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Pencils and critical locus on normal surfaces

F. Delgado * H. Maugendre †

Abstract

We study linear pencils of curves on normal surface singularities. Using the minimal good resolution of the pencil, we describe the topological type of generic elements of the pencil and characterize the behaviour of special elements. Then we show that the critical locus associated to the pencil is linked to the special elements. This gives a decomposition of the critical locus through the minimal good resolution and as a consequence, information on the topological type of the critical locus.

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1 Introduction

Let (Z,z) be a complex analytic normal surface, and let $\pi : (Z,z) \to (\mathbb{C}^2,0)$ be a finite complex analytic morphism germ. We choose coordinates (u,v) in $(\mathbb{C}^2,0)$ and denote $f := u \circ \pi$ and $g := v \circ \pi$. We consider the meromorphic function $h := f/g$ defined in a punctured neighbourhood V of z in Z. It can be seen as a map $h : V \to \mathbb{CP}^1$ defined by $h(x) := (f(x) : g(x))$. For $w = (w_1 : w_2) \in \mathbb{CP}^1$, the closure of $h^{-1}(w)$ defines the curve $w_2 f - w_1 g = 0$ on the surface (Z,z). The set $\Lambda := \{w_2 f - w_1 g, w_1, w_2 \in \mathbb{C}\}$ is the *pencil* defined by f and g. We denote ϕ_w the element of the pencil Λ equal to $w_2 f - w_1 g$. Its (non reduced) zero locus, denoted by Φ_w, is called the *fiber* defined by ϕ_w.

Such linear families of curves have been studied independently and through different approach for (Z,z) equal to $(\mathbb{C}^2,0)$ in [11], [7] and [16]. In the general case (it means (Z,z) a germ of normal complex analytic surface which is not smooth anymore), Lê Dũng Tràng and R. Bondil give in [3] a definition of general elements of the pencil which are characterized by the minimality of their Milnor number. In [2] R. Bondil gives an algebraic μ-constant theorem for linear families of plane curves. Other results have been obtained in the case where π is the restriction to (Z,z) of a linear projection of $(\mathbb{C}^n,0)$ onto $(\mathbb{C}^2,0)$

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(see [1], [4], [18]). At last, the topology of the morphism π has been studied in [13] and [14]. In [13], the authors define rational quotients which are topological invariants of (π, u, v) and give different ways to compute them. In [14], F. Michel presents another proof of the topological invariance of this set of rational numbers and moreover she gives a decomposition of the critical locus of π in bunches linked to the set of invariants.

Let $\rho : (X, E) \to (Z, z)$ be a *good* resolution of the singularity (Z, z). It is a resolution of the singularity (Z, z) such that the exceptional divisor is a union of smooth projective curves with normal crossings. In particular three irreducible components of the exceptional divisor do not meet at the same point. The lifting $h \circ \rho$ is a meromorphic function defined in a suitable neighbourhood of E in X but in a finite set of points.

A *good resolution* ρ of the pencil Λ is a good resolution of the singularity (Z, z) in which $h \circ \rho$ is a morphism and the exceptional divisor is a union of smooth projective curves with normal crossings. A good resolution of the pencil Λ is said to be *minimal* if and only if by the contraction of any rational component of self-intersection -1 of the exceptional divisor we do not obtain a good resolution of Λ anymore. We will see in section 2 that there exists a unique minimal good resolution of Λ.

An irreducible component E_α of E is called *dicritical* if the restriction of $h \circ \rho$ to E_α is not constant.

Considering the minimal good resolution $\rho : (Y, E) \to (Z, z)$ of the pencil Λ, we define *special* and *generic* values of Λ as follows. Let us denote $\widehat{h} = h \circ \rho$ and $\mathcal{D}$ the union of the dicritical components of E. We define the set of special zones $SZ(\Lambda) = \{\Delta_i, i \in I\}$ where I is a finite set, and Δ_i is either a connected components of $\overline{E \setminus \mathcal{D}}$, either a critical point of the restriction of $\widehat{h}$ to $\mathcal{D}$, or an intersection point between two dicritical components. Notice that $\widehat{h}_{|_{\Delta_i}}$ is constant.

Definition: The set of *special values* of Λ is constituted of the values $\widehat{h}(\Delta_i)$ for $i \in I$. A fiber associated to a special value is called a *special fiber* of Λ.

The other values of $\mathbb{CP}^1$ are called *generic values* for the pencil Λ. A fiber associated to a generic value is called a *generic fiber* of Λ.

We prove the following results.

Theorem 1 *Let w, w' be generic values for the pencil Λ, then the fibers Φ_w and $\Phi_{w'}$ have the same topological type.*

Moreover, if $e \in \mathbb{CP}^1$ is a special value for the pencil Λ, then the fibers Φ_w and Φ_e do not have the same topological type.

The above definition and theorem generalize some of the results contained in [11] (see theorem 4.1) where the authors study pencils defined on $\mathbb{C}^2$. Going on studying the topology of the pencil we prove the following result which extend to the case of normal surfaces the second item of theorems 1, 2, 3 of [7] which deals with pencils defined on $\mathbb{C}^2$.

Theorem 2 *Let ρ be the minimal good resolution of the pencil Λ, $\Delta \in SZ(\Lambda)$, and let $e \in \mathbb{CP}^1$. Then, the strict transform of Φ_e by ρ intersects Δ if and only if Φ_e is special and $\widehat{h}(\Delta) = e$.*

In a second part we are interested in understanding the behaviour of the critical locus of the map π. We denote by $I_z(\ ,\ )$ the local intersection multiplicity at z (see section 2.1). We prove the following result which generalize the third item of theorems 1, 2, 3 of [7].

Theorem 3 *Let $\rho : (Y, E) \to (Z, z)$ be the minimal good resolution of Λ. For each element $\Delta \in SZ(\Lambda)$ there exists an irreducible component of the critical locus $C(\pi)$ of π such that its strict transform by ρ intersects Δ.*

Moreover for each branch Γ of $C(\pi)$ there exist $\Delta \in SZ(\Lambda)$ such that the strict transform of Γ by ρ intersects Δ and the value $e = \widehat{h}(\Delta)$ is the unique one such that $I_z(\phi_e, \Gamma) > I_z(\phi_w, \Gamma)$ for all $w \neq e$.

A consequence of these results is Theorem 4:

Theorem 4 *Let Φ_w be a fiber of Λ. Then the three following properties are equivalent:*

1. *Φ_e is a special fiber of Λ.*
2. *$I_z(\phi_e, C(\pi)) > \min_{\phi \in \Lambda} I_z(\phi, C(\pi))$.*
3. *$\mu(\phi_e) > \min_{\phi \in \Lambda} \mu(\phi)$.*

In section 2, once we have set some preliminary results, we construct and study the minimal good resolution of Λ. In section 3, we prove Theorem 1 and 2 and in section 4 we prove Theorems 3 and 4. To finish, in section 5, we present some examples.

2 Preliminary results and notations

Let (Z, z) be a normal surface singularity and let $\rho : (X, E) \to (Z, z)$ be a good resolution of it. That means that ρ is a resolution of the singularity (Z, z) such that the exceptional divisor $E = \rho^{-1}(z)$ is a union of smooth projective curves $E = \bigcup_{\alpha \in G(\rho)} E_\alpha$ with normal crossings, in particular three of them have empty intersection.

For $\alpha \in G(\rho)$ and for each holomorphic function $f : (Z, z) \to (\mathbb{C}, 0)$ let denote by $\nu_\alpha(f)$ the vanishing order of $\overline{f} = f \circ \rho : X \to \mathbb{C}$ along the irreducible exceptional curve E_α (ν_α is just the divisorial valuation defined by E_α). The divisor $(\overline{f})$ defined by $\overline{f} = f \circ \rho$ on X could be written as

$$(\overline{f}) = (\widetilde{f}) + \sum_{\alpha \in G(\rho)} \nu_\alpha(f) E_\alpha$$

where, the local part $(\widetilde{f})$ is the strict transform of the germ $\{f = 0\}$. For each $\beta \in G(\rho)$ one has the known Mumford formula (see [15]):

$$(\overline{f}) \cdot E_\beta = (\widetilde{f}) \cdot E_\beta + \sum_{\alpha} \nu_\alpha(f)(E_\alpha \cdot E_\beta) = 0 \, . \tag{1}$$

(Here "$\cdot$" stand for the intersection form on the smooth surface X). Notice that the intersection matrix $(E_\alpha \cdot E_\beta)$ is negative definite and so $\{\nu_\alpha(f)\}$ is the unique solution of the linear system defined by the equations (1) above.

2.1 Intersection multiplicity

Let $C \subset (Z, z)$ be an irreducible germ of curve in (Z, z) and let $f \in \mathcal{O}_{Z,z}$ be a function. Let $\varphi : (\mathbb{C}, 0) \to (C, z)$ be a parametrization (uniformization) of (C, z), then we define the intersection multiplicity of $\{f = 0\} \subset Z$ and C at $z \in C$ as $I_z(f, C) = \mathrm{ord}_\tau(f \circ \varphi(\tau))$ (τ is the parameter in $\mathbb{C}$). Notice that the normalization $\overline{\mathcal{O}_{C,z}}$ of the local ring $\mathcal{O}_{C,z}$ of the germ C at $z \in C$ is a discrete valuation ring, so $\overline{\mathcal{O}_{C,z}} \simeq \mathbb{C}\{t\}$ for a uniformizing parameter

t and the valuation v_C is defined by the order function on t, i.e. $v_C(g) = \mathrm{ord}_t(g(t))$ for $g \in \mathcal{O}_{C,z} \subset \mathbb{C}\{t\}$. One has also that $I_z(f, C) = v_C(f)$. The intersection multiplicity $I_z(f, C)$ could be also understood as the degree $\deg(f|C)$ of the composition map of $f|_C : C \setminus \{z\} \to \mathbb{C}^*$ and the map from $\mathbb{C}^*$ into the unit circle S^1 which sends a non-zero complex number t, onto $t/|t|$. Obviously the above definition could be extended by linearity to define the intersection multiplicity of a f with a (local) divisor $\sum_{i=1}^k n_i C_i$ as $I_z(f, \sum n_i C_i) = \sum n_i I_z(f, C_i)$.

