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Triangle-free graphs that do not contain an induced subdivision of K_4 are 3-colorable

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Abstract

We show that triangle-free graphs that do not contain an induced subgraph isomorphic to a subdivision of K_4 are 3-colorable. This proves a conjecture of Trotignon and Vušković [5].

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1 Introduction

All graphs in this paper are finite and simple. Let G be a graph. For a vertex $v \in V(G)$, we denote its set of neighbors by $N(v)$, and we let $N[v] = \{v\} \cup N(v)$. For $X, Y \subseteq V(G)$, we say that X is *complete* to Y if every vertex in X is adjacent to every vertex in Y ; X is *anticomplete* to Y if every vertex in X is non-adjacent to every vertex in Y . A vertex $v \in V(G)$ is *complete* (*anticomplete*) to $X \subseteq V(G)$ if $\{v\}$ is complete (anticomplete) to X . A set $X \subseteq V(G)$ is a *cutset* for G if there is a partition (X, Y, Z) of $V(G)$ with $Y, Z \neq \emptyset$ and Y anticomplete to Z . The cutset X is a *clique cutset* if X is a (possibly empty) clique in G . For a graph H , we say that G *contains* H if H is isomorphic to an induced subgraph of G , and otherwise, G is *H -free*. For a family \mathcal{F} of graphs, we say that G is *\mathcal{F} -free* if G is F -free for every graph $F \in \mathcal{F}$.

For a graph G and $X \subseteq V(G)$, $G|X$ denotes the induced subgraph of G with vertex set X . For $X \subseteq V(G)$, we let $G \setminus X = G|(V(G) \setminus X)$ and for $x \in V(G)$, we let $G \setminus x = G|(V(G) \setminus \{x\})$. By a *path* in a graph we mean an induced path. Let C be a cycle in G . The *length* of C is $|V(C)|$. The *girth* of G is the length of a shortest cycle, and is defined to be ∞ if G has no cycle. A *hole* in a graph is an induced cycle of length at least four. An ISK_4 is a graph that is isomorphic to a subdivision of K_4 .

In [5] two of us studied the structure of ISK_4 -free graphs, and proposed the following conjecture (and proved several special cases of it):

Conjecture 1. *If G is $\{\text{ISK}_4, \text{triangle}\}$ -free, then $\chi(G) \leq 3$.*

In [2], Conjecture 1 was proved with 3 replaced by 4.

The main result of the present paper is the proof of Conjecture 1. In fact, we prove a stronger statement, from which Conjecture 1 easily follows:

Theorem 2. *Let G be an $\{\text{ISK}_4, \text{triangle}\}$ -free graph. Then either G has a clique cutset, G is complete bipartite, or G has a vertex of degree at most two.*

For an induced subgraph H of G we write $v \in H$ to mean $v \in V(H)$. We use the same convention if H is a path or a hole. For a path $P = p_1 - \dots - p_k$ we call the set $V(P) \setminus \{p_1, p_k\}$ the *interior* of P , and denote it by P^* .

A *wheel* in a graph is a pair $W = (C, x)$ where C is a hole and x has at least three neighbors in $V(C)$. We call C the *rim* of the wheel, and x the *center*. The neighbors of x in $V(C)$ are called the *spokes* of W . Maximal paths of C that do not contain any spokes in their interior are called the *sectors* of W . We write $V(W)$ to mean $V(C) \cup \{x\}$.

A graph is *series-parallel* if it does not contain a subdivision of K_4 as a (not necessarily induced) subgraph.

Theorem 3 ([1]). *Let G be a series-parallel graph. Then G is ISK_4 -free, wheel-free, and $K_{3,3}$ -free, and G contains a vertex of degree at most two.*

The following two facts were proved in [3]:

Theorem 4 ([3]). *Let G be an $\{\text{ISK}_4, \text{triangle}\}$ -free graph. Then either G is series-parallel, or G contains a $K_{3,3}$ subgraph, or G contains a wheel. If G contains a subdivision of $K_{3,3}$ as an induced subgraph, then G contains a $K_{3,3}$.*

Theorem 5 ([3]). *If G is an $\{\text{ISK}_4, \text{triangle}\}$ -free graph and G contains $K_{3,3}$, then either G is complete bipartite, or G has a clique cutset.*

Thus to prove Theorem 2 we need to analyze $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graphs that contain wheels. This approach was already explored in [5], but we were able to push it further, as follows.

A wheel $W = (C, x)$ is *proper* if for every $v \in V(G) \setminus V(W)$

- there is a sector S of W such that $N(v) \cap V(C) \subseteq V(S)$.
- If v has at least three neighbors in $V(C)$, then v is adjacent to x .

(Please note that this definition is different from the one in [5].) We prove:

Theorem 6. *Let G be an $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph, and let x be the center of a proper wheel in G . If $W = (C, x)$ is a proper wheel with a minimum number of spokes subject to having center x , then*

1. *every component of $V(G) \setminus N(x)$ contains the interior of at most one sector of W , and*
2. *for every $u \in N(x)$, the component D of $V(G) \setminus (N(x) \setminus \{u\})$ such that $u \in V(D)$ contains the interiors of at most two sectors of W , and if S_1, S_2 are sectors with $S_i^* \subseteq V(D)$ for $i = 1, 2$, then $V(S_1) \cap V(S_2) \neq \emptyset$.*

Using Theorem 6 we can prove a variant of a conjecture from [5] that we now explain. For a graph G and $x, y \in V(G)$, we say that (x, y) is a *non-center pair* for G if neither x nor y is the center of a proper wheel in G , and $x = y$ or $xy \in E(G)$. We prove:

Theorem 7. *Let G be an $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph which is not series-parallel, and let (x, y) be a non-center pair for G . Then some $v \in V(G) \setminus (N[x] \cup N[y])$ has degree at most two.*

Here is the outline of the proof; the full proof is given in Section 5. We assume that G is a counterexample to Theorem 7 with $|V(G)|$ minimum. Since G is not series-parallel, it follows from Theorem 4 that G contains a wheel, and we show in Lemma 9 that G contains a proper wheel. Let $s \in V(G)$ be the center of a proper wheel chosen as in Theorem 6, and let C_1, \dots, C_k be the components of $G \setminus N[s]$. By Theorem 6, it follows that $k > 1$. For each i , let N_i be the set of vertices of $N(s)$ with a neighbor in $V(C_i)$, and let $G_i = G|(V(C_i) \cup N_i \cup \{s\})$. We analyze the structure of the graphs G_i using the minimality of $|V(G)|$. It turns out that at most one G_i is not series-parallel, and that (by contracting C_i 's) there is at most one value of i for which $|V(C_i)| > 1$. Also, if $|V(C_i)| > 1$, then $\{x, y\} \cap V(C_i) \neq \emptyset$. We may assume that $|V(C_i)| = 1$ for all $i \in \{1, \dots, k-1\}$, and that $\{x, y\} \cap V(C_k) \neq \emptyset$. Now consider the bipartite graph G' , which (roughly speaking) is the graph obtained from $G \setminus \{s\}$ by contracting $V(C_k) \cup N_k$ to a single vertex z if $|V(C_k)| > 1$. It turns out that G' is $\{\text{ISK}_4, K_{3,3}\}$ -free and has girth at least 6, while cycles that do not contain z must be even longer. Now either there is an easy win, or we find a cycle in G' that contains a long path P of vertices all of degree two in G' and with $V(P) \subseteq V(G) \setminus (N[x] \cup N[y])$. Further analysis shows that at least one of these vertices has degree two in G , and Theorem 7 follows.

This paper is organized as follows. In Section 2 we prove Theorem 6. Section 3 contains technical tools that we need to deduce that the graph G' described above has various useful properties. In Section 4 we develop techniques to produce a cycle with a long path of vertices of degree two. In Section 5 we put all of our knowledge together to prove Theorems 2 and 7 and deduce Conjecture 1.

Let us finish this section with an easy fact about ISK_4 -free graphs. Given a hole C and a vertex $v \notin C$, v is *linked* to C if there are three paths P_1, P_2, P_3 such that

- $P_1^* \cup P_2^* \cup P_3^* \cup \{v\}$ is disjoint from C ;
- each P_i has one end v and the other end in C , and there are no other edges between P_i and C ;
- for $i, j \in \{1, 2, 3\}$ with $i \neq j$, $V(P_i) \cap V(P_j) = \{v\}$;
- if $x \in P_i$ is adjacent to $y \in P_j$ then either $v \in \{x, y\}$ or $\{x, y\} \subseteq V(C)$; and
- if v has a neighbor $c \in C$, then $c \in P_i$ for some i .

Lemma 8. *If G is ISK_4 -free, then no vertex of G can be linked to a hole.*

2 Wheels

Lemma 9. *Let G be an $\{\text{ISK}_4, \text{triangle}\}$ -free graph that contains a wheel. Then there is a proper wheel in G .*

Proof. Let G be an $\{\text{ISK}_4, \text{triangle}\}$ -free graph. Let $W = (C, x)$ be a wheel in G with $|V(C)|$ minimum. We claim that W is a proper wheel. Suppose $v \in V(G) \setminus V(W)$ violates the definition of a proper wheel.

If v has at least three neighbors in the hole $x - S - x$ for some sector S of W , then $(x - S - x, v)$ is a wheel with shorter rim than W , a contradiction. So v has at most two neighbors in every sector of W (and at most one if v is adjacent to x). Therefore there exist sectors S_1, S_2 of W such that v has a neighbor in $V(S_1) \setminus V(S_2)$ and a neighbor in $V(S_2) \setminus V(S_1)$. Also by the minimality of $|V(C)|$, every path of C whose ends are in $N(v)$ and with interior disjoint from $N(v)$ contains at most two spokes of W , and we can choose S_1, S_2 and for $i = 1, 2$, label the ends of S_i as a_i, b_i such that either $b_1 = a_2$, or b_1, a_2 are the ends of a third sector S_3 of W and v has no neighbor in S_3^* . If possible, we choose S_1, S_2 such that $b_1 = a_2$. If v has two neighbors in S_1 , denote them s, t such that a_1, t, s, b_1 are in order in C . If v has a unique neighbor in S_1 , denote it by s . Let z be the neighbor of v in S_2 closest to a_2 .

Assume first that v is non-adjacent to x . Suppose $b_1 \neq a_2$. By Lemma 8, x cannot be linked to the hole $z - S_2 - a_2 - S_3 - b_1 - S_1 - s - v - z$, and it follows that $z \neq b_2$. If v has two neighbors in S_1 , then v can be linked to $x - S_3 - x$ via the paths $v - s - S_1 - b_1, v - t - S_1 - a_1 - x, v - z - S_2 - a_2$; and if v has a unique neighbor in S_1 , then s can be linked to $x - S_3 - x$ via the paths $s - S_1 - b_1, s - S_1 - a_1 - x, s - v - z - S_2 - a_2$ (note that by the choice of S_1, S_2 and since $b_1 \neq a_2$, it follows that $s \neq b_1$). In both cases, this is contrary to Lemma 8. This proves that $b_1 = a_2$. Let y be the neighbor of v in S_2 closest to b_2 . Now if v has two neighbors in S_1 , then v can be linked to $x - S_1 - x$ via the paths $v - s, v - t, v - y - S_2 - b_2 - x$, contrary to Lemma 8. So v has a unique neighbor in S_1 , and similarly a unique neighbor in S_2 . It follows that s, b_1 and z are all distinct. Now we can link x to $s - S_1 - b_1 - S_2 - z - v - s$ via the paths $x - b_1, x - a_1 - S_1 - s$, and $x - b_2 - S_2 - z$, contrary to Lemma 8.

This proves that v is adjacent to x , and so v has at most one neighbor in every sector of W . If $b_1 \neq a_2$, then v can be linked to $x - S_3 - x$ via the paths $v - s - S_1 - b_1, v - x, v - z - S_2 - a_2$, and if $b_1 = a_2$, then s, b_1 and z are all distinct and hence x can be linked to the hole $s - S_1 - b_1 - S_2 - z - v - s$ via the paths $x - b_1, x - v$, and $x - b_2 - S_2 - z$; in both cases contrary to

Lemma 8. This proves that every $v \in V(G) \setminus V(W)$ satisfies the condition in the definition, and so W is a proper wheel in G . \square

Let $W = (C, v)$ be a wheel. We call x *proper* for W if either $x \in V(C) \cup \{v\}$; or

- all neighbors of x in $V(C)$ are in one sector of W ; and
- if x has more than two neighbors in $V(C)$, then x is adjacent to v .

A vertex x is *non-offensive* for a wheel $W = (C, v)$ if there exist two sectors S_1, S_2 of W such that

- x is adjacent to v ;
- x has neighbors in S_1 and in S_2 ;
- $N(x) \cap V(C) \subseteq V(S_1) \cup V(S_2)$;
- S_1 and S_2 are consecutive; and
- if $u \in V(G) \setminus V(W)$ is adjacent to x , then $N(u) \cap V(C) \subseteq V(S_1) \cup V(S_2)$.

If $V(S_1) \cap V(S_2) = \{a\}$, we also say that x is *a-non-offensive*.

