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# Triangle-free graphs that do not contain an induced subdivision of $K_4$ are 3-colorable

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## Abstract

We show that triangle-free graphs that do not contain an induced subgraph isomorphic to a subdivision of  $K_4$  are 3-colorable. This proves a conjecture of Trotignon and Vušković [5].

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# 1 Introduction

All graphs in this paper are finite and simple. Let  $G$  be a graph. For a vertex  $v \in V(G)$ , we denote its set of neighbors by  $N(v)$ , and we let  $N[v] = \{v\} \cup N(v)$ . For  $X, Y \subseteq V(G)$ , we say that  $X$  is *complete* to  $Y$  if every vertex in  $X$  is adjacent to every vertex in  $Y$ ;  $X$  is *anticomplete* to  $Y$  if every vertex in  $X$  is non-adjacent to every vertex in  $Y$ . A vertex  $v \in V(G)$  is *complete* (*anticomplete*) to  $X \subseteq V(G)$  if  $\{v\}$  is complete (anticomplete) to  $X$ . A set  $X \subseteq V(G)$  is a *cutset* for  $G$  if there is a partition  $(X, Y, Z)$  of  $V(G)$  with  $Y, Z \neq \emptyset$  and  $Y$  anticomplete to  $Z$ . The cutset  $X$  is a *clique cutset* if  $X$  is a (possibly empty) clique in  $G$ . For a graph  $H$ , we say that  $G$  *contains*  $H$  if  $H$  is isomorphic to an induced subgraph of  $G$ , and otherwise,  $G$  is  *$H$ -free*. For a family  $\mathcal{F}$  of graphs, we say that  $G$  is  *$\mathcal{F}$ -free* if  $G$  is  $F$ -free for every graph  $F \in \mathcal{F}$ .

For a graph  $G$  and  $X \subseteq V(G)$ ,  $G|X$  denotes the induced subgraph of  $G$  with vertex set  $X$ . For  $X \subseteq V(G)$ , we let  $G \setminus X = G|(V(G) \setminus X)$  and for  $x \in V(G)$ , we let  $G \setminus x = G|(V(G) \setminus \{x\})$ . By a *path* in a graph we mean an induced path. Let  $C$  be a cycle in  $G$ . The *length* of  $C$  is  $|V(C)|$ . The *girth* of  $G$  is the length of a shortest cycle, and is defined to be  $\infty$  if  $G$  has no cycle. A *hole* in a graph is an induced cycle of length at least four. An  $\text{ISK}_4$  is a graph that is isomorphic to a subdivision of  $K_4$ .

In [5] two of us studied the structure of  $\text{ISK}_4$ -free graphs, and proposed the following conjecture (and proved several special cases of it):

**Conjecture 1.** *If  $G$  is  $\{\text{ISK}_4, \text{triangle}\}$ -free, then  $\chi(G) \leq 3$ .*

In [2], Conjecture 1 was proved with 3 replaced by 4.

The main result of the present paper is the proof of Conjecture 1. In fact, we prove a stronger statement, from which Conjecture 1 easily follows:

**Theorem 2.** *Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}\}$ -free graph. Then either  $G$  has a clique cutset,  $G$  is complete bipartite, or  $G$  has a vertex of degree at most two.*

For an induced subgraph  $H$  of  $G$  we write  $v \in H$  to mean  $v \in V(H)$ . We use the same convention if  $H$  is a path or a hole. For a path  $P = p_1 - \dots - p_k$  we call the set  $V(P) \setminus \{p_1, p_k\}$  the *interior* of  $P$ , and denote it by  $P^*$ .

A *wheel* in a graph is a pair  $W = (C, x)$  where  $C$  is a hole and  $x$  has at least three neighbors in  $V(C)$ . We call  $C$  the *rim* of the wheel, and  $x$  the *center*. The neighbors of  $x$  in  $V(C)$  are called the *spokes* of  $W$ . Maximal paths of  $C$  that do not contain any spokes in their interior are called the *sectors* of  $W$ . We write  $V(W)$  to mean  $V(C) \cup \{x\}$ .

A graph is *series-parallel* if it does not contain a subdivision of  $K_4$  as a (not necessarily induced) subgraph.

**Theorem 3** ([1]). *Let  $G$  be a series-parallel graph. Then  $G$  is  $\text{ISK}_4$ -free, wheel-free, and  $K_{3,3}$ -free, and  $G$  contains a vertex of degree at most two.*

The following two facts were proved in [3]:

**Theorem 4** ([3]). *Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}\}$ -free graph. Then either  $G$  is series-parallel, or  $G$  contains a  $K_{3,3}$  subgraph, or  $G$  contains a wheel. If  $G$  contains a subdivision of  $K_{3,3}$  as an induced subgraph, then  $G$  contains a  $K_{3,3}$ .*

**Theorem 5** ([3]). *If  $G$  is an  $\{\text{ISK}_4, \text{triangle}\}$ -free graph and  $G$  contains  $K_{3,3}$ , then either  $G$  is complete bipartite, or  $G$  has a clique cutset.*

Thus to prove Theorem 2 we need to analyze  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graphs that contain wheels. This approach was already explored in [5], but we were able to push it further, as follows.

A wheel  $W = (C, x)$  is *proper* if for every  $v \in V(G) \setminus V(W)$

- there is a sector  $S$  of  $W$  such that  $N(v) \cap V(C) \subseteq V(S)$ .
- If  $v$  has at least three neighbors in  $V(C)$ , then  $v$  is adjacent to  $x$ .

(Please note that this definition is different from the one in [5].) We prove:

**Theorem 6.** *Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph, and let  $x$  be the center of a proper wheel in  $G$ . If  $W = (C, x)$  is a proper wheel with a minimum number of spokes subject to having center  $x$ , then*

1. *every component of  $V(G) \setminus N(x)$  contains the interior of at most one sector of  $W$ , and*
2. *for every  $u \in N(x)$ , the component  $D$  of  $V(G) \setminus (N(x) \setminus \{u\})$  such that  $u \in V(D)$  contains the interiors of at most two sectors of  $W$ , and if  $S_1, S_2$  are sectors with  $S_i^* \subseteq V(D)$  for  $i = 1, 2$ , then  $V(S_1) \cap V(S_2) \neq \emptyset$ .*

Using Theorem 6 we can prove a variant of a conjecture from [5] that we now explain. For a graph  $G$  and  $x, y \in V(G)$ , we say that  $(x, y)$  is a *non-center pair* for  $G$  if neither  $x$  nor  $y$  is the center of a proper wheel in  $G$ , and  $x = y$  or  $xy \in E(G)$ . We prove:

**Theorem 7.** *Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph which is not series-parallel, and let  $(x, y)$  be a non-center pair for  $G$ . Then some  $v \in V(G) \setminus (N[x] \cup N[y])$  has degree at most two.*

Here is the outline of the proof; the full proof is given in Section 5. We assume that  $G$  is a counterexample to Theorem 7 with  $|V(G)|$  minimum. Since  $G$  is not series-parallel, it follows from Theorem 4 that  $G$  contains a wheel, and we show in Lemma 9 that  $G$  contains a proper wheel. Let  $s \in V(G)$  be the center of a proper wheel chosen as in Theorem 6, and let  $C_1, \dots, C_k$  be the components of  $G \setminus N[s]$ . By Theorem 6, it follows that  $k > 1$ . For each  $i$ , let  $N_i$  be the set of vertices of  $N(s)$  with a neighbor in  $V(C_i)$ , and let  $G_i = G|(V(C_i) \cup N_i \cup \{s\})$ . We analyze the structure of the graphs  $G_i$  using the minimality of  $|V(G)|$ . It turns out that at most one  $G_i$  is not series-parallel, and that (by contracting  $C_i$ 's) there is at most one value of  $i$  for which  $|V(C_i)| > 1$ . Also, if  $|V(C_i)| > 1$ , then  $\{x, y\} \cap V(C_i) \neq \emptyset$ . We may assume that  $|V(C_i)| = 1$  for all  $i \in \{1, \dots, k-1\}$ , and that  $\{x, y\} \cap V(C_k) \neq \emptyset$ . Now consider the bipartite graph  $G'$ , which (roughly speaking) is the graph obtained from  $G \setminus \{s\}$  by contracting  $V(C_k) \cup N_k$  to a single vertex  $z$  if  $|V(C_k)| > 1$ . It turns out that  $G'$  is  $\{\text{ISK}_4, K_{3,3}\}$ -free and has girth at least 6, while cycles that do not contain  $z$  must be even longer. Now either there is an easy win, or we find a cycle in  $G'$  that contains a long path  $P$  of vertices all of degree two in  $G'$  and with  $V(P) \subseteq V(G) \setminus (N[x] \cup N[y])$ . Further analysis shows that at least one of these vertices has degree two in  $G$ , and Theorem 7 follows.

This paper is organized as follows. In Section 2 we prove Theorem 6. Section 3 contains technical tools that we need to deduce that the graph  $G'$  described above has various useful properties. In Section 4 we develop techniques to produce a cycle with a long path of vertices of degree two. In Section 5 we put all of our knowledge together to prove Theorems 2 and 7 and deduce Conjecture 1.

Let us finish this section with an easy fact about  $\text{ISK}_4$ -free graphs. Given a hole  $C$  and a vertex  $v \notin C$ ,  $v$  is *linked* to  $C$  if there are three paths  $P_1, P_2, P_3$  such that

- $P_1^* \cup P_2^* \cup P_3^* \cup \{v\}$  is disjoint from  $C$ ;
- each  $P_i$  has one end  $v$  and the other end in  $C$ , and there are no other edges between  $P_i$  and  $C$ ;
- for  $i, j \in \{1, 2, 3\}$  with  $i \neq j$ ,  $V(P_i) \cap V(P_j) = \{v\}$ ;
- if  $x \in P_i$  is adjacent to  $y \in P_j$  then either  $v \in \{x, y\}$  or  $\{x, y\} \subseteq V(C)$ ; and
- if  $v$  has a neighbor  $c \in C$ , then  $c \in P_i$  for some  $i$ .

**Lemma 8.** *If  $G$  is  $\text{ISK}_4$ -free, then no vertex of  $G$  can be linked to a hole.*

## 2 Wheels

**Lemma 9.** *Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}\}$ -free graph that contains a wheel. Then there is a proper wheel in  $G$ .*

*Proof.* Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}\}$ -free graph. Let  $W = (C, x)$  be a wheel in  $G$  with  $|V(C)|$  minimum. We claim that  $W$  is a proper wheel. Suppose  $v \in V(G) \setminus V(W)$  violates the definition of a proper wheel.

If  $v$  has at least three neighbors in the hole  $x - S - x$  for some sector  $S$  of  $W$ , then  $(x - S - x, v)$  is a wheel with shorter rim than  $W$ , a contradiction. So  $v$  has at most two neighbors in every sector of  $W$  (and at most one if  $v$  is adjacent to  $x$ ). Therefore there exist sectors  $S_1, S_2$  of  $W$  such that  $v$  has a neighbor in  $V(S_1) \setminus V(S_2)$  and a neighbor in  $V(S_2) \setminus V(S_1)$ . Also by the minimality of  $|V(C)|$ , every path of  $C$  whose ends are in  $N(v)$  and with interior disjoint from  $N(v)$  contains at most two spokes of  $W$ , and we can choose  $S_1, S_2$  and for  $i = 1, 2$ , label the ends of  $S_i$  as  $a_i, b_i$  such that either  $b_1 = a_2$ , or  $b_1, a_2$  are the ends of a third sector  $S_3$  of  $W$  and  $v$  has no neighbor in  $S_3^*$ . If possible, we choose  $S_1, S_2$  such that  $b_1 = a_2$ . If  $v$  has two neighbors in  $S_1$ , denote them  $s, t$  such that  $a_1, t, s, b_1$  are in order in  $C$ . If  $v$  has a unique neighbor in  $S_1$ , denote it by  $s$ . Let  $z$  be the neighbor of  $v$  in  $S_2$  closest to  $a_2$ .

Assume first that  $v$  is non-adjacent to  $x$ . Suppose  $b_1 \neq a_2$ . By Lemma 8,  $x$  cannot be linked to the hole  $z - S_2 - a_2 - S_3 - b_1 - S_1 - s - v - z$ , and it follows that  $z \neq b_2$ . If  $v$  has two neighbors in  $S_1$ , then  $v$  can be linked to  $x - S_3 - x$  via the paths  $v - s - S_1 - b_1, v - t - S_1 - a_1 - x, v - z - S_2 - a_2$ ; and if  $v$  has a unique neighbor in  $S_1$ , then  $s$  can be linked to  $x - S_3 - x$  via the paths  $s - S_1 - b_1, s - S_1 - a_1 - x, s - v - z - S_2 - a_2$  (note that by the choice of  $S_1, S_2$  and since  $b_1 \neq a_2$ , it follows that  $s \neq b_1$ ). In both cases, this is contrary to Lemma 8. This proves that  $b_1 = a_2$ . Let  $y$  be the neighbor of  $v$  in  $S_2$  closest to  $b_2$ . Now if  $v$  has two neighbors in  $S_1$ , then  $v$  can be linked to  $x - S_1 - x$  via the paths  $v - s, v - t, v - y - S_2 - b_2 - x$ , contrary to Lemma 8. So  $v$  has a unique neighbor in  $S_1$ , and similarly a unique neighbor in  $S_2$ . It follows that  $s, b_1$  and  $z$  are all distinct. Now we can link  $x$  to  $s - S_1 - b_1 - S_2 - z - v - s$  via the paths  $x - b_1, x - a_1 - S_1 - s$ , and  $x - b_2 - S_2 - z$ , contrary to Lemma 8.

This proves that  $v$  is adjacent to  $x$ , and so  $v$  has at most one neighbor in every sector of  $W$ . If  $b_1 \neq a_2$ , then  $v$  can be linked to  $x - S_3 - x$  via the paths  $v - s - S_1 - b_1, v - x, v - z - S_2 - a_2$ , and if  $b_1 = a_2$ , then  $s, b_1$  and  $z$  are all distinct and hence  $x$  can be linked to the hole  $s - S_1 - b_1 - S_2 - z - v - s$  via the paths  $x - b_1, x - v$ , and  $x - b_2 - S_2 - z$ ; in both cases contrary to

Lemma 8. This proves that every  $v \in V(G) \setminus V(W)$  satisfies the condition in the definition, and so  $W$  is a proper wheel in  $G$ .  $\square$

Let  $W = (C, v)$  be a wheel. We call  $x$  *proper* for  $W$  if either  $x \in V(C) \cup \{v\}$ ; or

- all neighbors of  $x$  in  $V(C)$  are in one sector of  $W$ ; and
- if  $x$  has more than two neighbors in  $V(C)$ , then  $x$  is adjacent to  $v$ .

A vertex  $x$  is *non-offensive* for a wheel  $W = (C, v)$  if there exist two sectors  $S_1, S_2$  of  $W$  such that

- $x$  is adjacent to  $v$ ;
- $x$  has neighbors in  $S_1$  and in  $S_2$ ;
- $N(x) \cap V(C) \subseteq V(S_1) \cup V(S_2)$ ;
- $S_1$  and  $S_2$  are consecutive; and
- if  $u \in V(G) \setminus V(W)$  is adjacent to  $x$ , then  $N(u) \cap V(C) \subseteq V(S_1) \cup V(S_2)$ .

If  $V(S_1) \cap V(S_2) = \{a\}$ , we also say that  $x$  is *a-non-offensive*.