Let $\rho : (X, E) \to (Z, z)$ be a good resolution of the normal singularity (Z, z) and $E = \bigcup_{\alpha \in G(\rho)} E_\alpha$ be the exceptional divisor. Let $\widetilde{C} := \overline{\rho^{-1}(C \setminus \{z\})}$ be the strict transform of C by ρ. Then (see [15])

$$I_z(f, C) = (\overline{f}) \cdot \widetilde{C} = (\widetilde{f}) \cdot \widetilde{C} + \sum_{\alpha \in G(\rho)} \nu_\alpha(f)(E_\alpha \cdot \widetilde{C}) \, .$$

Let us take now a good resolution ρ such that $\widetilde{C}$ is smooth and transversal to E at a smooth point P and also with the condition $(\widetilde{f}) \cdot \widetilde{C} = 0$. This resolution could be obtained by a finite number of point blowing ups starting on (say) the minimal good resolution of (Z, z). Let $\alpha(C) \in G(\rho)$ be the (unique) component of E such that $\widetilde{C} \cap E_{\alpha(C)} = P$. Then one has $I_z(f, C) = \nu_{\alpha(C)}(f) = I_P(f \circ \rho, \widetilde{C})$. Here $I_P(-, -)$ coincides with the usual local intersection multiplicity of two germs at the smooth local surface (X, P). Notice that $\widetilde{C}$ is a curvetta at the point $P \in E_{\alpha(C)}$, $\widetilde{C}$ is the normalization of C and $\rho|_{\widetilde{C}} : \widetilde{C} \to C$ is a uniformization of C.

Let f, g be analytic functions on (Z, z) and let $\Lambda = \langle f, g \rangle = \{\phi_w = w_2 f - w_1 g \,|\, w = (w_1 : w_2) \in \mathbb{CP}^1\}$ be the pencil of analytic functions defined by f and g. As in the case of plane branches (see [6]), one has the following easy and useful result:

Proposition 1 *Let $C \subset (Z, z)$ be an irreducible germ of curve. Then there exists a unique $w_0 \in \mathbb{CP}^1$ such that $I_z(\phi_w, C)$ is constant for all $w \in \mathbb{CP}^1 \setminus w_0$ and $I_z(\phi_{w_0}, C) > I_z(\phi_w, C)$.*

Proof. The statement is trivial taking into account that the valuation defined by C, ν_C, is the order of the series in $\mathbb{C}\{t\}$.

2.2 Resolution of pencils

Let $\pi = (f, g) : (Z, z) \to (\mathbb{C}^2, 0)$ be finite complex analytic morphism germ, let $\Lambda = \langle f, g \rangle = \{w_2 f - w_1 g \,|\, w = (w_1 : w_2) \in \mathbb{CP}^1\}$ be the pencil of analytic functions defined by f and g and let $h = (f/g) : V \to \mathbb{CP}^1$ be the meromorphic function defined by f/g in a suitable punctured neighbourhood of $z \in Z$.

A *good* resolution of (f, g) is a good resolution $\rho : (Y, E) \to (Z, z)$ of (Z, z) such that the (reduced) divisor $|(fg \circ \rho)^{-1}(0)|$ has normal crossings. It means in particular that three irreducible components of $|(fg \circ \rho)^{-1}(0)|$ doesn't meet at a same point. Starting on the minimal good resolution of (Z, z) one can produce a good resolution of (f, g) by a sequence of blowing-ups of points in the corresponding smooth surface (essentially resolving the singularities of the reduced total transform of the curve $\{fg = 0\}$). We also call it a good resolution of the corresponding curves $\Phi_{(0:1)} \cup \Phi_{(1:0)}$. Such a good resolution is minimal if and only if the contraction of any rational component of self-intersection -1 of the exceptional divisor does not give a good resolution anymore.

As defined in the introduction, a *good* resolution of the pencil Λ is a good resolution $\rho : (X, E) \to (Z, z)$ of the singularity (Z, z), such that the lifting $\widehat{h} = h \circ \rho$ is a morphism on X.

Let $\rho : (X, E) \to (Z, z)$ be a good resolution of (Z, z) and E_α an irreducible component of E. The *Hironaka quotient* of (f, g) on E_α is the following rational number:

$$q(E_\alpha) := \frac{\nu_\alpha(f)}{\nu_\alpha(g)}.$$

If $q(E_\alpha) > 1$ (resp. $q(E_\alpha) < 1$) then the component E_α belongs to the zero divisor (resp. pole divisor) of $h \circ \rho$. Note that if E_α is a dicritical component of E then $q(E_\alpha) = 1$. Notice that there may exists irreducible components E_α of E which are not dicritical and for which $q(E_\alpha) = 1$. Those are all components for which the restriction of $h \circ \rho$ is constant on E_α and E_α does not belong to the zero divisor nor to the pole divisor.

Proposition 2 *There exists a (unique) minimal good resolution of* Λ.

Proof. Let $\rho' : (Y', E') \to (Z, z)$ be the minimal good resolution of (f, g). The indetermination points of $h \circ \rho'$ are the intersection points of irreducible components E_α and E_β of the total transform $|(fg \circ \rho')^{-1}(0)|$ for which one has $q(E_\alpha) > 1$ and $q(E_\beta) < 1$. Here one of the components, E_α or E_β, is allowed to be the strict transform $\widetilde{\xi}$ of a branch ξ of $\{f = 0\}$ (in such a case we put $q(\widetilde{\xi}) > 1$) or $\{g = 0\}$ (respectively $q(\widetilde{\xi}) < 1$). Let P be such an indetermination point. Blowing-up at P one creates a divisor E_η of genus 0 and one has that $\nu_\eta(f) = \nu_\alpha(f) + \nu_\beta(f)$ and $\nu_\eta(g) = \nu_\alpha(g) + \nu_\beta(g)$. (If E_β is a branch ξ of $\{f = 0\}$ of multiplicity r, we have $\nu_\beta(f) = r$ and $\nu_\beta(g) = 0$. We use similar conventions for the case in which E_β is a branch of $\{g = 0\}$.) If $q(E_\eta) = 1$, then neither $E_\alpha \cap E_\eta$ nor $E_\beta \cap E_\eta$ is an indetermination point and moreover E_η is a dicritical divisor. Else if $q(E_\eta) > 1$ (resp. $q(E_\eta) < 1$) then $E_\beta \cap E_\eta$ (resp. $E_\alpha \cap E_\eta$) is an indetermination point and we iterate the process. After a finite number of blow-ups there does not subsist indetermination points and so we have constructed a good resolution $\rho'' : (Y'', E'') \to (Z, z)$ of Λ.

Now, to obtain a minimal good resolution of Λ, we have to contract some rational component of self-intersection -1 of the exceptional divisor (see theorem 5.9 of [9]). By the above construction the new components (specially the last one which is dicritical and with self-intersection -1) can not be contracted because in such a case we have an indetermination point. As a consequence a minimal good resolution of Λ is obtained from ρ'' by iterated contraction of the rational component of self-intersection -1 of the exceptional divisor which are not dicritical. Uniqueness follows as in the case of the usual minimal resolution (see for example [5] th. 6.2 p. 86).

Let consider $\rho : (Y, E) \to (Z, z)$ the minimal good resolution of the pencil Λ and $\widehat{h} = h \circ \rho$. For $w \in \mathbb{CP}^1$ let $\widehat{h}^{-1}(w) = \widetilde{\Phi_w}$ be the strict transform of the fiber Φ_w. For D a dicritical component of E, we will denote by $\deg(\widehat{h}_{|D})$ the degree of the restriction of $\widehat{h}$ to D, $\widehat{h}_{|D} : D \to \mathbb{CP}^1$.

Proposition 3 *Let w be a generic value for the pencil Λ, then*

a) The resolution ρ is a good resolution of ϕ_w.

b) $\widetilde{\Phi_w}$ intersects E only at smooth points of $\mathcal{D}$.

c) If $D \in \mathcal{D}$, the number of intersection points of $\widetilde{\Phi_w}$ and D is equal to $\deg(\widehat{h}_{|D})$.

Moreover, the minimal good resolution of Λ is the minimal good resolution of any pair of generic elements of Λ.

Proof. By definition of a generic value, $\widetilde{\Phi_w}$ meets the exceptional divisor E only at smooth points of $\mathcal{D}$. Let D be an irreducible component of $\mathcal{D}$ and P a point of $\widetilde{\Phi_w} \cap D$. Then, as P is not a critical point for $\widehat{h}$, $\widetilde{\Phi_w}$ is smooth and transversal to D at P. This implies also that

$$\deg\,(\widehat{h}_{|D}) = \sum_{P \in D} I_P(\widetilde{\phi_w}, D)$$

So, one has $\deg(\widehat{h}_{|D}) = \#(\widetilde{\Phi_w} \cap D)$.