Lemma 10. *Let G be an $\{\text{ISK}_4, \text{triangle}\}$ -free graph. Let $W = (C, v)$ be a wheel in G . Let S_1, S_2 be consecutive sectors of W , and let $x \in N(v) \setminus V(W)$ be a vertex such that $N(x) \cap V(C) \subseteq V(S_1) \cup V(S_2)$ and $N(x) \cap V(S_1), N(x) \cap V(S_2) \neq \emptyset$. Then $N(x) \cap V(C) \subseteq S_1^* \cup S_2^*$ and for $\{i, j\} = \{1, 2\}$, $|N(x) \cap (V(S_i) \setminus V(S_j))| \geq 3$.*

Proof. Since x is adjacent to v and G is triangle-free, it follows that $N(x) \cap V(C) \subseteq S_1^* \cup S_2^*$. Suppose for a contradiction that $|N(x) \cap (V(S_1) \setminus V(S_2))| \leq 2$. If $|N(x) \cap (V(S_1) \setminus V(S_2))| = 2$, then x has exactly three neighbors in the hole $v - S_1 - v$, contrary to Lemma 8. It follows that $|N(x) \cap (V(S_1) \setminus V(S_2))| = 1$. Let z denote the neighbor of x in $V(S_1)$. Let $\{w\} = V(S_1) \cap V(S_2)$, and let y denote the neighbor of x in $V(S_2)$ closest to w along S_2 . Then x can be linked to the hole $v - S_1 - v$ via the three paths $x - v$, $x - z$, and $x - y - S_2 - w$. This is a contradiction to Lemma 8, and the result follows. \square

We say that wheel $W = (C, v)$ is *k-almost proper* if there are spokes x_1, \dots, x_k of W and a set $X \subseteq V(G) \setminus V(W)$ such that

- no two spokes in $\{x_1, \dots, x_k\}$ are consecutive;

- W is proper in $G \setminus X$;
- for every x in X there exists i such that x is x_i -non-offensive.

Lemma 11. *Let G be an $\{\text{ISK}_4, \text{triangle}\}$ -free graph, and let $W = (C, v)$ be a 1-almost proper wheel in G . Let x_1 and X be as in the definition of a 1-almost proper wheel, and let S_1 and S_2 be the sectors of W containing x_1 .*

Then there exists a proper wheel W' in G with center v and the same number of spokes as W . Moreover, either $W = W'$, or $V(W') \setminus V(W) = \{x^\}$ where x^* is a non-offensive vertex for W , and $V(W) \setminus V(W') \subseteq V(S_1^*) \cup V(S_2^*) \cup \{x_1\}$.*

Proof. We may assume that $X \neq \emptyset$, for otherwise W is proper in G . For $x \in X$, let $P(x)$ denote the longest path in $G[V(S_1) \cup V(S_2)]$ starting and ending in a neighbor of x . Let $x^* \in X$ be a vertex with $|V(P(x^*))|$ maximum among vertices in X , and let Y denote interior of $P(x^*)$. Let $C' = G[(V(C) \cup \{x^*\}) \setminus Y]$. It follows that $W' = (C', v)$ is a wheel. Moreover, $N(v) \cap V(C') = ((N(v) \cap V(C)) \setminus \{x_1\}) \cup \{x^*\}$, and therefore W' has the same number of spokes as W .

If W' is proper, the result follows. Therefore, we may assume that there is a vertex $y \in V(G) \setminus V(W')$ that is not proper for W' . Let S'_1, S'_2 denote the sectors of W' containing $S_1 \setminus Y$ and $S_2 \setminus Y$, respectively.

Suppose first that $y \in V(W)$, and consequently $y \in Y$. Since x^* has at least two neighbors in each of S_1 and S_2 by Lemma 10, it follows that $|V(P(x^*))| \geq 4$. Consequently, either $N(y) \cap V(C') \subseteq S'_1$ or $N(y) \cap V(C') \subseteq S'_2$. Moreover, $N(y) \cap V(C') \subseteq N[x]$, and therefore $|N(y) \cap V(C')| \leq 1$. This implies that y is proper for W , a contradiction. This proves that $y \notin V(W)$.

Next, we suppose that $y \in X$. It follows that $N(y) \cap V(C') \subseteq V(S'_1) \cup V(S'_2)$. Since y is not proper for W' , but y is adjacent to v , it follows that $N(y) \cap V(S'_1), N(y) \cap V(S'_2) \neq \emptyset$. By Lemma 10, y has at least two neighbors in S'_1 . But then $V(P(x^*)) \subsetneq V(P(y))$, a contradiction to the choice of x^* . This proves that $y \in V(G) \setminus (X \cup V(W))$.

If $y \notin N(x^*)$, then $N(y) \cap V(C') \subseteq N(y) \cap V(C)$, and since $y \notin X$, it follows that y is proper for W and thus y is proper for W' . Consequently $y \in N(x^*)$. Since x^* is non-offensive for W , it follows that $N(y) \cap V(C) \subseteq V(S_1) \cup V(S_2)$. Since y is proper for W , we may assume by symmetry that $N(y) \cap V(C) \subseteq V(S_1)$. It follows that $N(y) \cap V(C') \subseteq V(S'_1)$. Since y is adjacent to x^* and G is triangle-free, it follows that y is non-adjacent to v , and hence $|N(y) \cap V(C)| \leq 2$. This implies that $|N(y) \cap V(C')| \leq 3$, and by Lemma 8, it is impossible for y to have exactly three neighbors in C' since G is ISK_4 -free. Therefore, $|N(y) \cap V(C')| \leq 2$, and therefore y is proper

for W' . This is a contradiction, and it follows that W' is proper in G , and hence W' is the desired wheel. \square

Lemma 12. *Let G be an $\{\text{ISK}_4, \text{triangle}\}$ -free graph, and let $W = (C, v)$ be a 2-almost proper wheel in G . Then there exists a proper wheel in G with center v and at most the same number of spokes as W .*

Proof. Let x_1, x_2 and X be as in the definition of a 2-almost proper wheel, and let S_1, S_2 be the sectors of W containing x_1 . Let X_1 denote the set x_1 -non-offensive vertices in X , and let $X_2 = X \setminus X_1$. We may assume that X_1, X_2 are both non-empty, for otherwise the result follows from Lemma 11. It follows that W is 1-almost proper, but not proper, in $G \setminus X_2$. Let W', x^* be as in Lemma 11. So W' is a proper wheel in $G \setminus X_2$. If W' is 1-almost proper in G , then the result of the Lemma follows from Lemma 11. So we may assume that W' is not 1-almost proper in G . Since every vertex of $V(G) \setminus X_2$ is proper for W' , we deduce that some vertex of $x \in X_2$ is not proper and not x_2 -non-offensive for W' .

By the definition of X_1 , and since W is 2-almost proper in G , it follows that $N(x) \cap V(C)$ is contained in the sectors S_3, S_4 of W containing x_2 . Since x_1 and x_2 are not consecutive, $S_3, S_4 \notin \{S_1, S_2\}$, and so by Lemma 11, $V(S_3) \cup V(S_4) \subseteq V(W')$. Consequently, S_3 and S_4 are sectors of W' . Since G is triangle-free and every vertex in X is adjacent to v , it follows that x is not adjacent to x^* . Therefore, $N(x) \cap V(C') \subseteq V(S_3) \cup V(S_4)$, x has both a neighbor in S_3 and a neighbor in S_4 , and S_3, S_4 are the sectors of W' containing x_2 . Let s_3 denote the neighbor of x in S_3 furthest from x_2 , and let s_4 denote the neighbors of x in S_4 furthest from x_2 . We may assume that among all vertices of X_2 that are not x_2 -non-offensive for W' , x is chosen so that the path of C from s_3 to s_4 containing x_2 is maximal.

Since x is not x_2 -non-offensive for W' , there exists a vertex $u \in N(x) \setminus V(W')$ with a neighbor in $V(C') \setminus (V(S_3) \cup V(S_4))$. Since x is x_2 -non-offensive for W , it follows that u has a neighbor in $V(C') \setminus V(C) = \{x^*\}$, and so u is adjacent to x and x^* .

Since x and x^* are non-offensive for W , it follows that $N(u) \cap V(C) \subseteq (V(S_1) \cup V(S_2)) \cap (V(S_3) \cup V(S_4))$. Since G is triangle-free, u is non-adjacent to v , and therefore $u \notin X$. Consequently, u is proper for W , and all the neighbors of u in C belong to one sector of W . It follows that u has at most one neighbor in $V(C)$. Suppose that u has exactly one neighbor in $V(C)$. Then u has three neighbors in the cycle arising from C' by replacing $s_3 - S_3 - x_2 - S_4 - s_4$ by $s_3 - x - s_4$, contrary to Lemma 8. It follows that u has no neighbors in $V(C)$.

Let P'_1 denote the path of C' from s_3 to x^* not containing x_2 , and let P_1 be $x - s_3 - P'_1 - x^*$. Let P'_2 denote the path of C' from s_4 to x^* not containing x_2 , and let P_2 be $x - s_4 - P'_2 - x^*$. Let $D = G[V(P_1) \cup \{u\}]$. Since x_1 and x_2 are not consecutive, each of P_1^*, P_2^* contains at least one neighbor of v , and so $W'' = (D, v)$ is a wheel with fewer spokes than W . Let S'_3 denote the sector of W'' containing x but not containing u . If W'' is proper in G , then the result follows. Therefore, we may assume that there is a vertex $y \in V(G) \setminus V(W'')$ that is not proper for W'' .

Since every vertex in $V(W') \setminus V(W'')$ and every vertex in $V(W) \setminus V(W'')$ has at most one neighbor in $V(W'')$, it follows that $y \notin V(W) \cup V(W')$. Suppose that $y \notin N(u)$. If $y \in X_2$, then $N(y) \cap V(D) \subseteq (V(S_3) \cup \{x\}) \cap V(D)$, and so y is proper for W'' , since y is adjacent to v , a contradiction. Thus $y \notin X_2$, and so y is proper for W' . If $y \notin N(x)$, then $N(y) \cap V(D) \subseteq N(y) \cap V(C')$, and again y is proper for W'' , a contradiction. Thus $y \in N(x)$, but since x is non-offensive for W , $N(y) \cap V(C) \subseteq V(S_3) \cup V(S_4)$, and so $N(y) \cap V(D) \subseteq V(S'_3)$. Since y is not proper for W'' , y is non-adjacent to v and has at least three neighbors in S'_3 . But y is proper for W' , and so y has at most two neighbors in S_3 ; thus y has exactly three neighbors in S'_3 and hence in D contrary to Lemma 8. This contradiction implies that $y \in N(u)$.

Since y is not proper for W'' , it follows that y has a neighbor in P_1^* , and since G is triangle-free, it follows that y is non-adjacent to x, x^* . We claim that y has no neighbor in P_2 . Suppose that it does. If $y \in X_2$, then, since y is adjacent to u and has a neighbor in P_1^* , we deduce that y is not x_2 -non-offensive for W' , and the claim follows from the maximality of the path of C from s_3 to s_4 containing x_2 . Thus we may assume that $y \notin X_2$. Consequently, y is proper for W' , a contradiction. This proves the claim.

Let z_1 be the neighbor of y in $V(P_1)$ closest to x along P_1 , and let z_2 be the neighbor of y in $V(P_1)$ closest to x^* along P_1 . Let D' be the hole $x^* - P_2 - x - u - x^*$. If $z_1 \neq z_2$, we can link y to D' via the paths $y - z_1 - P_1 - x$, $y - z_2 - P_1 - x^*$ and $y - u$, and if $z_1 = z_2$, then we can link z_1 to D' via the paths $z_1 - P_1 - x$, $z_1 - P_1 - x^*$ and $z_1 - y - u$, in both cases contrary to Lemma 8. This proves Lemma 12. \square

Throughout the remainder of this section G is an $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph, and $W = (C, x)$ is a proper wheel in G with minimum number of spokes (subject to having center x).

Lemma 13. *Let $P = p_1 - \dots - p_k$ be a path such that p_1, p_k have neighbors in $V(C)$, $V(P) \subseteq V(G) \setminus V(W)$, and there are no edges between P^* and $V(C)$. Assume that no sector of W contains $(N(p_1) \cup N(p_k)) \cap V(C)$. For $i \in \{1, k\}$, if x is non-adjacent to p_i , then p_i has a unique neighbor in C .*

Proof. Let S_1, S_2 be distinct sectors of W such that $N(p_1) \cap V(C) \subseteq V(S_1)$, and $N(p_k) \cap V(C) \subseteq V(S_2)$. We may assume that p_1 is non-adjacent to x , and so p_1 has at most two neighbors in C . Since p_1 cannot be linked to the hole C (or the hole obtained from C by rerouting S_2 through p_k) via two one-edge paths and P , it follows that p_1 has a unique neighbor in C . \square

Theorem 14. *Let $P = p_1 - \dots - p_k$ be a path with $V(P) \subseteq V(G) \setminus V(W)$ such that x has at most one neighbor in P .*

1. *If P contains no neighbor of x , then there is a sector S of W such that every edge from P to C has an end in $V(S)$.*
2. *If P contains exactly one neighbor of x , then there are two sectors S_1, S_2 of W such that $V(S_1) \cap V(S_2) \neq \emptyset$, and every edge from P to C has an end in $V(S_1) \cup V(S_2)$ (where possibly $S_1 = S_2$).*

Proof. Let P be a path violating the assertions of the theorem and assume that P is chosen with k minimum. Since W is proper, it follows that $k > 1$. Our first goal is to show that x has a neighbor in $V(P)$.

