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# Inverse problems for quasi-variational inequalities

Akhtar A. Khan<sup>1</sup> · Dumitru Motreanu<sup>2</sup>

**Abstract** In this short note, our aim is to investigate the inverse problem of parameter identification in quasi-variational inequalities. We develop an abstract nonsmooth regularization approach that subsumes the total variation regularization and permits the identification of discontinuous parameters. We study the inverse problem in an optimization setting using the output-least squares formulation. We prove the existence of a global minimizer and give convergence results for the considered optimization problem. We also discretize the identification problem for quasi-variational inequalities and provide the convergence analysis for the discrete problem. We give an application to the gradient obstacle problem.

**Keywords** Inverse problems · Regularization · Output least-squares · Quasi-variational inequalities

**Mathematics Subject Classification** 35R30 · 49N45 · 65J20 · 65J22 · 65M30

## 1 Introduction

Applied models frequently lead to variational problems involving parameters characterizing physical features of the model. The direct problem in this setting is to solve the variational problem. By contrast, an inverse problem seeks the identification of the parameters from a measurement of a solution of the variational problem. In recent years, the field of inverse problems emerged as one of the most vibrant and expanding branches of applied mathematics.

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Probably the main reason behind this is the increasing number of real-world situations that are being modeled and studied in a unified framework of inverse problems. However, the theoretical aspects of inverse problems are also quite challenging and require a delicate blending of various branches of mathematics.

In this paper, our aim is to study, for the first time, the inverse problem of parameter identification in quasi-variational inequalities. In our approach, the parameter identification in the considered quasi-variational inequality, which provides the solvability of the stated inverse problem, is achieved through the global minimization problem (see (9)) constructed with the regularization map. The global minimizers are found in Theorem 3.1.

The theory of quasi-variational inequalities has now established itself as one of the most promising areas of applied mathematics, offering a powerful mathematical apparatus for investigating a broad range of problems arising in diverse disciplines. Applications of quasi-variational inequalities can be found in material science [12], equilibrium models [2,22], financial models [6], frictional elastostatic contact [19], image processing [16], sand-piles formation [3], and numerous others.

We denote the parameter space by  $B$  which is a Banach space. We denote the set of admissible parameters by  $A$  which is a nonempty, closed, and convex subset of  $B$ . We will pose quasi-variational inequality in a Hilbert space  $V$  which we identify with its dual  $V^*$ . We take the measured data in a Hilbert space  $Z$  such that  $V$  continuously embeds in  $Z$ . We specify the strong convergence by  $\rightarrow$  and the weak convergence by  $\rightharpoonup$ . Assume that  $C$  is a nonempty, closed, and convex subset of  $V$ ,  $K : C \rightrightarrows C$  is a set-valued map such that for every  $u \in C$ , the set  $K(u)$  is a nonempty, closed, and convex subset of  $C$ , and  $m \in V^*$  is fixed. Assume that there is a bounded set  $C_0 \subset V$  such that  $K(v) \cap C_0 \neq \emptyset$ , for every  $v \in C$ . We define a trilinear form  $T : B \times V \times V \rightarrow \mathbb{R}$ , with  $T(a, u, v)$  symmetric in  $u$  and  $v$ , and assume that there are constants  $\alpha > 0$  and  $\beta > 0$  such that

$$T(a, u, v) \leq \beta \|a\|_B \|u\|_V \|v\|_V, \quad \text{for all } u, v \in V, a \in B, \quad (1)$$

$$T(a, u, u) \geq \alpha \|u\|_V^2, \quad \text{for all } u \in V, a \in A. \quad (2)$$

We formulate the quasi-variational inequality: Given  $a \in A$ , find  $u \in K(u)$  such that

$$T(a, u, v - u) \geq \langle m, v - u \rangle, \quad \text{for every } v \in K(u). \quad (3)$$

Given  $a \in A$ , the quasi-variational inequality of finding  $u = u(a)$  constitutes the direct problem. In contrast, the inverse problem seeks to identify the coefficient  $a$  from a measurement  $z$  of a solution  $u$  of the quasi-variational inequality. Quasi-variational inequality (3), introduced in connection with an impulse control problem (see Bensoussan and Lions [4], Mosco [18]), is convenient for many applications such as implicit obstacle problem, dam problems, and others. See [8, 13, 14] for more details.