Now, let w' be another generic value. Notice that the strict transforms of $\widetilde{\Phi_w}$ and $\widetilde{\Phi_{w'}}$ intersect in the same number of points each dicritical divisor D, so both fibers have the same number of branches, just $\sum_{D \in \mathcal{D}} \deg(\widehat{h}_{|D})$. Moreover, $\widetilde{\Phi_w}$ and $\widetilde{\Phi_{w'}}$ do not intersect $\mathcal{D}$ at the same points because $\widehat{h}$ is a morphism. As a consequence the minimal good resolution of Λ is a good resolution of any pair of generic fibers. It leaves to show that it is the minimal one.

By definition of the minimal good resolution of Λ, the irreducible components of the exceptional divisor of self-intersection -1 we have to contract in the minimal good resolution of the pencil Λ, to reach the minimal good resolution of $(\phi_w, \phi_{w'})$, lie in the dicritical components (see the proof of proposition 2; it is a consequence of the construction of the minimal good resolution of Λ). Contracting a dicritical component we obtain a map ρ'' such that the strict transforms of Φ_w and $\Phi_{w'}$ by ρ'' intersect an irreducible component of the exceptional divisor at the same point and so the strict transform by ρ'' of $\{\phi_w \phi_{w'} = 0\}$ has not normal crossings. Consequently the minimal good resolution of Λ is the minimal good resolution of the pair $(\phi_w, \phi_{w'})$.

2.3 Hironaka quotients

In 2.2 we have defined the Hironaka quotient of (f, g) on an irreducible component E_α of the exceptional divisor of a good resolution of (Z, z). In the same way we can define the Hironaka quotient of $(\phi_w, \phi_{w'})$ on E_α for any pair $(\phi_w, \phi_{w'})$ of elements of $\Lambda = \langle f, g \rangle$ as the rational number

$$q^w_{w'}(E_\alpha) := \frac{\nu_\alpha(\phi_w)}{\nu_\alpha(\phi_{w'})}.$$

In this way $q(E_\alpha) = q^0_\infty(E_\alpha)$ (here $0 = (0:1) \in \mathbb{CP}^1$, $\infty = (1:0) \in \mathbb{CP}^1$) but to simplify the notations we will still write $q(E_\alpha)$ for the Hironaka quotient of (f, g).

Notice that an irreducible component E_α of E is dicritical if and only if $q^w_{w'}(E_\alpha) = 1$ for any pair (w, w') of elements of $\mathbb{CP}^1$.

Corollary 1 *The Hironaka quotient of any pair of generic elements of Λ associated to any irreducible component of the exceptional divisor of the minimal good resolution of Λ is equal to one.*

Proof. Let $w, w' \in \mathbb{CP}^1$ be a pair of generic values of Λ and $D \in \mathcal{D}$, then $(\widetilde{\phi_w}) \cdot D = (\widetilde{\phi_{w'}}) \cdot D = \deg(\widehat{h}_{|D})$ (see proposition 3). On the other hand, if E_β is a non-dicritical component of E then one has $(\widetilde{\phi_w}) \cdot E_\beta = (\widetilde{\phi_{w'}}) \cdot E_\beta = 0$. Now, the system of linear equations given by the formula (1) for ϕ_w and $\phi_{w'}$ is the same and so the solutions $\{\nu_\alpha(\phi_w)\}$ and $\{\nu_\alpha(\phi_{w'})\}$ are the same. Thus, $\nu_\alpha(\phi_w) = \nu_\alpha(\phi_{w'})$ and $q^w_{w'}(E_\alpha) = 1$ for any $\alpha \in G(\rho)$.

Remark. Let E_α be a non dicritical component of the exceptional divisor of the minimal good resolution of the pencil Λ and let C be a curvet in E_α (an irreducible smooth curve

germ whose strict transform intersects E_α in a smooth point) such that $P = \widetilde{C} \cap E_\alpha$ does not belong to the strict transform of any fiber Φ of Λ. One has $I_z(\phi, C) = \nu_\alpha(\phi)$ for any $\phi \in \Lambda$ and by Proposition 1 there exists a unique $e \in \mathbb{CP}^1$ such that $\nu_\alpha(\phi_w)$ is constant for all $w \in \mathbb{CP}^1 \backslash \{e\}$ and $\nu_\alpha(\phi_e) > \nu_\alpha(\phi_w)$. Moreover, the above value $e \in \mathbb{CP}^1$ must be a special value of Λ.

Let $b : (Z_I, E_I) \to (Z, z)$ be the normalized blow-up of the ideal $I = (f, g)$. In [2] and [3] an element $\phi \in I$ is defined to be *general* if it is *superficial* and the strict transform of $\Phi = \{\phi = 0\}$ by b is smooth and transverse to the exceptional divisor at smooth points. (See definition 2.1 of [2]). Proposition 2.2 of [2] allows to characterize general elements in terms of any good resolution of Z_I, in particular one can use a good resolution $\rho : (Y, E) \to (Z, z)$ of the pencil Λ. In this terms one has that $\phi \in \Lambda$ is general if for any α

$$\nu_\alpha(\phi) = \nu_\alpha(I) = \min_{\psi \in I}\{\nu_\alpha(\psi)\} = \min_{\psi \in \Lambda}\{\nu_\alpha(\psi)\}$$

and moreover, the strict transform of Φ by ρ is smooth and transversal to E. By using the definition of the Milnor number of a germ of curve given in [8], from Theorem 1 and 2 of [3] one has that $\phi \in \Lambda$ is general if and only if

$$\mu(\phi) = \mu(I) := \min_{\psi \in I}\{\mu(\psi)\} = \min_{\psi \in \Lambda}\{\mu(\psi)\} \ .$$

Using proposition 3 and the above results about Hironaka quotients we have that Φ_w is a generic fiber if and only if ϕ_w is *general*. Moreover, one has also that $\mu(\phi_w) = \min_{\phi \in \Lambda}\{\mu(\phi)\}$ if and only if ϕ_w is generic, so, $\mu(\phi_{w_0}) > \min_{\phi \in \Lambda}\{\mu(\phi)\}$ if and only if w_0 is a special value of Λ. This is the equivalence of 1 and 3 in Theorem 4.

3 Topology of special fibers

3.1 Dual graph and topology

Let $M := Z \cap S_\varepsilon^{2n-1}$ where S_ε^{2n-1} represents the boundary of the small ball of radius ε of $\mathbb{C}^n$ centered at z. The manifold M is called the *link* (see [15] and also [20]) of the singularity (Z, z).

Let ϕ_w be an element of Λ and $K_{\phi_w} := \phi_w^{-1}(0) \cap M$. The *multilink* $\mathbf{K}_{\phi_w}$ *of* ϕ_w is the oriented link K_{ϕ_w} weighted by the multiplicities of the irreducible components of ϕ_w. For ε small enough, the topology of the multilink $\mathbf{K}_{\phi_w}$ in M does not depend on the choice of ε.

The fibers Φ_w and $\Phi_{w'}$ are said to be *topologically equivalent* if and only if there exists a diffeomorphism of M that send $\mathbf{K}_{\phi_w}$ on $\mathbf{K}_{\phi_{w'}}$ respecting orientations and weights (see [13]).

Let $\rho_w : (X, E) \to (Z, z)$ be the minimal good resolution of (Z, z) such that the divisor $(\phi_w \circ \rho_w)$ has normal crossings. From Neumann (see [17]), the topology of the multilink $\mathbf{K}_{\phi_w}$ determines the minimal good resolution ρ_w, where the irreducible components of the strict transform of Φ_w by ρ_w are weighted with their multiplicity and taking into account the self-intersections and genus of the irreducible components of the exceptional divisor. Conversely, the Mumford formula ((1) in [15]) and the fact that the intersection matrix $(E_\alpha \cdot E_\beta)_{\alpha,\beta \in G(\rho_w)}$ is negative definite (so invertible) imply that the set $\{\nu_\alpha(\phi_w), \alpha \in G(\rho_w)\}$ is uniquely defined and so the divisor $(\overline{\phi_w})$ (see section 2) is uniquely determinate on X

from the set $\{(\widetilde{\phi_w}) \cdot E_\alpha \,|\, \alpha \in G(\rho_w)\}$. As a consequence the minimal good resolution ρ_w characterizes the topology of the multilink $\mathbf{K}_{\phi_w}$.

Let $\rho : (X, E) \to (Z, z)$ be a good resolution of the normal surface singularity (Z, z), $E = \bigcup_{\alpha \in G(\rho)} E_\alpha$ its exceptional divisor. It is useful to encode the information of the resolution ρ by means of the so called *dual graph* of ρ. The set of vertices of this graph is the set $G(\rho)$, each vertex α is pondered by $(\alpha, E_\alpha^2, g(E_\alpha))$ where E_α^2 is the self-intersection of E_α, and $g(E_\alpha)$ its genus. An intersection point between E_α and E_β is represented by an edge linking the vertices α and β.

If we take ρ as a good resolution of the local curve $C = \sum_{i=1}^{\ell} n_i C_i$ (in particular if $C = \{\varphi = 0\}$ for some function φ) one add an arrow for each irreducible component C_i of C weighted by the multiplicity n_i. In the case in which we deal with a good resolution of pair of functions (f, g), in the graph of $fg = 0$ one mark with different colors the arrows corresponding to branches of $\{f = 0\}$ and those of $\{g = 0\}$ (another possibility is to use different kinds of marks, say for example arrows for f and stars for g). The sharp extremities of the arrows are considered as somekind of special vertices of the graph. The notations $\mathcal{G}(\rho)$, $\mathcal{G}(\rho, \varphi)$ and $\mathcal{G}(\rho, f, g)$ will be used for the dual graph in each situation. Note that the case of a good resolution ρ of the pencil $\Lambda = \langle f, g\rangle$ is encoded by the dual graph $\mathcal{G}(\rho, \phi_w, \phi_{w'})$ for a pair of generic fibers.