Suppose that x is anticomplete to $V(P)$. Then, by the minimality of k , there exist two sectors S_1, S_2 of W such that every edge from $\{p_1, \dots, p_{k-1}\}$ to $V(C)$ has an end in $V(S_1)$, and every edge from $\{p_2, \dots, p_k\}$ to $V(C)$ has an end in $V(S_2)$. It follows that $S_1 \neq S_2$. Then p_1 has a neighbor in $V(S_1) \setminus V(S_2)$, and p_k has a neighbor in $V(S_2) \setminus V(S_1)$, and every edge from P^* to $V(C)$ has an end in $V(S_1) \cap V(S_2)$. For $i = 1, 2$ let a_i, b_i be the ends of S_i . We may assume that a_1, b_1, a_2, b_2 appear in C in this order and that $a_1 \neq b_2$. Let Q_1 be the path of C from b_2 to a_1 not using b_1 , and let Q_2 be the path of C from b_1 to a_2 not using a_1 . We can choose S_1, S_2 with $|V(Q_2)|$ minimum (without changing P). Let s be the neighbor of p_1 in S_1 closest to a_1 , t the neighbor of p_1 in S_1 closest to b_1 , y the neighbor of p_k in S_2 closest to a_2 and z the neighbor of p_k in S_2 closest to b_2 . Then $s \neq b_1$ and $z \neq a_2$. It follows that $V(Q_2) \cap \{s, z\} = \emptyset$. Moreover, if $V(S_1) \cap V(S_2) \neq \emptyset$, then $b_1 = a_2$ and $V(Q_2) = \{b_1\}$, and in all cases $V(Q_1)$ is anticomplete to P^* .

Now $D_1 = s - p_1 - P - p_k - z - S_2 - b_2 - Q_1 - a_1 - S_1 - s$ is a hole.

- (1) $W_1 = (D_1, x)$ is a wheel with fewer spokes than W .

Since $V(Q_2) \cap V(D_1) = \emptyset$ and $V(Q_2)$ contains a neighbor of x , it follows that x has fewer neighbors in D_1 than it does in C . It now suffices to show that x has at least three neighbors in Q_1 . Since $a_1, b_2 \in V(Q_1)$, we may assume that x has no neighbor in Q_1^* , and Q_1 is a sector of W . Since not every edge between $V(P)$ and $V(C)$ has an end in $V(Q_1)$, it follows that

$t \neq a_1$ or $y \neq b_2$. By symmetry, we may assume that $t \neq a_1$. Since x cannot be linked to W by Lemma 8, it follows that x has at least four neighbors in $V(C)$, and therefore $V(S_1) \cap V(S_2) = \emptyset$. Consequently, P^* is anticomplete to $V(C)$. It follows from Lemma 13 that $s = t$. Now we can link s to the hole $a_1 - Q_1 - b_2 - x - a_1$ via the paths $s - S_1 - a_1$, $s - p_1 - P - p_k - z - S_2 - b_2$ and $s - S_1 - b_1 - x$, contrary to Lemma 8. This proves that $W_1 = (D_1, x)$ is a wheel with fewer spokes than W . This proves (1).

By the choice of W , it follows from (1) that W_1 is not proper. Let S_0 be the sector $a_1 - S_1 - s - p_1 - P - p_k - z - S_2 - b_2$ of (D_1, x) . Since W is a proper wheel and W_1 is not a proper wheel, we have that:

There exists $v \in V(G) \setminus V(W_1)$ such that either

- (2)
- *v is non-adjacent to x , and v has at least three neighbors in S_0 and $N(v) \cap V(D_1) \subseteq V(S_0)$, or*
 - *there is a sector S_3 of W with $V(S_3) \subseteq V(Q_1)$, such that v has a neighbor in $V(S_3) \setminus V(S_0)$ and a neighbor in $V(S_0) \setminus V(S_3)$, and $N(v) \cap V(C) \subseteq V(S_3)$.*

Let v be as in (2). First we show that $v \notin V(C)$. The only vertices of C that may have more than one neighbor in D_1 are b_1 and a_2 , and that only happens if $b_1 = a_2$. But $N(b_1) \cap V(D_1) \subseteq V(S_0)$ and b_1 is adjacent to x , so b_1 does not satisfy the conditions described in the bullets. Thus $v \notin V(C)$.

- (3) *v has a unique neighbor in P .*

If the first case of (2) holds, then the statement of (3) follows immediately from the minimality of k (since v is non-adjacent to x), and so we may assume that the second case of (2) holds. Observe that no vertex of $V(Q_1)$ is contained both in a sector with end a_1 and in a sector with end b_2 , and therefore we may assume that v has a neighbor in a sector that does not have end b_2 . If v is non-adjacent to x , we get a contradiction to the minimality of k . So we may assume that v is adjacent to x , and therefore v has a neighbor in S_3^* , and $b_2 \notin V(S_3)$. Let S_4 be the sector of D_1 such that $b_2 \in V(S_4)$ and $V(S_4) \subseteq V(Q_1)$. Suppose that $y \neq b_2$ or $V(S_3) \cap V(S_4) = \emptyset$. Let $i \in \{1, \dots, k\}$ be maximum such that v is adjacent to p_i . Now the path $v - p_i - P - p_k$ violates the assertions of the theorem, and so it follows from the minimality of k that $N(v) \cap V(P) \subseteq \{p_1, p_2\}$. Therefore, since G is triangle-free, it follows that v has a unique neighbor in P , and (3)

holds. So we may assume that $y = b_2$ and there exists $a_3 \in V(C)$ such that $V(S_4) \cap V(S_3) = \{a_3\}$. Let R be the path from v to a_3 with $R^* \subseteq S_3^*$. Now we can link v to $x - S_4 - x$ via the paths $v - x$, $v - R - a_3$ and $v - p_i - P - p_k - b_2$, where i is maximum such that v is adjacent to p_i , contrary to Lemma 8. This proves (3).

In view of (3) let $N(v) \cap V(P) = \{p_j\}$. In the case of the first bullet of (2), since v cannot be linked to the hole $x - S_0 - x$ by Lemma 8, it follows that v has at least four neighbors in S_0 , and therefore at least three neighbors in $V(S_1) \cup V(S_2)$, contrary to the fact that W is proper. So the case of the second bullet of (2) holds. Since W is proper $N(v) \cap (V(S_0) \setminus V(S_3)) \subseteq V(P)$, and $N(v) \cap V(D_1) \subseteq V(S_0) \cup V(S_3)$.

(4) *There are edges between P^* and $V(C)$.*

Suppose not. By Lemma 13, $s = t$ and $y = z$. We claim that in this case $b_1 \neq a_2$, for if $b_1 = a_2$, then b_1 can be linked to the hole $x - a_1 - S_1 - s - p_1 - P - p_k - z - S_2 - b_2 - x$ via the paths $b_1 - x$, $b_1 - S_1 - s$ and $b_1 - S_2 - z$, contrary to Lemma 8. If v has a unique neighbor r in C , then p_j can be linked to C via the paths $p_j - P - p_1 - s$, $p_j - P - p_k - z$ and $p_j - v - r$, contrary to Lemma 8, so v has at least two neighbors in C . Recall that $N(v) \cap V(C) \subseteq V(S_3)$. Let D be the hole obtained from C by rerouting S_3 through v . Then $s, z \in V(D)$, and p_j can be linked to D via the paths $p_j - P - p_1 - s$, $p_j - P - p_k - z$ and $p_j - v$, contrary to Lemma 8. This proves (4).

It follows from (4) that $b_1 = a_2$ and b_1 has neighbors in P^* . Now, by considering the path from a neighbor of b_1 in P^* to v with interior in P^* if v has a neighbor in P^* , and the paths $v - p_1$ or $v - p_k$ if v has no neighbor in P^* , the minimality of k implies that v is adjacent to x and one of a_1, b_2 belongs to S_3 .

By symmetry we may assume $a_1 \in V(S_3)$. Let R be the path from v to a_1 with $R^* \subseteq V(S_3)$. Now x can be linked to the hole $v - R - a_1 - S_1 - s - p_1 - P - p_j - v$ via the paths $x - v$, $x - a_1$ and $x - b_2 - S_2 - z - p_k - P - p_j$, contrary to Lemma 8.

In summary, we have now proved:

(5) *If P' is a path violating the assertion of the theorem and $|V(P')| = k$, then x has a neighbor in $V(P')$.*

By (5), x has a neighbor in $V(P)$, say x is adjacent to p_i . Then p_i is the unique neighbor of x in $V(P)$. By the minimality of k , there exist two

distinct sectors S_1, S_2 of W such that p_1 has a neighbor in $V(S_1) \setminus V(S_2)$, and p_k has a neighbor in $V(S_2) \setminus V(S_1)$. By (5), if $1 < i < j$, then every edge from $\{p_1, \dots, p_{i-1}\}$ to $V(C)$ has an end in $V(S_1)$, and every edge from $\{p_{i+1}, \dots, p_k\}$ to $V(C)$ has an end in $V(S_2)$; if $i = 1$ then every edge from $V(P) \setminus \{p_1\}$ to $V(C)$ has an end in $V(S_2)$; and if $i = k$ then every edge from $V(P) \setminus \{p_k\}$ to $V(C)$ has an end in $V(S_1)$.

For $j = 1, 2$, let a_j, b_j be the ends of S_j .

One of the following statements holds:

- *there are no edges between $V(C)$ and P^* , or*
- (6)
 - *we can choose S_1, S_2 such that a_1, b_1, a_2, b_2 appear in C in order and there is a sector S_3 with ends b_1, a_2 , and every edge between $V(C)$ and P^* is from b_1 to $\{p_2, \dots, p_{i-1}\}$ or from p_i to S_3^* , or from a_2 to $\{p_{i+1}, \dots, p_{k-1}\}$.*

Suppose (6) is false. It follows that there are edges between P^* and $V(C)$. Since G is triangle-free, p_i is anticomplete to $N(x) \cap V(C)$. Suppose that there is sector S_3 of W and an edge from S_3^* to P^* . By the minimality of k we deduce that $S_3 \notin \{S_1, S_2\}$, $1 < i < k$ and p_i has a neighbor in S_3^* . Again by the minimality of k it follows that there exist sectors S'_1, S'_2 such that $V(S'_j) \cap V(S_3) \neq \emptyset$ for $j = 1, 2$ and every edge from $\{p_1, \dots, p_{i-1}\}$ to C has an end in S'_1 , and every edge from $\{p_{i+1}, \dots, p_k\}$ to C has an end in S'_2 . Now we can choose $S_1 = S'_1$ and $S_2 = S'_2$. We may assume that a_1, b_1, a_2, b_2 appear in C in this order, and so b_1 and a_2 are the ends of S_3 . Since p_i has a neighbor in S_3^* , the minimality of k implies that $\{p_2, \dots, p_i\}$ is anticomplete to $V(S_1) \setminus \{b_1\}$, and $\{p_i, \dots, p_{k-1}\}$ is anticomplete to $V(S_2) \setminus \{a_2\}$, and the second bullet is satisfied. So P^* is anticomplete to $V(C) \setminus N(x)$. Since there are edges between P^* and $V(C)$, and since p_i is anticomplete to $N(x) \cap V(C)$, by symmetry we may assume that there is an edge between $\{p_2, \dots, p_{i-1}\}$ and $t \in N(x) \cap V(C)$. Then $t \in V(S_1)$. Let S_3 be the other sector of W incident with t . By the minimality of k it follows that S_2 can be chosen so that $V(S_3) \cap V(S_2) \neq \emptyset$, and again the case of the second bullet holds. This proves (6).

If the second bullet of (6) holds, let Q_1 be the path of C from b_2 to a_1 not using b_1 , and let $Q_2 = S_3$. To define Q_1 and Q_2 , let us now assume that the case of the first bullet holds. We may assume that a_1, b_1, a_2, b_2 appear in C in this order. Also, a_1, b_1, a_2, b_2 are all distinct, since P violates the assertion of the theorem. Let Q_1 be the path of C from b_2 to a_1 not using

b_1 , and let Q_2 be the path of C from b_1 to a_2 not using a_1 . We may assume that S_1, S_2 are chosen with $|V(Q_2)|$ minimum (without changing P).

Since W is proper, it follows that $N(p_1) \cap V(C) \subseteq V(S_1)$ and $N(p_k) \cap V(C) \subseteq V(S_2)$. Let s be the neighbor of p_1 in S_1 closest to a_1 , t the neighbor of p_1 in S_1 closest to b_1 , y the neighbor of p_k in S_2 closest to a_2 and z the neighbor of p_k in S_2 closest to b_2 . Then $s \neq b_1$ and $z \neq a_2$.

