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# Smooth solutions for nonlinear elastic waves with softening

Harold Berjamine\*, Stéphane Junca†, Bruno Lombard‡

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## Abstract

A new hyperbolic softening model has been proposed for wave propagation in damaged solids [*Proc. R. Soc. A*, **473** (2017), 20170024]. The linear elasticity becomes nonlinear through an additional internal variable. This thermodynamically relevant model yields a dissipative energy. The  $3 \times 3$  nonlinear hyperbolic system so-obtained is totally linearly degenerate like the well-known Kerr-Debye system. Existence of global smooth solutions is studied here thanks to the Kawashima condition. Moreover, shocks never appear with smooth initial data. Thus, the only possible blow-up of smooth solutions is the blow-up in  $L^\infty$  as for ODEs.

**AMS Classification.** Primary: 35L45, 35B65; Secondary: 74D10, 74J30.

**Key words:** Damaged solids; nonlinear balance laws; linearly degenerate flux; conservation law; finite-volume method.

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# 1 Introduction

The system of interest has been introduced in [2] to model nonlinear wave propagation in solids:

$$\partial_t \varepsilon - \partial_x v = 0, \quad (1)$$

$$\rho_0 \partial_t v - \partial_x \sigma = 0, \quad (2)$$

$$\partial_t g = \frac{1}{\tau} (W(\varepsilon) - \phi'(g)), \quad (3)$$

where the constants are  $\rho_0 > 0$  the density,  $E > 0$  the Young modulus,  $\tau > 0$  the relaxation time. The variables are  $\varepsilon > -1$  the strain,  $v$  the velocity,  $\sigma = (1 - g)E\varepsilon$  the stress,  $W(\varepsilon) = \frac{1}{2}E\varepsilon^2$  the strain energy,  $g$  an internal variable representing the damage. The system is completed with three initial data at time  $t = 0$ :  $\varepsilon_0(x)$ ,  $v_0(x)$ ,  $0 \leq g_0(x) < 1$ .

The storage function  $\phi(g)$  has to satisfy  $\phi'(0) = 0$  (to preserve the equilibrium  $(\varepsilon, g) = (0, 0)$  and to keep  $g \geq 0$ ),  $\phi' \geq 0$  and  $\phi'' > 0$  (to ensure the stability of constant equilibrium). Moreover,  $g < 1$  is required since for  $g = 1$  the solid is broken. An example of function  $\phi$  to ensure these constraints is  $\phi(g) = -\frac{1}{2}\gamma \ln(1 - g^2)$  with  $\gamma > 0$ .

The initial-value problem for the system of balance laws (1)-(3) can be rewritten in vectorial form with  $c = c(g) = \bar{c}\sqrt{1 - g}$  and  $\bar{c} = \sqrt{E/\rho_0}$ :

$$\partial_t U + \partial_x F(U) = G(U), \quad (4)$$

$$U = (\varepsilon, v, g)^\top, \quad F(U) = (-v, -c^2\varepsilon, 0)^\top, \quad G(U) = \frac{1}{\tau} (0, 0, W(\varepsilon) - \phi'(g))^\top.$$

The nonincreasing total energy  $\mathcal{E}$  and the internal energy  $e$  are:

$$\mathcal{E} := \rho_0(v^2/2 + e) = \rho_0(v^2/2 + (1 - g)W(\varepsilon) + \phi(g)). \quad (5)$$

For smooth solutions the dissipation of the energy is

$$\frac{d}{dt} \int_{\mathbb{R}} \mathcal{E} dx = -\frac{\rho_0}{\tau} \int_{\mathbb{R}} (W(\varepsilon) - \phi'(g))^2 dx = -\rho_0 \tau \int_{\mathbb{R}} (\partial_t g)^2 dx \leq 0. \quad (6)$$

It is a partially dissipative system [8, 1] which may ensure existence of global smooth solutions [16] under the Kawashima condition [15].