**Lemma 10.** *Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}\}$ -free graph. Let  $W = (C, v)$  be a wheel in  $G$ . Let  $S_1, S_2$  be consecutive sectors of  $W$ , and let  $x \in N(v) \setminus V(W)$  be a vertex such that  $N(x) \cap V(C) \subseteq V(S_1) \cup V(S_2)$  and  $N(x) \cap V(S_1), N(x) \cap V(S_2) \neq \emptyset$ . Then  $N(x) \cap V(C) \subseteq S_1^* \cup S_2^*$  and for  $\{i, j\} = \{1, 2\}$ ,  $|N(x) \cap (V(S_i) \setminus V(S_j))| \geq 3$ .*

*Proof.* Since  $x$  is adjacent to  $v$  and  $G$  is triangle-free, it follows that  $N(x) \cap V(C) \subseteq S_1^* \cup S_2^*$ . Suppose for a contradiction that  $|N(x) \cap (V(S_1) \setminus V(S_2))| \leq 2$ . If  $|N(x) \cap (V(S_1) \setminus V(S_2))| = 2$ , then  $x$  has exactly three neighbors in the hole  $v - S_1 - v$ , contrary to Lemma 8. It follows that  $|N(x) \cap (V(S_1) \setminus V(S_2))| = 1$ . Let  $z$  denote the neighbor of  $x$  in  $V(S_1)$ . Let  $\{w\} = V(S_1) \cap V(S_2)$ , and let  $y$  denote the neighbor of  $x$  in  $V(S_2)$  closest to  $w$  along  $S_2$ . Then  $x$  can be linked to the hole  $v - S_1 - v$  via the three paths  $x - v$ ,  $x - z$ , and  $x - y - S_2 - w$ . This is a contradiction to Lemma 8, and the result follows.  $\square$

We say that wheel  $W = (C, v)$  is *k-almost proper* if there are spokes  $x_1, \dots, x_k$  of  $W$  and a set  $X \subseteq V(G) \setminus V(W)$  such that

- no two spokes in  $\{x_1, \dots, x_k\}$  are consecutive;

- $W$  is proper in  $G \setminus X$ ;
- for every  $x$  in  $X$  there exists  $i$  such that  $x$  is  $x_i$ -non-offensive.

**Lemma 11.** *Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}\}$ -free graph, and let  $W = (C, v)$  be a 1-almost proper wheel in  $G$ . Let  $x_1$  and  $X$  be as in the definition of a 1-almost proper wheel, and let  $S_1$  and  $S_2$  be the sectors of  $W$  containing  $x_1$ .*

*Then there exists a proper wheel  $W'$  in  $G$  with center  $v$  and the same number of spokes as  $W$ . Moreover, either  $W = W'$ , or  $V(W') \setminus V(W) = \{x^*\}$  where  $x^*$  is a non-offensive vertex for  $W$ , and  $V(W) \setminus V(W') \subseteq V(S_1^*) \cup V(S_2^*) \cup \{x_1\}$ .*

*Proof.* We may assume that  $X \neq \emptyset$ , for otherwise  $W$  is proper in  $G$ . For  $x \in X$ , let  $P(x)$  denote the longest path in  $G[V(S_1) \cup V(S_2)]$  starting and ending in a neighbor of  $x$ . Let  $x^* \in X$  be a vertex with  $|V(P(x^*))|$  maximum among vertices in  $X$ , and let  $Y$  denote interior of  $P(x^*)$ . Let  $C' = G[(V(C) \cup \{x^*\}) \setminus Y]$ . It follows that  $W' = (C', v)$  is a wheel. Moreover,  $N(v) \cap V(C') = ((N(v) \cap V(C)) \setminus \{x_1\}) \cup \{x^*\}$ , and therefore  $W'$  has the same number of spokes as  $W$ .

If  $W'$  is proper, the result follows. Therefore, we may assume that there is a vertex  $y \in V(G) \setminus V(W')$  that is not proper for  $W'$ . Let  $S'_1, S'_2$  denote the sectors of  $W'$  containing  $S_1 \setminus Y$  and  $S_2 \setminus Y$ , respectively.

Suppose first that  $y \in V(W)$ , and consequently  $y \in Y$ . Since  $x^*$  has at least two neighbors in each of  $S_1$  and  $S_2$  by Lemma 10, it follows that  $|V(P(x^*))| \geq 4$ . Consequently, either  $N(y) \cap V(C') \subseteq S'_1$  or  $N(y) \cap V(C') \subseteq S'_2$ . Moreover,  $N(y) \cap V(C') \subseteq N[x]$ , and therefore  $|N(y) \cap V(C')| \leq 1$ . This implies that  $y$  is proper for  $W$ , a contradiction. This proves that  $y \notin V(W)$ .

Next, we suppose that  $y \in X$ . It follows that  $N(y) \cap V(C') \subseteq V(S'_1) \cup V(S'_2)$ . Since  $y$  is not proper for  $W'$ , but  $y$  is adjacent to  $v$ , it follows that  $N(y) \cap V(S'_1), N(y) \cap V(S'_2) \neq \emptyset$ . By Lemma 10,  $y$  has at least two neighbors in  $S'_1$ . But then  $V(P(x^*)) \subsetneq V(P(y))$ , a contradiction to the choice of  $x^*$ . This proves that  $y \in V(G) \setminus (X \cup V(W))$ .

If  $y \notin N(x^*)$ , then  $N(y) \cap V(C') \subseteq N(y) \cap V(C)$ , and since  $y \notin X$ , it follows that  $y$  is proper for  $W$  and thus  $y$  is proper for  $W'$ . Consequently  $y \in N(x^*)$ . Since  $x^*$  is non-offensive for  $W$ , it follows that  $N(y) \cap V(C) \subseteq V(S_1) \cup V(S_2)$ . Since  $y$  is proper for  $W$ , we may assume by symmetry that  $N(y) \cap V(C) \subseteq V(S_1)$ . It follows that  $N(y) \cap V(C') \subseteq V(S'_1)$ . Since  $y$  is adjacent to  $x^*$  and  $G$  is triangle-free, it follows that  $y$  is non-adjacent to  $v$ , and hence  $|N(y) \cap V(C)| \leq 2$ . This implies that  $|N(y) \cap V(C')| \leq 3$ , and by Lemma 8, it is impossible for  $y$  to have exactly three neighbors in  $C'$  since  $G$  is  $\text{ISK}_4$ -free. Therefore,  $|N(y) \cap V(C')| \leq 2$ , and therefore  $y$  is proper



for  $W'$ . This is a contradiction, and it follows that  $W'$  is proper in  $G$ , and hence  $W'$  is the desired wheel.  $\square$

**Lemma 12.** *Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}\}$ -free graph, and let  $W = (C, v)$  be a 2-almost proper wheel in  $G$ . Then there exists a proper wheel in  $G$  with center  $v$  and at most the same number of spokes as  $W$ .*

*Proof.* Let  $x_1, x_2$  and  $X$  be as in the definition of a 2-almost proper wheel, and let  $S_1, S_2$  be the sectors of  $W$  containing  $x_1$ . Let  $X_1$  denote the set  $x_1$ -non-offensive vertices in  $X$ , and let  $X_2 = X \setminus X_1$ . We may assume that  $X_1, X_2$  are both non-empty, for otherwise the result follows from Lemma 11. It follows that  $W$  is 1-almost proper, but not proper, in  $G \setminus X_2$ . Let  $W', x^*$  be as in Lemma 11. So  $W'$  is a proper wheel in  $G \setminus X_2$ . If  $W'$  is 1-almost proper in  $G$ , then the result of the Lemma follows from Lemma 11. So we may assume that  $W'$  is not 1-almost proper in  $G$ . Since every vertex of  $V(G) \setminus X_2$  is proper for  $W'$ , we deduce that some vertex of  $x \in X_2$  is not proper and not  $x_2$ -non-offensive for  $W'$ .

By the definition of  $X_1$ , and since  $W$  is 2-almost proper in  $G$ , it follows that  $N(x) \cap V(C)$  is contained in the sectors  $S_3, S_4$  of  $W$  containing  $x_2$ . Since  $x_1$  and  $x_2$  are not consecutive,  $S_3, S_4 \notin \{S_1, S_2\}$ , and so by Lemma 11,  $V(S_3) \cup V(S_4) \subseteq V(W')$ . Consequently,  $S_3$  and  $S_4$  are sectors of  $W'$ . Since  $G$  is triangle-free and every vertex in  $X$  is adjacent to  $v$ , it follows that  $x$  is not adjacent to  $x^*$ . Therefore,  $N(x) \cap V(C') \subseteq V(S_3) \cup V(S_4)$ ,  $x$  has both a neighbor in  $S_3$  and a neighbor in  $S_4$ , and  $S_3, S_4$  are the sectors of  $W'$  containing  $x_2$ . Let  $s_3$  denote the neighbor of  $x$  in  $S_3$  furthest from  $x_2$ , and let  $s_4$  denote the neighbors of  $x$  in  $S_4$  furthest from  $x_2$ . We may assume that among all vertices of  $X_2$  that are not  $x_2$ -non-offensive for  $W'$ ,  $x$  is chosen so that the path of  $C$  from  $s_3$  to  $s_4$  containing  $x_2$  is maximal.

Since  $x$  is not  $x_2$ -non-offensive for  $W'$ , there exists a vertex  $u \in N(x) \setminus V(W')$  with a neighbor in  $V(C') \setminus (V(S_3) \cup V(S_4))$ . Since  $x$  is  $x_2$ -non-offensive for  $W$ , it follows that  $u$  has a neighbor in  $V(C') \setminus V(C) = \{x^*\}$ , and so  $u$  is adjacent to  $x$  and  $x^*$ .

Since  $x$  and  $x^*$  are non-offensive for  $W$ , it follows that  $N(u) \cap V(C) \subseteq (V(S_1) \cup V(S_2)) \cap (V(S_3) \cup V(S_4))$ . Since  $G$  is triangle-free,  $u$  is non-adjacent to  $v$ , and therefore  $u \notin X$ . Consequently,  $u$  is proper for  $W$ , and all the neighbors of  $u$  in  $C$  belong to one sector of  $W$ . It follows that  $u$  has at most one neighbor in  $V(C)$ . Suppose that  $u$  has exactly one neighbor in  $V(C)$ . Then  $u$  has three neighbors in the cycle arising from  $C'$  by replacing  $s_3 - S_3 - x_2 - S_4 - s_4$  by  $s_3 - x - s_4$ , contrary to Lemma 8. It follows that  $u$  has no neighbors in  $V(C)$ .

Let  $P'_1$  denote the path of  $C'$  from  $s_3$  to  $x^*$  not containing  $x_2$ , and let  $P_1$  be  $x - s_3 - P'_1 - x^*$ . Let  $P'_2$  denote the path of  $C'$  from  $s_4$  to  $x^*$  not containing  $x_2$ , and let  $P_2$  be  $x - s_4 - P'_2 - x^*$ . Let  $D = G[V(P_1) \cup \{u\}]$ . Since  $x_1$  and  $x_2$  are not consecutive, each of  $P_1^*, P_2^*$  contains at least one neighbor of  $v$ , and so  $W'' = (D, v)$  is a wheel with fewer spokes than  $W$ . Let  $S'_3$  denote the sector of  $W''$  containing  $x$  but not containing  $u$ . If  $W''$  is proper in  $G$ , then the result follows. Therefore, we may assume that there is a vertex  $y \in V(G) \setminus V(W'')$  that is not proper for  $W''$ .

Since every vertex in  $V(W') \setminus V(W'')$  and every vertex in  $V(W) \setminus V(W'')$  has at most one neighbor in  $V(W'')$ , it follows that  $y \notin V(W) \cup V(W')$ . Suppose that  $y \notin N(u)$ . If  $y \in X_2$ , then  $N(y) \cap V(D) \subseteq (V(S_3) \cup \{x\}) \cap V(D)$ , and so  $y$  is proper for  $W''$ , since  $y$  is adjacent to  $v$ , a contradiction. Thus  $y \notin X_2$ , and so  $y$  is proper for  $W'$ . If  $y \notin N(x)$ , then  $N(y) \cap V(D) \subseteq N(y) \cap V(C')$ , and again  $y$  is proper for  $W''$ , a contradiction. Thus  $y \in N(x)$ , but since  $x$  is non-offensive for  $W$ ,  $N(y) \cap V(C) \subseteq V(S_3) \cup V(S_4)$ , and so  $N(y) \cap V(D) \subseteq V(S'_3)$ . Since  $y$  is not proper for  $W''$ ,  $y$  is non-adjacent to  $v$  and has at least three neighbors in  $S'_3$ . But  $y$  is proper for  $W'$ , and so  $y$  has at most two neighbors in  $S_3$ ; thus  $y$  has exactly three neighbors in  $S'_3$  and hence in  $D$  contrary to Lemma 8. This contradiction implies that  $y \in N(u)$ .

Since  $y$  is not proper for  $W''$ , it follows that  $y$  has a neighbor in  $P_1^*$ , and since  $G$  is triangle-free, it follows that  $y$  is non-adjacent to  $x, x^*$ . We claim that  $y$  has no neighbor in  $P_2$ . Suppose that it does. If  $y \in X_2$ , then, since  $y$  is adjacent to  $u$  and has a neighbor in  $P_1^*$ , we deduce that  $y$  is not  $x_2$ -non-offensive for  $W'$ , and the claim follows from the maximality of the path of  $C$  from  $s_3$  to  $s_4$  containing  $x_2$ . Thus we may assume that  $y \notin X_2$ . Consequently,  $y$  is proper for  $W'$ , a contradiction. This proves the claim.

Let  $z_1$  be the neighbor of  $y$  in  $V(P_1)$  closest to  $x$  along  $P_1$ , and let  $z_2$  be the neighbor of  $y$  in  $V(P_1)$  closest to  $x^*$  along  $P_1$ . Let  $D'$  be the hole  $x^* - P_2 - x - u - x^*$ . If  $z_1 \neq z_2$ , we can link  $y$  to  $D'$  via the paths  $y - z_1 - P_1 - x$ ,  $y - z_2 - P_1 - x^*$  and  $y - u$ , and if  $z_1 = z_2$ , then we can link  $z_1$  to  $D'$  via the paths  $z_1 - P_1 - x$ ,  $z_1 - P_1 - x^*$  and  $z_1 - y - u$ , in both cases contrary to Lemma 8. This proves Lemma 12.  $\square$

Throughout the remainder of this section  $G$  is an  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph, and  $W = (C, x)$  is a proper wheel in  $G$  with minimum number of spokes (subject to having center  $x$ ).

**Lemma 13.** *Let  $P = p_1 - \dots - p_k$  be a path such that  $p_1, p_k$  have neighbors in  $V(C)$ ,  $V(P) \subseteq V(G) \setminus V(W)$ , and there are no edges between  $P^*$  and  $V(C)$ . Assume that no sector of  $W$  contains  $(N(p_1) \cup N(p_k)) \cap V(C)$ . For  $i \in \{1, k\}$ , if  $x$  is non-adjacent to  $p_i$ , then  $p_i$  has a unique neighbor in  $C$ .*

*Proof.* Let  $S_1, S_2$  be distinct sectors of  $W$  such that  $N(p_1) \cap V(C) \subseteq V(S_1)$ , and  $N(p_k) \cap V(C) \subseteq V(S_2)$ . We may assume that  $p_1$  is non-adjacent to  $x$ , and so  $p_1$  has at most two neighbors in  $C$ . Since  $p_1$  cannot be linked to the hole  $C$  (or the hole obtained from  $C$  by rerouting  $S_2$  through  $p_k$ ) via two one-edge paths and  $P$ , it follows that  $p_1$  has a unique neighbor in  $C$ .  $\square$

**Theorem 14.** *Let  $P = p_1 - \dots - p_k$  be a path with  $V(P) \subseteq V(G) \setminus V(W)$  such that  $x$  has at most one neighbor in  $P$ .*

1. *If  $P$  contains no neighbor of  $x$ , then there is a sector  $S$  of  $W$  such that every edge from  $P$  to  $C$  has an end in  $V(S)$ .*
2. *If  $P$  contains exactly one neighbor of  $x$ , then there are two sectors  $S_1, S_2$  of  $W$  such that  $V(S_1) \cap V(S_2) \neq \emptyset$ , and every edge from  $P$  to  $C$  has an end in  $V(S_1) \cup V(S_2)$  (where possibly  $S_1 = S_2$ ).*

*Proof.* Let  $P$  be a path violating the assertions of the theorem and assume that  $P$  is chosen with  $k$  minimum. Since  $W$  is proper, it follows that  $k > 1$ . Our first goal is to show that  $x$  has a neighbor in  $V(P)$ .