Let D_1 be the hole $a_1 - S_1 - s - p_1 - P - p_k - z - S_2 - b_2 - Q_1 - a_1$. Then $W_1 = (D_1, x)$ is a wheel with fewer spokes than W . We may assume that (subject to the minimality of k) P was chosen so that $V(Q_1)$ is (inclusion-wise) minimal. By Lemma 8, x has a neighbor in $V(D_1) \setminus \{a_1, b_1, p_i\}$, and so x has a neighbor in Q_1^* .

Let S_0 be the sector $a_1 - S_1 - s - p_1 - P - p_i$, and let T_0 be the sector $p_i - P - p_k - z - b_2$ of (D_1, x) .

- (7) *No vertex $v \in V(G) \setminus V(W_1)$ has both a neighbor in $V(S_0) \setminus V(T_0)$ and a neighbor in $V(T_0) \setminus V(S_0)$.*

Suppose (7) is false, and let $v \in V(G) \setminus V(W_1)$ be such that v has a neighbor in $V(S_0) \setminus V(T_0)$ and a neighbor in $V(T_0) \setminus V(S_0)$.

First we claim that v is adjacent to x . Suppose v has a neighbor in $V(a_1 - S_1 - s)$. Since W is proper and $a_1, s \notin V(S_2)$ (because P violates the statement of the theorem), it follows that v has no neighbor in $V(z - S_2 - b_2)$. Consequently v has a neighbor in $V(T_0) \setminus (V(S_2) \cup V(S_0))$. Let j be maximum such that v is adjacent to p_j , then $j > i$. Now applying (5) to the path $v - p_j - P - p_k$ we deduce that v is adjacent to x , as required. Thus we may assume that $N(v) \cap (V(S_0) \cup V(T_0)) \subseteq V(P)$. Let j be minimum and l maximum such that v is adjacent to p_j, p_l . Then $j < i$ and $l > i$. Applying (5) to the path $p_1 - P - p_j - v - p_l - P - p_k$, we again deduce that x is adjacent to v . This proves the claim.

In view of the claim, Lemma 10 implies that v has at least two neighbors in $V(T_0) \setminus V(S_0)$ and at least two neighbors in $V(S_0) \setminus V(T_0)$. But now, rerouting P through v (as in the previous paragraph), we get a contradiction to the minimality of k . This proves (7).

- (8) *Every non-offensive vertex for W_1 is either a_1 -non-offensive or b_2 -non-offensive.*

Let v be a non-offensive vertex for W_1 . Since W is proper, it follows that $N(v) \cap V(C)$ is included in a unique sector of W . Consequently, v

is either a_1 -non-offensive, or b_2 -non-offensive, or p_i non-offensive. However, (7) implies that v is not p_i -non-offensive, and (8) follows.

Let X be the set of all non-offensive vertices for W_1 . It follows from Lemma 12 that W_1 is not proper in $V(G) \setminus X$.

There exists $v \in V(G) \setminus (V(W_1) \cup X)$ such that one of the following holds:

- v is non-adjacent to x , and v has at least three neighbors in S_0 , and $N(v) \cap V(D_1) \subseteq V(S_0)$.
 - v is non-adjacent to x , and v has at least three neighbors in T_0 , and $N(v) \cap V(D_1) \subseteq V(T_0)$.
- (9)
- v has a neighbor in $V(S_0) \setminus V(T_0)$ and a neighbor in $V(T_0) \setminus V(S_0)$, and $N(v) \cap V(D_1) \subseteq V(S_0) \cup V(T_0)$.
 - (possibly with the roles of S_0 and T_0 exchanged) there is a sector S_4 of W with $V(S_4) \subseteq V(Q_1)$ such that v has a neighbor in $V(S_4) \setminus (V(S_0) \cup V(T_0))$, v has a neighbor in $V(S_0) \setminus V(S_4)$, v does not have a neighbor in $V(T_0) \setminus (V(S_0) \cup V(S_4))$, and $N(v) \cap V(C) \subseteq V(S_4)$.

We may assume that the first three bullets of (9) do not hold. Since W is proper and W_1 is not, (possibly switching the roles of S_0 and T_0) there exists $v \in V(G) \setminus V(W_1)$ and a sector S_4 of W with $V(S_4) \subseteq V(Q_1)$, such that v has a neighbor in $V(S_4) \setminus V(S_0)$, v has a neighbor in $V(S_0) \setminus V(S_4)$, and $N(v) \cap V(C) \subseteq V(S_4)$. But now (7) implies that the last bullet of (9) holds. This proves (9).

Let $v \in V(G)$ be as in (9). Next we show that:

- (10) v has a unique neighbor in $V(P)$.

Suppose that v has at least two neighbors in P . In the first two cases of (9) we get a contradiction to the minimality of k . The third case is impossible by (7). Thus we may assume that the case of the fourth bullet of (9) holds. We may assume that $N(v) \cap V(P) \subseteq V(S_0)$, and in particular v has a neighbor in $\{p_1, \dots, p_{i-1}\}$. Suppose first that v is non-adjacent to x . Since v has a neighbor in $V(S_4) \setminus V(S_0)$, the minimality of k implies that $t = a_1$ and $a_1 \in V(S_4)$, and also that $b_2 \in V(S_4)$, contrary to the fact that x

has a neighbor in Q_1^* . So v is adjacent to x , and therefore v has a neighbor in S_0^* .

Since W is proper, $N(v) \cap (V(S_0) \setminus V(S_4)) \subseteq V(P)$. Let Q be the path from v to p_1 with $Q^* \subseteq V(P)$. Suppose first that $a_1 \notin V(S_4)$. Let S_5 be the sector of W with end a_1 and such that $V(S_5) \subseteq V(Q_1)$, and let b_3 be the other end of S_5 . Since Q is shorter than P , it follows from the minimality of k that $V(S_4) \cap V(S_5) = \{b_3\}$ and $t = a_1$. Let R be the path from v to b_3 with $R^* \subseteq S_4^*$. Then x has exactly three neighbors in the hole $v - R - b_3 - S_5 - a_1 - p_1 - Q - v$, contrary to Lemma 8. This proves that $a_1 \in V(S_4)$.

Let b_3 be the other end of S_4 , let S_5 be the second sector of W incident with b_3 , and let a_3 be the other end of S_5 . Since $v \notin X$, it follows that v has a neighbor $u \in V(G) \setminus V(W_1)$ such that u has a neighbor in $V(D_1) \setminus (V(S_4) \cup V(S_0))$. Since G is triangle-free, u is non-adjacent to x .

Suppose first that u has a neighbor in $V(Q_1) \setminus V(S_4)$. Since G is triangle-free and v has at least two neighbors in $V(P)$, it follows that $i \geq 4$, and therefore $k \geq 4$. Consequently, the path $u - v$ is shorter than P , and so it follows from the minimality of k that $N(u) \cap V(C) \subseteq V(S_5)$. Let R be the path from v to b_3 with $R^* \subseteq S_4^*$, and let D_2 be the hole $v - R - b_3 - x - v$. Let p be the neighbor of u in $V(S_5)$ closest to b_3 , and let q be the neighbor of u in $V(S_5)$ closest to a_3 . If $p \neq q$, we can link u to D_2 via the paths $u - p - S_5 - b_3$, $u - q - S_5 - a_3 - x$ and $u - v$, and if $p = q$ we can link p to D_2 via the paths $p - u - v$, $p - S_5 - b_3$ and $p - S_5 - a_3 - x$, in both cases contrary to Lemma 8. This proves that u has no neighbor in $V(Q_1) \setminus V(S_4)$, and therefore u has a neighbor in $V(T_0) \setminus V(S_0)$.

Next we define a new path Q . If u has a neighbor in $V(T_0) \cap V(S_2)$, let Q be the path $u - v$. If u is anticomplete to $V(T_0) \cap V(S_2)$, let j be maximum such that u is adjacent to p_j ; then $j > i$; let Q be the path $v - u - p_j - P - p_k$. Since $i > 4$, in both cases $|V(Q)| < k$ and x has a unique neighbor in $V(Q)$. It follows from the minimality of k that $z = y = b_2 = a_3$. Since P violates the theorem, it follows that p_1 has a neighbor in $V(S_1) \setminus \{a_1\}$.

Let T be the path from v to a_1 with $T^* \subseteq V(S_4)$. Suppose that $s \neq t$. Let D_3 be the hole $x - a_1 - S_1 - s - p_1 - t - S_1 - b_1 - x$. Now v can be linked to D_3 via the paths $v - x$, $v - Q - p_1$ (short-cutting through a neighbor of b_1 if possible) and $v - T - a_1$, contrary to Lemma 8. Thus $s = t$, and therefore $s \neq a_1$. Now we can link v to $x - S_1 - x$ via the paths $v - x$, $v - Q - p_1 - s$

(short-cutting through a neighbor of b_1 if possible) and $v - T - a_1$, contrary to Lemma 8. This proves (10).

In view of (10) let p_j be the unique neighbor of v in $V(P)$.

(11) *The fourth case of (9) holds.*

Suppose first that the case of the first bullet of (9) happens. Then by Lemma 8 v has at least four neighbors in the hole $x - S_0 - x$, and so, in view of (10), x has at least three neighbors in the path $a_1 - S_1 - s$, contrary to the fact that W is proper. By symmetry it follows that the cases of first two bullets of (9) do not happen. Suppose that the case of the third bullet of (9) happens. Since by (10) v has a unique neighbor in $V(P)$, it follows that v has a neighbor in $(V(S_0) \cup V(T_0)) \setminus V(P)$. By symmetry we may assume that v has a neighbor in $z - S_2 - b_2$, and, since W is proper, v is anticomplete to $V(S_0) \setminus V(P)$. Consequently, $p_j \in V(S_0) \setminus V(T_0)$, and so $j < i$. By the minimality of k (applied to the path $p_1 - P - p_j - v$), it follows that $j = k - 1$, and therefore $i = k$. Then $\{v, p_k\}$ is anticomplete to $V(C) \setminus V(S_2)$, since W is proper. By (5) v is adjacent to x . But now we get a contradiction to Lemma 10 applied to v and W_1 . This proves (11).

In the next claim we further restrict the structure of P .

One of the following statements holds:

- *there are edges between P^* and $V(C)$, or*
- (12)
- *$j = 1$ and we can choose S_4 so that $a_1 \in V(S_4)$, or*
 - *$j = k$ and we can choose S_4 so that $b_2 \in V(S_4)$.*

Suppose that (12) is false. Assume first that $j \notin \{1, k\}$. Then p_j is anticomplete to $V(C)$, since by assumption, there are no edges between P^* and C . If $s = t$, $y = z$ and v has a unique neighbor r in S_4 , then $r \in V(S_4) \setminus (V(S_1) \cup V(S_2))$, and p_j can be linked to C via the paths $p_j - P - p_k - z$, $p_j - v - r$ and $p_j - P - p_1 - s$, contrary to Lemma 8. If some of p_1, p_k, v have several neighbors in C , then similar linkages work for the holes obtained from C by rerouting S_1 through p_1 , S_2 through p_k , and S_4 through v , respectively. This proves that $j \in \{1, k\}$, and by symmetry may assume that $j = 1$. Then S_4 cannot be chosen so that $a_1 \notin V(S_4)$, for otherwise (12) holds. By the minimality of k and by (5), since S_4 cannot be chosen so that $a_1 \in V(S_4)$, it follows that x is adjacent to one of p_1, v and

$k = 2$. Since G is triangle-free, x has exactly one neighbor in $\{p_1, v\}$. Let R be the path from v to a_1 with $R^* \subseteq V(C) \setminus \{b_1\}$. Let Q'_1 be the subpath of R from an end of S_4 to a_1 . Then $V(Q'_1) \subseteq V(Q_1)$ and $b_2 \notin V(Q'_1)$, and so the path $p_1 - v$ contradicts the choice of P . This proves (12).

The goal of the next two claims is to obtain more information about i and j .

(13) $i = j$.