In Section 2, the totally linearly degenerate  $3 \times 3$  homogeneous system deduced from (4) is studied. The Kawashima condition for the full system with the source term are directly related to the function  $\phi$  in Section 3. The comparison with the Kerr-Debye system and the non existence of shock wave for the system (4) are in Section 4. Finally, numerical simulations of smooth solutions for the system (4) conclude the paper in Section 5.

## 2 The linearly degenerate homogeneous system

Consider the system (4) with no source:  $G = 0$ . The Jacobian matrix of the flux  $F$  has three eigenvectors  $r_-, r_0, r_+$  associated to 3 linearly degenerate eigenvalues:  $-c, 0, +c$  in

the hyperbolic region  $g < 1$ :

$$A = DF(U) = \begin{pmatrix} 0 & -1 & 0 \\ -c^2 & 0 & \varepsilon \bar{c}^2 \\ 0 & 0 & 0 \end{pmatrix}, \quad r_{\pm} = \begin{pmatrix} 1 \\ \mp c \\ 0 \end{pmatrix}, \quad r_0 = \begin{pmatrix} \varepsilon \bar{c}^2 \\ 0 \\ c^2 \end{pmatrix}. \quad (7)$$

Many things are known for  $2 \times 2$  totally linearly degenerate system [13, 14]. Less is known for  $3 \times 3$  system except under special conditions as in [11].

The homogeneous version of (3) means simply  $g \equiv g_0$  thus the nonlinear system (1)-(2) gives a linear wave equation with the variable sound speed  $c_0(x) = \bar{c}\sqrt{1 - g_0(x)}$ :

$$\partial_t^2 \varepsilon - \partial_x^2 (c_0^2(x) \varepsilon) = 0. \quad (8)$$

Physically, it corresponds simply to the linear elasticity with a varying Young modulus depending only on the space variable  $x$ . Then a proof using the Riemann invariants [9] of the elastodynamics yields the existence of global smooth solutions:

**Proposition 1 (Global smooth solution for the homogeneous system)**

*Let us assume that the initial data at time  $t = 0$ :  $\varepsilon_0(x), v_0(x)$  belongs to the space  $Lip_{loc}(\mathbb{R}, \mathbb{R})$  of locally Lipschitz-functions,  $g_0(x) \in Lip(\mathbb{R}, \mathbb{R})$ ,  $\sup_{\mathbb{R}} g_0(x) < 1$  and  $\partial_x g_0 \in Lip_{loc}(\mathbb{R}, \mathbb{R})$ . Then the homogeneous hyperbolic system admits a unique global smooth solution  $(\varepsilon, v)$  with the same regularity in space as the initial data:*

$$\varepsilon, v \in L_{loc}^{\infty}([0, +\infty[, Lip_{loc}(\mathbb{R}, \mathbb{R})) \cap C^1([0, +\infty[, L_{loc}^{\infty}(\mathbb{R}, \mathbb{R})).$$

A proof using the Riemann invariants of the  $2 \times 2$  system of linear elastodynamics is proposed. They are not Riemann invariants for the full  $3 \times 3$  system, nevertheless some computations are possible involving  $\partial_x g$ . Since  $g(t, x) = g_0(x)$  the term  $\partial_x g$  is easily controlled by the Lipschitz initial datum  $g_0(x)$ . For the system with the source term, the following proof fails because  $\partial_x g$  cannot be estimated so easily.