Suppose that  $x$  is anticomplete to  $V(P)$ . Then, by the minimality of  $k$ , there exist two sectors  $S_1, S_2$  of  $W$  such that every edge from  $\{p_1, \dots, p_{k-1}\}$  to  $V(C)$  has an end in  $V(S_1)$ , and every edge from  $\{p_2, \dots, p_k\}$  to  $V(C)$  has an end in  $V(S_2)$ . It follows that  $S_1 \neq S_2$ . Then  $p_1$  has a neighbor in  $V(S_1) \setminus V(S_2)$ , and  $p_k$  has a neighbor in  $V(S_2) \setminus V(S_1)$ , and every edge from  $P^*$  to  $V(C)$  has an end in  $V(S_1) \cap V(S_2)$ . For  $i = 1, 2$  let  $a_i, b_i$  be the ends of  $S_i$ . We may assume that  $a_1, b_1, a_2, b_2$  appear in  $C$  in this order and that  $a_1 \neq b_2$ . Let  $Q_1$  be the path of  $C$  from  $b_2$  to  $a_1$  not using  $b_1$ , and let  $Q_2$  be the path of  $C$  from  $b_1$  to  $a_2$  not using  $a_1$ . We can choose  $S_1, S_2$  with  $|V(Q_2)|$  minimum (without changing  $P$ ). Let  $s$  be the neighbor of  $p_1$  in  $S_1$  closest to  $a_1$ ,  $t$  the neighbor of  $p_1$  in  $S_1$  closest to  $b_1$ ,  $y$  the neighbor of  $p_k$  in  $S_2$  closest to  $a_2$  and  $z$  the neighbor of  $p_k$  in  $S_2$  closest to  $b_2$ . Then  $s \neq b_1$  and  $z \neq a_2$ . It follows that  $V(Q_2) \cap \{s, z\} = \emptyset$ . Moreover, if  $V(S_1) \cap V(S_2) \neq \emptyset$ , then  $b_1 = a_2$  and  $V(Q_2) = \{b_1\}$ , and in all cases  $V(Q_1)$  is anticomplete to  $P^*$ .

Now  $D_1 = s - p_1 - P - p_k - z - S_2 - b_2 - Q_1 - a_1 - S_1 - s$  is a hole.

- (1)  $W_1 = (D_1, x)$  is a wheel with fewer spokes than  $W$ .

Since  $V(Q_2) \cap V(D_1) = \emptyset$  and  $V(Q_2)$  contains a neighbor of  $x$ , it follows that  $x$  has fewer neighbors in  $D_1$  than it does in  $C$ . It now suffices to show that  $x$  has at least three neighbors in  $Q_1$ . Since  $a_1, b_2 \in V(Q_1)$ , we may assume that  $x$  has no neighbor in  $Q_1^*$ , and  $Q_1$  is a sector of  $W$ . Since not every edge between  $V(P)$  and  $V(C)$  has an end in  $V(Q_1)$ , it follows that

$t \neq a_1$  or  $y \neq b_2$ . By symmetry, we may assume that  $t \neq a_1$ . Since  $x$  cannot be linked to  $W$  by Lemma 8, it follows that  $x$  has at least four neighbors in  $V(C)$ , and therefore  $V(S_1) \cap V(S_2) = \emptyset$ . Consequently,  $P^*$  is anticomplete to  $V(C)$ . It follows from Lemma 13 that  $s = t$ . Now we can link  $s$  to the hole  $a_1 - Q_1 - b_2 - x - a_1$  via the paths  $s - S_1 - a_1$ ,  $s - p_1 - P - p_k - z - S_2 - b_2$  and  $s - S_1 - b_1 - x$ , contrary to Lemma 8. This proves that  $W_1 = (D_1, x)$  is a wheel with fewer spokes than  $W$ . This proves (1).

By the choice of  $W$ , it follows from (1) that  $W_1$  is not proper. Let  $S_0$  be the sector  $a_1 - S_1 - s - p_1 - P - p_k - z - S_2 - b_2$  of  $(D_1, x)$ . Since  $W$  is a proper wheel and  $W_1$  is not a proper wheel, we have that:

*There exists  $v \in V(G) \setminus V(W_1)$  such that either*

- (2)
  - *$v$  is non-adjacent to  $x$ , and  $v$  has at least three neighbors in  $S_0$  and  $N(v) \cap V(D_1) \subseteq V(S_0)$ , or*
  - *there is a sector  $S_3$  of  $W$  with  $V(S_3) \subseteq V(Q_1)$ , such that  $v$  has a neighbor in  $V(S_3) \setminus V(S_0)$  and a neighbor in  $V(S_0) \setminus V(S_3)$ , and  $N(v) \cap V(C) \subseteq V(S_3)$ .*

Let  $v$  be as in (2). First we show that  $v \notin V(C)$ . The only vertices of  $C$  that may have more than one neighbor in  $D_1$  are  $b_1$  and  $a_2$ , and that only happens if  $b_1 = a_2$ . But  $N(b_1) \cap V(D_1) \subseteq V(S_0)$  and  $b_1$  is adjacent to  $x$ , so  $b_1$  does not satisfy the conditions described in the bullets. Thus  $v \notin V(C)$ .

- (3)  *$v$  has a unique neighbor in  $P$ .*

If the first case of (2) holds, then the statement of (3) follows immediately from the minimality of  $k$  (since  $v$  is non-adjacent to  $x$ ), and so we may assume that the second case of (2) holds. Observe that no vertex of  $V(Q_1)$  is contained both in a sector with end  $a_1$  and in a sector with end  $b_2$ , and therefore we may assume that  $v$  has a neighbor in a sector that does not have end  $b_2$ . If  $v$  is non-adjacent to  $x$ , we get a contradiction to the minimality of  $k$ . So we may assume that  $v$  is adjacent to  $x$ , and therefore  $v$  has a neighbor in  $S_3^*$ , and  $b_2 \notin V(S_3)$ . Let  $S_4$  be the sector of  $D_1$  such that  $b_2 \in V(S_4)$  and  $V(S_4) \subseteq V(Q_1)$ . Suppose that  $y \neq b_2$  or  $V(S_3) \cap V(S_4) = \emptyset$ . Let  $i \in \{1, \dots, k\}$  be maximum such that  $v$  is adjacent to  $p_i$ . Now the path  $v - p_i - P - p_k$  violates the assertions of the theorem, and so it follows from the minimality of  $k$  that  $N(v) \cap V(P) \subseteq \{p_1, p_2\}$ . Therefore, since  $G$  is triangle-free, it follows that  $v$  has a unique neighbor in  $P$ , and (3)

holds. So we may assume that  $y = b_2$  and there exists  $a_3 \in V(C)$  such that  $V(S_4) \cap V(S_3) = \{a_3\}$ . Let  $R$  be the path from  $v$  to  $a_3$  with  $R^* \subseteq S_3^*$ . Now we can link  $v$  to  $x - S_4 - x$  via the paths  $v - x$ ,  $v - R - a_3$  and  $v - p_i - P - p_k - b_2$ , where  $i$  is maximum such that  $v$  is adjacent to  $p_i$ , contrary to Lemma 8. This proves (3).

In view of (3) let  $N(v) \cap V(P) = \{p_j\}$ . In the case of the first bullet of (2), since  $v$  cannot be linked to the hole  $x - S_0 - x$  by Lemma 8, it follows that  $v$  has at least four neighbors in  $S_0$ , and therefore at least three neighbors in  $V(S_1) \cup V(S_2)$ , contrary to the fact that  $W$  is proper. So the case of the second bullet of (2) holds. Since  $W$  is proper  $N(v) \cap (V(S_0) \setminus V(S_3)) \subseteq V(P)$ , and  $N(v) \cap V(D_1) \subseteq V(S_0) \cup V(S_3)$ .

(4) *There are edges between  $P^*$  and  $V(C)$ .*

Suppose not. By Lemma 13,  $s = t$  and  $y = z$ . We claim that in this case  $b_1 \neq a_2$ , for if  $b_1 = a_2$ , then  $b_1$  can be linked to the hole  $x - a_1 - S_1 - s - p_1 - P - p_k - z - S_2 - b_2 - x$  via the paths  $b_1 - x$ ,  $b_1 - S_1 - s$  and  $b_1 - S_2 - z$ , contrary to Lemma 8. If  $v$  has a unique neighbor  $r$  in  $C$ , then  $p_j$  can be linked to  $C$  via the paths  $p_j - P - p_1 - s$ ,  $p_j - P - p_k - z$  and  $p_j - v - r$ , contrary to Lemma 8, so  $v$  has at least two neighbors in  $C$ . Recall that  $N(v) \cap V(C) \subseteq V(S_3)$ . Let  $D$  be the hole obtained from  $C$  by rerouting  $S_3$  through  $v$ . Then  $s, z \in V(D)$ , and  $p_j$  can be linked to  $D$  via the paths  $p_j - P - p_1 - s$ ,  $p_j - P - p_k - z$  and  $p_j - v$ , contrary to Lemma 8. This proves (4).

It follows from (4) that  $b_1 = a_2$  and  $b_1$  has neighbors in  $P^*$ . Now, by considering the path from a neighbor of  $b_1$  in  $P^*$  to  $v$  with interior in  $P^*$  if  $v$  has a neighbor in  $P^*$ , and the paths  $v - p_1$  or  $v - p_k$  if  $v$  has no neighbor in  $P^*$ , the minimality of  $k$  implies that  $v$  is adjacent to  $x$  and one of  $a_1, b_2$  belongs to  $S_3$ .

By symmetry we may assume  $a_1 \in V(S_3)$ . Let  $R$  be the path from  $v$  to  $a_1$  with  $R^* \subseteq V(S_3)$ . Now  $x$  can be linked to the hole  $v - R - a_1 - S_1 - s - p_1 - P - p_j - v$  via the paths  $x - v$ ,  $x - a_1$  and  $x - b_2 - S_2 - z - p_k - P - p_j$ , contrary to Lemma 8.

In summary, we have now proved:

(5) *If  $P'$  is a path violating the assertion of the theorem and  $|V(P')| = k$ , then  $x$  has a neighbor in  $V(P')$ .*

By (5),  $x$  has a neighbor in  $V(P)$ , say  $x$  is adjacent to  $p_i$ . Then  $p_i$  is the unique neighbor of  $x$  in  $V(P)$ . By the minimality of  $k$ , there exist two

distinct sectors  $S_1, S_2$  of  $W$  such that  $p_1$  has a neighbor in  $V(S_1) \setminus V(S_2)$ , and  $p_k$  has a neighbor in  $V(S_2) \setminus V(S_1)$ . By (5), if  $1 < i < j$ , then every edge from  $\{p_1, \dots, p_{i-1}\}$  to  $V(C)$  has an end in  $V(S_1)$ , and every edge from  $\{p_{i+1}, \dots, p_k\}$  to  $V(C)$  has an end in  $V(S_2)$ ; if  $i = 1$  then every edge from  $V(P) \setminus \{p_1\}$  to  $V(C)$  has an end in  $V(S_2)$ ; and if  $i = k$  then every edge from  $V(P) \setminus \{p_k\}$  to  $V(C)$  has an end in  $V(S_1)$ .

For  $j = 1, 2$ , let  $a_j, b_j$  be the ends of  $S_j$ .

*One of the following statements holds:*

- *there are no edges between  $V(C)$  and  $P^*$ , or*
- (6)
  - *we can choose  $S_1, S_2$  such that  $a_1, b_1, a_2, b_2$  appear in  $C$  in order and there is a sector  $S_3$  with ends  $b_1, a_2$ , and every edge between  $V(C)$  and  $P^*$  is from  $b_1$  to  $\{p_2, \dots, p_{i-1}\}$  or from  $p_i$  to  $S_3^*$ , or from  $a_2$  to  $\{p_{i+1}, \dots, p_{k-1}\}$ .*

Suppose (6) is false. It follows that there are edges between  $P^*$  and  $V(C)$ . Since  $G$  is triangle-free,  $p_i$  is anticomplete to  $N(x) \cap V(C)$ . Suppose that there is sector  $S_3$  of  $W$  and an edge from  $S_3^*$  to  $P^*$ . By the minimality of  $k$  we deduce that  $S_3 \notin \{S_1, S_2\}$ ,  $1 < i < k$  and  $p_i$  has a neighbor in  $S_3^*$ . Again by the minimality of  $k$  it follows that there exist sectors  $S'_1, S'_2$  such that  $V(S'_j) \cap V(S_3) \neq \emptyset$  for  $j = 1, 2$  and every edge from  $\{p_1, \dots, p_{i-1}\}$  to  $C$  has an end in  $S'_1$ , and every edge from  $\{p_{i+1}, \dots, p_k\}$  to  $C$  has an end in  $S'_2$ . Now we can choose  $S_1 = S'_1$  and  $S_2 = S'_2$ . We may assume that  $a_1, b_1, a_2, b_2$  appear in  $C$  in this order, and so  $b_1$  and  $a_2$  are the ends of  $S_3$ . Since  $p_i$  has a neighbor in  $S_3^*$ , the minimality of  $k$  implies that  $\{p_2, \dots, p_i\}$  is anticomplete to  $V(S_1) \setminus \{b_1\}$ , and  $\{p_i, \dots, p_{k-1}\}$  is anticomplete to  $V(S_2) \setminus \{a_2\}$ , and the second bullet is satisfied. So  $P^*$  is anticomplete to  $V(C) \setminus N(x)$ . Since there are edges between  $P^*$  and  $V(C)$ , and since  $p_i$  is anticomplete to  $N(x) \cap V(C)$ , by symmetry we may assume that there is an edge between  $\{p_2, \dots, p_{i-1}\}$  and  $t \in N(x) \cap V(C)$ . Then  $t \in V(S_1)$ . Let  $S_3$  be the other sector of  $W$  incident with  $t$ . By the minimality of  $k$  it follows that  $S_2$  can be chosen so that  $V(S_3) \cap V(S_2) \neq \emptyset$ , and again the case of the second bullet holds. This proves (6).

If the second bullet of (6) holds, let  $Q_1$  be the path of  $C$  from  $b_2$  to  $a_1$  not using  $b_1$ , and let  $Q_2 = S_3$ . To define  $Q_1$  and  $Q_2$ , let us now assume that the case of the first bullet holds. We may assume that  $a_1, b_1, a_2, b_2$  appear in  $C$  in this order. Also,  $a_1, b_1, a_2, b_2$  are all distinct, since  $P$  violates the assertion of the theorem. Let  $Q_1$  be the path of  $C$  from  $b_2$  to  $a_1$  not using

$b_1$ , and let  $Q_2$  be the path of  $C$  from  $b_1$  to  $a_2$  not using  $a_1$ . We may assume that  $S_1, S_2$  are chosen with  $|V(Q_2)|$  minimum (without changing  $P$ ).

Since  $W$  is proper, it follows that  $N(p_1) \cap V(C) \subseteq V(S_1)$  and  $N(p_k) \cap V(C) \subseteq V(S_2)$ . Let  $s$  be the neighbor of  $p_1$  in  $S_1$  closest to  $a_1$ ,  $t$  the neighbor of  $p_1$  in  $S_1$  closest to  $b_1$ ,  $y$  the neighbor of  $p_k$  in  $S_2$  closest to  $a_2$  and  $z$  the neighbor of  $p_k$  in  $S_2$  closest to  $b_2$ . Then  $s \neq b_1$  and  $z \neq a_2$ .

Let  $D_1$  be the hole  $a_1 - S_1 - s - p_1 - P - p_k - z - S_2 - b_2 - Q_1 - a_1$ . Then  $W_1 = (D_1, x)$  is a wheel with fewer spokes than  $W$ . We may assume that (subject to the minimality of  $k$ )  $P$  was chosen so that  $V(Q_1)$  is (inclusion-wise) minimal. By Lemma 8,  $x$  has a neighbor in  $V(D_1) \setminus \{a_1, b_1, p_i\}$ , and so  $x$  has a neighbor in  $Q_1^*$ .