Suppose not; by symmetry we may assume that $j < i$. Suppose first that x is non-adjacent to v . By (5) and the minimality of k , it follows that the first assertion of the theorem holds for the path $p_1 - P - p_j - v$; therefore $a_1 = t$ and S_4 can be chosen so that $a_1 \in V(S_4)$. Since W is proper it follows that v has at most two neighbors in S_4 . If v has exactly two neighbors, then, in view of (6), v can be linked to $x - S_4 - x$ via two one-edge paths and the path $v - p_j - P - p_i - x$, contrary to Lemma 8. Therefore v has a unique neighbor r in S_4 . Now, again in view of (6), p_j can be linked to $x - S_4 - x$ via the paths $p_j - v - r$, $p_j - P - p_1 - a_1$ and $p_j - P - p_i - x$, again contrary to Lemma 8. This proves that v is adjacent to x , and, since G is triangle-free, v has a neighbor in S_4^* . It follows that the choice of S_4 is unique. Let R be the path from v to a_1 with $R^* \subseteq V(C) \setminus \{b_1\}$. Suppose $a_1 \in V(S_4)$. Then $R^* \subseteq S_4^*$. In this case, because of (6) and since $b_1 \neq s$, p_j can be linked to the hole $v - R - a_1 - x - v$ via the path $p_j - v$, $p_j - P - p_1 - s - S_1 - a_1$ and $p_j - P - p_i - x$, contrary to Lemma 8. Thus $a_1 \notin V(S_4)$. Let S_5 be the sector of W with end a_1 such that $V(S_5) \subseteq V(Q_1)$. If $t = a_1$ and $V(S_4) \cap V(S_5) \neq \emptyset$, then x has exactly three neighbors in the hole $v - R - a_1 - p_1 - P - p_j - v$, contrary to Lemma 8. Therefore the path $p_1 - P - p_j - v$ violates the assertion of the theorem, and so the minimality of k implies that $j = k - 1$ and consequently $i = k$. Then by (6) a_2 is anticomplete to $V(P) \setminus \{p_k\}$. Since $j \neq k$ and $a_1 \notin V(S_4)$ (the choice of S_4 is now unique), it follows from (12) that there are edges between P^* and $V(C)$. Now by (6) there is a sector S_3 of W with ends a_2, b_1 , and b_1 has a neighbor in P^* . Then there is a path T from b_1 to p_k with $T^* \subseteq P^*$, $b_1 - S_3 - a_2 - S_2 - y - p_k - T - b_1$ is a hole and x has exactly three neighbors in it, contrary to Lemma 8 (observe that $y \neq b_2$ because G has no triangles). This proves (13).

Since G is triangle-free, (13) implies that x is non-adjacent to v .

(14) $i \leq 2$ and $i \geq k - 1$.

Suppose (14) is false. By symmetry we may assume that $k - i > 1$. Consequently $k > 2$. Suppose that S_4 can be chosen so that $a_1 \in V(S_4)$. If v has a unique neighbor r in $V(S_4)$, then, since $s \neq b_1$, p_i can be linked to $x - S_4 - x$ via the paths $p_i - v - r$, $p_i - x$ and $p_i - P - p_1 - s - S_1 - a_1$, a contradiction. Thus v has at least two neighbors in $V(S_4)$. Now, again using the fact that $s \neq b_1$, p_i can be linked to the hole obtained from $x - S_4 - x$ by rerouting S_4 through v via the paths $p_i - v$, $p_i - x$ and $p_i - P - p_1 - s - S_1 - a_1$, again contrary to Lemma 8. Thus S_4 cannot be chosen so that $a_1 \in V(S_4)$. Let S_5 be the sector of W with end a_1 such that $V(S_5) \subseteq V(Q_1)$. Since $i \leq k - 2$, the minimality of k applied to the path $p_1 - P - p_i - v$ implies that $t = a_1$ and $V(S_4) \cap V(S_5) \neq \emptyset$. In particular, $i \neq 1$. It follows from (12) that there are edges between P^* and $V(C)$, and by (6) there is a sector S_3 of W with ends b_1, a_2 and every edge from p_i to $V(C)$ have an end in S_3^* . Together with the minimality of k (using the path $p_i - v$), this implies that p_i is anticomplete to $V(C)$. If v has a unique neighbor r in S_4 (and therefore $r \neq b_2$) and p_k has a unique neighbor in S_2 , then p_i can be linked to C via the paths $p_i - v - r$, $p_i - P - p_1 - a_1$ (short-cutting through neighbors of b_1 if possible), and $p_i - P - p_k - z$ (short-cutting through neighbors of a_2 if possible). If v has at least two neighbors in $V(S_4)$ or p_k has at least two neighbors in $V(S_2)$, then the same linkage works rerouting S_4 through v , and S_2 through p_k , respectively. This proves (14).

It follows from (13) and (14) that either

- $k = 3$ and $i = j = 2$, or
- $k = 2$.

If $k = 3$ and $i = j = 2$, then by (12) there are edges between P^* and $V(C)$, and so by (6) there is a sector S_3 with ends a_2, b_1 , so that p_2 has neighbors in S_3^* ; now the path $p_2 - v$ contradicts the minimality of k . Thus $k = 2$, and we may assume that $i = 1$, by symmetry. Since G is triangle-free, it follows that p_1 is non-adjacent to a_1, b_1 . Since now $P^* = \emptyset$ is anticomplete to $V(C)$, it follows from (12) that we can choose S_4 with $a_1 \in V(S_4)$. Since v is non-adjacent to x and W is proper, it follows that v has at most two neighbors in S_4 . If v has exactly two neighbors in S_4 , then v can be linked to the hole $x - S_4 - x$ via two one-edge paths, and the path $v - p_1 - x$, contrary to Lemma 8. Thus v has a unique neighbor r in $V(S_4)$. Now p_1 can be linked to $x - S_4 - x$ via the paths $p_1 - v - r$, $p_1 - x$ and $p_1 - s - S_1 - a_1$, again contrary to Lemma 8. This completes the proof of Theorem 14. \square

We can now prove Theorem 6 which we restate:

Theorem 15. *Let G be an $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph, and let x be the center of a proper wheel in G . If $W = (C, x)$ is a proper wheel with a minimum number of spokes subject to having center x , then*

1. *every component of $V(G) \setminus N(x)$ contains the interior of at most one sector of W , and*
2. *for every $u \in N(x)$, the component D of $V(G) \setminus (N(x) \setminus \{u\})$ such that $u \in V(D)$ contains the interiors of at most two sectors of W , and if S_1, S_2 are sectors with $S_i^* \subseteq V(D)$ for $i = 1, 2$, then $V(S_1) \cap V(S_2) \neq \emptyset$.*

Proof. To prove the first statement, we observe that if some component of $V(G) \setminus N(x)$ contains the interiors of two sectors of W , then this component contains a path violating the first assertions of Theorem 14.

For the second statement, suppose D contains the interiors of two disjoint sectors S_1, S_2 of W . Since $|D \cap N(x)| = 1$, we get a path in D violating the second assertion of Theorem 14. This proves Theorem 15. \square

3 Proper Wheel Centers

In the proof of our main theorem, we perform manipulations on $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graphs; in this section, we show that this preserves being $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free, and that no vertex becomes the center of a proper wheel.

Lemma 16. *Let G be an $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph, $s \in V(G)$, K a component of $G \setminus N[s]$, and N the set of vertices in $N(s)$ with a neighbor in K . Let $H = G[(V(K) \cup N \cup \{s\})]$. Then s is not the center of a proper wheel in H , and for $v \in V(H) \setminus \{s\}$, if v is the center of a proper wheel in H , then v is the center of a proper wheel in G .*

Proof. Since $H \setminus N[s]$ is connected, it follows that s is not the center of a proper wheel in H by Theorem 15. Let $v \in V(H) \setminus \{s\}$ be the center of a proper wheel $W = (C, v)$ in H . For all $w \in V(G) \setminus V(H)$, $N(w) \cap V(C) \subseteq N[s]$, and since G is triangle-free, it follows that every vertex $w \in V(G) \setminus V(H)$ either has at most one neighbor in $V(C)$, or $N(w) \cap V(C) \subseteq N(s)$.

Suppose that W is not proper in G . Then there exists a vertex w such that either w has more than two neighbors in a sector of W , but w is not adjacent to v , or w has neighbors in at least two sectors of W . It follows that w has more than one neighbor in $V(C)$, and thus in $N(s)$. Suppose that w has three distinct neighbors a, b, c in $V(C) \cap N(s)$. Let P be a shortest

path connecting two of a, b, c , say a and b , with interior in K ; then s is anticomplete to P^* . If c is anticomplete to P , then $G|(V(P) \cup \{w, s, a, b, c\})$ is an ISK_4 . Otherwise, by the minimality of $|V(P)|$, P^* consists of a single vertex x , and $\{w, s, x, a, b, c\}$ induces a $K_{3,3}$ subgraph in G , a contradiction. So w has exactly two neighbors a and b in $V(C)$, and thus a and b are in different sectors of W . Since $a, b \in N(s)$ and W is proper in H , it follows that $s \in V(C)$ and s is a spoke of W ; let S, S' be the two sectors of W containing s . But then v can be linked to the cycle $s - a - w - b - s$ via $v - s$ and the two paths with interiors in $S \setminus s$ and $S' \setminus s$. This is a contradiction by Lemma 8 and it follows that W is proper in G . This concludes the proof. \square

We use the following well-known lemma, which we prove for completeness.

Lemma 17. *Let G be a connected graph, $a, b, c \in V(G)$ with $d(a) = d(b) = d(c) = 1$, and let H be a connected induced subgraph of G containing a, b, c with $V(H)$ minimal subject to inclusion. Then either H is a subdivision of $K_{1,3}$ with a, b, c as the vertices of degree one, or H contains a triangle.*

Proof. Let G, a, b, c, H be as in the statement of the theorem. Let P be a shortest $a - b$ -path in H , and let Q be a shortest path from c to a vertex d with a neighbor in $V(P)$. By the minimality of $V(H)$, it follows that $V(H) = V(P) \cup V(Q)$. Moreover, P and Q are induced paths and no vertex of $Q \setminus d$ has a neighbor in $V(P)$. If d has exactly one neighbor in $V(P)$, then the result follows. If d has two consecutive neighbors in $V(P)$, then H contains a triangle. Otherwise, let $w \in V(P)$ such that d has a neighbor both on the subpath of P from w to a and on the subpath of P from w to b . It follows that $w \notin \{a, b, c\}$, and that $H \setminus w$ is connected and contains a, b, c . This contradicts the minimality of $V(H)$, and the result follows. \square

Lemma 18. *Let G be an $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph, s the center of a proper wheel in G , K a component of $G \setminus N[s]$, and N the set of vertices in $N(s)$ with a neighbor in K . Let G' arise from G by contracting $V(K)$ to a new vertex z . If $G|(V(K) \cup N \cup \{s\})$ is series-parallel, then G' is $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free.*

Proof. G' does not contain a triangle, because $N_{G'}(z) \subseteq N_G(s)$ is stable, and hence z is not in a triangle in G' . Suppose that H is an induced subgraph of G' which is a $K_{3,3}$ or a subdivision of K_4 . Then $z \in V(H)$. If z has degree two in H (and so H is an ISK_4), let a, b denote its neighbors; we can replace $a - z - b$ by an $a - b$ -path P with interior in K and obtain a subdivision of H , which is an ISK_4 , as an induced subgraph of G , a

contradiction. Thus z has degree three in H ; let a, b, c denote the neighbors of z in H . Let P be a shortest $a - b$ -path with interior in K . Then c has at most one neighbor on P , for otherwise $G|(V(P) \cup \{a, b, c, s\})$ is a wheel, contrary to the fact that $G|(V(K) \cup N \cup \{s\})$ is series-parallel and does not contain a wheel by Theorem 3. Let Q be a shortest path from c to $V(P) \setminus \{a, b\}$ with interior in K ; then each of a, b, c has a unique neighbor in $V(Q) \cup V(P)$ by symmetry. Let H' be a minimal connected induced subgraph of $G|(V(P) \cup V(Q))$ containing a, b, c . Since $G|(V(K) \cup N \cup \{s\})$ is series-parallel, it follows that H' is a subdivision of $K_{1,3}$ with a, b, c as the vertices of degree one by Lemma 17. Therefore, $G|(V(H \setminus z) \cup V(H'))$ is a subdivision of H , and by Theorem 4, it contains an ISK_4 or a $K_{3,3}$ subgraph in G . This is a contradiction, and the result is proved. \square

Lemma 19. *Let G be an $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph, s the center of a proper wheel in G , K a component of $G \setminus N[s]$, and N the set of vertices in $N(s)$ with a neighbor in K , and let $H = G|(V(K) \cup N \cup \{s\})$ be series-parallel. Let G' arise from G by contracting $V(K)$ to a new vertex z . Then z is not the center of a proper wheel in G' , and for $v \in V(G') \setminus \{s, z\}$, if v is the center of a proper wheel in G' , then v is the center of a proper wheel in G .*

Proof. Since $N_{G'}(z) \subseteq N_{G'}(s)$, it follows that z is not the center of a proper wheel in G' , for otherwise s would have a neighbor in every sector of such a wheel. This proves the first statement of the lemma.

Throughout the proof, let $v \in V(G') \setminus \{s, z\}$ be the center of a proper wheel in G' , and let $W = (C, v)$ be such a wheel with a minimum number of spokes. Since G' is $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free by Lemma 18, it follows that W satisfies the hypotheses of Theorem 6. Our goal is to show that v is the center of a proper wheel in G .

(15) *If $z \in V(C)$, then v is the center of a proper wheel in G .*

Suppose that $z \in V(C)$. Let a, b denote the neighbors of z in $V(C)$. Let P be a shortest $a - b$ -path with interior in K . Then every vertex in $V(K)$ has at most two neighbors in $V(P)$. Let $W' = (C', v)$ be the wheel in G that arises from W by replacing the subpath $a - z - b$ of C by $a - P - b$ to obtain C' .