Proof: The dimensionless  $3 \times 3$  nonlinear system is simply rewritten as a linear  $2 \times 2$  system with a variable coefficient:

$$\partial_t \varepsilon - \partial_x v = 0, \quad (9)$$

$$\partial_t v - \partial_x [(1 - g_0(x)) \varepsilon] = 0, \quad (10)$$

or in a short way:

$$\partial_t U + \partial_x F(x, U) = 0, \quad U = (\varepsilon, v)^{\top}, \quad F(x, U) = (-v, (g_0(x) - 1)\varepsilon)^{\top}.$$

Let  $A$  and  $G$  be the  $2 \times 2$  variable matrices which depend on the space variable  $x$ :

$$A(x) = \partial_U F = \begin{pmatrix} 0 & -1 \\ g_0(x) - 1 & 0 \end{pmatrix}, \quad G(x, U) = \partial_x F = \begin{pmatrix} 0 \\ (\partial_x g_0(x)) \varepsilon \end{pmatrix}.$$

The gradient of the Riemann invariants  $\nabla_U z_{\pm} = (\mp c_0, 1)$  are the left eigenvectors of the matrix  $A$ . Thus, the Riemann invariants are  $Z = (z_+, z_-)^\top$ . The system (9)-(10) reads:

$$\partial_t U + A(x) \partial_x U = -G(x, U), \quad (11)$$

$$\begin{aligned} \partial_t z_{\pm} \pm c_0 \partial_x z_{\pm} &= -\nabla_U z_{\pm} \cdot G(x, U) \\ &= -(\partial_x g_0) \varepsilon \\ &= -(\partial_x g_0) \frac{z_- - z_+}{2c_0 g_0}. \end{aligned} \quad (12)$$

Notice that  $\nabla_U z_{\pm}$ ,  $c_0$  and  $g_0$  are only functions of  $x$ . The function  $\sigma$  is linear with respect to  $\varepsilon$ , thus Lipschitz with respect to  $Z$ .

The system (11) with the variable  $U$  has been rewritten with the vector  $Z$  for the  $2 \times 2$  system (12), which variable coefficients are smooth while  $g_0 < 1$ . Using the characteristics  $X_{\pm}(t, x)$  and the Riemann invariants evaluated along the characteristics  $z_{\pm}(X_{\pm}(t, x), t)$ , the  $2 \times 2$  linear PDE system (12) becomes a family of the  $4 \times 4$  nonlinear ODE system parametrized by  $x \in \mathbb{R}$  and involving changes of variables:

$$\frac{dX_{\pm}}{dt} = \pm c_0(X_{\pm}) = \pm \sqrt{1 - g_0(X_{\pm})}, \quad X_{\pm}(0, x) = x, \quad (13)$$

$$\frac{dz_{\pm}}{dt} = - \left( (\partial_x g_0) \frac{z_- - z_+}{2c_0 g_0} \right) (X_{\pm}, t), \quad Z(x, 0) = Z_0(x). \quad (14)$$

This system is block triangular. The first two equations (13) are decoupled: the characteristics are global smooth functions since  $g_0$  is globally Lipschitz and  $\sup g_0 < 1$ , so that the function  $c_0$  is also globally Lipschitz. The characteristics have at most an exponential growth with respect to the time  $t$ .

Let us turn to the two coupled last equations (14). Notice that the coupling involves change of variables between  $X_-$  and  $X_+$ :  $z_{\mp}(X_{\pm}, t)$  instead of  $z_{\pm}(X_{\pm})$ . This is classical [6, 7] and can be managed by a fixed point strategy. Then the global existence follows.  $\square$

For the full system with a source term the situation is more intricate.

### 3 The (SK) condition for the complete system

The famous (SK) condition, defined by Shizuta and Kawashima in [15], yields existence of global smooth solutions near an equilibrium [16].

Consider an equilibrium  $U_e$  of the system (4), that is a constant solution:  $G(U_e) = 0$ . The (SK) condition writes at  $U_e$ :

$$\mathbf{Ker} DG(U_e) \cap \{\text{eigenvectors of } DF(U_e)\} = \{0\}. \quad (15)$$

The equilibrium  $U_e = (\varepsilon_e, v_e, g_e)$  for the system of interest is given by the equation  $W(\varepsilon_e) = \phi'(g_e)$ . When  $g_e > 0$ , there are two equilibrium  $(\varepsilon_e, v_e, g_e)$ , with

$$\varepsilon_e = \pm \sqrt{2\phi'(g_e)/E} \quad (16)$$

and without restriction on  $v_e$ . When  $g_e = 0$  the equilibrium is  $(0, v_e, 0)$ .