Following Neumann, one has:

Statement: The fiber Φ_w and $\Phi_{w'}$ are topologically equivalent if and only if the graphs $\mathcal{G}(\rho_w, \phi_w)$ and $\mathcal{G}(\rho_{w'}, \phi_{w'})$ are the same.

Let $\rho : (X, E) \to (Z, z)$ be a good resolution of (f, g) and let E_α be an irreducible component of E. We denote $\overset{\circ}{E}_\alpha$ the set of smooth points of E_α in the reduced total transform $|(fg \circ \rho)^{-1}(0)|$. An irreducible component E_α (or its corresponding vertex α) of E is a *rupture component* if $\chi(\overset{\circ}{E}_\alpha) < 0$, where χ is the Euler characteristic. Note that $\chi(\overset{\circ}{E}_\alpha)$ is equal to $2 - 2g(E_\alpha) - v(\alpha)$, where $v(\alpha)$ is the number of intersection points of E_α with other components of the total transform of $fg = 0$. Thus, the rupture components are all the rational ones with at least three different edges or arrows and all the non-rational irreducible components. We will say that α is an *end* when $\chi(\overset{\circ}{E}_\alpha) = 1$. Obviously α is an end if and only if E_α is rational and one has only one edge on it.

The *neighbouring-set* of E_α in X is the set constituted of E_α union the irreducible components of the exceptional divisor and of the strict transform of $\{fg = 0\}$ that intersect E_α. We denote it $st(E_\alpha)$.

A *chain* of length r, $r \geq 3$, in E is a connected part of E constituted of a finite set of irreducible components $E_{\alpha_1}, \ldots, E_{\alpha_r}$ satisfying:

- $\chi(\overset{\circ}{E}_{\alpha_i}) = 0$, for $2 \leq i \leq r - 1$, and
- $st(E_{\alpha_i}) = \{E_{\alpha_{i-1}}, E_{\alpha_i}, E_{\alpha_{i+1}}\}$ for $2 \leq i \leq r - 1$.

Notice that the strict transform of $\{fg = 0\}$ does not intersect $\{E_{\alpha_2}, \ldots, E_{\alpha_{r-1}}\}$.

A *cycle* of length r, $r \geq 3$, in E is a chain such that $st(E_{\alpha_r}) = \{E_{\alpha_{r-1}}, E_{\alpha_r}, E_{\alpha_1}\}$. A *cycle* of length 2 in E is a connected part of E constituted of two irreducible components $E_{\alpha_1}, E_{\alpha_2}$ such that $\chi(\overset{\circ}{E}_{\alpha_2}) = 0$ and $st(E_{\alpha_2}) = \{E_{\alpha_1}, E_{\alpha_2}\}$.

The following result is a direct generalization of proposition 1 and corollary 1 of [7].

Proposition 4 *Let $\rho : (X, E) \to (Z, z)$ be a good resolution of (f, g). Let E_α be an irreducible component of the exceptional divisor such that the strict transform of $\{fg = 0\}$ does not intersect E_α. Then there exists E_β in $st(E_\alpha)$ such that $q(E_\beta) > q(E_\alpha)$ if and only if there exists E_γ in $st(E_\alpha)$ such that $q(E_\gamma) < q(E_\alpha)$.*

Moreover, if $\{E_{\alpha_1}, \ldots, E_{\alpha_r}\}$, $r \geq 3$ is a chain, then one of the following facts is true:

- *$q(E_{\alpha_i}) < q(E_{\alpha_{i+1}})$ for $1 \leq i \leq r-1$.*
- *$q(E_{\alpha_i}) > q(E_{\alpha_{i+1}})$ for $1 \leq i \leq r-1$.*
- *$q(E_{\alpha_i})$ is constant for $1 \leq i \leq r$.*

In particular, if E_{α_r} is an end, then $q(E_{\alpha_i})$ is constant for $1 \leq i \leq r$ and if $\{E_{\alpha_1}, \ldots, E_{\alpha_r}\}$ is a cycle, then $q(E_{\alpha_i})$ is constant for $1 \leq i \leq r$.

The proof almost repeats the proof of the refereed Proposition by using the equations (1) for f and the divisor E_α as well as the same equation for g. As proposition 4 is a key result, we give back the proof for the first statement.

Proof. By using equation (1) for f we have:

$$0 = (\overline{f}) \cdot E_\alpha = (\widetilde{f}) \cdot E_\alpha + \sum_\gamma \nu_\gamma(f)(E_\gamma \cdot E_\alpha) = \sum_{\eta \in st(E_\alpha)} \nu_\eta(f)(E_\eta \cdot E_\alpha) \, .$$

The same equation is true for g instead f and thus one has:

$$\begin{aligned} \sum_{E_\eta \in st(E_\alpha), \eta \neq \alpha} \nu_\eta(f)(E_\eta \cdot E_\alpha) &= (-E_\alpha^2) \, \nu_\alpha(f) \\ \sum_{E_\eta \in st(E_\alpha), \eta \neq \alpha} \nu_\eta(g)(E_\eta \cdot E_\alpha) &= (-E_\alpha^2) \, \nu_\alpha(g) \end{aligned} \tag{2}$$

Let suppose that $q(E_\eta) \geq q(E_\alpha)$ for each $E_\eta \in st(E_\alpha)$. This condition is equivalent to:

$$(E_\eta \cdot E_\alpha)\nu_\eta(f)\nu_\alpha(g) \geq (E_\eta \cdot E_\alpha)\nu_\alpha(f)\nu_\eta(g) \, .$$

As $q(E_\beta) > q(E_\alpha)$, we obtain:

$$\nu_\alpha(g) \sum_{E_\eta \in st(E_\alpha), \eta \neq \alpha} (E_\eta \cdot E_\alpha)\nu_\eta(f) > \nu_\alpha(f) \sum_{E_\eta \in st(E_\alpha), \eta \neq \alpha} (E_\eta \cdot E_\alpha)\nu_\eta(g) \, .$$

However, by using the equations (2), both sides of the above inequality are equal to $(-E_\alpha^2)\nu_\alpha(f)\nu_\alpha(g)$ and so we reach a contradiction.

The others statements of the proposition are direct consequences of this result.

3.2 Proof of Theorems 1 and 2

Let $\rho : (Y, E) \to (Z, z)$ be the minimal good resolution of the pencil Λ, $\widehat{h} = h \circ \rho$. If w and w' are generic values for the pencil Λ, the Proposition 3, together with the above Statement give

Corollary 2 *Let $w, w' \in \mathbb{CP}^1$ be generic values of Λ. Then, the fibers Φ_w and $\Phi_{w'}$ are topologically equivalent.*

Thus, in order to finish the proof of Theorem 1 it only remains to show that a special fiber Φ_e is not topologically equivalent to a generic one.

Let Δ be an element of $SZ(\Lambda)$ and $e = \widehat{h}(\Delta)$. We denote Φ_e the fiber of Λ associated to e and by $\widetilde{\Phi_e}$ its strict transform by ρ. The remaining part of Theorem 1 and Theorem **??** are direct consequences of the three following lemmas.

Lemma 1 *If e is a special value of Λ associated to a connected component Δ of $\overline{E\backslash\mathcal{D}}$, then the strict transform of Φ_e by ρ intersects Δ.*

Proof. Let us assume that $\widetilde{\Phi_e}\cap\Delta = \emptyset$. Notice that if we change ρ by a good resolution of Λ such that it is also a good resolution of Φ_e then the connected set Δ remains unchanged. So, we can keep the notations we use for ρ for this new resolution.

Consider the Hironaka quotient with respect to e and w as a map $q^e_w : \mathcal{G}(\rho) \to \mathbb{Q}$. Note that for any E_α in Δ, we have $q^e_w(E_\alpha) > 1$. Let E_β be an irreducible component of Δ such that $q^e_w(E_\beta) \geq q^e_w(E_\alpha)$ for each E_α in Δ and let Δ' be the maximal connected subset of E such that $E_\beta \in \Delta'$ and $(q^e_w)_{|\Delta'}$ is constant and equal to $q^e_w(E_\beta)$. Notice that $E_\beta \subset \Delta' \subset \Delta$ because $q^e_w(E_\alpha) = 1$ for any E_α such that $E_\alpha \cap \Delta \neq \emptyset$ and $E_\alpha \not\subset \Delta$ (in fact such an E_α is a dicritical divisor). Let now $E_\gamma \subset \Delta'$ and such that $st(E_\gamma) \not\subset \Delta'$ and $E_\alpha \in st(E_\gamma)$, such that $E_\alpha \not\subset \Delta'$. One has $q^e_w(E_\beta) > q^e_w(E_\alpha) > 1$ if $E_\alpha \subset \Delta$ and $q^e_w(E_\beta) > q^e_w(E_\alpha) = 1$ otherwise. However, being $\Delta' \subset \Delta$, this contradicts Proposition 4 for the irreducible component E_γ.

As a consequence $\widetilde{\Phi_e} \cap \Delta \neq \emptyset$ and so Φ_e can not be topologically equivalent to Φ_w for a generic value w.

