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► **To cite this version:**

Philipp Braun, Christopher M Kellett, Luca Zaccarian. Complete control Lyapunov functions: Stability under state constraints. 11th IFAC Symposium on Nonlinear Control Systems, Sep 2019, Vienna, Austria. hal-01995208

HAL Id: hal-01995208

<https://hal.science/hal-01995208v1>

Submitted on 26 Jan 2019

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Complete control Lyapunov functions: Stability under state constraints [★]

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Abstract: Lyapunov methods are one of the main tools to investigate local and global stability properties of dynamical systems. Even though Lyapunov methods have been studied and applied over many decades to unconstrained systems, extensions to systems with more complicated state constraints have been limited. This paper proposes an extension of classical control Lyapunov functions (CLFs) for differential inclusions by incorporating in particular bounded (nonconvex) state constraints in the form of obstacles in the CLF formulation. We show that the extended CLF formulation, which is called a complete CLF (CCLF) in the following, implies obstacle avoidance and weak stability (or asymptotic controllability) of the equilibrium of the dynamical system. Additionally, the necessity of nonsmooth CCLFs is highlighted. In the last part we construct CCLFs for linear systems, highlighting the difficulties of constructing such functions.

Keywords: (nonsmooth) control Lyapunov functions; asymptotic controllability; obstacle avoidance; nonsmooth controller design.

1. INTRODUCTION

Lyapunov functions (LFs) or control Lyapunov functions (CLFs) are a well studied tool to investigate stability and controllability properties of equilibria of dynamical systems with and without input. While necessary and sufficient conditions for the existence of LFs and CLFs characterizing stability properties of a specific equilibria globally or locally (i.e., by restricting the domain to a sublevel set of the LF/CLF) have been derived, only a few papers concentrate on the extension of classical Lyapunov theory to incorporate more complicated constraints in the Lyapunov formulation to guarantee stability properties and obstacle avoidance simultaneously. A possible reason is the lack of constructive methods to design CLFs for nonlinear systems even in the unconstrained setting.

In this paper we propose an extension of classical CLFs, which we call complete CLFs (CCLFs) and which in particular allow us to consider bounded obstacles in the state space in the form of state constraints. We show that the existence of a CCLF guarantees asymptotic stabilizability of the origin for the dynamical system and simultaneous obstacle avoidance. Due to the topological obstruction of bounded obstacles in the state space, discontinuous feedback laws and subsequently nonsmooth CLFs and CCLFs need to be considered to guarantee asymptotic stability and obstacle avoidance simultaneously (Liberzon, 2003, Chapter 4). Related approaches combining Lyapunov func-

tions with control barrier functions or methods using so-called artificial potential fields concentrate on smooth representations and consequently do not consider discontinuous feedback laws.

Control barrier functions in the context of dynamical systems were introduced in Wieland and Allgöwer (2007) as a certificate to ensure obstacle avoidance, or equivalently to ensure that a given set of unsafe states is never entered. Since in general control barrier functions are assumed to be smooth, articles combining ideas of Lyapunov functions and control barrier functions (such as Ames et al. (2017), Ngo et al. (2005) or Tee et al. (2009)) cannot be used for controller design guaranteeing obstacle avoidance of bounded obstacles and asymptotic stability. Artificial potential fields in the robotics literature date back to the work in Khatib (1985), Khatib (1990), and with a recent relevant contribution in Paternain et al. (2018). Artificial potential fields define smooth Lyapunov-like functions whose gradient is used to ensure obstacle avoidance and asymptotic stability. However, due to the assumption of a smooth function, it is well known and acknowledged in the literature that avoidance and stability can at best be guaranteed for sets excluding a set of measure zero in the state space but not for every initial value. Additionally, artificial potential fields are generally constructed for fully actuated systems. Underactuated systems where it is only possible to manipulate the dynamics in the direction of subspaces of \mathbb{R}^n are only partially covered.

The contributions of this paper are as follows. In Section 2.2 nonsmooth CCLFs as an extension of CLFs are intro-

^{*} P. Braun and C. M. Kellett are supported by the Australian Research Council (Grant number: ARC-DP160102138).

duced providing the possibility to incorporate (bounded) obstacles in the formulation of dynamical systems. Additionally, we show that the existence of a CCLF implies obstacle avoidance and asymptotic stability of the dynamical system before we discuss the necessity of nonsmooth CCLFs via several examples. In Section 3 we propose a possible form of a CCLF by combining several smooth functions. Additionally we show under which conditions this construction leads to a CCLF for linear systems but also highlight the difficulties in the design process.

We use the following notation. For $x, \hat{x} \in \mathbb{R}^n$ we define $|x| = \sqrt{\sum_{i=1}^n x_i^2}$ and $|x|_{\hat{x}} = |x - \hat{x}|$. $B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$, $r > 0$, denotes an open ball and $\overline{B}_r(x)$ denotes its closure. \mathcal{K} , \mathcal{K}_∞ and \mathcal{KL} are the standard notations for comparison functions and \mathcal{P} denotes the class of positive definite functions (see Braun et al. (2018a), for example).

2. NONSMOOTH COMPLETE CONTROL LYAPUNOV FUNCTIONS

2.1 Mathematical setting

We consider differential inclusions

$$\dot{x} \in F(x) \quad (1)$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. An absolutely continuous function $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ is a solution of the differential inclusion (1) from initial condition $x(0) \in \mathbb{R}^n$ if $\dot{x}(t) \in F(x(t))$ for almost all $t \in \mathbb{R}_{\geq 0}$. In a common abuse of notation, we use x and $x(\cdot)$ to denote points in \mathbb{R}^n and solutions to (1). Additionally, $\mathcal{S}(x)$ denotes the set of solutions starting at $x = x(0)$. To ensure existence of solutions of (1) we make the following standard assumptions.