Let  $S_0$  be the sector  $a_1 - S_1 - s - p_1 - P - p_i$ , and let  $T_0$  be the sector  $p_i - P - p_k - z - b_2$  of  $(D_1, x)$ .

(7) *No vertex  $v \in V(G) \setminus V(W_1)$  has both a neighbor in  $V(S_0) \setminus V(T_0)$  and a neighbor in  $V(T_0) \setminus V(S_0)$ .*

Suppose (7) is false, and let  $v \in V(G) \setminus V(W_1)$  be such that  $v$  has a neighbor in  $V(S_0) \setminus V(T_0)$  and a neighbor in  $V(T_0) \setminus V(S_0)$ .

First we claim that  $v$  is adjacent to  $x$ . Suppose  $v$  has a neighbor in  $V(a_1 - S_1 - s)$ . Since  $W$  is proper and  $a_1, s \notin V(S_2)$  (because  $P$  violates the statement of the theorem), it follows that  $v$  has no neighbor in  $V(z - S_2 - b_2)$ . Consequently  $v$  has a neighbor in  $V(T_0) \setminus (V(S_2) \cup V(S_0))$ . Let  $j$  be maximum such that  $v$  is adjacent to  $p_j$ , then  $j > i$ . Now applying (5) to the path  $v - p_j - P - p_k$  we deduce that  $v$  is adjacent to  $x$ , as required. Thus we may assume that  $N(v) \cap (V(S_0) \cup V(T_0)) \subseteq V(P)$ . Let  $j$  be minimum and  $l$  maximum such that  $v$  is adjacent to  $p_j, p_l$ . Then  $j < i$  and  $l > i$ . Applying (5) to the path  $p_1 - P - p_j - v - p_l - P - p_k$ , we again deduce that  $x$  is adjacent to  $v$ . This proves the claim.

In view of the claim, Lemma 10 implies that  $v$  has at least two neighbors in  $V(T_0) \setminus V(S_0)$  and at least two neighbors in  $V(S_0) \setminus V(T_0)$ . But now, rerouting  $P$  through  $v$  (as in the previous paragraph), we get a contradiction to the minimality of  $k$ . This proves (7).

(8) *Every non-offensive vertex for  $W_1$  is either  $a_1$ -non-offensive or  $b_2$ -non-offensive.*

Let  $v$  be a non-offensive vertex for  $W_1$ . Since  $W$  is proper, it follows that  $N(v) \cap V(C)$  is included in a unique sector of  $W$ . Consequently,  $v$

is either  $a_1$ -non-offensive, or  $b_2$ -non-offensive, or  $p_i$  non-offensive. However, (7) implies that  $v$  is not  $p_i$ -non-offensive, and (8) follows.

Let  $X$  be the set of all non-offensive vertices for  $W_1$ . It follows from Lemma 12 that  $W_1$  is not proper in  $V(G) \setminus X$ .

*There exists  $v \in V(G) \setminus (V(W_1) \cup X)$  such that one of the following holds:*

- $v$  is non-adjacent to  $x$ , and  $v$  has at least three neighbors in  $S_0$ , and  $N(v) \cap V(D_1) \subseteq V(S_0)$ .
  - $v$  is non-adjacent to  $x$ , and  $v$  has at least three neighbors in  $T_0$ , and  $N(v) \cap V(D_1) \subseteq V(T_0)$ .
- (9)
- $v$  has a neighbor in  $V(S_0) \setminus V(T_0)$  and a neighbor in  $V(T_0) \setminus V(S_0)$ , and  $N(v) \cap V(D_1) \subseteq V(S_0) \cup V(T_0)$ .
  - (possibly with the roles of  $S_0$  and  $T_0$  exchanged) there is a sector  $S_4$  of  $W$  with  $V(S_4) \subseteq V(Q_1)$  such that  $v$  has a neighbor in  $V(S_4) \setminus (V(S_0) \cup V(T_0))$ ,  $v$  has a neighbor in  $V(S_0) \setminus V(S_4)$ ,  $v$  does not have a neighbor in  $V(T_0) \setminus (V(S_0) \cup V(S_4))$ , and  $N(v) \cap V(C) \subseteq V(S_4)$ .

We may assume that the first three bullets of (9) do not hold. Since  $W$  is proper and  $W_1$  is not, (possibly switching the roles of  $S_0$  and  $T_0$ ) there exists  $v \in V(G) \setminus V(W_1)$  and a sector  $S_4$  of  $W$  with  $V(S_4) \subseteq V(Q_1)$ , such that  $v$  has a neighbor in  $V(S_4) \setminus V(S_0)$ ,  $v$  has a neighbor in  $V(S_0) \setminus V(S_4)$ , and  $N(v) \cap V(C) \subseteq V(S_4)$ . But now (7) implies that the last bullet of (9) holds. This proves (9).

Let  $v \in V(G)$  be as in (9). Next we show that:

- (10)  $v$  has a unique neighbor in  $V(P)$ .

Suppose that  $v$  has at least two neighbors in  $P$ . In the first two cases of (9) we get a contradiction to the minimality of  $k$ . The third case is impossible by (7). Thus we may assume that the case of the fourth bullet of (9) holds. We may assume that  $N(v) \cap V(P) \subseteq V(S_0)$ , and in particular  $v$  has a neighbor in  $\{p_1, \dots, p_{i-1}\}$ . Suppose first that  $v$  is non-adjacent to  $x$ . Since  $v$  has a neighbor in  $V(S_4) \setminus V(S_0)$ , the minimality of  $k$  implies that  $t = a_1$  and  $a_1 \in V(S_4)$ , and also that  $b_2 \in V(S_4)$ , contrary to the fact that  $x$



has a neighbor in  $Q_1^*$ . So  $v$  is adjacent to  $x$ , and therefore  $v$  has a neighbor in  $S_0^*$ .

Since  $W$  is proper,  $N(v) \cap (V(S_0) \setminus V(S_4)) \subseteq V(P)$ . Let  $Q$  be the path from  $v$  to  $p_1$  with  $Q^* \subseteq V(P)$ . Suppose first that  $a_1 \notin V(S_4)$ . Let  $S_5$  be the sector of  $W$  with end  $a_1$  and such that  $V(S_5) \subseteq V(Q_1)$ , and let  $b_3$  be the other end of  $S_5$ . Since  $Q$  is shorter than  $P$ , it follows from the minimality of  $k$  that  $V(S_4) \cap V(S_5) = \{b_3\}$  and  $t = a_1$ . Let  $R$  be the path from  $v$  to  $b_3$  with  $R^* \subseteq S_4^*$ . Then  $x$  has exactly three neighbors in the hole  $v - R - b_3 - S_5 - a_1 - p_1 - Q - v$ , contrary to Lemma 8. This proves that  $a_1 \in V(S_4)$ .

Let  $b_3$  be the other end of  $S_4$ , let  $S_5$  be the second sector of  $W$  incident with  $b_3$ , and let  $a_3$  be the other end of  $S_5$ . Since  $v \notin X$ , it follows that  $v$  has a neighbor  $u \in V(G) \setminus V(W_1)$  such that  $u$  has a neighbor in  $V(D_1) \setminus (V(S_4) \cup V(S_0))$ . Since  $G$  is triangle-free,  $u$  is non-adjacent to  $x$ .

Suppose first that  $u$  has a neighbor in  $V(Q_1) \setminus V(S_4)$ . Since  $G$  is triangle-free and  $v$  has at least two neighbors in  $V(P)$ , it follows that  $i \geq 4$ , and therefore  $k \geq 4$ . Consequently, the path  $u - v$  is shorter than  $P$ , and so it follows from the minimality of  $k$  that  $N(u) \cap V(C) \subseteq V(S_5)$ . Let  $R$  be the path from  $v$  to  $b_3$  with  $R^* \subseteq S_4^*$ , and let  $D_2$  be the hole  $v - R - b_3 - x - v$ . Let  $p$  be the neighbor of  $u$  in  $V(S_5)$  closest to  $b_3$ , and let  $q$  be the neighbor of  $u$  in  $V(S_5)$  closest to  $a_3$ . If  $p \neq q$ , we can link  $u$  to  $D_2$  via the paths  $u - p - S_5 - b_3$ ,  $u - q - S_5 - a_3 - x$  and  $u - v$ , and if  $p = q$  we can link  $p$  to  $D_2$  via the paths  $p - u - v$ ,  $p - S_5 - b_3$  and  $p - S_5 - a_3 - x$ , in both cases contrary to Lemma 8. This proves that  $u$  has no neighbor in  $V(Q_1) \setminus V(S_4)$ , and therefore  $u$  has a neighbor in  $V(T_0) \setminus V(S_0)$ .

Next we define a new path  $Q$ . If  $u$  has a neighbor in  $V(T_0) \cap V(S_2)$ , let  $Q$  be the path  $u - v$ . If  $u$  is anticomplete to  $V(T_0) \cap V(S_2)$ , let  $j$  be maximum such that  $u$  is adjacent to  $p_j$ ; then  $j > i$ ; let  $Q$  be the path  $v - u - p_j - P - p_k$ . Since  $i > 4$ , in both cases  $|V(Q)| < k$  and  $x$  has a unique neighbor in  $V(Q)$ . It follows from the minimality of  $k$  that  $z = y = b_2 = a_3$ . Since  $P$  violates the theorem, it follows that  $p_1$  has a neighbor in  $V(S_1) \setminus \{a_1\}$ .

Let  $T$  be the path from  $v$  to  $a_1$  with  $T^* \subseteq V(S_4)$ . Suppose that  $s \neq t$ . Let  $D_3$  be the hole  $x - a_1 - S_1 - s - p_1 - t - S_1 - b_1 - x$ . Now  $v$  can be linked to  $D_3$  via the paths  $v - x$ ,  $v - Q - p_1$  (short-cutting through a neighbor of  $b_1$  if possible) and  $v - T - a_1$ , contrary to Lemma 8. Thus  $s = t$ , and therefore  $s \neq a_1$ . Now we can link  $v$  to  $x - S_1 - x$  via the paths  $v - x$ ,  $v - Q - p_1 - s$

(short-cutting through a neighbor of  $b_1$  if possible) and  $v - T - a_1$ , contrary to Lemma 8. This proves (10).

In view of (10) let  $p_j$  be the unique neighbor of  $v$  in  $V(P)$ .

(11) *The fourth case of (9) holds.*

Suppose first that the case of the first bullet of (9) happens. Then by Lemma 8  $v$  has at least four neighbors in the hole  $x - S_0 - x$ , and so, in view of (10),  $x$  has at least three neighbors in the path  $a_1 - S_1 - s$ , contrary to the fact that  $W$  is proper. By symmetry it follows that the cases of first two bullets of (9) do not happen. Suppose that the case of the third bullet of (9) happens. Since by (10)  $v$  has a unique neighbor in  $V(P)$ , it follows that  $v$  has a neighbor in  $(V(S_0) \cup V(T_0)) \setminus V(P)$ . By symmetry we may assume that  $v$  has a neighbor in  $z - S_2 - b_2$ , and, since  $W$  is proper,  $v$  is anticomplete to  $V(S_0) \setminus V(P)$ . Consequently,  $p_j \in V(S_0) \setminus V(T_0)$ , and so  $j < i$ . By the minimality of  $k$  (applied to the path  $p_1 - P - p_j - v$ ), it follows that  $j = k - 1$ , and therefore  $i = k$ . Then  $\{v, p_k\}$  is anticomplete to  $V(C) \setminus V(S_2)$ , since  $W$  is proper. By (5)  $v$  is adjacent to  $x$ . But now we get a contradiction to Lemma 10 applied to  $v$  and  $W_1$ . This proves (11).

In the next claim we further restrict the structure of  $P$ .

*One of the following statements holds:*

- *there are edges between  $P^*$  and  $V(C)$ , or*
- (12)
- *$j = 1$  and we can choose  $S_4$  so that  $a_1 \in V(S_4)$ , or*
  - *$j = k$  and we can choose  $S_4$  so that  $b_2 \in V(S_4)$ .*

Suppose that (12) is false. Assume first that  $j \notin \{1, k\}$ . Then  $p_j$  is anticomplete to  $V(C)$ , since by assumption, there are no edges between  $P^*$  and  $C$ . If  $s = t$ ,  $y = z$  and  $v$  has a unique neighbor  $r$  in  $S_4$ , then  $r \in V(S_4) \setminus (V(S_1) \cup V(S_2))$ , and  $p_j$  can be linked to  $C$  via the paths  $p_j - P - p_k - z$ ,  $p_j - v - r$  and  $p_j - P - p_1 - s$ , contrary to Lemma 8. If some of  $p_1, p_k, v$  have several neighbors in  $C$ , then similar linkages work for the holes obtained from  $C$  by rerouting  $S_1$  through  $p_1$ ,  $S_2$  through  $p_k$ , and  $S_4$  through  $v$ , respectively. This proves that  $j \in \{1, k\}$ , and by symmetry may assume that  $j = 1$ . Then  $S_4$  cannot be chosen so that  $a_1 \notin V(S_4)$ , for otherwise (12) holds. By the minimality of  $k$  and by (5), since  $S_4$  cannot be chosen so that  $a_1 \in V(S_4)$ , it follows that  $x$  is adjacent to one of  $p_1, v$  and

$k = 2$ . Since  $G$  is triangle-free,  $x$  has exactly one neighbor in  $\{p_1, v\}$ . Let  $R$  be the path from  $v$  to  $a_1$  with  $R^* \subseteq V(C) \setminus \{b_1\}$ . Let  $Q'_1$  be the subpath of  $R$  from an end of  $S_4$  to  $a_1$ . Then  $V(Q'_1) \subseteq V(Q_1)$  and  $b_2 \notin V(Q'_1)$ , and so the path  $p_1 - v$  contradicts the choice of  $P$ . This proves (12).

The goal of the next two claims is to obtain more information about  $i$  and  $j$ .

(13)  $i = j$ .

Suppose not; by symmetry we may assume that  $j < i$ . Suppose first that  $x$  is non-adjacent to  $v$ . By (5) and the minimality of  $k$ , it follows that the first assertion of the theorem holds for the path  $p_1 - P - p_j - v$ ; therefore  $a_1 = t$  and  $S_4$  can be chosen so that  $a_1 \in V(S_4)$ . Since  $W$  is proper it follows that  $v$  has at most two neighbors in  $S_4$ . If  $v$  has exactly two neighbors, then, in view of (6),  $v$  can be linked to  $x - S_4 - x$  via two one-edge paths and the path  $v - p_j - P - p_i - x$ , contrary to Lemma 8. Therefore  $v$  has a unique neighbor  $r$  in  $S_4$ . Now, again in view of (6),  $p_j$  can be linked to  $x - S_4 - x$  via the paths  $p_j - v - r$ ,  $p_j - P - p_1 - a_1$  and  $p_j - P - p_i - x$ , again contrary to Lemma 8. This proves that  $v$  is adjacent to  $x$ , and, since  $G$  is triangle-free,  $v$  has a neighbor in  $S_4^*$ . It follows that the choice of  $S_4$  is unique. Let  $R$  be the path from  $v$  to  $a_1$  with  $R^* \subseteq V(C) \setminus \{b_1\}$ . Suppose  $a_1 \in V(S_4)$ . Then  $R^* \subseteq S_4^*$ . In this case, because of (6) and since  $b_1 \neq s$ ,  $p_j$  can be linked to the hole  $v - R - a_1 - x - v$  via the path  $p_j - v$ ,  $p_j - P - p_1 - s - S_1 - a_1$  and  $p_j - P - p_i - x$ , contrary to Lemma 8. Thus  $a_1 \notin V(S_4)$ . Let  $S_5$  be the sector of  $W$  with end  $a_1$  such that  $V(S_5) \subseteq V(Q_1)$ . If  $t = a_1$  and  $V(S_4) \cap V(S_5) \neq \emptyset$ , then  $x$  has exactly three neighbors in the hole  $v - R - a_1 - p_1 - P - p_j - v$ , contrary to Lemma 8. Therefore the path  $p_1 - P - p_j - v$  violates the assertion of the theorem, and so the minimality of  $k$  implies that  $j = k - 1$  and consequently  $i = k$ . Then by (6)  $a_2$  is anticomplete to  $V(P) \setminus \{p_k\}$ . Since  $j \neq k$  and  $a_1 \notin V(S_4)$  (the choice of  $S_4$  is now unique), it follows from (12) that there are edges between  $P^*$  and  $V(C)$ . Now by (6) there is a sector  $S_3$  of  $W$  with ends  $a_2, b_1$ , and  $b_1$  has a neighbor in  $P^*$ . Then there is a path  $T$  from  $b_1$  to  $p_k$  with  $T^* \subseteq P^*$ ,  $b_1 - S_3 - a_2 - S_2 - y - p_k - T - b_1$  is a hole and  $x$  has exactly three neighbors in it, contrary to Lemma 8 (observe that  $y \neq b_2$  because  $G$  has no triangles). This proves (13).