Lemma 2 *If e is a special value of Λ associated to a smooth point P of D in $\mathcal{D}$ which is a critical point of $\widehat{h}$, then the strict transform of Φ_e by ρ intersects D at P. Moreover it is not smooth and transversal to D at P.*

Proof. Blowing-up at P we create a divisor E_α. As P lies in the zero locus of $(\phi_e/\phi_w)\circ\rho$, for any value $w \neq e$ we have $q^e_w(E_\alpha) > 1$. Moreover, as D is a dicritical component, $q^e_w(D) = 1$. Now, if we assume that $P \notin \widetilde{\Phi_e}$ then one can use Proposition 4 for the new divisor E_α and we reach a contradiction.

Assume that $\widetilde{\Phi_e}$ is smooth and transversal to D at the point P. In this case we can choose local coordinates $\{u, v\}$ on Y at P in such a way that $\widetilde{\Phi_w} = \{v = 0\}$ and $D = \{u = 0\}$ on a neighbourhood V of P. So, the function $\phi_e \circ \rho$ is $u^a v$ on V and, for a generic value w, $\phi_w \circ \rho$ is $u^b\eta(u, v)$ for a unit η. Note that $a = \nu_D(\phi_e) = \nu_D(\phi_w) = b$, being D dicritical, and so the expression of $\widehat{h}$ at P is $v\eta^{-1}(u, v)$. Now, the restriction of $\widehat{h}$ to D is given locally at P as the map $v \mapsto v$. Thus the point P is not a critical (ramified) point of $\widehat{h}_{|D} : D \to \mathbb{CP}^1$.

As a consequence $\widetilde{\Phi_e}$ is not smooth and transversal to D at P, in particular it can not be topologically equivalent to Φ_w for a generic value w.

Lemma 3 *If e is a special value of Λ associated to an intersection point P between two irreducible components of $\mathcal{D}$, then the strict transform of Φ_e by ρ intersects $\mathcal{D}$ at P.*

Proof. Let $P = E_{\alpha_1} \cap E_{\alpha_2}$ such that E_{α_1} and E_{α_2} are dicritical components. Let us assume that $P \notin \widetilde{\Phi_e}$. Blowing-up at P we create a divisor E_α satisfying $\{E_{\alpha_1}, E_\alpha, E_{\alpha_2}\} = st(E_\alpha)$. As $q^e_w(E_{\alpha_1}) = q^e_w(E_{\alpha_2}) = 1$ and $q^e_w(E_\alpha) > 1$, we reach a contradiction with proposition 4.

As a consequence, Φ_e is not resolved by ρ and so could not be topologically equivalent to a generic fiber Φ_w.

4 Behaviour of the critical locus

Let $\pi = (f, g) : (Z, z) \to (\mathbb{C}^2, 0)$ be a finite complex analytic morphism. Following Teissier ([19]), the critical locus of π is the analytic subspace defined by the zeroth Fitting ideal $F_0(\Omega_\pi)$ of the module Ω_π of relative differentials. The critical locus may have embedded components, however we are only interested in the components of dimension one. So, we denote by $C(\pi)$ the divisorial part of the critical set with its non-reduced structure, i.e. each of its components counted with its multiplicity, and we refer to $C(\pi)$ as the critical locus of π. Note that out of the singular point $z \in Z$, $C(\pi)$ is defined by the vanishing of the jacobian determinant and also that $C(\pi)$ depends on Λ and not on the pair of functions of Λ fixed to define the corresponding finite morphism, so we denote it also by $C(\Lambda)$. If we denote Γ_k, (resp. n_k) $k = 1, \ldots, \ell$ the irreducible components (branches) of $C(\Lambda)$ (resp. their multiplicity) then $C(\Lambda)$ is the local (Weil) divisor $C(\Lambda) = \sum_{k=1}^{\ell} n_k \Gamma_k$.

Before proving theorem 3 and 4, let us first recall two results from [13] and [14].

Let $(\phi_w, \phi_{w'})$ be any pair of germs of the pencil Λ, let $\rho' : (Y', E') \to (Z, z)$ be the minimal good resolution of $(\phi_w, \phi_{w'})$, and denote $\Gamma(w, w') := (\Gamma_k)_{k \in K}, K \subset \{1, \ldots, \ell\}$, the set of irreducible components of $C(\Lambda)$ which are not sent to a coordinate axe by $(\phi_w, \phi_{w'})$. Let Z_r be the set constituted of the union of the smooth points of E' (smooth points of E' in $|(\phi_w \phi_{w'} \circ \rho)^{-1}(0)|$) contained in an irreducible component of E' with Hironaka quotients equal to r, and the intersection points of two irreducible components of E' of Hironaka quotient r. The set Z_r is called the *r-zone* of $G(\rho')$. A connected component of Z_r which contains at least one rupture vertex is called a *r-rupture zone*. Then from [13] we have:

Theorem A. *The set* $\left\{ \dfrac{I_z(\phi_w, \Gamma_k)}{I_z(\phi_{w'}, \Gamma_k)}, k \in K \right\}$ *is equal to the set of Hironaka quotients associated to the rupture components of* $G(\rho', \phi_w, \phi_{w'})$.

In [14] a repartition in bunches of the branches of $\Gamma(w, w')$ is given as follows:

Theorem B. *The intersection of the strict transform of* $\Gamma(w, w')$ *with a connected component of* Z_r *is not empty if and only if it is a r-rupture zone. Moreover if* Γ *is an irreducible component of* $\Gamma(w, w')$ *whose strict transform intersects a r-rupture zone then* $\dfrac{I_z(\phi_w, \Gamma)}{I_z(\phi_{w'}, \Gamma)} = r$.

Next Lemma treats the case of irreducible components of the critical locus which are also components of a fiber.

Lemma 4 *Let* $\Phi_e = \sum_{i=1}^{t} r_i \xi_i$; $\xi_1, \ldots, \xi_t$ *being the irreducible components of the fiber* Φ_e. *Then* $r_i > 1$ *if and only if* ξ_i *is an irreducible component of* $C(\Lambda)$.

Proof. Let ξ be an irreducible component of a fiber Φ_e. Let $w \in \mathbb{CP}^1$ be a generic value and let $\rho' : (Y', E') \to (Z, z)$ be the minimal good resolution of (ϕ_e, ϕ_w). Let $\widetilde{\xi}$ be the strict transform of ξ by ρ' and let P be the intersection point of $\widetilde{\xi}$ with the exceptional divisor E', $P = \widetilde{\xi} \cap E_\alpha = \widetilde{\xi} \cap E'$. We can choose a local system of coordinates (u, v) in a neighbourhood $U \subset Y'$ of $P = (0, 0)$ such that $u = 0$ is an equation of E_α, $v = 0$ is a equation of $\widetilde{\xi}$ and the equation of $\widetilde{\Phi_\gamma}$ at P is $u^a v^k$ where $a = \nu_\alpha(\phi_e)$ and k is the

multiplicity of the branch ξ in Φ_e. On the other hand the equation of $\widetilde{\Phi_w}$ at P is $u^b\eta(u,v)$, $b=\nu_\alpha(\phi_w)$ and $\eta(u,v)$ a unit. So, the expression of $\widehat{h}$ at $P\in U$ is $u^{a-b}v^k(\eta(u,v))^{-1}$.

Let us first suppose that ξ belongs to $C(\Lambda)$. Let Q be a point of $\widetilde{\xi}\backslash\{P\}$, say Q has local coordinates $(u_0,0)$. The restriction of $\widehat{h}$ on a small disc $D(u_0,0)$ centered at Q in $u=u_0$ is $v^k\eta_0(u_0,v)$ with $\eta_0(u_0,v)$ a unit and $k>1$ because ξ lies in the ramification locus. So, as k is the multiplicity of ξ in Φ_e, ξ is non reduced.

Conversely, if ξ is an irreducible component of a fiber Φ_e which is not reduced, the multiplicity k of ξ in Φ_e satisfies $k>1$. Moreover the local equation of $\widehat{h}$ on any small disc $D(t,0)$ centered at any point of local coordinates $(t,0)$ in U is $v^k\eta(t,v)$ with $\eta(t,v)$ a unit. As $k>1$, each point $(t,0)$ is a ramification point and so $\widetilde{\xi}$ lies in the ramification locus. Hence ξ is an irreducible component of $C(\Lambda)$.

4.1 Proof of theorem 3 for singular points of $\mathcal{D}$ and critical points of the restriction of $\widehat{h}$ to $\mathcal{D}$

In the sequel $\rho:(Y,E)\to(Z,z)$ is the minimal good resolution of Λ and $\mathcal{D}$ the dicritical locus of E.

Proposition 5 *Let $P\in\mathcal{D}$ be such that $P\notin\overline{E\backslash\mathcal{D}}$. Then, P is a singular point of $\mathcal{D}$ or a critical point of $\widehat{h}_{|\mathcal{D}}$ if and only if there exists an irreducible component Γ of $C(\Lambda)$ such that its strict transform intersects $\mathcal{D}$ at P. Moreover if $\widehat{h}(P)=e$ then $I_z(\phi_e,\Gamma)>I_z(\phi_w,\Gamma)$ for all $w\neq e$.*

Proof. Let us assume that there exists an irreducible component Γ of $C(\Lambda)$ whose strict transform intersects $\mathcal{D}$ at P. Let $e=\widehat{h}(P)$ and denote by D the irreducible component of $\mathcal{D}$ such that $P\in D$. If Γ is a branch of Φ_e then it must be a multiple irreducible component of it by the above Lemma and as a consequence the point P is a critical point of $\widehat{h}|\mathcal{D}$.

So, let us consider the case in which Γ is not a branch of Φ_e and assume that P is not a singular point of $\mathcal{D}$, i.e. P is a smooth point of $\mathcal{D}$ in the exceptional divisor E.