Assumption 1. Consider the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with $0 \in F(0)$. We impose the following conditions on F :

- (i) (Regularity) F has nonempty, compact, and convex values on \mathbb{R}^n , and is upper semicontinuous, i.e., for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\xi \in B_\delta(x)$ implies $F(\xi) \subset F(x) + B_\varepsilon(0)$.
- (ii) (Local boundedness) For each $r > 0$ there exists $M > 0$ such that $|x| < r$ implies $\sup_{w \in F(x)} |w| \leq M$.
- (iii) (Local Lipschitz continuity) For each $x \in \mathbb{R}^n \setminus \{0\}$ there exists a constant $L > 0$ and a neighborhood $x \in \mathcal{O} \subset \mathbb{R}^n$ such that $F(x_1) \subset F(x_2) + B_{L|x_1-x_2|}(0)$ for all $x_1, x_2 \in \mathcal{O}$. \lrcorner

Assumption 1(i) ensures that the set of solutions $\mathcal{S}(x)$ is nonempty. Since we will discuss nonsmooth functions in the following, we use the (lower right) Dini derivative to extend the directional derivative to nondifferentiable functions. For a Lipschitz continuous function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ the Dini derivative in direction $v \in \mathbb{R}^n$, is defined as

$$d\varphi(x; v) = \liminf_{t \searrow 0} \frac{1}{t} (\varphi(x + tv) - \varphi(x)).$$

If $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable on a neighborhood containing $x \in \mathbb{R}^n$, then $d\varphi(x; v) = \langle \nabla \varphi(x), v \rangle$ holds. A Lipschitz continuous function is continuously differentiable almost everywhere due to Rademacher's theorem. It was shown in Sontag (1983) that weak \mathcal{KL} -stability (phrased as an asymptotic controllability property of a controlled differential equation) is equivalent to the existence of a continuous CLF. Subsequently, and independently, Rifford (2000) and Kellett and Teel (2000) (see also

Kellett and Teel (2004)) demonstrated the equivalence of weak \mathcal{KL} -stability and the existence of a locally Lipschitz CLF.

Definition 1. The differential inclusion (1) is weakly \mathcal{KL} -stable with respect to the origin if there exists $\beta \in \mathcal{KL}$ such that, for each $x(0) = \xi \in \mathbb{R}^n$, there exists $x(\cdot) \in \mathcal{S}(\xi)$ so that $|x(t)| \leq \beta(|\xi|, t)$ for all $t \geq 0$. \lrcorner

Theorem 1. (Nonsmooth CLFs). Suppose F satisfies Assumption 1. Then the differential inclusion (1) is weakly \mathcal{KL} -stable with respect to the origin if and only if there exists a Lipschitz continuous CLF $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and $\rho \in \mathcal{P}$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n \quad (2)$$

and for each $x \in \mathbb{R}^n$ there exists $w \in F(x)$ such that

$$dV(x; w) \leq -\rho(|x|). \quad (3)$$

2.2 Complete control Lyapunov functions

In this subsection we extend Definition 1 to incorporate avoidance properties in the stability definition.

Definition 2. Let $\mathcal{O} \subset \mathbb{R}^n$, $0 \notin \mathcal{O}$, be open. The differential inclusion (1) is weakly \mathcal{KL} -stable with respect to the origin with avoidance property with respect to \mathcal{O} , if there exists $\beta \in \mathcal{KL}$ such that, for each $\xi \in \mathbb{R}^n \setminus \mathcal{O}$, there exists $x(\cdot) \in \mathcal{S}(\xi)$ so that

$$|x(t)| \leq \beta(|\xi|, t) \quad \text{and} \quad x(t) \notin \mathcal{O} \quad \forall t \geq 0. \quad \lrcorner$$

Though not stated explicitly here, we assume that \mathcal{O} is nonempty. In the case where $\mathcal{O} = \emptyset$, Definition 2 reduces to Definition 1. Since weak \mathcal{KL} -stability can be equivalently concluded from the existence of a CLF, we consider an extension of CLFs appropriate for Definition 2.

Definition 3. (CCLF). Suppose that F satisfies Assumption 1. For $i \in [1 : N]$, $N \in \mathbb{N}$, let $\mathcal{O}_i \subset \mathbb{R}^n$ be open sets and let $V_C : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Lipschitz continuous function. Assume there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{P}$ such that the following properties are satisfied:

- (i) For all $i \in [1 : N]$, there exist $c_i \in \mathbb{R}$ such that
$$V_C(x) = c_i \quad \forall x \in \partial \mathcal{O}_i \quad \text{and} \quad c_i \leq \min_{x \in \mathcal{O}_i} V_C(x) \quad (4)$$

- (ii) V_C is positive definite and radially unbounded, i.e.,
$$\alpha_1(|x|) \leq V_C(x) \leq \alpha_2(|x|). \quad (5)$$

- (iii) For each $x \in \mathbb{R}^n \setminus (\cup_{i=1}^N \mathcal{O}_i)$ there exists $w \in F(x)$ such that

$$dV_C(x; w) \leq -\rho(x). \quad (6)$$

Then V_C is called a Complete Control Lyapunov Function (CCLF). \lrcorner

Without loss of generality, we assume that $c_i > 0$ for all $i \in [1 : N]$. If $c_i = 0$ for some $i \in [1 : N]$, then the radial unboundedness of V_C implies that $\partial \mathcal{O}_i = \{0\}$ or $\partial \mathcal{O}_i = \emptyset$ and thus $\mathcal{O}_i = \mathbb{R}^n \setminus \{0\}$ or $\mathcal{O}_i = \mathbb{R}^n$. In both cases, the assumptions on the functions involved in Definition 2 and 3 can be trivially satisfied.

