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Viable trajectories for nonconvex differential inclusions with constant delay

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Abstract: In this paper, we consider a nonconvex differential inclusion with constant delay. We study the existence of viable solutions when the state is constrained to the closure of an open subset of \( \mathbb{R}^n \). The main contribution is a relaxation result stating that, under some assumptions, each “viable solution” of the convexified inclusion can be approximated by “viable solutions” of the original one. This result is obtained thanks to an extension of the celebrated Filippov’s theorem to the case of delay differential inclusions.

Keywords: Delay differential inclusions, relaxation, state constraints, inward pointing conditions.

1. INTRODUCTION

Time delay systems are convenient to model some complex systems arising in population dynamics or engineering sciences. Delays appear naturally in the state variable or in the control when dealing with models involving control systems, and even in both control and state variables. Different approaches have been developed in the literature in order to study stability, controllability, observability and optimality problems for such systems (see, e.g., Fliss and Moumier (1998); Göllmann et al. (2009); Niculescu (2001)).

Differential inclusions is a convenient tool to work with various types of control systems (see, e.g., Aubin and Frankowska (1990)). For instance, a closed loop system can be written as a differential inclusion where, at each state, the set-valued map is defined by the set of all possible feedback controls at this state. Also, differential inclusions are helpful to study control systems with uncertainties, where the set-valued map incorporates model errors.

In the presence of state constraints, the investigation of such differential inclusions becomes very difficult and their analysis has occupied a considerable attention in the literature (see, e.g., Aubin and Mounier (1998); Gollmann et al. (2009); Niculescu (2001)).

Differential inclusions is a convenient tool to work with various types of control systems (see, e.g., Aubin and Frankowska (1990)). For instance, a closed loop system can be written as a differential inclusion where, at each state, the set-valued map is defined by the set of all possible feedback controls at this state. Also, differential inclusions are helpful to study control systems with uncertainties, where the set-valued map incorporates model errors.

Weak solutions to (1) are common in the literature (see, e.g., Aubin (1991); Haddad (1981) for more details). We underline that conditions like (4) are also given for a general type of convex-valued functional differential inclusions (see Aubin (1991); Haddad (1981) for more details).

The existence of viable trajectories for (1) is closely related to the convexity of the values of the set-valued map \( F \). In fact, even in the absence of delays, nonconvex-valued differential inclusions may not have viable solutions, while it is not the case of their convexification (see, e.g., (Aubin, 1991, Example p. 89)). When the set-valued map \( F \) is convex-valued, thanks to the viability theory of Aubin (1991), the existence of viable trajectories is completely characterized by a necessary and sufficient condition linking the geometry of the constraint set \( K \) to the set-valued map \( F \). In the absence of delays, and under some regularity assumptions on \( F \), this condition is as follows

\[ \forall t \in [0,T], x(t) \in K \forall x \in K, \ F(t,x) \cap T_K(x) \neq \emptyset, \tag{4} \]

where \( T_K(x) \) is the contingent cone to \( K \) at \( x \). If (4) holds true, then the constraint set \( K \) is called viability domain.

We underline that conditions like (4) are also given for a general type of convex-valued functional differential inclusions (see Aubin (1991); Haddad (1981) for more details).

The convexity hypothesis on the set-valued map \( F \) is very restrictive for some mathematical models. In order to compensate the lack of convexity, stronger regularity on \( F \) and stronger tangential conditions are needed. In the absence of delays, many works were devoted to various inward pointing conditions allowing to approximate
relaxed feasible trajectories by feasible trajectories and provide estimates on the distance of a given trajectory of unconstrained control system from the set of its feasible trajectories, see for instance Bettiol et al. (2010, 2012); Forcellini and Rampazzo (1999); Frankowska et al. (2016); Frankowska and Mazzola (2013); Frankowska and Rampazzo (2000); Frankowska and Vinter (2000). In the literature, these estimates have been referred to as neighboring feasible trajectory (NFT) estimates.

In general, for various practical examples, condition like (4) is not fulfilled. In this case, the constraint set \( K \) is not a viability domain and the largest viable subset of \( K \) (called viability kernel) is considered. In the absence of delay, viability algorithms allowing the computation of the viability kernel have been conceived for convex-valued differential inclusions (see, e.g., Frankowska and Quincampoix (1991); Saint-Pierre (1994)). Based on these algorithms, numerical methods have been developed (see, e.g., Rouquier et al. (2015)) and used to find viability kernels for numerous examples coming from different fields (see, e.g., Aubin et al. (2011); Haidar et al. (2017)). In order to develop similar numerical tools in the case of delay differential inclusions, viability algorithms and relaxation theorems under state constraints are crucial. The latter point is the purpose of this paper.