The linearized source term is a rank one matrix with the equation of the kernel,

$$\tau DG(U_e) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ E\varepsilon_e & 0 & -\phi''(g_e) \end{pmatrix}, \quad E\varepsilon_e\varepsilon = \phi''(g_e)g.$$

The eigenvectors  $r_{\pm}$  of  $DF(U_e)$  associated to the eigenvalues  $\pm c_e$  do not belong to  $\mathbf{Ker} DG(U_e)$ . The only problem remains for the eigenvector  $r_0$  in the kernel of  $DF(U_e)$ . It also belongs to  $\mathbf{Ker} DG(U_e)$  if and only if  $E\varepsilon_e^2 = (1 - g_e)\phi''(g_e)$ . Replacing the left hand side thanks to (16) yields only one equation to check when the (SK) condition is not fulfilled,  $(0 < g_e < 1)$ :

$$2\phi'(g_e) = (1 - g_e)\phi''(g_e). \quad (17)$$

Since  $\phi'' > 0$  the case  $g_e = 0$  is excluded.

Let us consider the example  $\phi(g) = -\frac{1}{2}\gamma \ln(1 - g^2)$ . A simple computation yields

$$\phi'(g) = \gamma \frac{g}{1 - g^2}, \quad \phi''(g) = \gamma \frac{1 + g^2}{(1 - g^2)^2} \geq \gamma.$$

Thus, the (SK) condition is fulfilled except when  $g_e = \sqrt{2} - 1 \simeq 0.414$ .

As a consequence of the previous study, with the condition  $\phi'(0) = 0$  and  $\phi'' > 0$ , the (SK) condition is always fulfilled if and only if,  $\forall g \in ]0, 1[$ ,

$$2\phi'(g) < (1 - g)\phi''(g). \quad (18)$$

### Lemma 3.1 (Loss of (SK) condition)

*If  $\lim_{g \rightarrow 1} (1 - g)\phi(g) = 0$  then the (SK) condition is not always satisfied.*

Proof: Notice that inequality (18) is always satisfied near  $g = 0$  since  $\phi'(0) = 0$  and  $\phi''(0) > 0$ . Let  $g_0$  belong to  $]0, 1[$ . It suffices to integrate the differential inequality (18) to get for all  $g \in ]g_0, 1[$ ,

$$\phi'(g) > \phi'(g_0) \left( \frac{1 - g_0}{1 - g} \right)^2, \quad \text{then} \quad \phi(g) > \phi(g_0) + \phi'(g_0) \frac{1 - g_0}{1 - g} (g - g_0).$$

Thus  $\liminf_{g \rightarrow 1} (1 - g)\phi(g) > \phi'(g_0)(1 - g_0)^2 > 0$  and the lemma follows by contradiction.  $\square$

A family of examples satisfying always the (SK) condition is given by  $\phi(g) = \frac{1}{2}\gamma g^2(1 - g)^{-\alpha}$ , with  $\alpha > 1$ . The condition  $\alpha > 1$  is necessary and sufficient. The proof is direct and needs only to check:  $\phi'(0) = 0$ ,  $\phi'' > 0$  on  $[0, 1[$  and (18).

As a direct consequence of (18) and the proof of Lemma 3.1, one gets the following lemma.