Theorem 2. Consider the differential inclusion (1) satisfying Assumption 1. Additionally, let \mathcal{O}_i , $i \in [1 : N]$, $N \in \mathbb{N}$, be open sets and let $V_C : \mathbb{R}^n \rightarrow \mathbb{R}$ be a CCLF according to Definition 3. Then the differential inclusion (1) is weakly \mathcal{KL} -stable with respect to the origin and has the avoidance property with respect to $\mathcal{O} = \cup_{i=1}^N \mathcal{O}_i$. \lrcorner

The proof of Theorem 2 can be obtained by making minimal changes to the proofs of (Braun et al., 2018a, Thm. 4.11 and Thm. 4.12). For completeness, we report the proof of Theorem 2 in Appendix A. Theorem 2 provides the opportunity to incorporate constraints in the Lyapunov framework. In particular if the state space is constrained and contains unsafe states or obstacles $\mathcal{D} \subset \mathbb{R}^n$ which need to be avoided, CCLFs provide a tool to ensure the existence of a solution $x(\cdot) \in \mathcal{S}(x)$ of system (1) such that $x(\cdot)$ converges to the target (the origin) while avoiding the obstacles or unsafe states $\mathcal{D} \subset \mathcal{O}$ for an appropriately chosen set \mathcal{O} . While the set of unsafe states \mathcal{D} is given, the open set \mathcal{O} can be used as a design parameter. If $\mathcal{D} \subset \mathcal{O}$ holds and if avoidance and convergence according to Definition 2 is satisfied for all initial values $x \in \mathbb{R}^n \setminus \mathcal{O}$, then it is also ensured that the set \mathcal{D} is never entered. An immediate application of CCLFs is obtained by considering level sets of CLFs. Let $\mathcal{D} \subset \mathbb{R}^n \setminus \{0\}$ be an arbitrary closed set and let V be a CLF satisfying the assumptions of Theorem 1 (i.e., with respect to the unconstrained setting). Since, \mathcal{D} is closed and $0 \notin \mathcal{D}$ by assumption, there exists $c > 0$ such that $\{x \in \mathbb{R}^n | V(x) \leq c\} \cap \mathcal{D} = \emptyset$. Thus we can define the open set $\mathcal{O} = \mathbb{R}^n \setminus \{x \in \mathbb{R}^n | V(x) \leq c\}$, which clearly satisfies $\mathcal{D} \subset \mathcal{O}$ and due to the properties of a CLF, V is a CCLF with respect to the open set \mathcal{O} . This common practice of restricting the domain of a LF or CLF to a sublevel set, and thus a forward invariant set, is quite straightforward. However, the definition of CCLFs in Definition 3 is more general and capable through nontrivial constructions, discussed in the next sections, to handle bounded open sets \mathcal{O} .

2.3 Necessity of nonsmooth CCLFs

In the case of bounded obstacles, i.e., bounded sets \mathcal{O}_i , $i \in [1 : N]$, the use of nonsmooth functions V_C is essential. This point will be made more precise in this section, before we propose a particular form of candidate CCLFs.

Lemma 1. Let $\mathcal{O}_i \subset \mathbb{R}^n$, $i \in [1 : N]$, $N \in \mathbb{N}$ be open. Assume that V_C is a CCLF according to Definition 3 and assume that there exists $i \in [1 : N]$ such that \mathcal{O}_i is bounded. Then there exists $x \in \mathbb{R}^n \setminus (\cup_{i=1}^N \mathcal{O}_i \cup \{0\})$ such that the gradient $\nabla V_C(x)$ is not defined. \square

The result is an immediate consequence of (Braun and Kellett, 2018, Thm. 1) which shows that a smooth function V_C satisfying (4) and (5) needs to have a critical point $\bar{x} \in \mathbb{R}^n \setminus (\cup_{i=1}^N \mathcal{O}_i \cup \{0\})$ with $\nabla V_C(\bar{x}) = 0$ and thus condition (6) cannot be satisfied. The necessity of a nonsmooth function due to topological obstructions is also discussed in (Liberzon, 2003, Chapter 4) in the necessity of discontinuous feedback laws. To illustrate this problem consider the function $V : \mathbb{R}^n \rightarrow \mathbb{R}$, $V(x) = |x|^2$ as a candidate CLF and an obstacle $\mathcal{D} = \{\hat{x}\}$ consisting of a single point $\hat{x} \in \mathbb{R}^n \setminus B_{\sqrt{\pi}}(0)$. Define the function

$$C_{\mathcal{D}}(x) = \begin{cases} \frac{1}{2}(1 + \cos(|x|_{\hat{x}}^2)), & \text{for } |x|_{\hat{x}}^2 \leq \pi \\ 0, & \text{for } |x|_{\hat{x}}^2 > \pi \end{cases}$$

which is continuously differentiable and has a global maximum $C_{\mathcal{D}}(\hat{x}) = 1$. With $\lambda > 0$, the linear combination

$$V_C(x) = V(x) + \lambda C_{\mathcal{D}}(x), \quad (7)$$

satisfies the properties (4) and (5). The set \mathcal{O}_1 can be defined as an open neighborhood around \hat{x} . Since V_C is

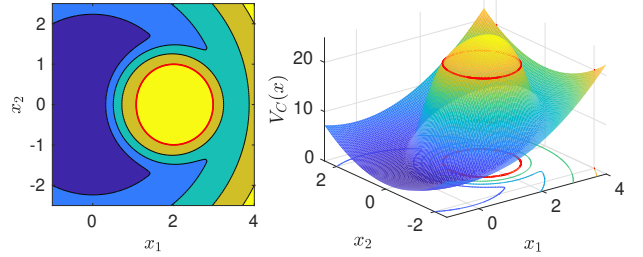


Fig. 1. Visualization of the continuously differentiable function V_C defined in (7) with a saddle point \bar{x} on the x_1 -axis. The boundary of a potential open set \mathcal{O}_1 is visualized in red.

continuously differentiable, the function V_C has a critical point in $\mathbb{R}^n \setminus (\mathcal{O}_1 \cup \{0\})$, where the gradient of V and the gradient of $\lambda C_{\mathcal{D}}$ cancel each other (see (Braun and Kellett, 2018, Thm. 1)). Thus condition (6) cannot be satisfied.