If  $K(u) = C$  for all  $u \in C$ , then (3) becomes the variational inequality: find  $u \in C$  such that

$$T(a, u, v - u) \geq \langle m, v - u \rangle, \quad \text{for every } v \in C. \quad (4)$$

Variational inequality (4) has been extensively studied in the literature and has found numerous applications. Identification problems in variational inequalities have also been studied (see [10]).

## 2 Solvability of quasi-variational inequalities

The existence theory for quasi-variational inequalities is challenging. Although some existence results exploit the ordering structure of the underlying spaces, one of the most commonly adopted techniques for solving quasi-variational inequalities is by finding fixed points of the associated variational selection.

To define the variational selection, we fix an arbitrary element  $w \in C$  and consider the following variational inequality: Given  $a \in A$ , find  $u \in K(w)$  such that

$$T(a, u, v - u) \geq \langle m, v - u \rangle, \quad \text{for every } v \in K(w). \quad (5)$$

The variational selection is a (set-valued, in general) map  $S : C \rightrightarrows C$  such that for any  $w \in C$ , the image  $S(w)$  is the set of all solutions of (5). Evidently, if  $u$  is a fixed point of the map  $S$ , that is,  $u \in S(u)$ , then  $u$  solves quasi-variational inequality (3).

We shall use the fixed point theorem of Kluge [15] to find a fixed point of the variational selection:

**Theorem 2.1** [15] *Let  $\mathcal{Z}$  be a reflexive Banach space and let  $C \subset \mathcal{Z}$  be nonempty, convex, and closed. Assume that  $\Psi : C \rightrightarrows C$  is a set-valued map such that for every  $u \in C$ , the set  $\Psi(u)$  is nonempty, closed, and convex, and the graph of  $\Psi$  is sequentially weakly closed. Assume that either the set  $C$  is bounded or the set  $\Psi(C)$  is bounded. Then the map  $\Psi$  has at least one fixed point in  $C$ .*

We have the following existence result for quasi-variational inequality (3).

**Theorem 2.2** *Besides the general setting described above, assume that the map  $K : C \rightrightarrows C$  is  $M$ -continuous, that is, it satisfies the following conditions:*

- (M1) *For any sequence  $\{x_n\} \subset C$  with  $x_n \rightharpoonup x$ , and for each  $y \in K(x)$ , there exists a sequence  $\{y_n\}$  such that  $y_n \in K(x_n)$  and  $y_n \rightarrow y$ .*
- (M2) *For  $y_n \in K(x_n)$  with  $x_n \rightharpoonup x$  and  $y_n \rightharpoonup y$ , we have  $y \in K(x)$ .*

*Then, for every  $a \in A$ , the quasi-variational inequality (3) has a nonempty solution set.*

*Proof* We will divide the proof into several parts. Our aim is to show that the variational selection  $S : C \rightrightarrows C$  satisfies the assumptions imposed on the map  $\Psi$  in Theorem 2.1.

For a given  $a \in A$  and a fixed  $w \in C$ , the parametric variational inequality (5) is uniquely solvable. Therefore, for the given  $a \in A$ , and each  $w \in C$ , the solution set  $S(w)$  is nonempty, and (trivially) closed and convex.

We shall now prove that the graph of  $S$  is sequentially weakly closed. Let  $\{(v_n, y_n)\} \subset C \times C$  be such that  $y_n = S(v_n)$  with  $y_n \rightharpoonup y$  and  $v_n \rightharpoonup v$ . We will show that  $y = S(v)$ . The set  $C$  being convex and closed is also weakly closed, and hence  $v \in C$ . Since  $y_n = S(v_n)$ , we infer that  $y_n \in K(v_n)$  and

$$T(a, y_n, z - y_n) \geq \langle m, z - y_n \rangle, \quad \text{for every } z \in K(v_n). \quad (6)$$

