), pp. 121–128.
- [30] ———, *The anyon fluid in the Bogoliubov approximation*, Phys. Rev. D, 45 (1992), pp. 3807–3817.
- [31] R. O. UMUCAL ILAR, E. MACALUSO, T. COMPARIN, AND I. CARUSOTTO, *Time-of-flight measurements as a possible method to observe anyonic statistics*, Phys. Rev. Lett., 120 (2018), p. 230403.
- [32] X. G. WEN AND A. ZEE, *Compressibility and superfluidity in the fractional-statistics liquid*, Phys. Rev. B, 41 (1990), pp. 240–253.
- [33] E. WESTERBERG, *Mean-field approximation for anyons in a magnetic field*, Int. J. Mod. Phys. B, 7 (1993), pp. 2177–2199.
- [34] F. WILCZEK, *Fractional Statistics and Anyon Superconductivity*, World Scientific, Singapore, 1990.
- [35] E. YAKABOYLU AND M. LEMESHKO, *Anyonic statistics of quantum impurities in two dimensions*, Phys. Rev. B, 98 (2018), p. 045402.
- [36] Y. ZHANG, G. J. SREEJITH, N. D. GEMELKE, AND J. K. JAIN, *Fractional angular momentum in cold atom systems*, Phys. Rev. Lett., 113 (2014), p. 160404.
- [37] Y. ZHANG, G. J. SREEJITH, AND J. K. JAIN, *Creating and manipulating non-Abelian anyons in cold atom systems using auxiliary bosons*, Phys. Rev. B., 92 (2015), p. 075116.
- [38] X. ANTOINE, A. LEVITT AND Q. TANG, *Efficient spectral computation of the stationary states of rotating Bose-Einstein condensates by preconditioned nonlinear conjugate gradient methods*, J. Comput. Phys., 343 (2017), pp. 92–109.
- [39] I. DANAILA AND B. PROTAS, *Computation of Ground States of the Gross-Pitaevskii Functional via Riemannian Optimization*, SIAM J. Scientific Computing, 39(6) (2017), p. B1102-B1129.