If the strict transform $\widetilde{\Phi}_e$ of Φ_e at P has normal crossings with $\mathcal{D}$, then there exists an irreducible branch ξ of Φ_e such that its strict transform $\widetilde{\xi}$ coincides with $(\widetilde{\Phi}_e)_P$, i.e. $\widetilde{\xi}$ is smooth, transversal to D and ξ is not a multiple branch of Φ_e by Lemma 4. By Theorem B there exists a r-rupture zone R in the minimal good resolution of (ϕ_e,ϕ_w) (here w is assumed to be a generic value) such that the strict transform of Γ intersects R and moreover $I_z(\phi_e,\Gamma)/I_z(\phi_w,\Gamma)=r$ with $r>1$ because $P\in\widetilde{\Gamma}\cap\widetilde{\Phi_e}$. Taking into account that $\widetilde{\Phi_e}$ is smooth and transversal to the dicritical divisor D, then one has that $P=\widetilde{\Gamma}\cap E\subset D\subset R$ and so, by Theorem A,

$$\frac{I_z(\phi_e,\Gamma)}{I_z(\phi_w,\Gamma)}=q_w^e(D)=\frac{\nu_D(\phi_e)}{\nu_D(\phi_w)}\ .$$

However this is impossible because the last quotient is equal to 1, being D dicritical. Thus, as a consequence, $(\widetilde{\Phi_e})_P$ must be singular or tangent to D. In both cases P is a critical point of $\widehat{h}|_{\mathcal{D}}$ (i.e. ϕ_e is a special function of Λ).

Conversely, let P be a singular point of $\mathcal{D}$ or a smooth point of $\mathcal{D}$ which is a critical point of $\widehat{h}_{|\mathcal{D}}$ and let $e=\widehat{h}(P)$, then from Theorem 2, Φ_e is a special fiber of Λ. If the

irreducible component of $\widetilde{\Phi}_e$ that intersects $\mathcal{D}$ at P is non reduced then from lemma 4 we have finished. Thus, we assume that $\widetilde{\Phi}_e$ is reduced at P.

First, note that $\widetilde{\Phi}_e$ has not normal crossings with E at P. Because if P is smooth on $\mathcal{D}$ then $\widetilde{\Phi}_e$ is either singular or tangent to $\mathcal{D}$, and in the other case, it means if P is a singular point of $\mathcal{D}$, then there are at least three components of the total transform intersecting at P.

Let $w, w' \in \mathbb{CP}^1$ be generic values and let $\rho' : (Y', E') \to (Z, z)$ be the minimal good resolution of $\phi_w \phi_{w'} \phi_e$. Note that $\rho' = \rho \circ \sigma$, where σ is a sequence of point blowing-ups on Y, each of them produces some new irreducible rational exceptional components. In particular $\Delta = \sigma^{-1}(P) \subset E'$ is a connected exceptional part and must contain a rupture component $E_\alpha \subset E'$. Notice that no component of Δ is contracted in the minimal good resolution $\rho'' : (Y'', E'') \to (Z, z)$ of the pair (ϕ_e, ϕ_w); i.e $\Delta \subset E''$ and in particular $E_\alpha \subset E''$ is also a rupture component in E''. Let R be the corresponding rupture zone in E'' which contains E_α. Note that for each $E_\beta \subset R \subset \Delta$ one has $r = q^e_w(E_\beta) = \nu_\beta(\phi_e)/\nu_\beta(\phi_w) > 1$.

Now, Theorem A implies that there exists a branch Γ of $C(\Lambda)$ such that its strict transform by ρ'' intersects Δ and also

$$\frac{I_z(\phi_e, \Gamma)}{I_z(\phi_w, \Gamma)} = \frac{\nu_\alpha(\phi_e)}{\nu_\alpha(\phi_w)} = r > 1.$$

Taking into account that $R \subset \Delta$ and $\sigma(\Delta) = P$, one has that the strict transform of Γ by ρ intersects E at the point P and moreover $I_z(\phi_e, \Gamma) > I_z(\phi_w, \Gamma)$. Note that the above inequality is true for any irreducible component Γ of $C(\Lambda)$ such that its strict transform by ρ intersects $\mathcal{D}$ at P. Thus, the special fiber ϕ_e is the unique fiber with the condition $I_z(\phi_e, \Gamma) > \min_w I_z(\phi_w, \Gamma)$.

Remark. Notice that if P is a smooth point of $\mathcal{D}$ which is a critical point of $\widehat{h}_{|\mathcal{D}}$ or if P is a singular point of $\mathcal{D}$, then for any fiber Φ_a and $\Phi_{a'}$ different from Φ_e, we have $q^a_{a'}(E_\alpha) = 1$ and then $I_z(\phi_a, \Gamma) = I_z(\phi_{a'}, \Gamma)$.

4.2 Proof of theorem 3 for the connected components of $\overline{E \backslash \mathcal{D}}$

Let us remind that $\rho : (Y, E) \to (Z, z)$ is the minimal good resolution of Λ and $\mathcal{D}$ the dicritical locus of E. Let Δ be a connected component of $\overline{E \backslash \mathcal{D}}$ such that $(h \circ \rho)(\Delta) = e$. Let w, w' be generic values of Λ and let us denote $\rho' : (Y', E') \to (Z, z)$ the minimal good resolution of $\phi_w \phi_{w'} \phi_e$. Let us denote by $\tau : (Y', E') \to (Y, E)$ the composition of point blowing-ups which produces Y' from (Y, E)

$$(Y', E') \xrightarrow{\tau} (Y, E) \xrightarrow{\rho} (Z, z)$$

Let Δ' by the pull-back of Δ by τ. Note that Δ' is a connected component of $\overline{E' \backslash \mathcal{D}'}$ because the dicritical locus $\mathcal{D}'$ on E' is just the strict transform of $\mathcal{D}$ by τ. We will distinguish two cases, depending on the existence of a rupture component E'_α in Δ' (with respect to ϕ_w and ϕ_e).

Case 1) There exist a rupture component E'_α in Δ'.

For each component $E_\beta \subset \Delta'$ one has $q^w_{w'}(E_\beta) = 1$ and $q^e_w(E_\beta) > 1$. Let R be the rupture zone of E' such that $E_\alpha \subset R$. Then $R \subset \Delta'$ because q^e_w is constant and > 1 on R

and moreover $q^e_w(D) = 1$ for any dicritical divisor, in particular for D dicritical such that $D \cap \Delta' \neq \emptyset$.

Now, from Theorem B, there exist a branch Γ of the critical locus $C(\Lambda)$ such that its strict transform by ρ', $\widetilde{\Gamma}$, intersect R. As consequence the strict transform of Γ by ρ, $\tau(\widetilde{\Gamma})$ intersects Δ. Again Theorem B implies that $q^e_w(E_\alpha) = I_z(\phi_e, \Gamma)/I_z(\phi_w, \Gamma)$ and so the special value e is the unique one such that $I_z(\phi_e, \Gamma) > I_z(\phi_{w'}, \Gamma)$ for any generic value w'.

Case 2) There are no rupture components in Δ'.

In this case $\Delta' = \{E_{\alpha_1}, \ldots, E_{\alpha_r}\}$ in such a way that there exists a dicritical component $D \in \mathcal{D}'$ such that $\{D = E_{\alpha_0}, E_{\alpha_1}, \ldots, E_{\alpha_r}\}$ is a chain and $\chi(E_{\alpha_r}) \geq 0$. Now, note that the strict transform of Φ_e intersects Δ' (see Theorem **??**), so the only way to avoid the existence of a rupture component with respect to $\phi_w\phi_e$ is that E_{α_r} is an end (i.e it is rational and is connected only with the previous one $E_{\alpha_{r-1}}$) and such that $\widetilde{\Phi_e}$, the strict transform of Φ_e by ρ', intersect E_{α_r}. Moreover, $\widetilde{\Phi_e}$ with its reduced structure is smooth and transversal to E_{α_r}. It means that the minimal good resolution of Λ is a resolution of the reduced irreducible component ξ_e of Φ_e whose strict transform meets Δ at E_{α_r}. Actually, otherwise to resolve ξ_e, we have to blow-up at $\xi_e \cap E_{\alpha_r}$ and this process produces a rupture component.