### Lemma 3.2 (Not (SK) fulfilled on a continuum)

*If the (SK) condition is not satisfied on an interval  $[g_1, g_2]$  then*

$$\phi(g) = \phi(g_1) + \phi'(g_1) \frac{1 - g_1}{1 - g} (g - g_1).$$

## 4 Comparison with the Kerr-Debye model

In this section we compare the system (1)-(3) to the Kerr-Debye system well-known in nonlinear optics. In the latter system smooth initial data in the Sobolev space  $H^2(\mathbb{R})$  yield global smooth solutions. The system (1)-(3) can be rewritten in a similar form as the Kerre-Debye system, except the source term which is modified. This modification prevents from transferring all known results on Kerr-Debye system to our case. In particular, we cannot deduce existence of global solutions. However, other results are known for the modified Kerr-Debye system [5] ensuring that no discontinuity can appear in finite time. The only catastrophe which can occur is a  $L^\infty$  blow-up as for the solutions of ODEs [12].

### The Kerr-Debye model

$$\partial_t d + \partial_x h = 0, \quad (19)$$

$$\partial_t h + \partial_x e = 0, \quad (20)$$

$$\partial_t \chi = \frac{1}{\tau} (e^2 - \chi), \quad (21)$$

where  $d = (1 + \chi)e$  and the initial condition  $(d, h, \chi)(x, 0) = (d_0, h_0, \chi_0)(x)$ . With  $\chi_0 \geq 0$  it follows immediatly that  $\chi \geq 0$ . The semilinear behavior of solutions of the system is proven in [4]. That means that a smooth solution is not global only if the solution blows up in sup-norm [12]. The global existence of all smooth solution is proven in [5]. This system is also endowed with a strictly convex partially dissipative energy,

$$\tilde{\mathcal{E}} = \frac{d^2}{1 + \chi} + h^2 + \frac{\chi^2}{2}, \quad \frac{d}{dt} \int_{\mathbb{R}} \tilde{\mathcal{E}} dx = -\tau \int_{\mathbb{R}} (\partial_t \chi)^2 dx \leq 0 \quad (22)$$

To prove (22) the system (19)-(21) is rewritten in variables  $W = (e, h, \chi)$ , to obtain a symmetric system. The semilinear behavior is proven by energy estimates. More precisely, if  $W$  is bounded in  $L^\infty([0, T * [ \times \mathbb{R})$  then  $W$  is also bounded in  $L^\infty([0, T * [, H^2(\mathbb{R}))$  which is enough to prevent the blow up of the gradient i.e. shock-wave.

### Our system rewritten in Kerr-Debye variables

Motivated by the previous results on the Kerr-Debye system, the system (1)-(3) is rewritten in Kerr-Debye variables:

$$d = \varepsilon, \quad h = -v, \quad \frac{d}{1 + \chi} = e = \sigma = (1 - g)\varepsilon \implies 1 + \chi = \frac{1}{1 - g}. \quad (23)$$

Thus,  $\chi_t = g_t(1 - g)^{-2} = (1 + \chi)^2 g_t = (1 + \chi)^2 (d^2/2 - \phi'(g))$  and our system becomes with  $\rho_0 = 1$ ,  $E = 1$  and  $\tau = 1$ :

$$\partial_t d + \partial_x h = 0, \quad (24)$$

$$\partial_t h + \partial_x e = 0, \quad (25)$$

$$\partial_t \chi = (1 + \chi)^2 \left( \frac{d^2}{2} - \phi'(g) \right) = (1 + \chi)^4 \frac{e^2}{2} - \psi'(\chi), \quad (26)$$

where  $\psi'$  is the increasing function defined by,  $\psi'(\chi) = (1+\chi)^2\phi'(1-(1+\chi)^{-1})$ . Comparing (21) with (26), there appears only two changes, the weight  $(1+\chi)^4$  and the function  $\psi$ . Conversely, the Kerr-Debye system can be rewritten in variables  $(\varepsilon, v, g)$ :

$$\partial_t \varepsilon = \partial_x v, \quad \partial_t v = \partial_x \sigma, \quad \partial_t g = (1-g)^4 \varepsilon^2 - \phi'(g).$$

Notice that  $\psi'$  linear – as for the Kerr-Debye system – corresponds to the following choice for  $\phi'$ :  $\phi'(g) = g(1-g)$ . For this choice,  $g < 1$  for all time since  $\chi > 0$  for all time. Unfortunately, this choice is not consistent with the requirement on  $\phi$ :  $\phi'' > 0$ .