It remains to show that W' is a proper wheel in G . Suppose that some vertex $x \in V(G) \setminus V(K)$ has two or more neighbors in P^* . Then $x \in N \subseteq V(H)$, and $H|(\{a, b, x, s\} \cup V(P))$ is a wheel in H with center x , a contradiction since H is series-parallel Theorem 3.

Since $v \notin V(K)$, it follows from the claim of the previous paragraph that v has at most one neighbor in P^* , and no neighbor unless v is adjacent to z , and therefore there are at most two sectors of W' intersecting P^* . We claim that if for a vertex x we have $|N_G(x) \cap V(C')| \geq 3$, then $x \notin V(K)$ and $|N_{G'}(x) \cap V(C)| \geq 3$. Suppose that x is a vertex violating this claim. If $x \in K$, then $N_G(x) \cap V(C') \subseteq V(P)$, and so $|N_G(x) \cap V(C')| \leq 2$ by the minimality of $|V(P)|$, a contradiction; it follows that $x \notin K$. Therefore, $|N_G(x) \cap P^*| \leq 1$, and thus $|N_G(x) \cap V(C')| - |N_{G'}(x) \cap V(C)| \leq 1$. But $|N_G(x) \cap V(C')| > 3$ by Lemma 8, and so $|N_{G'}(x) \cap V(C)| \geq 3$, a contradiction. So the claim holds.

Now suppose that there is a vertex x which is not proper for W' . If x has neighbors in at most one sector of W' , then $|N_G(x) \cap V(C')| \geq 3$, but we proved above that $|N_{G'}(x) \cap V(C)| \geq 3$, and so, since W is proper, x is adjacent to v , a contradiction. It follows that x has neighbors in more than one sector of W' . Since x is proper for W , it follows that x has a neighbor in P^* and thus, either $x \in V(K)$ or x is adjacent to z . Since x is proper for W , it follows that $N_G(x) \cap V(C')$ is contained in the sectors of W' intersecting P^* . In particular, there are exactly two such sectors S_1 and S_2 of W' , they are consecutive, and v has a neighbor in P^* . Consequently, v is adjacent to z and z is a spoke in W .

We claim that x has at most two neighbors in $V(C')$. If $x \in V(K)$ then $N_G(x) \cap V(C') \subseteq V(P)$ and we have already shown that every vertex of K has at most two neighbors in P . Thus we may assume that $x \notin K$, and so x is adjacent to z . Since G' is triangle-free by Lemma 18, it follows that x is not adjacent to v . Since x is proper for W , it follows that x has at most two neighbors in $V(C)$, and hence in $V(C')$, by our first claim. This proves our second claim. It follows that x has exactly one neighbor s_1 in $S_1 \setminus S_2$ and exactly one neighbor s_2 in $S_2 \setminus S_1$. If x is non-adjacent to v , then $G|(V(S_1) \cup V(S_2) \cup \{x, v\})$ is an ISK_4 in G , a contradiction. Therefore, x is adjacent to v and can be linked to the cycle $G|(V(S_1) \cup \{v\})$ via $x - v$, $x - s_1$, and a subpath of $x - s_2 - S_2$. Therefore W' is a proper wheel in G . This proves (15).

By (15), we may assume that $z \notin V(C)$. So W is a wheel in G . Since W is proper in G' , there is a sector S of W containing all neighbors of z in C . Then clearly the following holds.

(16) *For every $x \in K$, $N_G(x) \cap V(C) \subseteq N_{G'}(z) \cap V(C) \subseteq V(S)$.*

Next we claim the following.

(17) *If z is not adjacent to v , then W is a proper wheel in G .*

If $x \in G \setminus (V(C) \cup V(K))$, then x is proper for W in G as x is proper for W in G' . Now consider a vertex $x \in V(K)$. Since z is not adjacent to v , and W is proper in G' , it follows that $|N_{G'}(z) \cap V(C)| \leq 2$. Then by (16), $|N_G(x) \cap V(C)| \leq 2$, and hence x is proper for W in G . This proves 17.

By (17), we may assume that z is adjacent to v . Let a and b be the ends of S . We now define a sequence of wheels in G with center v . Let $W_1 = W$ and $S_1 = S$. Assume that wheels W_1, \dots, W_i have been defined, and define W_{i+1} as follows. If there is a vertex $x_i \in V(K)$ that is not adjacent to v and has at least three neighbors in S_i , then let S_{i+1} be the path from a to b in $G[V(S_i) \cup \{x_i\}]$ that contains x_i , and (by (16)) let W_{i+1} be the wheel obtained from W_i by replacing S_i by S_{i+1} . Since S_{i+1} is strictly shorter than S_i , this sequence must stop at some point; say it stops with wheel W_t . For $1 \leq i \leq t$, let C_i be the rim of W_i (so $V(C_i) = (V(C) \setminus V(S)) \cup V(S_i)$). Then $W_t = (C_t, v)$ is a wheel in G such that every vertex of K that has at least three neighbors in S_t is adjacent to v . We will show that W_t is a proper wheel in G , but first we show the following.

(18) *For $1 \leq i < t$, if a vertex y is proper for W_i , then y is proper for W_{i+1} .*

Suppose that y is proper for W_i and not proper for W_{i+1} . Then y is adjacent to x_i . Suppose first that y is non-adjacent to v and $|N_G(y) \cap V(C_{i+1})| \geq 3$. Since y cannot have three neighbors in C_{i+1} by Lemma 8, it follows that $|N_G(y) \cap V(C_{i+1})| > 3$. Moreover, since $N_G(y) \cap V(C_{i+1}) \subseteq \{x_i\} \cup (N_G(y) \cap V(C_i))$, it follows that $|N_G(y) \cap V(C_i)| \geq 3$. But then y is not proper for W_i , a contradiction. It follows that y has a neighbor in $C_{i+1} \setminus S_{i+1} = C \setminus S$, and thus $y \notin V(K)$. Therefore $y \in V(G')$, and since y is adjacent to x_i in G , it follows that y is adjacent to z in G' . Since z is adjacent to v in G' and G' is triangle-free by Lemma 18, it follows that y is non-adjacent to v . Note that since $i + 1 > 1$, it follows that a vertex of K has a neighbor in S^* , and therefore z has a neighbor in S^* . Since W satisfies the hypotheses of Theorem 6, and since $y - z$ is a path containing exactly one neighbor of v , it follows that the neighbors of y in C are in a sector S' of W consecutive with S . Since y is non-adjacent to v , it follows that y has at most two neighbors in S' . Note that since G' is triangle-free, and $N_{G'}(z) \cap V(C) \subseteq V(S)$, it follows that z has no neighbors in S' . If y has

exactly two neighbors in S' , then y can be linked in G' to the hole $v - S' - v$ via two one-edge paths and the path $y - z - v$. So y has exactly one neighbor r in S' that is in $V(S') \setminus V(S)$, and now z can be linked to $v - S' - v$ via the paths $z - v$, $z - y - r$, and a path with interior in S , contrary to Lemma 8. This concludes the proof of (18).

Every vertex in $G \setminus V(K)$ is proper for W and hence it is proper for W_t by (18). Suppose that there is a vertex $x \in V(K)$ that is not proper for W_t . By (16), $N_G(x) \cap V(C_t) \subseteq V(S_t)$. So x is non-adjacent to v and has at least three neighbors in S_t , contradicting the assumption that the wheel sequence terminates with W_t . Therefore, W_t is a proper wheel in G with center v . \square

4 Tools

In this section we develop tools for our main theorem for finding a vertex of degree one, or a cycle with all but a few vertices of degree two.

Lemma 20. *Let G be a graph, $x \in V(G)$, such that $G \setminus x$ is a forest. Then either $V(G) = N[x]$ and $G \setminus x$ is stable, or $V(G) \setminus N[x]$ contains a vertex of degree at most one in G , or G contains an induced cycle C containing x such that every vertex of $V(C) \setminus \{x\}$ except for possibly one has degree two in G .*

Proof. If every component of $G \setminus x$ contains exactly one vertex, then either $V(G) = N[x]$ or $V(G) \setminus N[x]$ contains a vertex of degree zero. Hence, we may assume that there exists a component T of $G \setminus x$ with at least two vertices, and T is a tree. Let A be the set of vertices of degree at least three in T . If A is non-empty, then let T' be the subtree of T that contains all vertices of A and minimal with respect to this property, and let a be a leaf of T' . There is a path $P = v - \dots - v'$ in T , whose ends are distinct leaves of T and P contains at most one vertex of degree three in T (namely a). This is trivial if A is empty, and follows from the definition of a otherwise.

If x is non-adjacent to v , then v is a vertex in $V(G) \setminus N[x]$ of degree one in G , so we may assume that x is adjacent to v , and similarly for v' . Now, let v'' be the neighbor of x in $P \setminus v$ closest to v along P . We set $C = x - v - P - v'' - x$ and observe that all vertices of C except possibly x and a have degree two in G . \square

Lemma 21. *Let G be a series-parallel graph, and let $x, y \in V(G)$ with $x = y$ or $xy \in E(G)$. If $G \setminus \{x, y\}$ contains a cycle, then there is an induced cycle C*

in G such that $V(C) \cap \{x, y\} = \emptyset$ and all but at most two vertices of C have degree two in G (and are thus anticomplete to $\{x, y\}$), or $V(G) \setminus (N[x] \cup N[y])$ contains a vertex of degree at most one in G .

Proof. By contracting the edge xy and deleting any parallel edges that may arise, we may assume that $x = y$. We may further assume that every vertex except for possibly x has degree at least two, because vertices of degree one in $N(x)$ can be deleted without affecting the hypotheses or the conclusions, and if there is a vertex of degree at most one in $V(G) \setminus N[x]$, then the conclusion holds.

Let C be a cycle in $G \setminus x$. Since G is series-parallel and by the definition of series-parallel graphs, it follows that there do not exist three paths from x to $V(C)$ that are vertex disjoint except for x in G . By Menger's theorem [4], it follows that there is a partition (X, Y, Z) of $V(G)$ with X of size at most two, and $Y, Z \neq \emptyset$ such that Y is anticomplete to Z in G , $V(C) \subseteq Y \cup X$ and $x \in Z$.

We choose a partition (X, Y, Z) with $|X|$ minimal, and subject to that, $|X \cup Y|$ minimal, such that Y is anticomplete to Z in G , $Y, Z \neq \emptyset$, $x \in Z$, and $G|(Y \cup X)$ contains a cycle. It follows that $|X| \leq 2$.

Suppose first that $X = \emptyset$. If $G|Y$ is an induced cycle, the result follows. Otherwise, since $G|Y$ contains a cycle, it follows that there is a vertex x' such that $G|(Y \setminus \{x'\})$ contains a cycle. By induction applied to $G|Y$ and the vertex x' , the result follows.

Next, suppose that $X = \{x'\}$. If $G|Y$ is a forest, then $x' \neq x$ and thus we obtain the desired result by applying Lemma 20 to $G|(X \cup Y)$. Otherwise, we apply induction to $G|(X \cup Y)$ and x' , and again, the result follows.

It follows that $X = \{x', y'\}$, and therefore, the component of $G|(Z \cup X)$ containing x contains x' and y' , for otherwise $\{x'\}$ or $\{y'\}$ would be a better choice of X for the partition. Suppose that $G|Y$ is connected. If there is a vertex z such that every $x' - y'$ -path with interior in $G|Y$ uses z , then $\{x', z\}$ or $\{y', z\}$ yields a better choice of X and partition. Therefore, by Menger's theorem [4], there are two disjoint paths P_1, P_2 from x' to y' with interior in Y , and since $G|Y$ is connected, it follows that there is path Q from P_1 to P_2 in Y . Moreover, there is a path R from x' to y' with interior in Z since the component of $G|(Z \cup X)$ containing x also contains x' and y' ; but $P_1 \cup P_2 \cup Q \cup R$ is a (not necessarily induced) subdivision of K_4 in G , contrary to the fact that G is series-parallel. Thus $G|Y$ is not connected. By the minimality of $X \cup Y$, for every component K of $G|Y$, the graph $G|((X \setminus \{x\}) \cup V(K))$ contains no cycle. However, $G|(X \cup Y)$ contains a cycle C not using x , and so C contains vertices from more than one component

of $G|Y$. It follows that $x', y' \in V(C)$, and thus $x \notin \{x', y'\}$. Therefore, for every component K of $G|Y$, the graph $G|(X \cup V(K))$ is a tree. Since K is connected, it follows that x', y' are leaves. If $G|(X \cup V(K))$ contains a leaf other than x', y' , then the result follows. So each component is a path from x' to y' , and no vertex of the path except for x', y' has further neighbors in G . But then the union of two of those paths (there are at least two, since $G|Y$ is not connected) yields the desired cycle; the result follows. \square

Theorem 22. *Let G be an $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph, $x, y \in V(G)$ with $x = y$ or $xy \in E(G)$. Then either*

- $V(G) = N[x] \cup N[y]$;
- there exists a vertex in $V(G) \setminus (N[x] \cup N[y])$ of degree at most one in G ;
- there exists an induced cycle C containing at least one of x, y such that at most one vertex v in $V(C) \setminus (N[x] \cup N[y])$ has $d(v) > 2$; or
- there exists an induced cycle C containing neither x nor y and a vertex $z \in V(C)$ such that at most one vertex v in $V(C) \setminus N[z]$ has $d(v) > 2$.