In Figure 1 the level sets of function (7) for $\lambda = 20$ and $\hat{x} = (\sqrt{\pi}, 0)^T$ are visualized. For the differential inclusion $\dot{x} \in F(x) = \bar{B}_{|x|}(0)$, (6) is satisfied for almost all $x \in \mathbb{R}^2$. However, on the x_1 -axis there exists a point $\bar{x} \in \mathbb{R}^2$ where $\nabla V_C(\bar{x}) = 0$ and thus a decrease cannot be obtained. This problem is closely related to the intuitive fact that on the x_1 -axis behind the set \mathcal{O}_1 , a decision needs to be taken to avoid \mathcal{O}_1 from above or from below. In (7), the goal is to define V so that it is a CLF for the differential inclusion without obstacles, whereas $C_{\mathcal{D}}$ is designed to contain the obstacle in its superlevel sets. As also observed in the literature on artificial potential fields, V and $C_{\mathcal{D}}$ cannot be constructed independently and highly depend on each other to avoid the existence of local minima in the function V_C . For example if the function $V(x) = |x|^2$ is replaced by $V(x) = x_1^2 + 4x_2^2$, due to the shape of V and $C_{\mathcal{D}}$, a local minimum (and not just a saddle point) is created on the x_1 -axis.

3. NONSMOOTH CANDIDATE CCLFS

In this section we propose a possible form of a nonsmooth candidate CCLF. The critical condition for a function to be a CCLF is condition (6), which is discussed in Section 4 below for linear systems. For the construction we use ideas from the papers Braun et al. (2018b,c), which propose hybrid avoidance control laws for linear systems. As in Braun et al. (2018b,c), the construction is based on avoidance points $\hat{x} \in \mathbb{R}^n \setminus \{0\}$. The open set \mathcal{O} around the unsafe point then represents the obstacle that should be avoided. Here, we concentrate on a single avoidance point, (i.e., $N = 1$ in Definition 3) but observe that the considerations in this section carry over to the case with multiple unsafe points (obstacles).

Following Braun et al. (2018b), given the unsafe point $\hat{x} \in \mathbb{R}^n \setminus \{0\}$, a parameter $\delta_\mu > 0$, and a direction $d \in \mathbb{R}^n$, $|d| = 1$, we define the two shifted points

$$c_p = \hat{x} - p\delta_\mu d, \quad \text{for } p \in \{-1, 1\}. \quad (8)$$

The direction d will be discussed in the subsequent section (see (20)). Consider the functions

$$C_1(x) = -\eta_1 |x|_{c_1}^2 + \eta_2, \quad C_{-1}(x) = -\eta_1 |x|_{c_{-1}}^2 + \eta_2,$$

where $\eta_1, \eta_2 \in \mathbb{R}_{>0}$, and $V(x) = |x|^2$, and combine them to form the candidate CCLF

$$V_C(x) = \max\{V(x), \min\{C_1(x), C_{-1}(x)\}\}. \quad (9)$$

We assume that η_1 and η_2 are chosen such that

$$V_C(0) = 0 \quad \text{and} \quad V_C(\hat{x}) > V(\hat{x}) \quad (10)$$

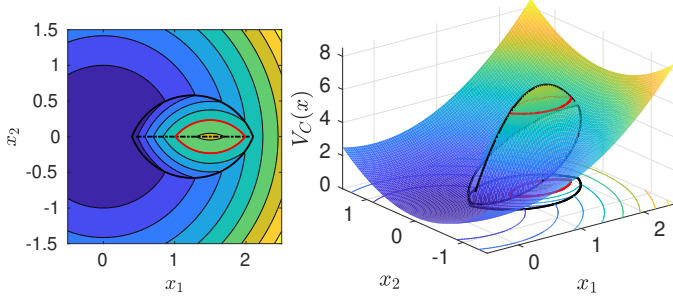


Fig. 2. Visualization of the function V_C defined in (9) for the parameters $\hat{x} = (1.5, 0)^T$, $\delta_\mu = 0.4$, $d = (0, 1)^T$, $\eta_1 = 5$ and $\eta_2 = 7$. The black lines indicate its nonsmooth domains. The boundary of a potential open set \mathcal{O} is visualized in red.

are satisfied. This is always possible for large enough η_1 and η_2 . Because of the “max” in (9), this guarantees that 0 is a strict global minimum and \hat{x} is a strict local maximum of V_C . In Figure 2 an example of the function (9) with $\hat{x} = (1.5, 0)^T$, $\delta_\mu = 0.4$, $d = (0, 1)^T$, $\eta_1 = 5$ and, $\eta_2 = 7$ is visualized. The function in (9) is continuously differentiable almost everywhere. As compared to the continuously differentiable function visualized in Figure 1, the function in Figure 2 does not have a saddle point and thus (under the additional assumption that the right-hand side $F(x)$ allows for it) provides the possibility to take a decision on the x_1 -axis using the Dini derivative as in (6). To compute the set where the gradient does not exist, we derive in the next lemma a second representation of the function V_C defined in (9).

Lemma 2. Consider the function V_C defined in (9) with $\eta_2 > \frac{\eta_1}{1+\eta_1}|c_p|^2$, $p \in \{-1, 1\}$. Additionally consider the scaled points

$$c_p^* := \frac{\eta_1}{1+\eta_1}c_p, \quad p \in \{-1, 1\} \quad (11)$$

and the radius

$$r^* := \sqrt{-\frac{\eta_1}{(1+\eta_1)^2}c_p^T c_p + \frac{\eta_2}{1+\eta_1}}. \quad (12)$$

Then $r^* > 0$ and the function V_C can be rewritten as

$$V_C(x) = \begin{cases} C_p(x), & \text{if } x \in C_p^*, \quad p \in \{-1, 1\} \\ V(x), & \text{if } x \in \mathcal{V}, \end{cases} \quad (13)$$

where

$$C_p^* = \{x \in B_{r^*}(c_p^*) | p(x - \hat{x})^T d \geq 0\}, \quad p \in \{-1, 1\}, \quad (14a)$$

$$\mathcal{V} = \mathbb{R}^n \setminus (C_1^* \cup C_{-1}^*). \quad (14b)$$

Proof. To establish the statements in the lemma we first compute the sets where the functions C_1 , C_{-1} and, V have the same value.