Since  $G$  is triangle-free, (13) implies that  $x$  is non-adjacent to  $v$ .

(14)  $i \leq 2$  and  $i \geq k - 1$ .

Suppose (14) is false. By symmetry we may assume that  $k - i > 1$ . Consequently  $k > 2$ . Suppose that  $S_4$  can be chosen so that  $a_1 \in V(S_4)$ . If  $v$  has a unique neighbor  $r$  in  $V(S_4)$ , then, since  $s \neq b_1$ ,  $p_i$  can be linked to  $x - S_4 - x$  via the paths  $p_i - v - r$ ,  $p_i - x$  and  $p_i - P - p_1 - s - S_1 - a_1$ , a contradiction. Thus  $v$  has at least two neighbors in  $V(S_4)$ . Now, again using the fact that  $s \neq b_1$ ,  $p_i$  can be linked to the hole obtained from  $x - S_4 - x$  by rerouting  $S_4$  through  $v$  via the paths  $p_i - v$ ,  $p_i - x$  and  $p_i - P - p_1 - s - S_1 - a_1$ , again contrary to Lemma 8. Thus  $S_4$  cannot be chosen so that  $a_1 \in V(S_4)$ . Let  $S_5$  be the sector of  $W$  with end  $a_1$  such that  $V(S_5) \subseteq V(Q_1)$ . Since  $i \leq k - 2$ , the minimality of  $k$  applied to the path  $p_1 - P - p_i - v$  implies that  $t = a_1$  and  $V(S_4) \cap V(S_5) \neq \emptyset$ . In particular,  $i \neq 1$ . It follows from (12) that there are edges between  $P^*$  and  $V(C)$ , and by (6) there is a sector  $S_3$  of  $W$  with ends  $b_1, a_2$  and every edge from  $p_i$  to  $V(C)$  have an end in  $S_3^*$ . Together with the minimality of  $k$  (using the path  $p_i - v$ ), this implies that  $p_i$  is anticomplete to  $V(C)$ . If  $v$  has a unique neighbor  $r$  in  $S_4$  (and therefore  $r \neq b_2$ ) and  $p_k$  has a unique neighbor in  $S_2$ , then  $p_i$  can be linked to  $C$  via the paths  $p_i - v - r$ ,  $p_i - P - p_1 - a_1$  (short-cutting through neighbors of  $b_1$  if possible), and  $p_i - P - p_k - z$  (short-cutting through neighbors of  $a_2$  if possible). If  $v$  has at least two neighbors in  $V(S_4)$  or  $p_k$  has at least two neighbors in  $V(S_2)$ , then the same linkage works rerouting  $S_4$  through  $v$ , and  $S_2$  through  $p_k$ , respectively. This proves (14).

It follows from (13) and (14) that either

- $k = 3$  and  $i = j = 2$ , or
- $k = 2$ .

If  $k = 3$  and  $i = j = 2$ , then by (12) there are edges between  $P^*$  and  $V(C)$ , and so by (6) there is a sector  $S_3$  with ends  $a_2, b_1$ , so that  $p_2$  has neighbors in  $S_3^*$ ; now the path  $p_2 - v$  contradicts the minimality of  $k$ . Thus  $k = 2$ , and we may assume that  $i = 1$ , by symmetry. Since  $G$  is triangle-free, it follows that  $p_1$  is non-adjacent to  $a_1, b_1$ . Since now  $P^* = \emptyset$  is anticomplete to  $V(C)$ , it follows from (12) that we can choose  $S_4$  with  $a_1 \in V(S_4)$ . Since  $v$  is non-adjacent to  $x$  and  $W$  is proper, it follows that  $v$  has at most two neighbors in  $S_4$ . If  $v$  has exactly two neighbors in  $S_4$ , then  $v$  can be linked to the hole  $x - S_4 - x$  via two one-edge paths, and the path  $v - p_1 - x$ , contrary to Lemma 8. Thus  $v$  has a unique neighbor  $r$  in  $V(S_4)$ . Now  $p_1$  can be linked to  $x - S_4 - x$  via the paths  $p_1 - v - r$ ,  $p_1 - x$  and  $p_1 - s - S_1 - a_1$ , again contrary to Lemma 8. This completes the proof of Theorem 14.  $\square$

We can now prove Theorem 6 which we restate:

**Theorem 15.** *Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph, and let  $x$  be the center of a proper wheel in  $G$ . If  $W = (C, x)$  is a proper wheel with a minimum number of spokes subject to having center  $x$ , then*

1. *every component of  $V(G) \setminus N(x)$  contains the interior of at most one sector of  $W$ , and*
2. *for every  $u \in N(x)$ , the component  $D$  of  $V(G) \setminus (N(x) \setminus \{u\})$  such that  $u \in V(D)$  contains the interiors of at most two sectors of  $W$ , and if  $S_1, S_2$  are sectors with  $S_i^* \subseteq V(D)$  for  $i = 1, 2$ , then  $V(S_1) \cap V(S_2) \neq \emptyset$ .*

*Proof.* To prove the first statement, we observe that if some component of  $V(G) \setminus N(x)$  contains the interiors of two sectors of  $W$ , then this component contains a path violating the first assertions of Theorem 14.

For the second statement, suppose  $D$  contains the interiors of two disjoint sectors  $S_1, S_2$  of  $W$ . Since  $|D \cap N(x)| = 1$ , we get a path in  $D$  violating the second assertion of Theorem 14. This proves Theorem 15.  $\square$

### 3 Proper Wheel Centers

In the proof of our main theorem, we perform manipulations on  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graphs; in this section, we show that this preserves being  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free, and that no vertex becomes the center of a proper wheel.

**Lemma 16.** *Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph,  $s \in V(G)$ ,  $K$  a component of  $G \setminus N[s]$ , and  $N$  the set of vertices in  $N(s)$  with a neighbor in  $K$ . Let  $H = G[(V(K) \cup N \cup \{s\})]$ . Then  $s$  is not the center of a proper wheel in  $H$ , and for  $v \in V(H) \setminus \{s\}$ , if  $v$  is the center of a proper wheel in  $H$ , then  $v$  is the center of a proper wheel in  $G$ .*

*Proof.* Since  $H \setminus N[s]$  is connected, it follows that  $s$  is not the center of a proper wheel in  $H$  by Theorem 15. Let  $v \in V(H) \setminus \{s\}$  be the center of a proper wheel  $W = (C, v)$  in  $H$ . For all  $w \in V(G) \setminus V(H)$ ,  $N(w) \cap V(C) \subseteq N[s]$ , and since  $G$  is triangle-free, it follows that every vertex  $w \in V(G) \setminus V(H)$  either has at most one neighbor in  $V(C)$ , or  $N(w) \cap V(C) \subseteq N(s)$ .

Suppose that  $W$  is not proper in  $G$ . Then there exists a vertex  $w$  such that either  $w$  has more than two neighbors in a sector of  $W$ , but  $w$  is not adjacent to  $v$ , or  $w$  has neighbors in at least two sectors of  $W$ . It follows that  $w$  has more than one neighbor in  $V(C)$ , and thus in  $N(s)$ . Suppose that  $w$  has three distinct neighbors  $a, b, c$  in  $V(C) \cap N(s)$ . Let  $P$  be a shortest

path connecting two of  $a, b, c$ , say  $a$  and  $b$ , with interior in  $K$ ; then  $s$  is anticomplete to  $P^*$ . If  $c$  is anticomplete to  $P$ , then  $G|(V(P) \cup \{w, s, a, b, c\})$  is an  $\text{ISK}_4$ . Otherwise, by the minimality of  $|V(P)|$ ,  $P^*$  consists of a single vertex  $x$ , and  $\{w, s, x, a, b, c\}$  induces a  $K_{3,3}$  subgraph in  $G$ , a contradiction. So  $w$  has exactly two neighbors  $a$  and  $b$  in  $V(C)$ , and thus  $a$  and  $b$  are in different sectors of  $W$ . Since  $a, b \in N(s)$  and  $W$  is proper in  $H$ , it follows that  $s \in V(C)$  and  $s$  is a spoke of  $W$ ; let  $S, S'$  be the two sectors of  $W$  containing  $s$ . But then  $v$  can be linked to the cycle  $s - a - w - b - s$  via  $v - s$  and the two paths with interiors in  $S \setminus s$  and  $S' \setminus s$ . This is a contradiction by Lemma 8 and it follows that  $W$  is proper in  $G$ . This concludes the proof.  $\square$

We use the following well-known lemma, which we prove for completeness.

**Lemma 17.** *Let  $G$  be a connected graph,  $a, b, c \in V(G)$  with  $d(a) = d(b) = d(c) = 1$ , and let  $H$  be a connected induced subgraph of  $G$  containing  $a, b, c$  with  $V(H)$  minimal subject to inclusion. Then either  $H$  is a subdivision of  $K_{1,3}$  with  $a, b, c$  as the vertices of degree one, or  $H$  contains a triangle.*

*Proof.* Let  $G, a, b, c, H$  be as in the statement of the theorem. Let  $P$  be a shortest  $a - b$ -path in  $H$ , and let  $Q$  be a shortest path from  $c$  to a vertex  $d$  with a neighbor in  $V(P)$ . By the minimality of  $V(H)$ , it follows that  $V(H) = V(P) \cup V(Q)$ . Moreover,  $P$  and  $Q$  are induced paths and no vertex of  $Q \setminus d$  has a neighbor in  $V(P)$ . If  $d$  has exactly one neighbor in  $V(P)$ , then the result follows. If  $d$  has two consecutive neighbors in  $V(P)$ , then  $H$  contains a triangle. Otherwise, let  $w \in V(P)$  such that  $d$  has a neighbor both on the subpath of  $P$  from  $w$  to  $a$  and on the subpath of  $P$  from  $w$  to  $b$ . It follows that  $w \notin \{a, b, c\}$ , and that  $H \setminus w$  is connected and contains  $a, b, c$ . This contradicts the minimality of  $V(H)$ , and the result follows.  $\square$

**Lemma 18.** *Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph,  $s$  the center of a proper wheel in  $G$ ,  $K$  a component of  $G \setminus N[s]$ , and  $N$  the set of vertices in  $N(s)$  with a neighbor in  $K$ . Let  $G'$  arise from  $G$  by contracting  $V(K)$  to a new vertex  $z$ . If  $G|(V(K) \cup N \cup \{s\})$  is series-parallel, then  $G'$  is  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free.*

*Proof.*  $G'$  does not contain a triangle, because  $N_{G'}(z) \subseteq N_G(s)$  is stable, and hence  $z$  is not in a triangle in  $G'$ . Suppose that  $H$  is an induced subgraph of  $G'$  which is a  $K_{3,3}$  or a subdivision of  $K_4$ . Then  $z \in V(H)$ . If  $z$  has degree two in  $H$  (and so  $H$  is an  $\text{ISK}_4$ ), let  $a, b$  denote its neighbors; we can replace  $a - z - b$  by an  $a - b$ -path  $P$  with interior in  $K$  and obtain a subdivision of  $H$ , which is an  $\text{ISK}_4$ , as an induced subgraph of  $G$ , a

contradiction. Thus  $z$  has degree three in  $H$ ; let  $a, b, c$  denote the neighbors of  $z$  in  $H$ . Let  $P$  be a shortest  $a - b$ -path with interior in  $K$ . Then  $c$  has at most one neighbor on  $P$ , for otherwise  $G|(V(P) \cup \{a, b, c, s\})$  is a wheel, contrary to the fact that  $G|(V(K) \cup N \cup \{s\})$  is series-parallel and does not contain a wheel by Theorem 3. Let  $Q$  be a shortest path from  $c$  to  $V(P) \setminus \{a, b\}$  with interior in  $K$ ; then each of  $a, b, c$  has a unique neighbor in  $V(Q) \cup V(P)$  by symmetry. Let  $H'$  be a minimal connected induced subgraph of  $G|(V(P) \cup V(Q))$  containing  $a, b, c$ . Since  $G|(V(K) \cup N \cup \{s\})$  is series-parallel, it follows that  $H'$  is a subdivision of  $K_{1,3}$  with  $a, b, c$  as the vertices of degree one by Lemma 17. Therefore,  $G|(V(H \setminus z) \cup V(H'))$  is a subdivision of  $H$ , and by Theorem 4, it contains an  $\text{ISK}_4$  or a  $K_{3,3}$  subgraph in  $G$ . This is a contradiction, and the result is proved.  $\square$

**Lemma 19.** *Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph,  $s$  the center of a proper wheel in  $G$ ,  $K$  a component of  $G \setminus N[s]$ , and  $N$  the set of vertices in  $N(s)$  with a neighbor in  $K$ , and let  $H = G|(V(K) \cup N \cup \{s\})$  be series-parallel. Let  $G'$  arise from  $G$  by contracting  $V(K)$  to a new vertex  $z$ . Then  $z$  is not the center of a proper wheel in  $G'$ , and for  $v \in V(G') \setminus \{s, z\}$ , if  $v$  is the center of a proper wheel in  $G'$ , then  $v$  is the center of a proper wheel in  $G$ .*

*Proof.* Since  $N_{G'}(z) \subseteq N_{G'}(s)$ , it follows that  $z$  is not the center of a proper wheel in  $G'$ , for otherwise  $s$  would have a neighbor in every sector of such a wheel. This proves the first statement of the lemma.

Throughout the proof, let  $v \in V(G') \setminus \{s, z\}$  be the center of a proper wheel in  $G'$ , and let  $W = (C, v)$  be such a wheel with a minimum number of spokes. Since  $G'$  is  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free by Lemma 18, it follows that  $W$  satisfies the hypotheses of Theorem 6. Our goal is to show that  $v$  is the center of a proper wheel in  $G$ .

(15) *If  $z \in V(C)$ , then  $v$  is the center of a proper wheel in  $G$ .*

Suppose that  $z \in V(C)$ . Let  $a, b$  denote the neighbors of  $z$  in  $V(C)$ . Let  $P$  be a shortest  $a - b$ -path with interior in  $K$ . Then every vertex in  $V(K)$  has at most two neighbors in  $V(P)$ . Let  $W' = (C', v)$  be the wheel in  $G$  that arises from  $W$  by replacing the subpath  $a - z - b$  of  $C$  by  $a - P - b$  to obtain  $C'$ .

It remains to show that  $W'$  is a proper wheel in  $G$ . Suppose that some vertex  $x \in V(G) \setminus V(K)$  has two or more neighbors in  $P^*$ . Then  $x \in N \subseteq V(H)$ , and  $H|(\{a, b, x, s\} \cup V(P))$  is a wheel in  $H$  with center  $x$ , a contradiction since  $H$  is series-parallel Theorem 3.