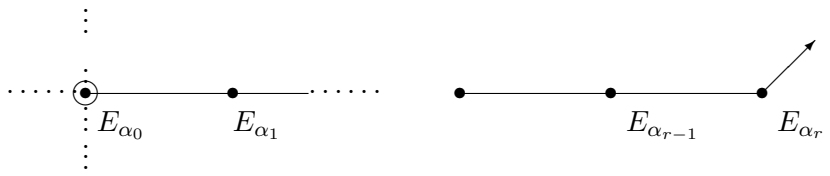

Figure 1: Graph in Case 2

Lemma 5 *Let $v_0, \ldots, v_r$, $e_1, \ldots, e_r$ be sequences of integers such that $v_{i-1} = e_i v_i - v_{i+1}$ for $i = 1, \ldots, r-1$. Let $q_0, \ldots, q_{r-1} \in \mathbb{Z}$ defined recursively as $q_0 = 1$, $q_1 = e_1$ and, for $i \geq 2$, $q_i = e_i q_{i-1} - q_{i-2}$. Then, for $i \geq 1$ one has $\gcd(q_i, q_{i-1}) = 1$ and $v_0 = q_i v_i - q_{i-1} v_{i+1}$.*

Proof. Obviously $\gcd(q_0, q_1) = 1$ and from the definition of q_i, if $\gcd(q_{i-1}, q_{i-2}) = 1$ then $\gcd(q_{i-1}, q_i) = 1$. The equality $v_0 = q_i v_i - q_{i-1} v_{i+1}$ is obvious for $i = 1$ and, by induction, using the equality $v_{i-1} = e_i v_i - v_{i+1}$ in the inductive hypothesis $v_0 = q_{i-1} v_{i-1} - q_{i-2} v_i$ one has

$$v_0 = q_{i-1} v_{i-1} - q_{i-2} v_i = q_{i-1}(e_i v_i - v_{i+1}) - q_{i-2} v_i = q_i v_i - q_{i-1} v_{i+1} \, .$$

Now, the proof of the case 2 is a consequence of the next:

Proposition 6 *The irreducible curve ξ_e is a branch of Φ_e with multiplicity bigger than 1. As a consequence ξ_e is also a branch of $C(\Lambda)$ and so $C(\Lambda)$ intersect Δ.*

Proof. Recall that w is a generic element of Λ. For the sake of simplicity let denote $v_i = \nu_{\alpha_i}(\phi_w)$ and $e_i = -E^2_{\alpha_i}$ for $i = 0, \ldots, r$. Then, by using the formula

$$\left((\widetilde{\phi_w}) + \sum_{\alpha \in G(\rho')} \nu_\alpha(\phi_w) E_\alpha \right) \cdot E_{\alpha_i} = 0 \tag{3}$$

for $i = 1, \ldots, r$ one has that

$$\begin{aligned} v_0 &= e_1 v_1 - v_2 \\ v_1 &= e_2 v_2 - v_3 \\ &\cdots \\ v_{r-2} &= e_{r-1} v_{r-1} - v_r \\ v_{r-1} &= e_r v_r \end{aligned} \tag{4}$$

By Lemma 5 one has $v_0 = q_r v_r$. Moreover, taking into account that $e_i = -E_{\alpha_i}^2 \geq 2$ one can easily prove that $q_r > q_{r-1} > \cdots > q_1 > q_0 = 1$.

Let us consider now the special fiber Φ_e and let us denote $v_i' = \nu_{E_{\alpha_i}}(\phi_e)$ for $i = 0, \ldots, r$. The equations (3) applied for ϕ_e instead of ϕ_w gives a sequence of equalities $v_{i-1}' = e_i v_i' - v_{i+1}'$, for $i = 1, \ldots, r-1$ (like in (4) above with v_i' instead v_i) together with the last one:

$$v_{r-1}' = e_r v_r' - (\widetilde{\phi_e}) \cdot E_{\alpha_r} = e_r v_r' - k \ .$$

Lemma 5 implies that $v_0' = q_r v_r' - q_{r-1} k$. Being $E_{\alpha_0} = D$ a dicritical divisor one has that $v_0' = \nu_{\alpha_0}(\phi_e) = \nu_{\alpha_0}(\phi_w) = v_0$, i.e.

$$q_r v_r = q_r v_r' - q_{r-1} k \ .$$

By Lemma 5 again, $\gcd(q_r, q_{r-1}) = 1$ and so q_r divides k. In particular $k = (\widetilde{\phi_e}) \cdot E_\sigma > 1$ and the irreducible germ ξ_e appears repeated k times in Φ_e.

4.3 Special fibers and critical locus

Let $C(\Lambda) = \sum_{i=1}^{\ell} n_i \Gamma_i$ be the decomposition of the critical locus in irreducible components. For each $i \in \{1, \ldots, \ell\}$ the intersection multiplicity $I_z(\phi, \Gamma_i)$ is constant but for exactly the unique special value $\varepsilon(\Gamma_i)(= \varepsilon(i))$ such that $I_z(\phi_{\varepsilon(i)}, \Gamma_i) > I_z(\phi, \Gamma_i)$, for $\phi \neq \phi_{\varepsilon(i)}$. So, as in [7], one has a surjective map $\varepsilon : \mathcal{B}(C(\Lambda)) \to Sp(\Lambda)$ from the set of branches of the critical locus to the set of special fibers of Λ.

If $w \in \mathbb{CP}^1$ is a generic value one has that

$$I_z(\phi_w, C(\Lambda)) = \sum_{i=1}^{\ell} n_i I_z(\phi_w, \Gamma_i) = \min\{I_z(\phi, C(\Lambda)) \, , \, \phi \in \Lambda\}$$

and, on the other hand, for a special value $e \in \mathbb{CP}^1$ one has

$$I_z(\phi_e, C(\Lambda)) = \sum_{i=1}^{\ell} n_i I_z(\phi_e, \Gamma_i) > \sum_{i=1}^{\ell} n_i I_z(\phi_w, \Gamma_i) = \min\{I_z(\phi, C(\Lambda)) \, , \, \phi \in \Lambda\} \ .$$

Thus, as a consequence one has the following

Corollary 3 *Φ_e is a special fiber of Λ if and only if*

$$I_z(\phi_e, C(\Lambda) > \min\{I_z(\phi, C(\Lambda)), \phi \in \Lambda\} \ .$$

Remark. As in [7] the map $\varepsilon : \mathcal{B}(C(\Lambda)) \to Sp(\Lambda)$, defined above, could be factorized through the set of special zones $SZ(\Lambda)$ as $\varepsilon = \xi \circ \psi$:

$$\mathcal{B}(C(\Lambda)) \stackrel{\psi}{\to} SZ(\Lambda) \stackrel{\xi}{\to} Sp(\Lambda)$$

The map ψ associates to the branch Γ the special zone Δ such that the strict transform of Γ in the minimal good resolution intersects Δ. In the same way the map ξ sends $\Delta \in SZ(\Lambda)$ to $\widehat{h}(\Delta)$.

By means of a good resolution of all the fibers of Λ $\rho' : (Y', E') \to (Z, z)$ (i.e. a good resolution of the product of all the special fibers and a pair of generic ones) and the determination of all the rupture zones in E' with respect to the pairs (ϕ_e, ϕ_w), being e special and w generic, one can determine a finer decomposition in bunches of the branches of the critical locus $C(\Lambda)$.

5 Examples

As seen in section 3.1, to the minimal good resolution ρ of the pencil Λ, one can associate its intersection graph $\mathcal{G}(\rho)$. The following examples illustrate theorems 1, 2 and 3 in terms of intersection graph. To construct $\mathcal{G}(\rho)$, we follow the method of Laufer described in [10], [12] and also [13]. It consists in first establishing the graph of the minimal resolution of the discriminant curve, which is the image by π of the critical locus $C(\pi)$ of π. Then we deduce the graph of the minimal good resolution of (Z, z) and then the one of $G(\rho)$, using in particular proposition 3.6.1 and 3.7.1 of [12]. As in the Figure 1 of Section 4 we use a different kind of mark for the vertices representing dicritical divisors.

5.1 Example 1

Let (Z, z) be defined by $z^3 = h(x, y)$ with $h(x, y) = (y + x^2)(y - x^2)(y + 2x^2)(x + y^2)(x - y^2)(x + 2y^2)$ and let π be the projection on the (x, y)-plane. Such a way $(u, v) = (x, y)$ and $f = u \circ \pi = x$ and $g = v \circ \pi = y$.

The discriminant curve of π is the curve $h(u, v) = 0$. The dual graph of its minimal embedded resolution is represented in the Figure 2.

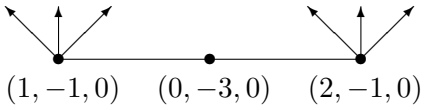

Figure 2: Graph of the discriminant of π.

From proposition 3.6.1 of [12] we deduce the graph of the minimal good resolution of (Z, z) (see Figure 3).

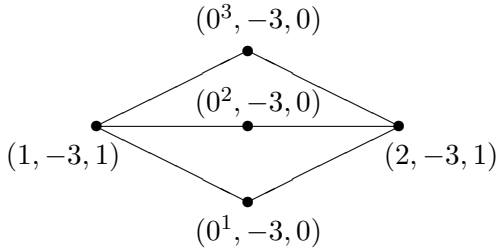

Figure 3: The graph of the minimal good resolution of (Z, z).

As the minimal embedded resolution of the discriminant curve $h(u,v)=0$ of π is also the minimal good resolution of the product $uv(\lambda u+\mu v)h(u,v)=0$, for $(\lambda:\mu)\in\mathbb{C}P^1$, from propositions 3.6.1 and 3.7.1 of [12] we can deduce the graph of the minimal good resolution of Λ (Figure 4), the one of (f,g) and as a consequence the one of the minimal good resolution of $(\phi_w\phi_{w'}fg)^{-1}(0)$ where w and w' are generic values of Λ (Figure 5). Notice that the minimal good resolution of Λ is also the minimal good resolution of (f,g).

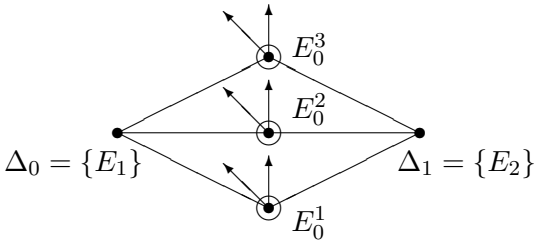

Figure 4: The graph of the minimal good resolution of Λ.