Case 1 ($C_1(x) = C_{-1}(x)$): Since the constants cancel, we are left with the condition $|x|_{c_1}^2 = |x|_{c_{-1}}^2$. With the definition of c_p , $p \in \{-1, 1\}$, in (8), the left and the right-hand sides can be expanded to

$$\begin{aligned} (x - c_p)^T(x - c_p) &= (x - \hat{x} + p\delta_\mu d)^T(x - \hat{x} + p\delta_\mu d) \\ &= (x - \hat{x})^T(x - \hat{x}) + 2p\delta_\mu(x - \hat{x})^T d + 2\delta_\mu^2 \end{aligned}$$

from which $|x|_{c_1}^2 = |x|_{c_{-1}}^2$ yields $2\delta_\mu(x - \hat{x})^T d = -2\delta_\mu(x - \hat{x})^T d$. Therefore, since $\delta_\mu > 0$,

$$(x - \hat{x})^T d = 0, \quad (15)$$

representing a hyperplane in \mathbb{R}^n orthogonal to d .

Case 2 ($V(x) = C_p(x)$ for $p \in \{-1, 1\}$): By definition, this set is characterized by

$$x^T x = -\eta_1(x - c_p)^T(x - c_p) + \eta_2$$

which is equivalent to

$$(1 + \eta_1)x^T x - 2\eta_1 x^T c_p = \eta_2 - \eta_1 c_p^T c_p.$$

Dividing by $1 + \eta_1 > 0$ and reordering the terms leads to

$$x^T x - 2x^T c_p^* = \frac{\eta_2}{1+\eta_1} - c_p^{*T} c_p^*$$

with c_p^* defined in (11). Adding $|c_p^*|^2$ to both sides we get

$$\begin{aligned} (x - c_p^*)^T(x - c_p^*) &= |c_p^*|^2 + \frac{\eta_2}{1+\eta_1} - c_p^{*T} c_p^* \\ &= -\frac{\eta_1}{(1+\eta_1)^2}c_p^{*T} c_p^* + \frac{\eta_2}{1+\eta_1} = (r^*)^2, \end{aligned} \quad (16)$$

which is positive by the condition on η_1 and η_2 and can always be achieved by selecting η_2 large enough. \square

Note that (16) describes a sphere with center $c_p^* = \frac{\eta_1}{1+\eta_1}c_p$ for $p \in \{-1, 1\}$. Based on Lemma 2, consider the strict inequality of the right bound (10) and the representation of V_C in Lemma 2, which implies that \hat{x} is in the interior of $C_{-1}^* \cup C_1^*$, and there exists $r \in (0, r^*)$ such that the sets

$$C_p = \{x \in B_r(c_p) | p(x - \hat{x})^T d \geq 0\}, \quad (17)$$

for $p \in \{-1, 1\}$ satisfy $C_p \subset C_p^*$.

We may select $\mathcal{O} = \text{int}(C_1 \cup C_{-1})$ because for all $x \in \partial\mathcal{O}$ we have (using $p = 1$ if $(x - \hat{x})^T d \geq 0$ and $p = -1$ otherwise)

$$V_C(x) = C_p(x) = \eta_2 - \eta_1 r^2 =: c$$

as required by (4).

Since radial unboundedness of V_C follows trivially from (9) and radial unboundedness of V , we have that V_C is a candidate CCLF by satisfying items (i) and (ii) of Definition 3. Since item (iii) depends on the right-hand side $F(x)$, in the next section we concentrate on item (iii) for a special class of differential inclusions.

4. CCLFS FOR LINEAR SYSTEMS

Linear control systems

$$\dot{x} = Ax + Bu, \quad x(0) \in \mathbb{R}^n \quad (18)$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ represent a special class of differential inclusions (1) where $F(\cdot)$ is defined by

$$F(x) = \overline{\text{conv}}(\{\xi \in \mathbb{R}^n | \xi = Ax + Bu, u \in \mathcal{U}(x)\})$$

for all $x \in \mathbb{R}^n$ and for $\mathcal{U}(x) \subset \mathbb{R}^m$ for all x . If the set of inputs $\mathcal{U}(x)$ is convex and compact for all $x \in \mathbb{R}^n$, then F satisfies Assumption 1.

Assume first that the linear system (18) is fully actuated, i.e., $B \in \mathbb{R}^{n \times n}$ has full rank. Since in this case, it is possible to move in any direction with an appropriate input u (large enough but bounded), V_C defined in (7) satisfies item (iii) of Definition 3 if it does not have local minima other than the origin. Expression (13) and the gradients of V and C_p , $p \in \{-1, 1\}$ reveal that no such local minimum can exist and thus V_C is a CCLF.

In a more challenging setting, consider a stabilizable linear system (18) with one-dimensional input $u \in \mathbb{R}$, i.e., $B \in \mathbb{R}^{n \times 1}$. Additionally, assume that $V(x) = x^T x$ is a Lyapunov function for the uncontrolled system and hence that matrix $A^T + A$ is Hurwitz. This assumption is not restrictive as argued in Braun et al. (2018b), because through an appropriate coordinate transformation and by redefining the input this condition can always be achieved for a stabilizable linear system.

We derive conditions guaranteeing that the candidate

CCLF defined in (9) is a CCLF for the linear system (18) and a neighborhood \mathcal{O} around a given point $\hat{x} \in \mathbb{R}^n$.