Our model can be seen as a nonlinear generalization of the Kerr-Debye system. The nonlinear generalization consists in the nonlinear relaxation with respect to the variable  $\chi$ . This additional nonlinearity prevents to obtain a global energy estimate for the derivatives of the solutions of (1)-(3) as in [5].

The mapping,  $g \mapsto \chi$  is increasing,  $g = 0 \Leftrightarrow \chi = 0$ ,  $g = 1 \Leftrightarrow \chi = +\infty$ . Thus, the constraint on  $g$  becomes a condition of no blow up for  $\chi$ . The equation (26) yields automatically the positivity of  $\chi > 0$  and then the constraint  $g < 1$  required by our model.

Moreover, the semilinear behavior for generalized Kerr-Debye systems is known, Theorem 4.1 in [5]. Thus, our system enjoys a semilinear behavior. It means that no shock can occur with smooth initial data:

**Corollary 1 (No shock)** *Let  $\varepsilon_0, v_0, g_0$  belong to  $H^2(\mathbb{R})$  and  $\sup_{\mathbb{R}} g_0 < 1$  then the solution of the system (1)-(3) remains in  $H^2$  as soon as it remains in  $L^\infty$ .*

The solution is then global smooth or blows up. The blow up means that  $\varepsilon$  or  $v$  blow up in  $L^\infty$  or  $g = 1$  in finite time.

Let us explain why the [5]'s proof does not work for the system (1)- (2)- (3). To obtain the  $L^\infty$  bound with energy estimayes it suffices to get a  $H^1$  bound. The  $L^2$  bound is free with the energy. So, let's go to the estimate on the derivatives. To this purpose, we rewrite the system in variables  $V = (e, h, \chi)$ , (forgetting the /2 in  $e^2/2$  to avoid fractions)

$$(1+\chi)\partial_t e + \partial_x h = -e\partial_t \chi = -e((1+\chi)^4 e^2 - \psi'(\chi)), \quad (27)$$

$$\partial_t h + \partial_x e = 0, \quad (28)$$

$$\partial_t \chi = (1+\chi)^4 e^2 - \psi'(\chi), \quad (29)$$

Multiplying the first two equations by  $(e, h)$ , integrating in space and after few computations we recover the energy:

$$\frac{d}{dt} \int_{\mathbb{R}} ((1+\chi)e^2 + h^2 + 2\psi(\chi)) dx = -(1+\chi)^{-4} \int_{\mathbb{R}} (\partial_t \chi)^2 dx \leq 0, \quad (30)$$

This yields the  $L^2$  estimate.

$H^1$  bound:

We rewrite the system with  $\partial_t \chi$  in the first equation,

$$(1+\chi)\partial_t e + e\partial_t \chi + \partial_x h =, \quad (31)$$

$$\partial_t h + \partial_x e = 0, \quad (32)$$

$$\partial_t \chi = e^2 - \psi'(\chi). \quad (33)$$



Differentiating with respect to  $t$ ,

$$(1 + \chi)\partial_{tt}e + 2\partial_t e \partial_t \chi + e\partial_{tt}\chi + \partial_t \partial_x h =, \quad (34)$$

$$\partial_{tt}h + \partial_t \partial_x e = 0, \quad (35)$$

$$\partial_{tt}\chi = 2e\partial_t e - \psi''(\chi)\partial_t \chi. \quad (36)$$

Taking the usual energy estimate by multiplying the first two equations by  $(\partial_t e, \partial_t h)$  we have as in [5]