Since  $v \notin V(K)$ , it follows from the claim of the previous paragraph that  $v$  has at most one neighbor in  $P^*$ , and no neighbor unless  $v$  is adjacent to  $z$ , and therefore there are at most two sectors of  $W'$  intersecting  $P^*$ . We claim that if for a vertex  $x$  we have  $|N_G(x) \cap V(C')| \geq 3$ , then  $x \notin V(K)$  and  $|N_{G'}(x) \cap V(C)| \geq 3$ . Suppose that  $x$  is a vertex violating this claim. If  $x \in K$ , then  $N_G(x) \cap V(C') \subseteq V(P)$ , and so  $|N_G(x) \cap V(C')| \leq 2$  by the minimality of  $|V(P)|$ , a contradiction; it follows that  $x \notin K$ . Therefore,  $|N_G(x) \cap P^*| \leq 1$ , and thus  $|N_G(x) \cap V(C')| - |N_{G'}(x) \cap V(C)| \leq 1$ . But  $|N_G(x) \cap V(C')| > 3$  by Lemma 8, and so  $|N_{G'}(x) \cap V(C)| \geq 3$ , a contradiction. So the claim holds.

Now suppose that there is a vertex  $x$  which is not proper for  $W'$ . If  $x$  has neighbors in at most one sector of  $W'$ , then  $|N_G(x) \cap V(C')| \geq 3$ , but we proved above that  $|N_{G'}(x) \cap V(C)| \geq 3$ , and so, since  $W$  is proper,  $x$  is adjacent to  $v$ , a contradiction. It follows that  $x$  has neighbors in more than one sector of  $W'$ . Since  $x$  is proper for  $W$ , it follows that  $x$  has a neighbor in  $P^*$  and thus, either  $x \in V(K)$  or  $x$  is adjacent to  $z$ . Since  $x$  is proper for  $W$ , it follows that  $N_G(x) \cap V(C')$  is contained in the sectors of  $W'$  intersecting  $P^*$ . In particular, there are exactly two such sectors  $S_1$  and  $S_2$  of  $W'$ , they are consecutive, and  $v$  has a neighbor in  $P^*$ . Consequently,  $v$  is adjacent to  $z$  and  $z$  is a spoke in  $W$ .

We claim that  $x$  has at most two neighbors in  $V(C')$ . If  $x \in V(K)$  then  $N_G(x) \cap V(C') \subseteq V(P)$  and we have already shown that every vertex of  $K$  has at most two neighbors in  $P$ . Thus we may assume that  $x \notin K$ , and so  $x$  is adjacent to  $z$ . Since  $G'$  is triangle-free by Lemma 18, it follows that  $x$  is not adjacent to  $v$ . Since  $x$  is proper for  $W$ , it follows that  $x$  has at most two neighbors in  $V(C)$ , and hence in  $V(C')$ , by our first claim. This proves our second claim. It follows that  $x$  has exactly one neighbor  $s_1$  in  $S_1 \setminus S_2$  and exactly one neighbor  $s_2$  in  $S_2 \setminus S_1$ . If  $x$  is non-adjacent to  $v$ , then  $G|(V(S_1) \cup V(S_2) \cup \{x, v\})$  is an  $ISK_4$  in  $G$ , a contradiction. Therefore,  $x$  is adjacent to  $v$  and can be linked to the cycle  $G|(V(S_1) \cup \{v\})$  via  $x - v$ ,  $x - s_1$ , and a subpath of  $x - s_2 - S_2$ . Therefore  $W'$  is a proper wheel in  $G$ . This proves (15).

By (15), we may assume that  $z \notin V(C)$ . So  $W$  is a wheel in  $G$ . Since  $W$  is proper in  $G'$ , there is a sector  $S$  of  $W$  containing all neighbors of  $z$  in  $C$ . Then clearly the following holds.

$$(16) \quad \text{For every } x \in K, N_G(x) \cap V(C) \subseteq N_{G'}(z) \cap V(C) \subseteq V(S).$$



Next we claim the following.

(17) *If  $z$  is not adjacent to  $v$ , then  $W$  is a proper wheel in  $G$ .*

If  $x \in G \setminus (V(C) \cup V(K))$ , then  $x$  is proper for  $W$  in  $G$  as  $x$  is proper for  $W$  in  $G'$ . Now consider a vertex  $x \in V(K)$ . Since  $z$  is not adjacent to  $v$ , and  $W$  is proper in  $G'$ , it follows that  $|N_{G'}(z) \cap V(C)| \leq 2$ . Then by (16),  $|N_G(x) \cap V(C)| \leq 2$ , and hence  $x$  is proper for  $W$  in  $G$ . This proves 17.

By (17), we may assume that  $z$  is adjacent to  $v$ . Let  $a$  and  $b$  be the ends of  $S$ . We now define a sequence of wheels in  $G$  with center  $v$ . Let  $W_1 = W$  and  $S_1 = S$ . Assume that wheels  $W_1, \dots, W_i$  have been defined, and define  $W_{i+1}$  as follows. If there is a vertex  $x_i \in V(K)$  that is not adjacent to  $v$  and has at least three neighbors in  $S_i$ , then let  $S_{i+1}$  be the path from  $a$  to  $b$  in  $G[V(S_i) \cup \{x_i\}]$  that contains  $x_i$ , and (by (16)) let  $W_{i+1}$  be the wheel obtained from  $W_i$  by replacing  $S_i$  by  $S_{i+1}$ . Since  $S_{i+1}$  is strictly shorter than  $S_i$ , this sequence must stop at some point; say it stops with wheel  $W_t$ . For  $1 \leq i \leq t$ , let  $C_i$  be the rim of  $W_i$  (so  $V(C_i) = (V(C) \setminus V(S)) \cup V(S_i)$ ). Then  $W_t = (C_t, v)$  is a wheel in  $G$  such that every vertex of  $K$  that has at least three neighbors in  $S_t$  is adjacent to  $v$ . We will show that  $W_t$  is a proper wheel in  $G$ , but first we show the following.

(18) *For  $1 \leq i < t$ , if a vertex  $y$  is proper for  $W_i$ , then  $y$  is proper for  $W_{i+1}$ .*

Suppose that  $y$  is proper for  $W_i$  and not proper for  $W_{i+1}$ . Then  $y$  is adjacent to  $x_i$ . Suppose first that  $y$  is non-adjacent to  $v$  and  $|N_G(y) \cap V(C_{i+1})| \geq 3$ . Since  $y$  cannot have three neighbors in  $C_{i+1}$  by Lemma 8, it follows that  $|N_G(y) \cap V(C_{i+1})| > 3$ . Moreover, since  $N_G(y) \cap V(C_{i+1}) \subseteq \{x_i\} \cup (N_G(y) \cap V(C_i))$ , it follows that  $|N_G(y) \cap V(C_i)| \geq 3$ . But then  $y$  is not proper for  $W_i$ , a contradiction. It follows that  $y$  has a neighbor in  $C_{i+1} \setminus S_{i+1} = C \setminus S$ , and thus  $y \notin V(K)$ . Therefore  $y \in V(G')$ , and since  $y$  is adjacent to  $x_i$  in  $G$ , it follows that  $y$  is adjacent to  $z$  in  $G'$ . Since  $z$  is adjacent to  $v$  in  $G'$  and  $G'$  is triangle-free by Lemma 18, it follows that  $y$  is non-adjacent to  $v$ . Note that since  $i + 1 > 1$ , it follows that a vertex of  $K$  has a neighbor in  $S^*$ , and therefore  $z$  has a neighbor in  $S^*$ . Since  $W$  satisfies the hypotheses of Theorem 6, and since  $y - z$  is a path containing exactly one neighbor of  $v$ , it follows that the neighbors of  $y$  in  $C$  are in a sector  $S'$  of  $W$  consecutive with  $S$ . Since  $y$  is non-adjacent to  $v$ , it follows that  $y$  has at most two neighbors in  $S'$ . Note that since  $G'$  is triangle-free, and  $N_{G'}(z) \cap V(C) \subseteq V(S)$ , it follows that  $z$  has no neighbors in  $S'$ . If  $y$  has

exactly two neighbors in  $S'$ , then  $y$  can be linked in  $G'$  to the hole  $v - S' - v$  via two one-edge paths and the path  $y - z - v$ . So  $y$  has exactly one neighbor  $r$  in  $S'$  that is in  $V(S') \setminus V(S)$ , and now  $z$  can be linked to  $v - S' - v$  via the paths  $z - v$ ,  $z - y - r$ , and a path with interior in  $S$ , contrary to Lemma 8. This concludes the proof of (18).

Every vertex in  $G \setminus V(K)$  is proper for  $W$  and hence it is proper for  $W_t$  by (18). Suppose that there is a vertex  $x \in V(K)$  that is not proper for  $W_t$ . By (16),  $N_G(x) \cap V(C_t) \subseteq V(S_t)$ . So  $x$  is non-adjacent to  $v$  and has at least three neighbors in  $S_t$ , contradicting the assumption that the wheel sequence terminates with  $W_t$ . Therefore,  $W_t$  is a proper wheel in  $G$  with center  $v$ .  $\square$

## 4 Tools

In this section we develop tools for our main theorem for finding a vertex of degree one, or a cycle with all but a few vertices of degree two.

**Lemma 20.** *Let  $G$  be a graph,  $x \in V(G)$ , such that  $G \setminus x$  is a forest. Then either  $V(G) = N[x]$  and  $G \setminus x$  is stable, or  $V(G) \setminus N[x]$  contains a vertex of degree at most one in  $G$ , or  $G$  contains an induced cycle  $C$  containing  $x$  such that every vertex of  $V(C) \setminus \{x\}$  except for possibly one has degree two in  $G$ .*

*Proof.* If every component of  $G \setminus x$  contains exactly one vertex, then either  $V(G) = N[x]$  or  $V(G) \setminus N[x]$  contains a vertex of degree zero. Hence, we may assume that there exists a component  $T$  of  $G \setminus x$  with at least two vertices, and  $T$  is a tree. Let  $A$  be the set of vertices of degree at least three in  $T$ . If  $A$  is non-empty, then let  $T'$  be the subtree of  $T$  that contains all vertices of  $A$  and minimal with respect to this property, and let  $a$  be a leaf of  $T'$ . There is a path  $P = v - \dots - v'$  in  $T$ , whose ends are distinct leaves of  $T$  and  $P$  contains at most one vertex of degree three in  $T$  (namely  $a$ ). This is trivial if  $A$  is empty, and follows from the definition of  $a$  otherwise.

If  $x$  is non-adjacent to  $v$ , then  $v$  is a vertex in  $V(G) \setminus N[x]$  of degree one in  $G$ , so we may assume that  $x$  is adjacent to  $v$ , and similarly for  $v'$ . Now, let  $v''$  be the neighbor of  $x$  in  $P \setminus v$  closest to  $v$  along  $P$ . We set  $C = x - v - P - v'' - x$  and observe that all vertices of  $C$  except possibly  $x$  and  $a$  have degree two in  $G$ .  $\square$

**Lemma 21.** *Let  $G$  be a series-parallel graph, and let  $x, y \in V(G)$  with  $x = y$  or  $xy \in E(G)$ . If  $G \setminus \{x, y\}$  contains a cycle, then there is an induced cycle  $C$*

in  $G$  such that  $V(C) \cap \{x, y\} = \emptyset$  and all but at most two vertices of  $C$  have degree two in  $G$  (and are thus anticomplete to  $\{x, y\}$ ), or  $V(G) \setminus (N[x] \cup N[y])$  contains a vertex of degree at most one in  $G$ .

*Proof.* By contracting the edge  $xy$  and deleting any parallel edges that may arise, we may assume that  $x = y$ . We may further assume that every vertex except for possibly  $x$  has degree at least two, because vertices of degree one in  $N(x)$  can be deleted without affecting the hypotheses or the conclusions, and if there is a vertex of degree at most one in  $V(G) \setminus N[x]$ , then the conclusion holds.

Let  $C$  be a cycle in  $G \setminus x$ . Since  $G$  is series-parallel and by the definition of series-parallel graphs, it follows that there do not exist three paths from  $x$  to  $V(C)$  that are vertex disjoint except for  $x$  in  $G$ . By Menger's theorem [4], it follows that there is a partition  $(X, Y, Z)$  of  $V(G)$  with  $X$  of size at most two, and  $Y, Z \neq \emptyset$  such that  $Y$  is anticomplete to  $Z$  in  $G$ ,  $V(C) \subseteq Y \cup X$  and  $x \in Z$ .

We choose a partition  $(X, Y, Z)$  with  $|X|$  minimal, and subject to that,  $|X \cup Y|$  minimal, such that  $Y$  is anticomplete to  $Z$  in  $G$ ,  $Y, Z \neq \emptyset$ ,  $x \in Z$ , and  $G|(Y \cup X)$  contains a cycle. It follows that  $|X| \leq 2$ .

Suppose first that  $X = \emptyset$ . If  $G|Y$  is an induced cycle, the result follows. Otherwise, since  $G|Y$  contains a cycle, it follows that there is a vertex  $x'$  such that  $G|(Y \setminus \{x'\})$  contains a cycle. By induction applied to  $G|Y$  and the vertex  $x'$ , the result follows.

Next, suppose that  $X = \{x'\}$ . If  $G|Y$  is a forest, then  $x' \neq x$  and thus we obtain the desired result by applying Lemma 20 to  $G|(X \cup Y)$ . Otherwise, we apply induction to  $G|(X \cup Y)$  and  $x'$ , and again, the result follows.

It follows that  $X = \{x', y'\}$ , and therefore, the component of  $G|(Z \cup X)$  containing  $x$  contains  $x'$  and  $y'$ , for otherwise  $\{x'\}$  or  $\{y'\}$  would be a better choice of  $X$  for the partition. Suppose that  $G|Y$  is connected. If there is a vertex  $z$  such that every  $x' - y'$ -path with interior in  $G|Y$  uses  $z$ , then  $\{x', z\}$  or  $\{y', z\}$  yields a better choice of  $X$  and partition. Therefore, by Menger's theorem [4], there are two disjoint paths  $P_1, P_2$  from  $x'$  to  $y'$  with interior in  $Y$ , and since  $G|Y$  is connected, it follows that there is path  $Q$  from  $P_1$  to  $P_2$  in  $Y$ . Moreover, there is a path  $R$  from  $x'$  to  $y'$  with interior in  $Z$  since the component of  $G|(Z \cup X)$  containing  $x$  also contains  $x'$  and  $y'$ ; but  $P_1 \cup P_2 \cup Q \cup R$  is a (not necessarily induced) subdivision of  $K_4$  in  $G$ , contrary to the fact that  $G$  is series-parallel. Thus  $G|Y$  is not connected. By the minimality of  $X \cup Y$ , for every component  $K$  of  $G|Y$ , the graph  $G|((X \setminus \{x\}) \cup V(K))$  contains no cycle. However,  $G|(X \cup Y)$  contains a cycle  $C$  not using  $x$ , and so  $C$  contains vertices from more than one component

of  $G|Y$ . It follows that  $x', y' \in V(C)$ , and thus  $x \notin \{x', y'\}$ . Therefore, for every component  $K$  of  $G|Y$ , the graph  $G|(X \cup V(K))$  is a tree. Since  $K$  is connected, it follows that  $x', y'$  are leaves. If  $G|(X \cup V(K))$  contains a leaf other than  $x', y'$ , then the result follows. So each component is a path from  $x'$  to  $y'$ , and no vertex of the path except for  $x', y'$  has further neighbors in  $G$ . But then the union of two of those paths (there are at least two, since  $G|Y$  is not connected) yields the desired cycle; the result follows.  $\square$

**Theorem 22.** *Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph,  $x, y \in V(G)$  with  $x = y$  or  $xy \in E(G)$ . Then either*

- $V(G) = N[x] \cup N[y]$ ;
- there exists a vertex in  $V(G) \setminus (N[x] \cup N[y])$  of degree at most one in  $G$ ;
- there exists an induced cycle  $C$  containing at least one of  $x, y$  such that at most one vertex  $v$  in  $V(C) \setminus (N[x] \cup N[y])$  has  $d(v) > 2$ ; or
- there exists an induced cycle  $C$  containing neither  $x$  nor  $y$  and a vertex  $z \in V(C)$  such that at most one vertex  $v$  in  $V(C) \setminus N[z]$  has  $d(v) > 2$ .