The containment  $y_n \in K(v_n)$ , by using the  $M$ -continuity of the map  $K$  and taking into account the convergence  $y_n \rightharpoonup y$  and  $v_n \rightharpoonup v$ , implies that  $y \in K(v)$  (see (M2)). Let  $z \in K(v)$  be arbitrary. Using the  $M$ -continuity once again (precisely, condition (M1)) ensure that there is a sequence  $\{z_n\}$  converging strongly to  $z$  and satisfying  $z_n \in K(v_n)$ . We insert  $z = z_n \in K(v_n)$  in (6) and rearrange the resulting inequality as follows

$$T(a, z_n, z_n - y_n) \geq T(a, y_n - z_n, y_n - z_n) + \langle m, z_n - y_n \rangle \geq \langle m, z_n - y_n \rangle,$$

which when passed to the limit  $n \rightarrow \infty$  implies that for every  $z \in K(v)$ , we have

$$T(a, z, z - y) \geq \langle m, z - y \rangle.$$

We now insert  $y + t(z - y)$  with  $t \in (0, 1)$  in place of  $z$  in the above inequality to get

$$T(a, y, z - y) + tT(a, z - y, z - y) \geq \langle m, z - y \rangle.$$

We pass in the above inequality to limit  $t \rightarrow 0$  obtaining

$$T(a, y, z - y) \geq \langle m, z - y \rangle, \quad \text{for every } z \in K(v),$$

which ensues that  $y \in S(v)$ . Finally, the boundedness of  $S(C)$  follows from the ellipticity of  $T$ . Theorem 2.1 then confirms that there is a fixed point for  $S$ , which solves (3). The proof is complete.  $\square$

*Remark 2.1* The linearity of  $T(a, u, v)$  with respect to  $a$  does not play any role in the above result.

### 3 Solvability of the inverse problem

We will now introduce an optimization framework to study the inverse problem of parameter identification in quasi-variational inequalities. To counter the adverse effects of the ill-posedness of the inverse problem, we develop an abstract regularization framework by assuming:

- (A1) The Banach space  $B$  is continuously embedded in a Banach space  $L$ . There is another Banach space  $\widehat{B}$  that is compactly embedded in  $L$ . The set  $A$  consists of real-valued functions and is a subset of  $B \cap \widehat{B}$ , closed and bounded in  $B$  and also closed in  $L$ .
- (A2) For any sequence  $\{b_k\} \subset B$  with  $b_k \rightarrow 0$  in  $L$ , any bounded sequence  $\{u_k\} \subset V$ , and fixed  $v \in V$ , we have

$$T(b_k, u_k, v) \rightarrow 0. \quad (7)$$

- (A3)  $R : \widehat{B} \rightarrow \mathbb{R}$  is convex, and lower-semicontinuous with respect to  $\|\cdot\|_L$  such that

$$R(a) \geq \tau_1 \|a\|_{\widehat{B}} - \tau_2, \quad \text{for every } a \in A, \quad \text{for some } \tau_1 > 0, \tau_2 > 0. \quad (8)$$

Using the above framework, which will be exemplified shortly, we consider the following regularized output-least-squares (OLS) optimization problem: Find  $a \in A$  by solving

$$\min_{a \in A} J_\kappa(a) := \frac{1}{2} \|u(a) - z\|_Z^2 + \kappa R(a), \quad (9)$$

where  $\kappa > 0$  is a regularization parameter,  $R$  is the regularization map introduced in (A3),  $u(a)$  is the solution of (3), and  $z \in Z$  is the measured data.

For recent developments in inverse problems, see [5, 7, 9, 11, 17, 21] and the cited references therein.

We have the following existence result:

**Theorem 3.1** *For  $\kappa > 0$ , regularized output-least-squares problem (9) has a nonempty solution set.*

*Proof* The functional  $J_\kappa$  is bounded from below and hence there exists a minimizing sequence. Let  $\{a_n\} \subset A$  be a minimizing sequence such that  $\lim_{n \rightarrow \infty} J_\kappa(a_n) =$

$\inf\{J_\kappa(b) \mid b \in A\}$ . Let  $u_n = u(a_n)$  be the solution of quasi-variational inequality (3) for the parameter  $a_n$ . Due to the definition of  $\{a_n\}$  and the inequality  $\kappa R(a_n) \leq J_\kappa(a_n)$ , the sequence  $\{a_n\}$  is bounded in  $\widehat{B}$ . Due to the compact embedding of  $\widehat{B}$  into  $L$ ,  $\{a_n\}$  has a subsequence which converges strongly in  $\|\cdot\|_L$ . Keeping the same notation for subsequences as well, let  $\{a_n\}$  be the subsequence converging in  $\|\cdot\|_L$  to some  $\bar{a} \in A$ .