Theorem 3. Consider a stabilizable linear system (18) with one-dimensional input $u \in \mathbb{R}$ and assume that $V(x) = x^T x$ is a Lyapunov function for the unconstrained and uncontrolled system. Let

$$\hat{x} \in \mathbb{R}^n \setminus \text{span}(B) \quad (19)$$

and define the projection and the corresponding direction

$$P_{\hat{x}} = I - \frac{1}{\hat{x}^T \hat{x}} \hat{x} \hat{x}^T, \quad d = \frac{P_{\hat{x}} B}{|P_{\hat{x}} B|}. \quad (20)$$

Then there exist parameters $\delta_\mu, \eta_1, \eta_2 \in \mathbb{R}_{>0}$ such that the control law

$$u(x) = -\frac{\langle x - c_p^*, Ax \rangle}{\langle x - c_p^*, B \rangle} \quad (21)$$

is well-defined for all $x \in \bar{\mathcal{V}} \cap \mathcal{C}_p^*$, $p \in \{-1, 1\}$. If additionally

$$0 > \max_{x \in \bar{\mathcal{V}} \cap \mathcal{C}_p^*} dV_C \left(x; -\frac{\langle x - c_p^*, Ax \rangle}{\langle x - c_p^*, B \rangle} \right) \quad (22)$$

for $p \in \{-1, 1\}$ then V_C defined in Lemma (13) is a CCLF according to Definition 3 for $\mathcal{O} \subset \mathbb{R}^n$ defined as a neighborhood around \hat{x} . \square

The direction d defines the orientation of the hyperplane (15) and, by construction, $\hat{x}^T d = 0$ which simplifies the representation of (15) to $\{x \in \mathbb{R}^n | x^T d = 0\}$ and similarly the expressions of $\mathcal{C}_p, \mathcal{C}_p^*, p \in \{-1, 1\}$, in (14) and (17). Moreover, d is aligned with the projection of B on the hyperplane $P_{\hat{x}} x$ for $x \in \mathbb{R}^n$. For suitable non-controllable (but stabilizable) systems, it can be shown that condition (19) is necessary for the existence of a CCLF. Condition (22) guarantees that V_C decreases along trajectories in $\bar{\mathcal{V}} \cap \mathcal{C}_p^*$, $p \in \{-1, 1\}$, if the control law (21) is used. A more explicit condition for (22) is given in Remark 1 after the proof of Theorem 3. A simplification of this condition is a goal of future research.

Proof. As a first step, we show that the control law (21) is well-defined and provide a motivation for the selection of u . For the sphere $\mathcal{S}^* := \{x \in \mathbb{R}^n | |x|_{c_p^*} = r^*\}$, Braun et al. (2018b) proposed the control law (21) based on the condition $\frac{d}{dt}|x(t)|_{c_p^*}^2 = 0$. For $x(0) \in \mathcal{S}^*$, by construction, the input u in (21) ensures that $x(t) \in \mathcal{S}^*$ for all $t \geq 0$, if $\langle x - c_p^*, b \rangle \neq 0$ is satisfied.

For (21) to be well-defined, it is sufficient if $\langle x - c_p^*, b \rangle \neq 0$ holds for all $x \in \mathcal{C}_p^*$, which can be achieved by making the intersection of the ball $B_{r^*}(c_p^*)$ and the half-space $p \cdot x^T d \geq 0$ small. To this end we define

$$\hat{x}^* = \frac{1}{2} (c_1^* + c_{-1}^*) = \frac{1}{2} \frac{\eta_1}{1 + \eta_1} (c_1 + c_{-1}) = \frac{\eta_1}{1 + \eta_1} \hat{x}.$$

which is a linear combination of \hat{x} and $\hat{x}^* \notin \text{span}(B)$. In the same way as δ_μ defines the distance from c_p to \hat{x} (and thus to the hyperplane $x^T d = 0$), we can define

$$\delta_\mu^* = |\hat{x}^* - c_p^*| = \left(\frac{\eta_1}{1 + \eta_1} \right) \delta_\mu \quad (23)$$

as the distance from c_p^* to the hyperplane $x^T d = 0$. Then, as visualized in Figure 3, for (21) to be well-defined in \mathcal{C}_p^* , $p \in \{-1, 1\}$, it is sufficient that

$$\frac{\delta_\mu^*}{r^*} > \left| \frac{\langle \hat{x}, B \rangle}{|\hat{x}| |B|} \right| \quad (24)$$

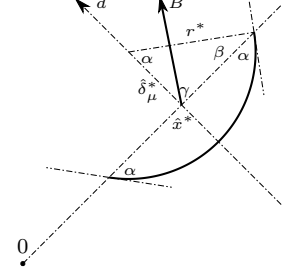


Fig. 3. Visualization of the relation between the angle γ and the angle α . To ensure that $\langle x - c_p^*, B \rangle \neq 0$ for all $x \in \mathcal{C}_p^*$ it is sufficient that $\alpha < \gamma$ holds.

be satisfied. To see this, observe that $\alpha := \arccos(\delta_\mu^*/r^*)$ defines the maximal angle of a tangent vector of $x \in \mathcal{C}_p^*$ and $x - c_p^*$, and

$$\gamma := \arccos \left(\left| \frac{\langle \hat{x}, B \rangle}{|\hat{x}| |B|} \right| \right) = \arccos \left(\left| \frac{\langle \hat{x}, B \rangle}{|\hat{x}| |B|} \right| \right)$$

defines the angle between \hat{x} and B . Thus $\alpha < \gamma$ is equivalent to condition (24) and ensures that \mathcal{C}_p^* does not contain points x such that $x - c_p^*$ and B are perpendicular. Note that r^* can be made arbitrarily small by increasing η_2 for fixed η_1 . This implies that $r^* - \delta_\mu^* > 0$ can be made arbitrarily small and therefore (24) can be achieved through an appropriate selection of δ_μ, η_1 and η_2 .

In Section 3 we have shown that item (i) and (ii) of Definition 3 are satisfied for V_C defined in (9) and appropriate parameters η_1, η_2 and here we have shown that additionally condition (24) can be satisfied. Thus, we concentrate on the decrease condition (6) in item (iii) of Definition 3 in the following. For the proof, we use the representation of V_C in (13) and in particular the definition of the sets (14) to verify (6) for all $x \in \mathbb{R}^n$.