The dicritical components of E are E_0^1,E_0^2,E_0^3. We have $SZ(\Lambda)=\{\Delta_1,\Delta_2\}$ with $\Delta_1=\{E_1\}$ and $\Delta_2=\{E_2\}$. The map $(f/g)\circ\rho$ has no critical point on $\mathcal{D}$ and $\mathcal{D}$ has no singular point neither. The special fiber associated to Δ_1 is $\{f=0\}$ and the one associated to Δ_2 is $\{g=0\}$. We conclude that Λ admits two special elements f and g; the special value associated to Δ_1 is $(0:1)$ and the one associated to Δ_2 is $(1:0)$. The Hironaka quotients are $q(E_1)=2$ and $q(E_2)=1/2$.

Moreover, using the minimal resolution of the discriminant curve (see Figure 2), we deduce that, for each Δ_i, there exists three irreducible components of the reduced critical locus of π whose strict transform intersects Δ_i.

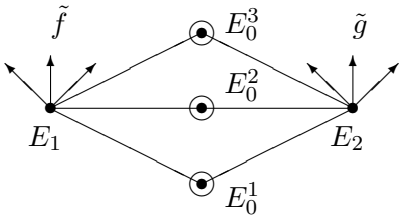

Figure 5: Minimal good resolution of (f,g).

5.2 Example 2

Let (Z,z) be the singularity D_6 defined by the equation $z^2=y(x^2+y^4)$. The graph of the minimal resolution of it is shown in Figure 6.

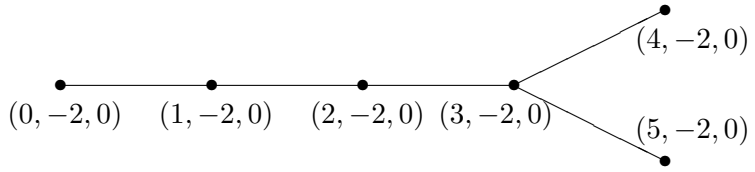

Figure 6: The graph of the minimal good resolution of D_6.

On this surface we will make two examples for two different projections (pencils). Firstly, let $\pi = (f, g) : (Z, z) \to (\mathbb{C}^2, 0)$ be defined by $f(x, y, z) = u \circ \pi = x$ and $g(x, y, z) = v \circ \pi = y$. The discriminant curve of π is the curve $v(u^2 + v^4) = 0$. Notice that this projection is not a generic one because the image of the curve $\{g = 0\}$ is an irreducible component of the discriminant curve and the image of $\{f = 0\}$ is tangent to the discriminant curve.

The minimal good resolution of Λ is just equal to the one of (Z, z) and there exists a unique dicritical component E_1: the divisor with weight $(1, -2, 0)$. Thus, one has two special zones, $SZ(\Lambda) = \{\Delta_0, \Delta_1\}$ with $\Delta_0 = \{E_0\}$ and $\Delta_1 = \{E_2, E_3, E_4, E_5\}$ (see Figure 7 for the notations). The Hironaka quotients corresponding to each vertex are: $q(E_0) = q(E_1) = 1$, $q(E_2) = 3/2$ and $q(E_3) = q(E_4) = q(E_5) = 2$.

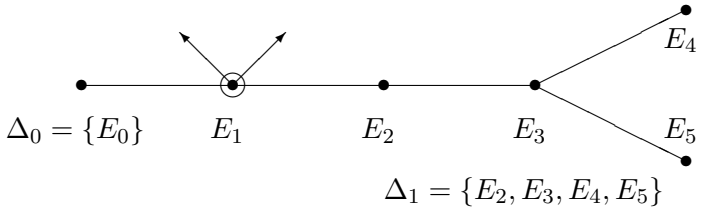

Figure 7: The graph of the minimal good resolution of Λ.

The connected component Δ_0 doesn't contain any rupture component and Δ_1 admits a rupture component of Hironaka quotient equal to 2. The special fiber associated to Δ_1 is $\{f = 0\}$ whose strict transform meets Δ_1 at E_3, and there are two irreducible components of $C(\pi)$ intersecting Δ_1 at E_4 and E_5. The special fiber of Λ associated to Δ_0 is $\{g = 0\}$ which is also a non reduced irreducible component of the critical locus. It intersects Δ_0 at E_0. The minimal good resolution of the pencil Λ is also the minimal good resolution of (f, g), so the corresponding graph of the minimal good resolution of $fg = 0$ is represented in figure 8.

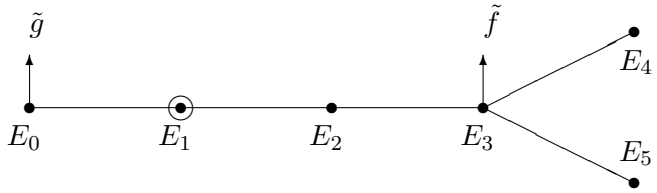

Figure 8: The graph of the minimal good resolution of (f, g).

For the second example on D_6, let the projection $\pi = (f, g) : (Z, z) \to (\mathbb{C}^2, 0)$ defined by $f(x, y, z) = x + y = u$ and $g(x, y, z) = x + 2iy^2 = v$. As in the previous one the minimal good resolution of Λ and the one of $\{fg = 0\}$ coincides with the minimal good resolution of (Z, z). However, now the graph of the minimal good resolution of π is sligthly different and it is represented in figure 9.

In this case f is a generic element of the pencil Λ and g is the special element associated to Δ_1. The special fiber of Λ associated to Δ_0 is $g - f = 0$. It is also a non reduced

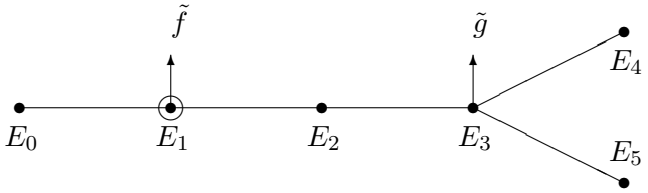

Figure 9: The graph of the minimal good resolution of (f, g).

irreducible component of the critical locus $C(\pi)$. In this case the Hironaka quotients are $q(E_0) = q(E_1) = 1$, $q(E_2) = 2/3$ and $q(E_3) = q(E_4) = q(E_5) = 1/2$.

5.3 Example 3

With this example, issued from [13], we illustrate the case where a special zone is a singular point of the dicritical locus.

Let (Z, z) be defined by $z^2 = (x^2 + y^5)(y^2 + x^3)$ and let $\pi = (f, g) : (Z, z) \to (\mathbb{C}^2, 0)$ be the projection on the (x, y)-plane. The dual graph of the minimal embedded resolution of the discriminant curve $(u^2 + v^5)(v^2 + u^3) = 0$ of π and the coordinate axes is shown in Figure 10.

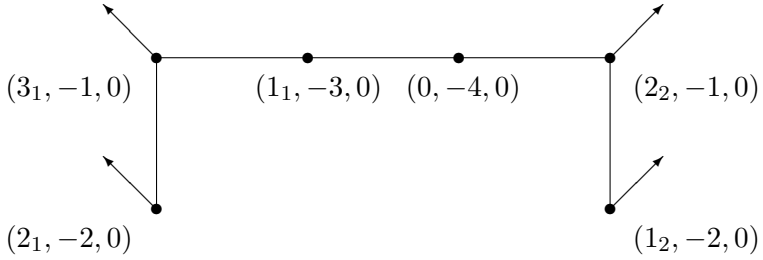

Figure 10: Graph of the discriminant of π and the coordinates axes.

The graph of the minimal good resolution of Λ is in figure 11. The components E_{0^1} and E_{0^2} are dicritical. Thus, there exists two special zones Δ_0 and Δ_1 with $\Delta_0 = \{E_{1^1}, E_{1^2}\}$ and $\Delta_1 = E_{0^1} \cap E_{0^2} = \{P\}$ where P is the singular point of $\mathcal{D}$.

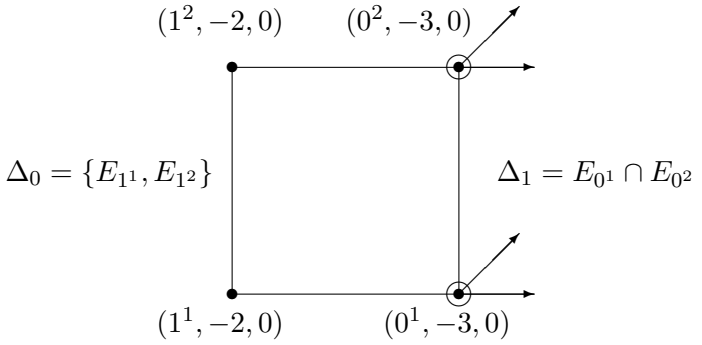

Figure 11: The graph of the minimal good resolution of Λ.

The special fibers associated to Δ_0 and Δ_1 are respectively $\{f = 0\}$ and $\{g = 0\}$ and

the graph of the minimal good resolution of (f, g) is in figure 12.

The Hironaka quotients of the rational components (of self-intersection -1) E_2 and E_3 are respectively $2/3$ and $5/2$ and there exists two irreducible components of $C(\pi)$ whose strict transform intersects E_2 and two others whose strict transform intersects E_3.

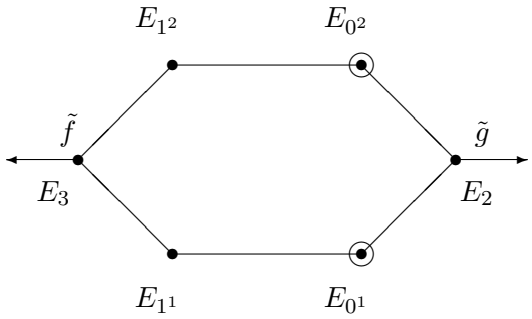

Figure 12: The graph of the minimal good resolution of (f, g).

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