On the other hand, by the definition of  $u_n$  (see (3)), we have  $u_n \in K(u_n)$  and

$$T(a_n, u_n, v - u_n) \geq \langle m, v - u_n \rangle, \quad \text{for every } v \in K(u_n). \quad (10)$$

Let  $\{u_n\}$  be a sequence of solutions of (10) that correspond to the subsequence of parameters  $\{a_n\}$  converging to  $\bar{a} \in A$ . We claim that  $\{u_n\}$  is bounded. By assumption made before (1) there exists a bounded sequences  $\{s_n\}$  with  $s_n \in K(u_n) \cap C_0$  for each  $n$ . By taking  $v = s_n$  in (10), we obtain

$$T(a_n, u_n, s_n - u_n) \geq \langle m, s_n - u_n \rangle,$$

which by using the ellipticity of  $T$  in (2) as well as (1) ensures that the sequence  $\{u_n\}$  is bounded.

Since  $\{u_n\}$  is bounded, there exists a weakly convergent subsequence in  $V$ . Keeping the same notation for the subsequence, let  $\{u_n\}$  be the subsequence which converges weakly to some  $\bar{u} \in V$ . We shall show that  $\bar{u} = u(\bar{a})$ . Since  $u_n \in K(u_n)$  and since  $u_n$  converges weakly to  $\bar{u}$ , by using condition (M2) of the  $M$ -continuity of the map  $K$ , we obtain  $\bar{u} \in K(\bar{u})$ . Let  $z \in K(\bar{u})$  be arbitrary. Using condition (M1) of the  $M$ -continuity, we ensure that there is a sequence  $\{z_n\}$  converging strongly to  $z$  in  $V$  and satisfies  $z_n \in K(u_n)$ . Setting  $v = z_n \in K(u_n)$  in (10) and using the identity

$$\begin{aligned} T(a_n, u_n, z_n - u_n) &= T(a_n, u_n - z_n, z_n - u_n) + T(a_n, z_n - z, z_n - u_n) \\ &\quad + T(a_n - \bar{a}, z, z_n - u_n) + T(\bar{a}, z, z_n - u_n) \end{aligned}$$

and the ellipticity of  $T$  in (2), we obtain

$$T(a_n, z_n - z, z_n - u_n) + T(a_n - \bar{a}, z, z_n - u_n) + T(\bar{a}, z, z_n - u_n) \geq \langle m, z_n - u_n \rangle,$$

which, by (7) and the symmetry of  $T(a, u, v)$  in  $u$  and  $v$ , when passed to the limit  $n \rightarrow \infty$ , yields

$$T(\bar{a}, z, z - \bar{u}) \geq \langle m, z - \bar{u} \rangle, \quad \text{for every } z \in K(\bar{u}).$$

We set  $\bar{u} + t(z - \bar{u})$ ,  $t \in (0, 1)$ , in place of  $z$  in the above inequality to get

$$T(\bar{a}, \bar{u} + t(z - \bar{u}), z - \bar{u}) \geq \langle m, z - \bar{u} \rangle.$$

We pass in the above inequality to limit  $t \rightarrow 0$  to get

$$T(\bar{a}, \bar{u}, z - \bar{u}) \geq \langle m, z - \bar{u} \rangle,$$

which, due to the fact that  $z \in K(\bar{u})$  was chosen arbitrarily, ensures that  $\bar{u}$  solves (3) for  $\bar{a}$ .

The weak lower-semicontinuity of the norm and lower-semicontinuity of  $R$  in  $\|\cdot\|_L$  yield

$$\begin{aligned}
J_\kappa(\bar{a}) &= \frac{1}{2}\|u(\bar{a}) - z\|_Z^2 + \kappa R(\bar{a}) \\
&\leq \liminf_{n \rightarrow \infty} \frac{1}{2}\|u_n - z\|_Z^2 + \liminf_{n \rightarrow \infty} \kappa R(a_n) \\
&\leq \liminf_{n \rightarrow \infty} \left\{ \frac{1}{2}\|u_n - z\|_Z^2 + \kappa R(a_n) \right\} \\
&= \lim_{n \rightarrow \infty} J_\kappa(a_n) = \inf \{J_\kappa(b) \mid b \in A\},
\end{aligned}$$

which confirms that  $\bar{a}$  is a solution of (9). The proof is complete.  $\square$