Case 1 ($x \in \mathcal{V}$): Since $A^T + A$ is Hurwitz by assumption, the input $u = 0$ satisfies $dV_C(x, Ax) = x^T (A^T + A)x < 0$ for all $x \in \mathcal{V}$. This implies that an appropriate function $\rho \in \mathcal{P}$ can be defined.

Case 2 ($x \in \bar{\mathcal{V}} \cap \mathcal{C}_p^*, p \in \{-1, 1\}$): If condition (24) is satisfied, the control law (21) is well-defined on \mathcal{C}_p^* and (22) ensures a decrease.

Case 3 ($x \in \mathcal{C}_p^* \setminus (\bar{\mathcal{V}} \cup \mathcal{C}_{-p}^*), p \in \{-1, 1\}$): Due to the selection of $r \in (0, r^*)$ to define \mathcal{C}_p in (17) and due the definition of δ_μ^* in (23) and property (24), the estimate

$$\frac{\delta_\mu}{r} > \frac{\delta_\mu^*}{r^*} > \left| \frac{\langle \hat{x}, B \rangle}{|\hat{x}| |B|} \right| \quad (25)$$

holds. Similar to control law (21), we define

$$u(x) = \frac{1 - \langle x - c_p, Ax \rangle}{\langle x - c_p, B \rangle}. \quad (26)$$

based on the condition $\frac{d}{dt}|x(t)|_{c_p}^2 = 2$, i.e., $u(x)$ is defined such that the distance to c_p increases, and thus $V_C(x(t)) = C_p(x(t))$ decreases along trajectories $x(t) \in \mathcal{C}_p^* \setminus (\bar{\mathcal{V}} \cup \mathcal{C}_{-p}^*)$. Inequality (25) ensures that the control law is well-defined for all $x(t) \in \mathcal{C}_p^*$.

Case 4 ($x \in \mathcal{C}_{-1}^* \cap \mathcal{C}_1^*$): On $\mathcal{C}_{-1}^* \cap \mathcal{C}_1^*$, the function V_C is concave (and thus, in particular, semiconcave, see (Clarke, 2011, Sec. 5)) as the minimum of two concave functions $C_{-1}(x)$ and $C_1(x)$. Thus, the control law (26) for $p \in \{1, -1\}$ arbitrary ensures that (6) is satisfied on $(\mathcal{C}_{-1}^* \cap \mathcal{C}_1^*) \setminus \hat{x}$ which concludes the proof. \square

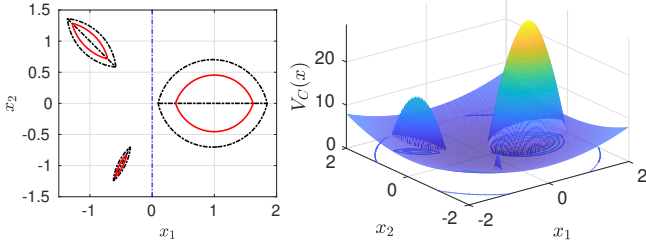


Fig. 4. Visualization of the function V_C defined in (13) on the right. On the left, boundaries of the sets (14) (black), boundaries of potential sets \mathcal{O}_i , $i = 1, 2, 3$, (red) and the subspace $\text{span}(B)$ (blue) are visualized.

Remark 1. Due to the properties of the control law (21), it holds that

$$dV_C\left(x; -\frac{\langle x - c_p^*, Ax \rangle}{\langle x - c_p^*, b \rangle}\right) = dV\left(x; -\frac{\langle x - c_p^*, Ax \rangle}{\langle x - c_p^*, b \rangle}\right), \quad \forall x \in \bar{\mathcal{V}} \cap C_p^*.$$

From the proof of Theorem 3 it follows that well-definedness of (21) in $\bar{\mathcal{V}} \cap C_p^*$ implies well-definedness of (21) in C_p^* . Thus, for (22) to be true, it is sufficient that

$$0 > \max_{x \in C_p^*} dV\left(x; -\frac{\langle x - c_p^*, Ax \rangle}{\langle x - c_p^*, B \rangle}\right) \quad (27)$$

$$= \max_{x \in C_p^*} x^T (A^T + A)x - 2x^T B \frac{\langle x - c_p^*, Ax \rangle}{\langle x - c_p^*, B \rangle} \quad (28)$$

be satisfied. \lrcorner

5. NUMERICAL EXAMPLE

Consider the linear system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

with three obstacles $\hat{x}_1 = (1, 0)^T$, $\hat{x}_2 = (-1, 1)^T$ and $\hat{x}_3 = (-0.5, -1)^T$. The matrix $A^T + A$ has the eigenvalues -1 and -3 , i.e., the origin of the uncontrolled system is stable and $V(x) = x^T x$ is a CLF. We define the CCLF (7) through the parameters $\eta_1 = 35$, $\eta_2 = 30$ for all three obstacles, and $\delta_{\mu,1} = 0.2$, $\delta_{\mu,2} = 0.7$ and $\delta_{\mu,3} = 0.85$. Condition (24) for the three obstacles is satisfied through

$$\frac{\delta_{\mu,1}^*}{r_1^*} = 0.2167, \quad \frac{\delta_{\mu,2}^*}{r_2^*} = 0.775, \quad \frac{\delta_{\mu,3}^*}{r_3^*} = 0.9357$$

and

$$\left| \frac{\langle \hat{x}_1, B \rangle}{|\hat{x}_1| |B|} \right| = 0, \quad \left| \frac{\langle \hat{x}_2, B \rangle}{|\hat{x}_2| |B|} \right| = 0.7071, \quad \left| \frac{\langle \hat{x}_3, B \rangle}{|\hat{x}_3| |B|} \right| = 0.8944.$$

The smaller the angle between the reference points \hat{x} and the direction B , the smaller the open domain \mathcal{O} needs to be. Condition (22) is verified by solving the optimization problem (28) using `fmincon` in Matlab and we can conclude that the function V_C is a CCLF. The function V_C is visualized in Figure 4 on the right. On the left, the boundaries of the sets defined in (14) are highlighted in black. Possible boundaries of the sets \mathcal{O} are shown in red and the blue line visualizes the subspace $\text{span}(B)$ in (19).