*Proof.* Suppose first that  $G$  is series-parallel. Define  $H = G$  and  $v = x$  if  $x = y$ , and define  $H$  as the graph that arises from contracting the edge  $xy$  to a new vertex  $v$  if  $x \neq y$ . Then  $H$  is series-parallel. Suppose that  $H \setminus v$  is a forest, and apply Lemma 20. If the first outcome of Lemma 20 holds, then  $V(H) = N_H(v)$ , and so  $V(G) = N_G(x) \cup N_G(y)$ . If the second outcome of Lemma 20 holds, then  $V(H) \setminus N_H[v]$  contains a vertex of degree at most one in  $H$ , and so  $V(G) \setminus (N_G[x] \cup N_G[y])$  contains a vertex of degree at most one in  $G$ . Finally, if the third outcome of Lemma 20 holds, then  $H$  contains an induced cycle  $C$  containing  $v$  such that every vertex of  $V(C) \setminus \{v\}$  except for at most one has degree two in  $H$ , and so there is an induced cycle  $C'$  in  $G$  containing at least one of  $x, y$  such that every vertex of  $V(C') \setminus \{x, y\}$  except for possibly one has degree two in  $G$ . This proves the result in the case that  $H \setminus v$  is a forest. So  $H \setminus v$  contains a cycle, and thus  $G \setminus \{x, y\}$  contains a cycle. By Lemma 21, either  $V(G) \setminus (N[x] \cup N[y])$  contains a vertex of degree at most one in  $G$ , or  $G$  contains a cycle  $C$  with  $V(C) \cap \{x, y\} = \emptyset$  and such that all but at most two vertices in  $V(C)$  have degree two in  $G$ . In the former case, the second outcome of this theorem holds; in the latter case, the fourth outcome of this theorem holds by choosing  $z \in V(C)$  with  $d_G(z)$  maximum among vertices in  $V(C)$ .

Thus we may assume that  $G$  contains a proper wheel by Lemma 9; let  $z$  be the center of a proper wheel (where possibly  $z \in \{x, y\}$ ). Let  $W$  be such a wheel with minimum number of spokes. Let  $Z = \{x, y\} \cap N(z)$ . Since  $x = y$  or  $xy \in E(G)$ , it follows that  $x$  and  $y$  are in the same component of  $G \setminus (N[z] \setminus Z)$ . Since  $N(z)$  is stable, it follows that  $|Z| \leq 1$ . Therefore, by Theorem 6, the component of  $G \setminus (N[z] \setminus Z)$  containing  $\{x, y\} \setminus \{z\}$  includes the interiors of at most two sectors of  $W$ . Again by Theorem 6, the interior of every other sector of  $W$  is contained in a separate component of  $G \setminus (N[z] \setminus Z)$ . Since  $W$  has at least four sectors by Lemma 8, there is a component  $K$  of  $G \setminus (N[z] \setminus Z)$  that does not contain  $x$  and  $y$ , and that contains no neighbor of  $x, y$ . Let  $N$  be the set of neighbors of  $z$  with a neighbor in  $K$ . Then, we apply induction to  $H = G|(V(K) \cup N \cup \{z\})$  and  $z$ . By the choice of  $H$  and  $z$ , the first outcome does not hold. If the second outcome holds for  $H$  and  $z$ , then it holds for  $G$  and  $x, y$  as well, since  $(N[x] \cup N[y]) \cap V(H) \subseteq N[z] \cap V(H)$ . If the third or fourth outcome holds for  $H$  and  $z$ , then the third or fourth outcome holds for  $G$  and  $x, y$ .  $\square$

## 5 Main Result

We say that  $(G, x, y)$  has the property  $\mathcal{P}$  if  $V(G) \setminus (N[x] \cup N[y])$  contains a vertex of degree at most two in  $G$ .

We can now prove Theorem 7 which we restate:

**Theorem 23.** *Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph which is not series-parallel, and let  $(x, y)$  be a non-center pair for  $G$ . Then  $(G, x, y)$  has the property  $\mathcal{P}$ .*

*Proof.* Suppose for a contradiction that the theorem does not hold, and let  $(G, x, y)$  be a counterexample with  $|V(G)|$  minimum. Then every vertex in  $V(G) \setminus (N[x] \cup N[y])$  has degree at least three in  $G$ . Since  $G$  is not series-parallel, and  $G$  is  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free, it follows from Theorem 4 that  $G$  contains a wheel and hence by Lemma 9, it follows that  $G$  contains a proper wheel  $W = (C, s)$ . Let  $C_1, \dots, C_k$  denote the components of  $V(G) \setminus N[s]$ . For  $i = 1, \dots, k$  let  $N_i$  denote the set of neighbors  $v$  of  $s$  such that  $v$  has a neighbor in  $C_i$ , and let  $G_i$  denote the induced subgraph of  $G$  with vertex set  $V(C_i) \cup N_i \cup \{s\}$ .

(19) *For  $i = 1, \dots, k$ , if  $G_i$  is series-parallel and  $G_i \setminus (N[s] \cap \{x, y, s\})$  contains a cycle, then  $\{x, y\} \cap V(C_i) \neq \emptyset$ .*

Let  $i \in \{1, \dots, k\}$  such that  $G_i$  is series-parallel, and let  $G_i \setminus (N[s] \cap \{x, y, s\})$  contain a cycle. Since  $G$  is triangle-free, it follows that  $1 \leq |N[s] \cap$

$\{x, y, s\} \mid \leq 2$ . By Lemma 21 applied to  $G_i$  and the vertices in  $N[s] \cap \{x, y, s\}$ , it follows that either there is a vertex in  $V(G_i) \setminus N[s]$  of degree at most one in  $G_i$  anticomplete to  $\{x, y\} \cap N(s)$ , or  $G_i \setminus (N[s] \cap \{x, y, s\})$  contains a cycle  $C'$  with at least two vertices of degree two in  $G_i$ . In both cases, there is a vertex  $z$  in  $V(G_i) \setminus N[s]$  of degree at most two in  $G_i$  and  $z$  is anticomplete to  $N[s] \cap \{x, y, s\}$ , and hence its degree in  $G$  is also at most two. Since  $(G, x, y)$  does not satisfy property  $\mathcal{P}$ , it follows that  $z \in N[x] \cup N[y]$ , and thus  $\{x, y\} \cap V(C_i) \neq \emptyset$ . This proves (19).

(20) *For  $i = 1, \dots, k$ , if  $G_i$  is series-parallel, then  $|V(C_i)| = 1$ .*

Let  $i \in \{1, \dots, k\}$  be such that  $G_i$  is series-parallel, and suppose that  $|V(C_i)| > 1$ . Let  $G'$  be the graph that arises from  $G$  by contracting  $V(C_i)$  to a new vertex  $z$ . We let  $x' = z$  if  $x \in V(C_i)$  and  $x' = x$  otherwise; and we let  $y' = z$  if  $y \in V(C_i)$  and  $y' = y$  otherwise. By Lemma 18,  $G'$  is  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free. By Lemma 19,  $(x', y')$  is a non-center pair for  $G'$ . By the minimality of  $|V(G)|$ , it follows that  $(G', x', y')$  has the property  $\mathcal{P}$ . Let  $v \in V(G') \setminus (N[x'] \cup N[y'])$  be a vertex of degree at most two in  $G'$ . From the definition of  $x'$  and  $y'$ , it follows that  $v \notin N[x] \cup N[y]$ . It follows that either  $v = z$ , or  $v \neq z$  and  $d_G(v) > 2$ , and so  $v \in N[z]$ .

Suppose first that  $v = z$ . Then  $z \notin N[x'] \cup N[y']$ , and so  $V(G_i) \cap \{x, y\} = \emptyset$ . By (19), it follows that  $G_i \setminus s$  is a tree. Since  $v$  has degree at most two in  $G'$ , it follows that  $|N_i| \leq 2$ , and since  $G_i \setminus N[s]$  is connected, it follows that every vertex of  $N_i$  is a leaf of  $G_i \setminus s$ . Thus, either  $V(C_i)$  contains a leaf of  $G_i \setminus s$ , or  $G_i \setminus s$  is a path with ends in  $N_i$ , and so in both cases  $V(C_i)$  contains a vertex of degree at most two in  $G$ . This is a contradiction since  $V(G_i) \cap \{x, y\} = \emptyset$ ; it follows that  $v \neq z$ .

It follows that  $v \in N(z)$ . Since  $d_G(v) > 2$ , it follows that  $d_{G'}(v) < d_G(v)$ , and thus  $v$  has more than one neighbor in  $V(C_i)$ . Let  $P$  be a path in  $C_i$  between two neighbors of  $v$ , then  $v - P - v$  is a cycle in  $G_i \setminus (N[s] \cap \{x, y, s\})$ . By (19), it follows that  $V(C_i) \cap \{x, y\} \neq \emptyset$ . But then  $z \in \{x', y'\}$ , and so  $v \in N[x'] \cup N[y']$ , a contradiction. This proves (20).

(21) *For  $i = 1, \dots, k$ , if  $G_i$  contains a wheel, then  $x \in V(C_i)$  or  $y \in V(C_i)$ .*

Suppose not, and let  $i \in \{1, \dots, k\}$  be such that  $G_i$  contains a wheel and  $V(C_i) \cap \{x, y\} = \emptyset$ . Since  $N_i$  is a stable set, it follows that  $|N_i \cap \{x, y\}| \leq 1$ , and by symmetry, we may assume that  $y \notin N_i$ . Let  $y' = s$ , and let  $x' = x$  if  $x \in N_i$  and  $x' = s$  otherwise. By Lemma 16,  $(x', y')$  is a non-center pair for  $G_i$ . Since  $G_i$  is an induced subgraph of  $G$ , it follows that  $G_i$  is

$\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free. Since  $G$  is a minimum counterexample, it follows that  $(G_i, x', y')$  has the property  $\mathcal{P}$ . Let  $v$  be a vertex of degree at most two in  $G_i$  with  $v \notin N[x'] \cup N[y']$ . Since  $v \notin N[s]$ , it follows that  $d_G(v) = d_{G_i}(v)$ . Therefore  $v \in N[x] \cup N[y]$ . Let  $w \in \{x, y\}$  be such that  $v \in N[w]$ . By assumption,  $w \notin V(C_i)$ . If  $w \in V(G_i)$ , then  $w \in N_i$ , and so  $w = x = x'$ , and thus  $v \notin N[w]$ , a contradiction. It follows that  $w \notin V(G_i)$ , and so  $N[w] \cap V(G_i) \subseteq N[s]$ , and again  $v \notin N[w]$ , a contradiction. This proves that  $\{x, y\} \cap V(C_i) \neq \emptyset$ , and (21) follows.

It follows from Theorem 3 together with (21) and (20) that there is at most one  $i \in \{1, \dots, k\}$  with  $|V(C_i)| > 1$ . We may assume  $|V(C_i)| = 1$  for all  $i \in \{1, \dots, k-1\}$ .

(22) *If  $|V(C_k)| > 1$ , let  $G' = G \setminus (V(C_k) \cup \{s\})$ ; otherwise let  $G' = G \setminus s$ . Then  $G'$  has girth at least eight.*

Observe that  $G'$  is bipartite with one side of the bipartition being  $N(s)$ .

Suppose that  $C$  is a cycle of length four in  $G'$ . Let  $V(C) = \{a, b, c, d\}$  and  $N(s) \cap V(C) = \{a, c\}$ . If  $d_G(b) \neq 2$ , let  $e$  be a neighbor of  $b$  which is not  $a, c$ . Note that  $e \in N(s)$ . Then  $\{a, b, c, d, e, s\}$  induces an  $\text{ISK}_4$  or a  $K_{3,3}$ , a contradiction. It follows that  $d_G(b) = d_G(d) = 2$ , and moreover,  $\{x, y\} \cap \{a, c\} \neq \emptyset$ . So, by symmetry, say  $x = a$ , and we may assume that  $d \neq y$ .

Observe that  $G \setminus b$  is not series-parallel by Theorem 3. By the minimality of  $|V(G)|$ , it follows that there exists  $v \in V(G) \setminus (N[x] \cup N[y])$  with  $d_{G \setminus \{b\}}(v) \leq 2$ . Since  $d_G(v') = d_{G \setminus \{b\}}(v')$  for all  $v' \in V(G) \setminus \{x, b, c\}$ , it follows that  $v = c$ , and so  $N_G(c) = \{b, d, s\}$ , and so  $\{s, a\}$  is a cutset in  $G$ . Let  $G'' = G \setminus \{b, c, d\}$ , and if  $y \in \{b, c, d\}$ , let  $y' = x$ , otherwise,  $y' = y$ . Then  $G''$  is not series-parallel. A proper wheel in  $G''$  is proper in  $G$ , because each vertex in  $\{b, c, d\}$  has at most one neighbor in the wheel,  $s$  or  $a$ . Therefore,  $(x, y')$  is a non-center pair for  $G''$ . By the minimality of  $|V(G)|$ , it follows that  $(G'', x, y')$  has the property  $\mathcal{P}$ . But this is a contradiction, since every vertex in  $V(G'') \setminus N[x]$  has the same degree in  $G$  and  $G''$ . This proves that  $G'$  contains no 4-cycle.

Suppose  $G'$  contains a 6-cycle  $C$ . Then, since exactly three vertices in  $V(C)$  are neighbors of  $s$ , it follows that  $G|(V(C) \cup \{s\})$  is an  $\text{ISK}_4$ , a contradiction. It follows that  $G'$  has girth at least eight, and so (22) is proved.

(23)  $|V(C_k)| > 1$ .

Suppose not, and let  $G' = G \setminus s$ . Then  $G'$  satisfies the hypotheses of Theorem 22. Since  $s$  is the center of a proper wheel in  $G$ , it follows that there exists a vertex  $z$  in  $G'$  that is not in  $N[x] \cup N[y] \cup N[s]$ , and so the first outcome of Theorem 22 does not hold. The second outcome does not hold, because every vertex in  $V(G') \setminus (N[x] \cup N[y])$  of degree one in  $G'$  has degree at most two in  $G$ , a contradiction.

Therefore, the third or fourth outcome of Theorem 22 holds, and hence there exists an induced cycle  $C$  in  $G'$  with vertices  $c_1 - \dots - c_t - c_1$ , and  $i, j \in \{1, \dots, t\}$ ,  $l \in \{0, \dots, 3\}$  such that all vertices of  $C$  except for  $c_i, \dots, c_{i+l}$  (where  $c_{t+1} = c_1$  and so on) and  $c_j$  have degree two in  $G'$ , do not coincide with  $x, y$  and are non-neighbors of  $x, y$ . By (22),  $t \geq 8$ . Consequently,  $G'$  contains two adjacent vertices in  $V(G') \setminus (N[x] \cup N[y])$  of degree two in  $G'$ . Since  $G$  is triangle-free, it follows that one of them has degree two in  $G$ , a contradiction. Thus,  $|V(C_k)| > 1$ , and (23) is proved.

By (20), (21) and (23) we may assume that  $x \in V(C_k)$ . Let  $G'$  arise from  $G$  by contracting  $V(C_k) \cup N_k$  to a single vertex  $z$ , and by deleting  $s$  and every vertex that is only adjacent to  $z$ . It follows that  $G'$  is bipartite. Our goal is to prove that  $G' \setminus z$  has girth at least 16, see (28). By (22), we know that  $G' \setminus z$  has girth at least eight.