The above framework simplifies nicely (and does not need  $\widehat{B}$  and  $L$ ), if the regularization space  $B$  and the parameter space are linked. The following result sheds some light on this aspect:

**Theorem 3.2** *Assume that a Hilbert space  $\widehat{H}$  is compactly embedded into the Banach space  $B$ ,  $A \subset \widehat{H}$  is nonempty, closed, and convex, the map  $R : \widehat{H} \rightarrow \mathbb{R}$  is convex, lower-semicontinuous, and there exist  $\alpha_1 > 0$  and  $\alpha_2 > 0$  such that  $R(a) \geq \alpha_1 \|a\|_{\widehat{H}}^2 - \alpha_2$ , for every  $a \in A$ . Then (9) is solvable.*

*Proof* The proof follows by the arguments used above but relying on (1) (instead of (7)).  $\square$

## 4 Finite-dimensional approximation

We shall discretize the identification problem for quasi-variational inequalities. Recall that we have been dealing with the quasi-variational inequality: Given  $a \in A$ , find  $u = u(a) \in K(u)$  such that

$$T(a, u, v - u) \geq \langle m, v - u \rangle, \quad \text{for every } v \in K(u). \quad (11)$$

We considered the following regularized optimization problem: Find  $a \in A$  by solving

$$\min_{a \in A} J_\kappa(a) = \frac{1}{2}\|u(a) - z\|_Z^2 + \kappa R(a), \quad (12)$$

where  $\kappa > 0$  is a regularization parameter,  $R$  is the regularization map,  $u(a)$  is the unique solution of (11), and  $z \in Z$  is the measured data.

We now proceed to describe the discretization framework. Assume that we have a family  $\{V_n\}$  of finite-dimensional subspaces of  $V$ , and for each  $n$ ,  $P_n : V \rightarrow V_n$  is a projection operator such that  $\|v - P_n v\|_V \rightarrow 0$  for each  $v \in V$ . We assume that  $\{B_n\}$  is a family of finite-dimensional subspaces of  $B$ . We set  $A_n = B_n \cap A$  and assume that  $\bigcap_n A_n \neq \emptyset$ . For each  $n \in \mathbb{N}$ , let  $C_n \subset V_n$  be a nonempty, closed, and convex set. Let  $K_n : C_n \rightrightarrows C_n$  be a set-valued map such that for every  $v \in C_n$ , the set  $K_n(v)$  is nonempty, closed, and convex. We assume that the maps  $K_n$  approximate  $K$  in the following sense:

(M3) For any sequence  $\{x_n\}$  with  $x_n \in C_n$  and  $x_n \rightarrow x$  in  $V$ , and for each  $y \in K(x)$ , there exists a sequence  $\{y_n\}$  such that  $y_n \in K_n(x_n)$  and  $y_n \rightarrow y$ .

(M4) For  $y_n \in K_n(x_n)$  with  $x_n \rightarrow x$  and  $y_n \rightarrow y$  in  $V$ , we have  $y \in K(x)$ .

We denote by  $R_n : A_n \rightarrow \mathbb{R}$  the discrete analogue of  $R$  and assume the following conditions:

1. For any  $a \in A$ , there exists a sequence  $\{a_n\}$  with  $a_n \in A_n$  such that  $a_n \rightarrow a$  in  $L$  and

$$\limsup_{n \rightarrow \infty} R_n(a_n) \leq R(a). \quad (13)$$

2. For every sequence  $\{a_n\}$  with  $a_n \in A_n$  and  $a_n \rightarrow a$  in  $L$ , we have

$$R(a) \leq \liminf_{n \rightarrow \infty} R_n(a_n). \quad (14)$$

The discrete analogue of (11) then reads: Given  $a_n \in A_n$ , find  $u_n \in K_n(u_n)$  such that

$$T(a_n, u_n, v - u_n) \geq \langle m, v_n - u_n \rangle, \quad \text{for every } v \in K_n(u_n). \quad (15)$$

Consider the following finite-dimensional minimization problem: Find  $\bar{a}_n \in A_n$  by solving

$$\min_{a_n \in A_n} J_\kappa^n(a_n) = \frac{1}{2} \|u_n(a_n) - z\|_Z^2 + \kappa R_n(a_n), \quad (16)$$

where  $u_n(a_n)$  is the solution of (15). Evidently, the data  $z$  could be replaced by discrete data  $z_n$  such that  $z_n \rightarrow z$  as  $n \rightarrow \infty$ .