Proof. Suppose first that G is series-parallel. Define $H = G$ and $v = x$ if $x = y$, and define H as the graph that arises from contracting the edge xy to a new vertex v if $x \neq y$. Then H is series-parallel. Suppose that $H \setminus v$ is a forest, and apply Lemma 20. If the first outcome of Lemma 20 holds, then $V(H) = N_H(v)$, and so $V(G) = N_G(x) \cup N_G(y)$. If the second outcome of Lemma 20 holds, then $V(H) \setminus N_H[v]$ contains a vertex of degree at most one in H , and so $V(G) \setminus (N_G[x] \cup N_G[y])$ contains a vertex of degree at most one in G . Finally, if the third outcome of Lemma 20 holds, then H contains an induced cycle C containing v such that every vertex of $V(C) \setminus \{v\}$ except for at most one has degree two in H , and so there is an induced cycle C' in G containing at least one of x, y such that every vertex of $V(C') \setminus \{x, y\}$ except for possibly one has degree two in G . This proves the result in the case that $H \setminus v$ is a forest. So $H \setminus v$ contains a cycle, and thus $G \setminus \{x, y\}$ contains a cycle. By Lemma 21, either $V(G) \setminus (N[x] \cup N[y])$ contains a vertex of degree at most one in G , or G contains a cycle C with $V(C) \cap \{x, y\} = \emptyset$ and such that all but at most two vertices in $V(C)$ have degree two in G . In the former case, the second outcome of this theorem holds; in the latter case, the fourth outcome of this theorem holds by choosing $z \in V(C)$ with $d_G(z)$ maximum among vertices in $V(C)$.

Thus we may assume that G contains a proper wheel by Lemma 9; let z be the center of a proper wheel (where possibly $z \in \{x, y\}$). Let W be such a wheel with minimum number of spokes. Let $Z = \{x, y\} \cap N(z)$. Since $x = y$ or $xy \in E(G)$, it follows that x and y are in the same component of $G \setminus (N[z] \setminus Z)$. Since $N(z)$ is stable, it follows that $|Z| \leq 1$. Therefore, by Theorem 6, the component of $G \setminus (N[z] \setminus Z)$ containing $\{x, y\} \setminus \{z\}$ includes the interiors of at most two sectors of W . Again by Theorem 6, the interior of every other sector of W is contained in a separate component of $G \setminus (N[z] \setminus Z)$. Since W has at least four sectors by Lemma 8, there is a component K of $G \setminus (N[z] \setminus Z)$ that does not contain x and y , and that contains no neighbor of x, y . Let N be the set of neighbors of z with a neighbor in K . Then, we apply induction to $H = G|(V(K) \cup N \cup \{z\})$ and z . By the choice of H and z , the first outcome does not hold. If the second outcome holds for H and z , then it holds for G and x, y as well, since $(N[x] \cup N[y]) \cap V(H) \subseteq N[z] \cap V(H)$. If the third or fourth outcome holds for H and z , then the third or fourth outcome holds for G and x, y . \square

5 Main Result

We say that (G, x, y) has the property \mathcal{P} if $V(G) \setminus (N[x] \cup N[y])$ contains a vertex of degree at most two in G .

We can now prove Theorem 7 which we restate:

Theorem 23. *Let G be an $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free graph which is not series-parallel, and let (x, y) be a non-center pair for G . Then (G, x, y) has the property \mathcal{P} .*

Proof. Suppose for a contradiction that the theorem does not hold, and let (G, x, y) be a counterexample with $|V(G)|$ minimum. Then every vertex in $V(G) \setminus (N[x] \cup N[y])$ has degree at least three in G . Since G is not series-parallel, and G is $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free, it follows from Theorem 4 that G contains a wheel and hence by Lemma 9, it follows that G contains a proper wheel $W = (C, s)$. Let C_1, \dots, C_k denote the components of $V(G) \setminus N[s]$. For $i = 1, \dots, k$ let N_i denote the set of neighbors v of s such that v has a neighbor in C_i , and let G_i denote the induced subgraph of G with vertex set $V(C_i) \cup N_i \cup \{s\}$.

(19) *For $i = 1, \dots, k$, if G_i is series-parallel and $G_i \setminus (N[s] \cap \{x, y, s\})$ contains a cycle, then $\{x, y\} \cap V(C_i) \neq \emptyset$.*

Let $i \in \{1, \dots, k\}$ such that G_i is series-parallel, and let $G_i \setminus (N[s] \cap \{x, y, s\})$ contain a cycle. Since G is triangle-free, it follows that $1 \leq |N[s] \cap$

$\{x, y, s\} \mid \leq 2$. By Lemma 21 applied to G_i and the vertices in $N[s] \cap \{x, y, s\}$, it follows that either there is a vertex in $V(G_i) \setminus N[s]$ of degree at most one in G_i anticomplete to $\{x, y\} \cap N(s)$, or $G_i \setminus (N[s] \cap \{x, y, s\})$ contains a cycle C' with at least two vertices of degree two in G_i . In both cases, there is a vertex z in $V(G_i) \setminus N[s]$ of degree at most two in G_i and z is anticomplete to $N[s] \cap \{x, y, s\}$, and hence its degree in G is also at most two. Since (G, x, y) does not satisfy property \mathcal{P} , it follows that $z \in N[x] \cup N[y]$, and thus $\{x, y\} \cap V(C_i) \neq \emptyset$. This proves (19).

(20) *For $i = 1, \dots, k$, if G_i is series-parallel, then $|V(C_i)| = 1$.*

Let $i \in \{1, \dots, k\}$ be such that G_i is series-parallel, and suppose that $|V(C_i)| > 1$. Let G' be the graph that arises from G by contracting $V(C_i)$ to a new vertex z . We let $x' = z$ if $x \in V(C_i)$ and $x' = x$ otherwise; and we let $y' = z$ if $y \in V(C_i)$ and $y' = y$ otherwise. By Lemma 18, G' is $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free. By Lemma 19, (x', y') is a non-center pair for G' . By the minimality of $|V(G)|$, it follows that (G', x', y') has the property \mathcal{P} . Let $v \in V(G') \setminus (N[x'] \cup N[y'])$ be a vertex of degree at most two in G' . From the definition of x' and y' , it follows that $v \notin N[x] \cup N[y]$. It follows that either $v = z$, or $v \neq z$ and $d_G(v) > 2$, and so $v \in N[z]$.

Suppose first that $v = z$. Then $z \notin N[x'] \cup N[y']$, and so $V(G_i) \cap \{x, y\} = \emptyset$. By (19), it follows that $G_i \setminus s$ is a tree. Since v has degree at most two in G' , it follows that $|N_i| \leq 2$, and since $G_i \setminus N[s]$ is connected, it follows that every vertex of N_i is a leaf of $G_i \setminus s$. Thus, either $V(C_i)$ contains a leaf of $G_i \setminus s$, or $G_i \setminus s$ is a path with ends in N_i , and so in both cases $V(C_i)$ contains a vertex of degree at most two in G . This is a contradiction since $V(G_i) \cap \{x, y\} = \emptyset$; it follows that $v \neq z$.

It follows that $v \in N(z)$. Since $d_G(v) > 2$, it follows that $d_{G'}(v) < d_G(v)$, and thus v has more than one neighbor in $V(C_i)$. Let P be a path in C_i between two neighbors of v , then $v - P - v$ is a cycle in $G_i \setminus (N[s] \cap \{x, y, s\})$. By (19), it follows that $V(C_i) \cap \{x, y\} \neq \emptyset$. But then $z \in \{x', y'\}$, and so $v \in N[x'] \cup N[y']$, a contradiction. This proves (20).

(21) *For $i = 1, \dots, k$, if G_i contains a wheel, then $x \in V(C_i)$ or $y \in V(C_i)$.*

Suppose not, and let $i \in \{1, \dots, k\}$ be such that G_i contains a wheel and $V(C_i) \cap \{x, y\} = \emptyset$. Since N_i is a stable set, it follows that $|N_i \cap \{x, y\}| \leq 1$, and by symmetry, we may assume that $y \notin N_i$. Let $y' = s$, and let $x' = x$ if $x \in N_i$ and $x' = s$ otherwise. By Lemma 16, (x', y') is a non-center pair for G_i . Since G_i is an induced subgraph of G , it follows that G_i is

$\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free. Since G is a minimum counterexample, it follows that (G_i, x', y') has the property \mathcal{P} . Let v be a vertex of degree at most two in G_i with $v \notin N[x'] \cup N[y']$. Since $v \notin N[s]$, it follows that $d_G(v) = d_{G_i}(v)$. Therefore $v \in N[x] \cup N[y]$. Let $w \in \{x, y\}$ be such that $v \in N[w]$. By assumption, $w \notin V(C_i)$. If $w \in V(G_i)$, then $w \in N_i$, and so $w = x = x'$, and thus $v \notin N[w]$, a contradiction. It follows that $w \notin V(G_i)$, and so $N[w] \cap V(G_i) \subseteq N[s]$, and again $v \notin N[w]$, a contradiction. This proves that $\{x, y\} \cap V(C_i) \neq \emptyset$, and (21) follows.

It follows from Theorem 3 together with (21) and (20) that there is at most one $i \in \{1, \dots, k\}$ with $|V(C_i)| > 1$. We may assume $|V(C_i)| = 1$ for all $i \in \{1, \dots, k-1\}$.

(22) *If $|V(C_k)| > 1$, let $G' = G \setminus (V(C_k) \cup \{s\})$; otherwise let $G' = G \setminus s$. Then G' has girth at least eight.*

Observe that G' is bipartite with one side of the bipartition being $N(s)$.

Suppose that C is a cycle of length four in G' . Let $V(C) = \{a, b, c, d\}$ and $N(s) \cap V(C) = \{a, c\}$. If $d_G(b) \neq 2$, let e be a neighbor of b which is not a, c . Note that $e \in N(s)$. Then $\{a, b, c, d, e, s\}$ induces an ISK_4 or a $K_{3,3}$, a contradiction. It follows that $d_G(b) = d_G(d) = 2$, and moreover, $\{x, y\} \cap \{a, c\} \neq \emptyset$. So, by symmetry, say $x = a$, and we may assume that $d \neq y$.

Observe that $G \setminus b$ is not series-parallel by Theorem 3. By the minimality of $|V(G)|$, it follows that there exists $v \in V(G) \setminus (N[x] \cup N[y])$ with $d_{G \setminus \{b\}}(v) \leq 2$. Since $d_G(v') = d_{G \setminus \{b\}}(v')$ for all $v' \in V(G) \setminus \{x, b, c\}$, it follows that $v = c$, and so $N_G(c) = \{b, d, s\}$, and so $\{s, a\}$ is a cutset in G . Let $G'' = G \setminus \{b, c, d\}$, and if $y \in \{b, c, d\}$, let $y' = x$, otherwise, $y' = y$. Then G'' is not series-parallel. A proper wheel in G'' is proper in G , because each vertex in $\{b, c, d\}$ has at most one neighbor in the wheel, s or a . Therefore, (x, y') is a non-center pair for G'' . By the minimality of $|V(G)|$, it follows that (G'', x, y') has the property \mathcal{P} . But this is a contradiction, since every vertex in $V(G'') \setminus N[x]$ has the same degree in G and G'' . This proves that G' contains no 4-cycle.

Suppose G' contains a 6-cycle C . Then, since exactly three vertices in $V(C)$ are neighbors of s , it follows that $G|(V(C) \cup \{s\})$ is an ISK_4 , a contradiction. It follows that G' has girth at least eight, and so (22) is proved.

(23) $|V(C_k)| > 1$.

Suppose not, and let $G' = G \setminus s$. Then G' satisfies the hypotheses of Theorem 22. Since s is the center of a proper wheel in G , it follows that there exists a vertex z in G' that is not in $N[x] \cup N[y] \cup N[s]$, and so the first outcome of Theorem 22 does not hold. The second outcome does not hold, because every vertex in $V(G') \setminus (N[x] \cup N[y])$ of degree one in G' has degree at most two in G , a contradiction.

Therefore, the third or fourth outcome of Theorem 22 holds, and hence there exists an induced cycle C in G' with vertices $c_1 - \dots - c_t - c_1$, and $i, j \in \{1, \dots, t\}$, $l \in \{0, \dots, 3\}$ such that all vertices of C except for c_i, \dots, c_{i+l} (where $c_{t+1} = c_1$ and so on) and c_j have degree two in G' , do not coincide with x, y and are non-neighbors of x, y . By (22), $t \geq 8$. Consequently, G' contains two adjacent vertices in $V(G') \setminus (N[x] \cup N[y])$ of degree two in G' . Since G is triangle-free, it follows that one of them has degree two in G , a contradiction. Thus, $|V(C_k)| > 1$, and (23) is proved.