(24) *Every vertex in  $V(G') \setminus \{z\}$  has at most one neighbor in  $N_k$  in  $G$ . There is no 4-cycle in  $G'$  containing  $z$ .*

Suppose first that there is a vertex  $v \in V(G') \setminus \{z\}$  with at least two neighbors  $a, b \in N_k$  in  $G$ . Since  $v \in V(G')$  and in  $G'$  there are no vertices of degree one adjacent to  $z$ , it follows that  $v$  has another neighbor  $c \in N(s) \setminus N_k$ . Let  $P$  be a path connecting  $a$  and  $b$  with interior in  $V(C_k)$ . Such a path exists, since  $a, b \in N_k$ . It follows that  $G|(V(P) \cup \{a, b, c, v, s\})$  is an  $\text{ISK}_4$  in  $G$ , a contradiction. This implies the first statement of (24).

Suppose that  $z$  is contained in a 4-cycle with vertex set  $\{a, b, c, z\}$  in  $G'$  such that  $a, c \in N_{G'}(z)$ . Note that  $a, c \notin N(s)$  and  $b \in N(s) \setminus N_k$ . By (22),  $G \setminus (\{s\} \cup V(C_k))$  contains no 4-cycle, and thus  $a$  and  $c$  have no common neighbor in  $N_k$ . Let  $a', c'$  be a neighbor of  $a$  and  $c$  in  $N_k$ , respectively;  $a'$  and  $c'$  exists since  $a, c \in N_{G'}(z)$ . Let  $P$  be a shortest path between  $a'$  and  $c'$  with interior in  $C_k$ . Since  $b \notin N_k$ , it follows that  $b$  is anticomplete to  $V(P)$ . Therefore,  $G|(\{a, b, c, s\} \cup V(P))$  is an  $\text{ISK}_4$  in  $G$ , a contradiction. This proves (24).

(25)  *$G'$  is  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free.*



Since  $G'$  is bipartite, it follows that  $G'$  is triangle-free. Suppose that  $G'$  contains an induced subgraph  $H$  which is either a  $K_{3,3}$  or an  $\text{ISK}_4$ . Since  $G$  is  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free, it follows that  $z \in V(H)$ . Suppose that  $z$  has degree two in  $H$ . By (24), the neighbors of  $z$  in  $V(H)$  do not have a common neighbor in  $N_k$ . Let  $P$  be a path in  $G$  connecting the neighbors of  $z$  in  $V(H)$  with interior in  $V(C_k) \cup N_k$  containing exactly two vertices in  $N_k$ . Then  $G|(V(H) \setminus \{z\}) \cup V(P)$  is an induced subdivision of  $H$  in  $G$ . By Theorem 4, it follows that  $G$  is not  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free, a contradiction.

It follows that  $z$  has degree three in  $H$ . Let  $a, b, c$  be the neighbors of  $z$  in  $H$ . By (24), each of  $a, b, c$  has a unique neighbor in  $N_k$ . Let  $a', b', c'$  be neighbors of  $a, b, c$  in  $N_k$ . Let  $H'$  be a minimal induced subgraph of  $G|(V(C_k) \cup \{a, b, c, a', b', c'\})$  which is connected and contains  $\{a, b, c\}$ . It follows that each of  $a, b, c$  has a unique neighbor (namely  $a', b', c'$ , respectively), in  $H'$ . By Lemma 17,  $H'$  is a subdivision of  $K_{1,3}$  in which  $a, b, c$  are the vertices of degree one. Consequently,  $G|(V(H \setminus z) \cup V(H'))$  is an induced subgraph of  $G$  which is a subdivision of  $H$ . But then  $G$  is not  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free by Theorem 4. Hence  $G'$  is  $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free. This proves (25).

(26)  $G'$  does not contain a proper wheel with center different from  $z$ .

Suppose  $v \neq z$  is the center of a proper wheel  $G'$ . By Theorem 6, there is a component  $C$  of  $G' \setminus N[v]$  that is disjoint from  $N[z]$ . Let  $N$  denote the set of vertices in  $N(v)$  with a neighbor in  $C$ .

Then  $H = G'|(N \cup V(C) \cup \{v\})$  satisfies the hypotheses of Theorem 22. Since  $V(C) \neq \emptyset$ , it follows that the first outcome of Theorem 22 does not hold. Moreover, every vertex in  $V(H) \setminus N[v]$  of degree one in  $H$  has degree at most two in  $G$ , since such a vertex belongs to  $C$  and  $C$  is disjoint from  $N[z]$ , and the only additional neighbor that such a vertex may have in  $G$  is  $v$ . Furthermore, such a vertex is in  $V(G') \setminus N[z]$ , and hence in  $V(G) \setminus (N[x] \cup N[y])$  as  $x \in V(C_k)$ . It follows that the second outcome of Theorem 22 does not hold.

Therefore, the third or fourth outcome of Theorem 22 holds, and hence there exists an induced cycle  $C'$  in  $H$  with vertices  $c_1 - \dots - c_t - c_1$ , and  $i, j \in \{1, \dots, t\}$ ,  $l \in \{0, \dots, 3\}$  such that all vertices of  $C'$  except for  $c_i, \dots, c_{i+l}$  (where  $c_{t+1} = c_1$  and so on) and  $c_j$  have degree two in  $H$ , do not coincide with  $v$  and are non-neighbors of  $v$ . By (22),  $t \geq 8$ , since  $z \notin V(H)$ . Consequently,  $G'$  contains two adjacent vertices in  $V(G') \setminus N[z]$  of degree two

in  $G'$ . Since  $G$  is triangle-free, it follows that one of them is non-adjacent to  $s$  and thus has degree two in  $G$ , a contradiction. Hence (26) is proved.

(27) *For every component  $K$  of  $G' \setminus N[z]$ ,  $G'|_{(V(K) \cup N(z))}$  is a forest.*

Suppose not, and let  $K$  be a component of  $G' \setminus N[z]$  such that  $G'|_{(V(K) \cup N(z))}$  is not a forest. Suppose first that  $H = G'|_{(V(K) \cup N[z])}$  is not series-parallel. Then  $H$  contains a proper wheel by Lemma 9. Let  $v$  be the center of a proper wheel in  $H$ . Since  $H \setminus N[z]$  is connected, it follows from Theorem 6 that  $v \neq z$ . By Lemma 16, it follows that  $v$  is the center of a proper wheel in  $G'$ , contrary to (26).

It follows that  $H$  is series-parallel, and by our assumption,  $H \setminus z$  contains a cycle. By applying Lemma 21 to  $H$  and  $z$ , it follows that there is either a vertex in  $V(H) \setminus N[z]$  of degree one, or a cycle  $C$  not containing  $z$ , with all but at most two vertices of degree two in  $H$ . In the latter case, since  $G' \setminus z$  has girth at least eight,  $C$  contains two adjacent vertices in  $V(H) \setminus N[z]$  of degree two in  $H$ , and thus of degree two in  $G'$ . Since  $G$  is triangle-free, it follows that in both cases  $G$  contains a vertex of degree at most two not in  $N[z]$ , and thus not in  $N[x] \cup N[y]$ . This is a contradiction, and (27) is proved.

(28) *The girth of  $G' \setminus z$  is at least 16.*

Suppose that this is false. Let  $C$  be an induced cycle in  $G' \setminus z$  of length less than 16. Since by (27), for every component  $K$  of  $G' \setminus N[z]$ , we have that  $G'|_{(V(K) \cup N(z))}$  is a forest, it follows that  $C \setminus N[z]$  has at least two components. Since  $z$  is not contained in a 4-cycle in  $G'$  by (24), and  $G'$  is bipartite, it follows that each component of  $C \setminus N[z]$  has at least three vertices. If  $C \setminus N[z]$  has at least four components, it follows that  $C$  has length at least 16. If  $C \setminus N[z]$  has exactly three components, then  $G'|_{(V(C) \cup \{z\})}$  is an  $\text{ISK}_4$ , a contradiction. So  $C \setminus N[z]$  has exactly two components. For every component  $K$  of  $G' \setminus N[z]$ , by (27) we have that  $G'|_{(V(K) \cup N(z))}$  is a forest. Therefore, the two components of  $C \setminus N[z]$  are contained in two different components of  $G' \setminus N[z]$ ; say  $A$  and  $B$ . Let  $N_A, N_B$  denote the set vertices in  $N(z)$  with a neighbor in  $A, B$ , respectively. Suppose that  $|N_A| \geq 3$ . Since  $|V(C) \cap N(z)| = 2$ , it follows that there is a path  $P$  from a vertex  $c$  in  $N_A \setminus V(C)$  to  $V(C)$  with interior in  $V(A)$ . Since  $G'|_{(V(A) \cup N_A)}$  and  $G'|_{(V(B) \cup N_B)}$  are trees, it follows that  $c$  has at most one neighbor in each component  $K$  of  $C \setminus N[z]$ . Therefore,  $G'|_{(V(P) \cup V(C) \cup \{z\})}$  contains an induced subgraph of  $G'$  which is either a subdivision of  $K_4$  or of  $K_{3,3}$ , a contradiction by Theorem 4 and (25). So  $|N_A| = 2$ . Since  $G'|_{(V(A) \cup N_A)}$

is a tree, it follows that either  $A$  contains a vertex of degree one in  $G'$ , non-adjacent to  $z$ , or  $G'|(V(A) \cup N_A)$  is a path containing at least five vertices, and hence  $A$  contains two adjacent vertices of degree two in  $G'$ , non-adjacent to  $z$ . Since  $G$  is triangle-free, it follows that in either case  $G$  contains a vertex of degree at most two not in  $N[z]$ , and thus not in  $N[x] \cup N[y]$ . This is a contradiction, and (28) is proved.

Recall that  $\{x, y\} \cap V(C_k) \neq \emptyset$ , and we may assume that  $x \in V(C_k)$ , and thus  $y \in V(C_k) \cup N_k$ . Let  $G''$  be the graph that arises from  $G$  by deleting  $\{s\} \cup (V(C_k) \setminus \{x\}) \cup (N_k \setminus \{y\})$ , and every vertex other than  $x$  with neighbors only in  $N_k$  (this last operation does not change the degree of any vertex in  $V(G'')$  except for possibly  $y$ ). Then  $N_{G''}(x) \subseteq \{y\}$ . It follows from (28) that  $G'' \setminus \{y\}$  has girth at least 16, and from (22) that  $G''$  has girth at least eight. If  $y \in V(G'')$ , let  $y' = y$ ; otherwise, let  $y' = x$ . It follows that if  $y' = y$ , then  $y \in N_k$ .

Since  $G''$  is an induced subgraph of  $G$ , it follows that  $G''$  and  $x, y'$  satisfy the hypotheses of Theorem 22.

Since  $s$  is the center of a proper wheel, it follows from Theorem 6 that there are at least two components of  $G'' \setminus N[s]$  in which  $y'$  has no neighbors. Consequently,  $V(G'') \neq N[x] \cup N[y']$ , and thus the first outcome of Theorem 22 does not hold.

The second outcome of Theorem 22 does not hold, because if  $G''$  contains a vertex  $v$  of degree one non-adjacent to  $y'$ , then  $v$  has degree at most two in  $G$ , and  $v \notin N[x] \cup N[y]$ , a contradiction.

Suppose that the third outcome holds, and so  $G''$  contains an induced cycle  $C$  containing  $y'$  (since  $d_{G''}(x) \leq 1$ ) such that at most one vertex in  $V(C) \setminus N[y]$  has degree more than two. Since  $G''$  has girth at least eight,  $|V(C)| \geq 8$ , and in particular  $C$  contains a vertex  $v$  of distance three from  $y$  in  $C$  and degree two in  $G''$ . Let  $y - a - b - v$  be the three-edge path from  $y$  to  $v$  in  $C$ . Then  $v$  is not adjacent to  $s$  in  $G$ , because  $G|(V(G'') \cup \{s\})$  is bipartite and  $ys \in E(G)$ . Moreover,  $v$  is anticomplete to  $N_k$ , because otherwise  $z - a - b - v - z$  is a 4-cycle in  $G'$  using  $z$ , contradicting (24). So  $v$  has degree two in  $G$  and is not in  $N[x] \cup N[y]$ , a contradiction.

Thus, the fourth outcome holds, and so  $G''$  contains an induced cycle  $C$  not containing  $x, y'$  and containing a vertex  $z'$  such that at most one vertex in  $V(C) \setminus N[z']$  has degree more than two in  $G''$ . Since  $|V(C)| \geq 16$ , it follows that  $C$  contains a path  $P = p_1 - \dots - p_6$  of six vertices, all of degree two in  $G''$  and non-adjacent to  $x, y'$ . We may assume that  $N(s) \cap V(P) \subseteq \{p_1, p_3, p_5\}$  by symmetry. Since  $z$  is not in a 4-cycle in  $G'$  by (24), not both  $p_2$  and  $p_4$

have a neighbor in  $N_k$ . It follows that either  $p_2$  or  $p_4$  has degree two in  $G$ , a contradiction. This completes the proof of Theorem 23.  $\square$

We can now prove Theorem 2 which we restate:

**Theorem 24.** *Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}\}$ -free graph. Then either  $G$  has a clique cutset,  $G$  is complete bipartite, or  $G$  has a vertex of degree at most two.*

*Proof.* Let  $G$  be an  $\{\text{ISK}_4, \text{triangle}\}$ -free graph. If  $G$  is series-parallel, then  $G$  contains a vertex of degree at most two Theorem 3. If  $G$  contains  $K_{3,3}$  as a subgraph, then by Theorem 5, either  $G$  is complete bipartite or  $G$  has a clique cutset. If  $G$  is not series-parallel and  $K_{3,3}$ -free, then  $G$  contains a vertex of degree at most two by Theorem 7 applied to the graph obtaining from  $G$  by adding an isolated vertex  $x$  with the non-center pair  $(x, x)$ . This implies the result.  $\square$

Note that the outcome of a clique cutset in Theorem 24 cannot be avoided, as the following example shows. Let  $G$  be any  $\{\text{ISK}_4, \text{triangle}\}$ -free graph (e. g. a  $C_5$ , or a wheel), and let  $H$  arise from  $G$  by adding  $|V(G)|$  disjoint copies of  $K_{3,3}$  to  $G$  and identifying each vertex of  $G$  with a vertex of a different copy of  $K_{3,3}$ . The resulting graph is  $\{\text{ISK}_4, \text{triangle}\}$ -free, not series-parallel, and not bipartite if  $G$  is not bipartite, and it contains no vertex of degree at most two.

Finally we are ready to prove the following.

**Theorem 25.** *If  $G$  is an  $\{\text{ISK}_4, \text{triangle}\}$ -free graph, then  $G$  is 3-colorable.*

*Proof.* The proof is by induction on  $|V(G)|$  using Theorem 2. If  $G$  is complete bipartite, then  $G$  is 2-colorable. If  $G$  has a vertex  $v$  of degree at most two, then, by induction,  $G \setminus v$  is 3-colorable, and hence  $G$  is 3-colorable. If  $G$  has a clique cutset  $C$  such that  $(A, B, C)$  is a partition of  $V(G)$  with  $A$  anticomplete to  $B$  and  $C$  a clique, then  $\chi(G) = \max\{\chi(G|(A \cup C)), \chi(G|(B \cup C))\}$ , and again by induction,  $G$  is 3-colorable.  $\square$

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## References

- [1] Duffin, Richard J. *Topology of series-parallel networks*. Journal of Mathematical Analysis and Applications 10, no. 2 (1965): 303–318.
- [2] Khang Le, Ngoc. *Chromatic number of  $ISK_4$ -free graphs*. arXiv preprint arXiv:1611.04279 (2016).
- [3] Lévêque, Benjamin, Frédéric Maffray, and Nicolas Trotignon. *On graphs with no induced subdivision of  $K_4$* . Journal of Combinatorial Theory, Series B 102, no. 4 (2012): 924–947.
- [4] Menger, Karl. *Zur allgemeinen Kurventheorie*. Fundamenta Mathematicae 10, No. 1 (1927): 96–115.
- [5] Trotignon, Nicolas, and Kristina Vušković. *On Triangle-Free Graphs That Do Not Contain a Subdivision of the Complete Graph on Four Vertices as an Induced Subgraph*. Journal of Graph Theory (2016).