The following is a convergence result for the finite-dimensional optimization problem (16):

**Theorem 4.1** *For each  $n$ , discrete problem (16) has a minimizer  $\bar{a}_n \in A_n$ . Moreover, there is a subsequence of  $\{\bar{a}_n\}$  that converges in  $\|\cdot\|_L$  to a solution of (12).*

*Proof* The solvability follows from the arguments used before. Due to the assumption  $\bigcap_n A_n \neq \emptyset$ , there is a constant  $c$  such that  $J_\kappa^n(\bar{a}_n) \leq c$ , for every  $n$ . Therefore,  $\{\bar{a}_n\}$  is bounded in  $\|\cdot\|_{\widehat{B}}$ . Due to the compact embedding of  $\widehat{B}$  into  $L$ , there is a subsequence that converges strongly, in  $\|\cdot\|_L$ , to an element of  $A$ . Keeping the same notation for the subsequences, we assume that  $\{\bar{a}_n\}$  converges, in  $\|\cdot\|_L$ , to some  $\bar{a} \in A$ . Let  $\bar{u}$  be the solution of quasi-variational inequality (11) for the parameter  $\bar{a}$  and let  $\bar{u}_n$  be the solution of quasi-variational inequality (15) for the parameter  $\bar{a}_n$ . It can be shown as in Theorem 3.1 that the sequence  $\{\bar{u}_n\}$  is bounded in  $V$ . Therefore, there exists a subsequence, still denoted by  $\{\bar{u}_n\}$ , such that  $\bar{u}_n$  converges weakly to some  $\bar{u} \in V$  as  $n \rightarrow \infty$ . By the definition of the solution of (15), we have  $\bar{u}_n \in K_n(\bar{u}_n)$  which implies that  $\bar{u} \in K(\bar{u})$  (see assumption (M4)). We will show that  $\bar{u} = u(\bar{a})$ .

By the definition of  $\bar{u}_n$ , we have  $\bar{u}_n \in K_n(\bar{u}_n)$  and

$$T(\bar{a}_n, \bar{u}_n, v - \bar{u}_n) \geq \langle m, v - \bar{u}_n \rangle, \quad \text{for every } v \in K_n(\bar{u}_n).$$

Let  $z \in K(\bar{u})$  be arbitrary and let  $\{z_n\}$  be a sequence with  $z_n \in K_n(\bar{u}_n)$  such that  $z_n \rightarrow z$  in  $V$  (see assumption (M3)). We set  $v = z_n$  in the above inequality, and after a rearrangement, obtain,

$$T(\bar{a}_n, \bar{u}_n - z, z_n - \bar{u}_n) + T(\bar{a}_n - \bar{a}, z, z_n - \bar{u}_n) + T(\bar{a}, z, z_n - \bar{u}_n) \geq \langle m, v - \bar{u}_n \rangle$$

which, in view of (2) and (7), when passed to the limit  $n \rightarrow \infty$ , implies that

$$T(\bar{a}, z, z - \bar{u}) \geq \langle m, z - \bar{u} \rangle, \quad \text{for every } z \in K(\bar{u})$$

ensuring that  $\bar{u} = u(\bar{a})$ .



This observation, in view of (14), yields

$$\begin{aligned}
J_\kappa(\bar{a}) &= \frac{1}{2} \|\bar{u} - z\|_Z^2 + \kappa R(\bar{a}) \\
&\leq \lim_{n \rightarrow \infty} \frac{1}{2} \|\bar{u}_n - z\|_Z^2 + \liminf_{h \rightarrow 0} \kappa R_n(\bar{a}_n) \\
&\leq \liminf_{n \rightarrow \infty} \left\{ \frac{1}{2} \|\bar{u}_n - z\|_Z^2 + \kappa R_n(\bar{a}_n) \right\}.
\end{aligned}$$