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} ((1 + \chi)(\partial_t e)^2 + (\partial_t h)^2) dx + 2 \int_{\mathbb{R}} \partial_t \chi (\partial_t e)^2 dx + \int_{\mathbb{R}} (\partial_{tt} \chi) e \partial_t e dx = 0, \quad (37)$$

since the term  $\partial_t e \partial_t \partial_x h + \partial_t h \partial_t \partial_x e = \partial_x (\partial_t e \partial_t h)$ . From (33) we have

$$\int_{\mathbb{R}} \partial_t \chi (\partial_t e)^2 dx = \int_{\mathbb{R}} e^2 (\partial_t e)^2 dx - \int_{\mathbb{R}} \psi'(\chi) (\partial_t e)^2 dx,$$

and from (36)  $2e\partial_t e = \partial_{tt}\chi + \psi''(\chi)\partial_t \chi$  so

$$\begin{aligned} 2 \int_{\mathbb{R}} (\partial_{tt} \chi) e \partial_t e dx &= \int_{\mathbb{R}} (\partial_{tt} \chi)^2 dx + \int_{\mathbb{R}} (\partial_{tt} \chi) \psi''(\chi) \partial_t \chi dx, \\ &= \int_{\mathbb{R}} (\partial_{tt} \chi)^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \psi''(\chi) (\partial_t \chi)^2 dx - \frac{1}{2} \int_{\mathbb{R}} \psi'''(\chi) (\partial_t \chi)^3 dx, \end{aligned}$$

since  $(\partial_{tt} \chi) \psi''(\chi) \partial_t \chi = \partial_t (\psi''(\chi) (\partial_t \chi)^2 / 2 - \psi'''(\chi) (\partial_t \chi)^3 / 2)$ .

Adding these equality we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \left( (1 + \chi)(\partial_t e)^2 + (\partial_t h)^2 + \frac{1}{2} \psi''(\chi) (\partial_t \chi)^2 \right) dx \quad (38)$$

$$+ \int_{\mathbb{R}} \left( e^2 (\partial_t e)^2 + \frac{1}{2} (\partial_{tt} \chi)^2 \right) dx \quad (39)$$

$$= \int_{\mathbb{R}} \psi'(\chi) (\partial_t e)^2 dx + \frac{1}{4} \int_{\mathbb{R}} \psi'''(\chi) (\partial_t \chi)^3 dx \quad (40)$$

Now, the main difficulty appears. It is the cubic term in (40). This term prevents the use of the Gronwall inequality. This term disappears for the Kerr-Debye system since  $\psi''' \equiv 0$ .

## 5 Numerical solution for the complete system

The system of balance laws (4) is solved numerically. Following Sec. 4.2 of [3], an explicit time-stepping formula is used, which involves the numerical flux of a finite-volume scheme (a flux-limiter method based on the Roe scheme). The initial data is chosen as follows:  $v_0(x)$  is zero,  $g_0(x) = g_e$  is constant, while the strain  $\varepsilon_0(x) = \varepsilon_e - 2VF(kx)$  has a smooth waveform  $F(x) = \frac{4}{3\sqrt{3}} (\sin(x) - \frac{1}{2} \sin(2x)) \mathbf{1}_{0 \leq x \leq 2\pi}$  with fundamental wavelength  $2\pi/k = 0.2$  and amplitude  $V$ . The domain  $x \in [-5, 5]$  is discretized with 20 000 points

and the Courant number is 0.95. Outflow conditions are implemented at the boundaries of the domain, as presented in Sec. 7.2.1 of [10]. In this section, the physical constants  $\rho_0$ ,  $E$  equal one, and  $\tau$ ,  $\gamma$  equal  $10^{-4}$  (SI).

Numerical results at the time  $t = 4.5$  are shown in Fig. 1 for  $\varepsilon_e = g_e = 0$ . In the small amplitude limit, the solution converges towards the solution obtained for linear elasticity ( $g \equiv 0$ ), where the initial data is transported at constant speed. As amplitudes are increased, wavefront steepening is observed, along with a diminution of the wave amplitude and of the speed of sound (delay). Nevertheless, the solution keeps smooth.