By (20), (21) and (23) we may assume that $x \in V(C_k)$. Let G' arise from G by contracting $V(C_k) \cup N_k$ to a single vertex z , and by deleting s and every vertex that is only adjacent to z . It follows that G' is bipartite. Our goal is to prove that $G' \setminus z$ has girth at least 16, see (28). By (22), we know that $G' \setminus z$ has girth at least eight.

(24) *Every vertex in $V(G') \setminus \{z\}$ has at most one neighbor in N_k in G . There is no 4-cycle in G' containing z .*

Suppose first that there is a vertex $v \in V(G') \setminus \{z\}$ with at least two neighbors $a, b \in N_k$ in G . Since $v \in V(G')$ and in G' there are no vertices of degree one adjacent to z , it follows that v has another neighbor $c \in N(s) \setminus N_k$. Let P be a path connecting a and b with interior in $V(C_k)$. Such a path exists, since $a, b \in N_k$. It follows that $G|(V(P) \cup \{a, b, c, v, s\})$ is an ISK_4 in G , a contradiction. This implies the first statement of (24).

Suppose that z is contained in a 4-cycle with vertex set $\{a, b, c, z\}$ in G' such that $a, c \in N_{G'}(z)$. Note that $a, c \notin N(s)$ and $b \in N(s) \setminus N_k$. By (22), $G \setminus (\{s\} \cup V(C_k))$ contains no 4-cycle, and thus a and c have no common neighbor in N_k . Let a', c' be a neighbor of a and c in N_k , respectively; a' and c' exists since $a, c \in N_{G'}(z)$. Let P be a shortest path between a' and c' with interior in C_k . Since $b \notin N_k$, it follows that b is anticomplete to $V(P)$. Therefore, $G|(\{a, b, c, s\} \cup V(P))$ is an ISK_4 in G , a contradiction. This proves (24).

(25) *G' is $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free.*

Since G' is bipartite, it follows that G' is triangle-free. Suppose that G' contains an induced subgraph H which is either a $K_{3,3}$ or an ISK_4 . Since G is $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free, it follows that $z \in V(H)$. Suppose that z has degree two in H . By (24), the neighbors of z in $V(H)$ do not have a common neighbor in N_k . Let P be a path in G connecting the neighbors of z in $V(H)$ with interior in $V(C_k) \cup N_k$ containing exactly two vertices in N_k . Then $G|(V(H) \setminus \{z\}) \cup V(P)$ is an induced subdivision of H in G . By Theorem 4, it follows that G is not $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free, a contradiction.

It follows that z has degree three in H . Let a, b, c be the neighbors of z in H . By (24), each of a, b, c has a unique neighbor in N_k . Let a', b', c' be neighbors of a, b, c in N_k . Let H' be a minimal induced subgraph of $G|(V(C_k) \cup \{a, b, c, a', b', c'\})$ which is connected and contains $\{a, b, c\}$. It follows that each of a, b, c has a unique neighbor (namely a', b', c' , respectively), in H' . By Lemma 17, H' is a subdivision of $K_{1,3}$ in which a, b, c are the vertices of degree one. Consequently, $G|(V(H \setminus z) \cup V(H'))$ is an induced subgraph of G which is a subdivision of H . But then G is not $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free by Theorem 4. Hence G' is $\{\text{ISK}_4, \text{triangle}, K_{3,3}\}$ -free. This proves (25).

(26) G' does not contain a proper wheel with center different from z .

Suppose $v \neq z$ is the center of a proper wheel G' . By Theorem 6, there is a component C of $G' \setminus N[v]$ that is disjoint from $N[z]$. Let N denote the set of vertices in $N(v)$ with a neighbor in C .

Then $H = G'|(N \cup V(C) \cup \{v\})$ satisfies the hypotheses of Theorem 22. Since $V(C) \neq \emptyset$, it follows that the first outcome of Theorem 22 does not hold. Moreover, every vertex in $V(H) \setminus N[v]$ of degree one in H has degree at most two in G , since such a vertex belongs to C and C is disjoint from $N[z]$, and the only additional neighbor that such a vertex may have in G is v . Furthermore, such a vertex is in $V(G') \setminus N[z]$, and hence in $V(G) \setminus (N[x] \cup N[y])$ as $x \in V(C_k)$. It follows that the second outcome of Theorem 22 does not hold.

Therefore, the third or fourth outcome of Theorem 22 holds, and hence there exists an induced cycle C' in H with vertices $c_1 - \dots - c_t - c_1$, and $i, j \in \{1, \dots, t\}$, $l \in \{0, \dots, 3\}$ such that all vertices of C' except for c_i, \dots, c_{i+l} (where $c_{t+1} = c_1$ and so on) and c_j have degree two in H , do not coincide with v and are non-neighbors of v . By (22), $t \geq 8$, since $z \notin V(H)$. Consequently, G' contains two adjacent vertices in $V(G') \setminus N[z]$ of degree two

in G' . Since G is triangle-free, it follows that one of them is non-adjacent to s and thus has degree two in G , a contradiction. Hence (26) is proved.

(27) *For every component K of $G' \setminus N[z]$, $G'|_{(V(K) \cup N(z))}$ is a forest.*

Suppose not, and let K be a component of $G' \setminus N[z]$ such that $G'|_{(V(K) \cup N(z))}$ is not a forest. Suppose first that $H = G'|_{(V(K) \cup N[z])}$ is not series-parallel. Then H contains a proper wheel by Lemma 9. Let v be the center of a proper wheel in H . Since $H \setminus N[z]$ is connected, it follows from Theorem 6 that $v \neq z$. By Lemma 16, it follows that v is the center of a proper wheel in G' , contrary to (26).

It follows that H is series-parallel, and by our assumption, $H \setminus z$ contains a cycle. By applying Lemma 21 to H and z , it follows that there is either a vertex in $V(H) \setminus N[z]$ of degree one, or a cycle C not containing z , with all but at most two vertices of degree two in H . In the latter case, since $G' \setminus z$ has girth at least eight, C contains two adjacent vertices in $V(H) \setminus N[z]$ of degree two in H , and thus of degree two in G' . Since G is triangle-free, it follows that in both cases G contains a vertex of degree at most two not in $N[z]$, and thus not in $N[x] \cup N[y]$. This is a contradiction, and (27) is proved.

(28) *The girth of $G' \setminus z$ is at least 16.*

Suppose that this is false. Let C be an induced cycle in $G' \setminus z$ of length less than 16. Since by (27), for every component K of $G' \setminus N[z]$, we have that $G'|_{(V(K) \cup N(z))}$ is a forest, it follows that $C \setminus N[z]$ has at least two components. Since z is not contained in a 4-cycle in G' by (24), and G' is bipartite, it follows that each component of $C \setminus N[z]$ has at least three vertices. If $C \setminus N[z]$ has at least four components, it follows that C has length at least 16. If $C \setminus N[z]$ has exactly three components, then $G'|_{(V(C) \cup \{z\})}$ is an ISK_4 , a contradiction. So $C \setminus N[z]$ has exactly two components. For every component K of $G' \setminus N[z]$, by (27) we have that $G'|_{(V(K) \cup N(z))}$ is a forest. Therefore, the two components of $C \setminus N[z]$ are contained in two different components of $G' \setminus N[z]$; say A and B . Let N_A, N_B denote the set vertices in $N(z)$ with a neighbor in A, B , respectively. Suppose that $|N_A| \geq 3$. Since $|V(C) \cap N(z)| = 2$, it follows that there is a path P from a vertex c in $N_A \setminus V(C)$ to $V(C)$ with interior in $V(A)$. Since $G'|_{(V(A) \cup N_A)}$ and $G'|_{(V(B) \cup N_B)}$ are trees, it follows that c has at most one neighbor in each component K of $C \setminus N[z]$. Therefore, $G'|_{(V(P) \cup V(C) \cup \{z\})}$ contains an induced subgraph of G' which is either a subdivision of K_4 or of $K_{3,3}$, a contradiction by Theorem 4 and (25). So $|N_A| = 2$. Since $G'|_{(V(A) \cup N_A)}$

is a tree, it follows that either A contains a vertex of degree one in G' , non-adjacent to z , or $G'|(V(A) \cup N_A)$ is a path containing at least five vertices, and hence A contains two adjacent vertices of degree two in G' , non-adjacent to z . Since G is triangle-free, it follows that in either case G contains a vertex of degree at most two not in $N[z]$, and thus not in $N[x] \cup N[y]$. This is a contradiction, and (28) is proved.

Recall that $\{x, y\} \cap V(C_k) \neq \emptyset$, and we may assume that $x \in V(C_k)$, and thus $y \in V(C_k) \cup N_k$. Let G'' be the graph that arises from G by deleting $\{s\} \cup (V(C_k) \setminus \{x\}) \cup (N_k \setminus \{y\})$, and every vertex other than x with neighbors only in N_k (this last operation does not change the degree of any vertex in $V(G'')$ except for possibly y). Then $N_{G''}(x) \subseteq \{y\}$. It follows from (28) that $G'' \setminus \{y\}$ has girth at least 16, and from (22) that G'' has girth at least eight. If $y \in V(G'')$, let $y' = y$; otherwise, let $y' = x$. It follows that if $y' = y$, then $y \in N_k$.

Since G'' is an induced subgraph of G , it follows that G'' and x, y' satisfy the hypotheses of Theorem 22.

Since s is the center of a proper wheel, it follows from Theorem 6 that there are at least two components of $G'' \setminus N[s]$ in which y' has no neighbors. Consequently, $V(G'') \neq N[x] \cup N[y']$, and thus the first outcome of Theorem 22 does not hold.

The second outcome of Theorem 22 does not hold, because if G'' contains a vertex v of degree one non-adjacent to y' , then v has degree at most two in G , and $v \notin N[x] \cup N[y]$, a contradiction.

Suppose that the third outcome holds, and so G'' contains an induced cycle C containing y' (since $d_{G''}(x) \leq 1$) such that at most one vertex in $V(C) \setminus N[y]$ has degree more than two. Since G'' has girth at least eight, $|V(C)| \geq 8$, and in particular C contains a vertex v of distance three from y in C and degree two in G'' . Let $y - a - b - v$ be the three-edge path from y to v in C . Then v is not adjacent to s in G , because $G|(V(G'') \cup \{s\})$ is bipartite and $ys \in E(G)$. Moreover, v is anticomplete to N_k , because otherwise $z - a - b - v - z$ is a 4-cycle in G' using z , contradicting (24). So v has degree two in G and is not in $N[x] \cup N[y]$, a contradiction.

Thus, the fourth outcome holds, and so G'' contains an induced cycle C not containing x, y' and containing a vertex z' such that at most one vertex in $V(C) \setminus N[z']$ has degree more than two in G'' . Since $|V(C)| \geq 16$, it follows that C contains a path $P = p_1 - \dots - p_6$ of six vertices, all of degree two in G'' and non-adjacent to x, y' . We may assume that $N(s) \cap V(P) \subseteq \{p_1, p_3, p_5\}$ by symmetry. Since z is not in a 4-cycle in G' by (24), not both p_2 and p_4

have a neighbor in N_k . It follows that either p_2 or p_4 has degree two in G , a contradiction. This completes the proof of Theorem 23. \square

We can now prove Theorem 2 which we restate:

Theorem 24. *Let G be an $\{\text{ISK}_4, \text{triangle}\}$ -free graph. Then either G has a clique cutset, G is complete bipartite, or G has a vertex of degree at most two.*

Proof. Let G be an $\{\text{ISK}_4, \text{triangle}\}$ -free graph. If G is series-parallel, then G contains a vertex of degree at most two Theorem 3. If G contains $K_{3,3}$ as a subgraph, then by Theorem 5, either G is complete bipartite or G has a clique cutset. If G is not series-parallel and $K_{3,3}$ -free, then G contains a vertex of degree at most two by Theorem 7 applied to the graph obtaining from G by adding an isolated vertex x with the non-center pair (x, x) . This implies the result. \square

Note that the outcome of a clique cutset in Theorem 24 cannot be avoided, as the following example shows. Let G be any $\{\text{ISK}_4, \text{triangle}\}$ -free graph (e. g. a C_5 , or a wheel), and let H arise from G by adding $|V(G)|$ disjoint copies of $K_{3,3}$ to G and identifying each vertex of G with a vertex of a different copy of $K_{3,3}$. The resulting graph is $\{\text{ISK}_4, \text{triangle}\}$ -free, not series-parallel, and not bipartite if G is not bipartite, and it contains no vertex of degree at most two.

Finally we are ready to prove the following.

Theorem 25. *If G is an $\{\text{ISK}_4, \text{triangle}\}$ -free graph, then G is 3-colorable.*

Proof. The proof is by induction on $|V(G)|$ using Theorem 2. If G is complete bipartite, then G is 2-colorable. If G has a vertex v of degree at most two, then, by induction, $G \setminus v$ is 3-colorable, and hence G is 3-colorable. If G has a clique cutset C such that (A, B, C) is a partition of $V(G)$ with A anticomplete to B and C a clique, then $\chi(G) = \max\{\chi(G|(A \cup C)), \chi(G|(B \cup C))\}$, and again by induction, G is 3-colorable. \square

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