6. CONCLUSIONS

In this paper we discuss an extension of classical Lyapunov theory to incorporate bounded obstacles in the formulation of CLFs. Our constructions for linear systems, in particular due to the consideration of nonsmooth functions, show the difficulties in the verification of candidate CCLFs. Future work will thus concentrate on constructive methods in the CCLF design for more general classes of systems.

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This implies that for all $x \in \mathbb{R}^n \setminus \cup_{i=1}^N \mathcal{O}_i$ there exists a $\phi(\cdot; x) \in \mathcal{S}(x)$ such that the decrease condition (A.3) is satisfied for all $t \in [0, \Gamma]$ where $\Gamma > 0$. Since this argument can be applied iteratively to the initial value $\phi(\Gamma; x)$, there exists a solution $\phi(\cdot, x) \in \mathcal{S}(x)$ such that (A.3) is satisfied for all $t \geq 0$. \square

Appendix A. PROOF OF THEOREM 2

We adapt the proofs given in (Braun et al., 2018a, Thm. 4.11, Thm. 4.12). The proofs in Braun et al. (2018a) themselves rely on arguments used in Clarke et al. (1998) and Lakshmikantham and Leela (1969). Additionally, we use the notation $\phi(\cdot, x) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ to denote a solution $\phi(\cdot, x) \in \mathcal{S}(x)$ of the differential inclusion (1).

Proof. We define the set-valued map $H : \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}^n$,

$$H(v) = \{x \in \mathbb{R}^n | V_C(x) = v\}, \quad (\text{A.1})$$

and the function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$

$$\tilde{\gamma}(v) = \min\{\rho(|x|) | x \in H(v) \setminus \{\cup_{i=1}^N \mathcal{O}_i\}\}. \quad (\text{A.2})$$

Since $\rho \in \mathcal{P}$ and $V_C(x) > 0$ for all $x \neq 0$ it holds that $\tilde{\gamma}(|x|) > 0$ for all $x \neq 0$ and $\tilde{\gamma}(0) = 0$. We define $\gamma \in \mathcal{P}$ such that $\gamma(v) \leq \tilde{\gamma}(v)$ for all $v \in \mathbb{R}_{\geq 0}$ (where $\gamma \in \mathcal{P}$ implies that γ is continuous). Since V_C is locally Lipschitz, $V_C(\phi(\cdot, x))$ is absolutely continuous. For $x \in \mathbb{R}^n \setminus \cup_{i=1}^N \mathcal{O}_i$ assume there exists $\phi(\cdot; x) \in \mathcal{S}(x)$ such that

$$\begin{aligned} \frac{d}{dt} V_C(\phi(t; x)) &= \langle \nabla V_C(\phi(t; x)), \dot{\phi}(t; x) \rangle \\ &\leq -\frac{1}{4} \rho(|\phi(t; x)|) \leq -\frac{1}{4} \gamma(V_C(\phi(t; x))) \end{aligned} \quad (\text{A.3})$$

for almost all $t \in \mathbb{R}_{\geq 0}$. We apply the comparison principle (see (Sontag and Wang, 2000, Lemma A.4), for example) which provides a function $\beta \in \mathcal{KL}$ such that $V_C(\phi(t; x)) \leq \beta(V_C(x); t)$ and

$$|\phi(t; x)| \leq \alpha_1^{-1} \circ \beta(\alpha_2(|x|), t) = \tilde{\beta}(|x|, t)$$

with $\tilde{\beta} \in \mathcal{KL}$.

To complete the proof, we need to show that the pointwise condition (6) ensures that for all $x \in \mathbb{R}^n$ there exists $\phi(\cdot; x) \in \mathcal{S}(x)$ such that the estimate (A.3) is satisfied for almost all $t \in \mathbb{R}_{\geq 0}$. We assume to the contrary, that there exists an $x \in \mathbb{R}^n \setminus \cup_{i=1}^N \mathcal{O}_i$ and a $\Gamma > 0$ such that all solutions $\phi(\cdot; x) \in \mathcal{S}(x)$ satisfy

$$\frac{d}{dt} V_C(\phi(t; x)) > -\frac{1}{4} \rho(|\phi(t; x)|) \quad (\text{A.4})$$

for all t in a set of non-zero measure contained in $[0, \Gamma]$.

We choose an $\varepsilon > 0$ such that $\frac{1}{2} \rho(y) < \rho(|x|)$ for all $y \in B_\varepsilon(x) \setminus \cup_{i=1}^N \mathcal{O}_i$. Due to condition (6), there exists $\tilde{w} \in F(x)$ such that

$$dV_C(x; \tilde{w}) \leq -\rho(|x|). \quad (\text{A.5})$$

Since F is Lipschitz continuous there exists a Lipschitz continuous function $w : [0, \Gamma] \rightarrow \mathbb{R}^n$ such that $\phi(\cdot; x) \in \mathcal{S}(x)$, $\dot{\phi}(t; x) = w(t)$ for almost all $t \in [0, \Gamma]$ and $w(0) = \tilde{w}$ (and $w(t) \in F(\phi(t; x))$). Note that $\phi(\cdot; x)$ is Lipschitz continuous. From the assumed condition (A.4) and $\frac{1}{2} \rho(|\phi(t; x)|) < \rho(|x|)$, we obtain the condition

$$\frac{1}{2} (V_C(\phi(t; x)) - V_C(\phi(0; x))) \geq -\frac{1}{2} \rho(x)$$

for all $t \in (0, \Gamma)$ such that $\phi(t; x) \in B_\varepsilon(x) \setminus \cup_{i=1}^N \mathcal{O}_i$. Since the left-hand side is Lipschitz continuous, we can take the limit inferior for $t \rightarrow 0$ on both sides, which contradicts (A.5) and thus the assumption (A.4) was wrong.