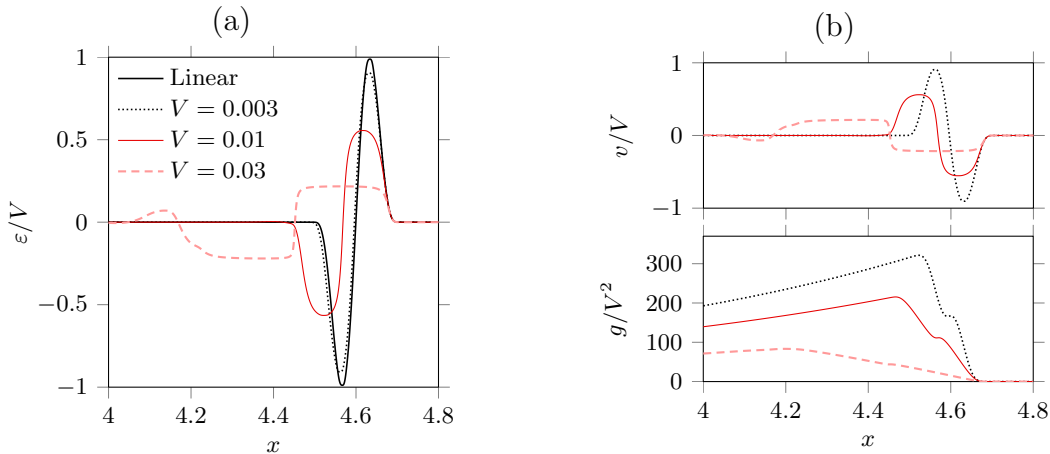


Figure 1: Equilibrium  $\varepsilon_e = g_e = 0$ . Numerical solution at  $t = 4.5$  for several amplitudes  $V$ . (a) Normalized strain  $\varepsilon/V$ ; (b) normalized velocity  $v/V$  and softening  $g/V^2$ .

At  $\varepsilon_e = \pm\sqrt{\gamma/E}$  and  $g_e = \sqrt{2} - 1$ , the (SK) condition is no longer satisfied. However, the stability of the equilibrium is verified numerically. This is illustrated in Fig. 2, which displays the numerical solution for  $V = 0.001$  at various times. The dynamics of the system seems to be driven by its stable equilibrium points.

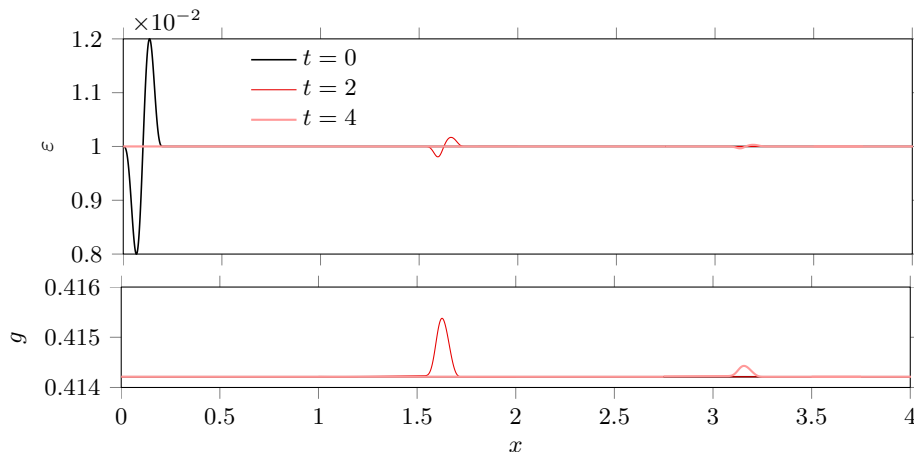


Figure 2: Equilibrium  $\varepsilon_e = \sqrt{\gamma/E}$ ,  $g_e = \sqrt{2} - 1$ . Numerical solution for  $V = 0.001$  at several times. Strain  $\varepsilon$  (top); softening  $g$  (bottom).

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