Let  $a \in A$  be arbitrary. Then there exists a sequence  $\{a_n\}$  such that  $a_n \in A_n$ ,  $a_n \rightarrow a$  in  $L$  and  $R_n(a_n) \rightarrow R(a)$  as  $n \rightarrow \infty$  (see (13) and (14)). We have shown above that the corresponding solutions  $\{u_n\}$  with  $u_n = u_n(a_n)$  converges weakly to  $u(a)$ . We need that in fact the convergence is strong. We note that  $u_n \in K_n(u_n)$  and

$$T(a_n, u_n, z - u_n) \geq \langle m, z - u_n \rangle, \quad \text{for every } z \in K_n(u_n).$$

Due to the containment  $u_n \in K_n(u_n)$  and the fact that  $u_n$  converges weakly to some  $u$ , we have  $u \in K(u)$  (see assumption (M4)). Moreover, there exists  $w_n \in V_n$  such that  $w_n \in K_n(u_n)$  and  $w_n \rightarrow u$  strongly (see assumption (M3)). By setting  $z = w_n$  in the above inequality, we obtain

$$T(a_n, u_n, w_n - u_n) \geq \langle m, w_n - u_n \rangle,$$

which can be rearranged as follows

$$T(a_n, u_n - u, w_n - u) + T(a_n, u_n - u, u - u_n) + T(a_n, u, w_n - u_n) + \langle m, u_n - w_n \rangle \geq 0.$$

By (2), this implies that

$$T(a_n, u_n - u, w_n - u) + T(a_n, u, w_n - u_n) + \langle m, u_n - w_n \rangle \geq \alpha \|u_n - u\|^2,$$

or equivalently

$$\begin{aligned}
&T(a_n, u_n - u, w_n - u) + T(a_n - a, u, w_n - u_n) + T(a, u, w_n - u_n) \\
&\quad + \langle m, u_n - w_n \rangle \geq \alpha \|u_n - u\|^2
\end{aligned}$$

which by (7) when passed to the limit  $n \rightarrow \infty$  implies that  $\{u_n\}$  converges strongly to  $u$ .

In view of the above observations, we have

$$\begin{aligned}
J_\kappa(\bar{a}) &\leq \liminf_{n \rightarrow \infty} \left\{ \frac{1}{2} \|u(\bar{a}_n) - z\|_Z^2 + \kappa R_n(\bar{a}_n) \right\} \\
&\leq \liminf_{h \rightarrow 0} \left\{ \frac{1}{2} \|u_n(a_n) - z\|_Z^2 + \kappa R_n(a_n) \right\} \\
&= \frac{1}{2} \|u(a) - z\|_Z^2 + \kappa R(a) = J_\kappa(a),
\end{aligned}$$

and hence  $\bar{a}$  is a solution of (12). This completes the proof.  $\square$

## 5 An application

We now justify our framework by identifying a parameter in a gradient obstacle problem. Let  $\Omega$  be a bounded domain in  $R^N$ , where  $1 \leq N < \infty$ , with a sufficiently smooth boundary. We set  $V = H_0^1(\Omega)$  and  $B = L^\infty(\Omega)$ . Let  $\kappa^*$  be a constant, and let  $\kappa_c$  be a Lipschitz continuous

real function on  $\mathbb{R}$  such that  $0 < \kappa_c(r) \leq \kappa^*$ , for every  $r \in \mathbb{R}$ . Define a convex set  $C$  and  $K : C \rightrightarrows C$  by

$$\begin{aligned} C &= \{w \in V \mid |\nabla w| \leq \kappa^*, \text{ a.e. on } \Omega\}, \\ K(v) &= \{w \in V \mid |\nabla w| \leq \kappa_c(v), \text{ a.e. on } \Omega\}, \end{aligned}$$

which satisfy conditions (M1) and (M2) of Theorem 2.2 (see [12]).

We focus on identifying  $a$  in the quasi-variational inequality: Find  $u \in K(u)$  such that

$$T(a, u, v - u) \geq \langle m, v - u \rangle, \quad \text{for every } v \in K(u),$$

where  $T(a, u, v) = \int_{\Omega} a \nabla u \nabla v$ .

To introduce the regularization space, we recall that the total variation of  $f \in L^1(\Omega)$  is given by

$$\text{TV}(f) = \sup \left\{ \int_{\Omega} f(\nabla \cdot g) : g \in (C^1(\Omega))^N, |g(x)| \leq 1 \text{ for all } x \in \Omega \right\}$$

where  $|\cdot|$  represents the Euclidean norm. Clearly, if  $f \in W^{1,1}(\Omega)$ , then  $\text{TV}(f) = \int_{\Omega} |\nabla f|$ .

If  $f \in L^1(\Omega)$  satisfies  $\text{TV}(f) < \infty$ , then  $f$  is said to have bounded variation, and the Banach space  $\text{BV}(\Omega)$  is defined by  $\text{BV}(\Omega) = \{f \in L^1(\Omega) : \text{TV}(f) < \infty\}$  being endowed with the norm  $\|f\|_{\text{BV}(\Omega)} = \|f\|_{L^1(\Omega)} + \text{TV}(f)$ . The functional  $\text{TV}(\cdot)$  is a seminorm on  $\text{BV}(\Omega)$  and is often called the BV-seminorm.

We set  $L = L^1(\Omega)$ ,  $\widehat{B} = \text{BV}(\Omega)$ , and  $R(a) = TV(a)$ , and define two sets

$$\begin{aligned} A_1 &= \{a \in L^\infty(\Omega) \mid c_1 \leq a(x) \leq c_2, \text{ a.e. in } \Omega, \}, \\ A_2 &= \{a \in L^\infty(\Omega) \mid c_1 \leq a(x) \leq c_2, \text{ a.e. in } \Omega, \text{TV}(a) \leq c_3\}, \end{aligned}$$

where  $c_1, c_2$  and  $c_3$  are positive constants. Clearly, both sets are compact in  $L$ , whereas  $A_2$  is bounded in  $\|\cdot\|_{\widehat{B}}$ . It is known that  $L^\infty(\Omega)$  is continuously embedded in  $L^1(\Omega)$ ,  $\text{BV}(\Omega)$  is compactly embedded in  $L^1(\Omega)$ , and  $TV(\cdot)$  is convex and lower-semicontinuous in  $L^1(\Omega)$ -norm, see [1,20].

Let  $u, v \in H_0^1(\Omega)$  be fixed and let  $\{a_n\}$  be a bounded sequence in  $L^\infty(\Omega)$ . If  $a_n \rightarrow 0$  in the  $L^1(\Omega)$  norm, then, by the Lebesgue dominated convergence theorem,  $\int_{\Omega} a_n \nabla u \cdot \nabla v \rightarrow 0$ . Furthermore, for any sequence  $\{v_n\}$ , bounded in  $V$ , and for any  $v \in V$ , we have

$$\begin{aligned} \left| \int_{\Omega} a_n \nabla v_n \cdot \nabla v \right| &\leq \int_{\Omega} |a_n \nabla v_n \cdot \nabla v| = \int_{\Omega} (|a_n|^{1/2} \nabla v) \cdot (|a_n|^{1/2} \nabla v_n) \\ &\leq \left[ \int_{\Omega} |a_n| |\nabla v_n \cdot \nabla v_n| \right]^{1/2} \left[ \int_{\Omega} |a_n| |\nabla v \cdot \nabla v| \right]^{1/2}, \end{aligned}$$

which confirms (7). Here we use the fact that  $a_n \rightarrow 0$  in the  $L^1(\Omega)$  implies  $\left[ \int_{\Omega} |a_n| |\nabla v \cdot \nabla v| \right]^{1/2} \rightarrow 0$ , whereas  $\left[ \int_{\Omega} |a_n| |\nabla v_n \cdot \nabla v_n| \right]^{1/2}$  remains bounded due to the boundedness of  $v_n$  in  $V$  and the boundedness of  $a_n$  in  $L^\infty$ . We note that the boundedness of  $a_n$  in  $L^\infty(\Omega)$  has been obtained by optimally choosing the feasible set (see the sets  $A_1$  and  $A_2$  given above). Conditions (1) and (2) are also verified by standard arguments. Summarizing, the developed framework is applicable and ensures the identification of a discontinuous parameter in the considered gradient obstacle problem.

## 6 Concluding remarks

As a first step towards developing a theory for identifying variable parameters in quasi-variational inequalities, we provided new existence, convergence, and discretization results. It remains an open question to establish a penalization approach for quasi-variational inequality so that an optimization problem with an equality constraint can be considered. Of course, derivation of implementable necessary optimality conditions remains a priority as well. Note the parameter-to-solution map does not exhibit any smoothness and hence some smoothing needs to be done. Numerical results are of utmost importance, and some derivative-free methods can be used as the initial step.

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