Let \( \lambda > 0 \). Consider the following inward pointing condition:

\[
(IPC^\lambda_{rel}) \quad \forall t \in [0,T], \forall x \in \partial K, \forall y \in x + \tau \lambda B, \\
\forall v \in F(t,x,y) \text{ such that } \max_{n \in N^1_K(x)} (n,v) \geq 0, \\
\exists \bar{w} \in \text{Liminf}_{(t,x,y)\rightarrow (t,x,y)} \text{co} F(s,z,\xi) \\
\text{satisfying } \max_{n \in N^1_K(x)} (n,w-v) < 0,
\]

where \( \text{Liminf} \) denotes the Kuratowski lower set limit (see Aubin and Frankowska (1990)), \( N^1_K(x) := N_K(x) \cap S^{n-1} \), \( S^{n-1} \) is the unit sphere and \( N_K(x) \) denotes the Clarke normal cone to \( K \) at \( x \) (see Clarke (1990)). Assuming \( (IPC^\lambda_{rel}) \), we give a relaxation result stating that the set of feasible trajectories of (1) is dense in the set of relaxed feasible trajectories of (2). This is proved by using several preliminary results. The first one is an extension of the Filippov theorem from Filippov (1967) to delay differential inclusions, which is an essential step to construct feasible trajectories. Then, we provide NFT estimates on the distance of a given trajectory from the set of feasible trajectories.

The paper is organized as follows. Section 2 presents the list of notations, definitions and assumptions in use. In Section 3 we state our main results. The proofs can be found in Frankowska and Haidar (2017). The application of our relaxation theorems in optimal control is discussed found in Frankowska and Haidar (2017). The application of these results to constructing similar numerical tools in the case of delay differential inclusions is presented in Rouquier et al. (2015) and used to find viability algorithms, numerical methods have been developed (see, e.g., Frankowska and Haidar (2017)). In order to develop similar numerical tools in the case of delay differential inclusions, viability algorithms allowing the computation of the viability kernel have been conceived for convex-valued differential inclusions (see, e.g., Aubin et al. (2011); Haidar et al. (2017)). In order to develop similar numerical tools in the case of delay differential inclusions, viability algorithms allowing the computation of the viability kernel have been conceived for convex-valued differential inclusions (see, e.g., Aubin et al. (2011); Haidar et al. (2017)). In order to develop similar numerical tools in the case of delay differential inclusions, viability algorithms allowing the computation of the viability kernel have been conceived for convex-valued differential inclusions (see, e.g., Aubin et al. (2011); Haidar et al. (2017)).

2. PRELIMINARIES

In this section we list the notations and the main assumptions in use.

2.1 Notations and definitions

Consider the Euclidean space \((\mathbb{R}^n, \| \cdot \|)\), where \( n \) is a positive integer. We denote by \( B(x,r) \) the closed ball of center \( x \in \mathbb{R}^n \) and radius \( r > 0 \) and by \( B \) the closed unit ball in \( \mathbb{R}^n \) centered at \( 0 \). Let \( coA \) stands for the convex hull of a subset \( A \subset \mathbb{R}^n \). Given interval \( I \subset \mathbb{R} \), \( (C(I,\mathbb{R}^m), \| \cdot \|_c) \) denotes the Banach space of continuous functions from \( I \) into \( \mathbb{R}^m \), where \( \| \cdot \|_c \) is the norm of uniform convergence. We denote by \( L^1(I,\mathbb{R}^m) \) the space of Lebesgue integrable functions from \( I \) to \( \mathbb{R}^m \). Let \( K \) be a nonempty closed subset of \( \mathbb{R}^n \). Int \( K \) be its interior and \( \partial K \) its boundary, \( d_K(x) = \inf_{y \in K} \| x - y \| \) is the distance from \( x \) to \( K \).

We will use the following notion of solution:

**Definition 1.** Let \( 0 \leq t_0 \leq T, \tau > 0 \) and \( \varphi \in C([-\tau,0],\mathbb{R}^n) \). A function \( x \in C([t_0-\tau,T],\mathbb{R}^n) \) is called an \( F \)-trajectory, if \( x(\cdot) \) is absolutely continuous on \([t_0,T]\) and satisfies (1).

An \( F \)-trajectory which verifies the state constraint (3) is called feasible \( F \)-trajectory. A trajectory associated to the relaxed differential inclusion (2) is called relaxed \( F \)-trajectory, and relaxed feasible \( F \)-trajectory if in addition (3) holds true.

2.2 Assumptions

Let \( 0 \leq t_0 \leq T, \tau > 0 \) and \( F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \sim \mathbb{R}^n \) be a set-valued map with non-empty closed images. In our main theorems, we will assume the following regularity conditions on \( F \):

(A1) for every \( X = (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \) the set-valued map \( F(\cdot,X) \) is measurable;

(A2) the set-valued map \( F(t,\cdot) \) is locally Lipschitz, i.e. \( \forall R > 0, \exists \kappa_R(\cdot) \in L^1([t_0,T],\mathbb{R}^+)_r \) such that, for a.e. \( t \in [t_0,T] \) and any \( X = (x_1,y_1) = (x_2,y_2) \in RB \times RB \)

\[
F(t,X) \subset F(t,Y) + \kappa_R(t)\|X-Y\|B;
\]

(A3) the set-valued map \( F \) has a sublinear growth, i.e. there exists \( \sigma > 0 \) such that, for a.e. \( t \in [t_0,T] \) and any \( X = (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \)

\[
F(t,X) \subset \sigma(1 + \| X \|)B;
\]

(A4) for a given \( \lambda > 0 \), the set-valued map \( F \) is upper semicontinuous on \([t_0,T] \times \partial K \times (\partial K + \tau \lambda B) \) i.e. for all \( t \in [t_0,T] \) and all \( X \in \partial K \times (\partial K + \tau \lambda B) \), we have \( F(t,X) \neq \emptyset \) and for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
F(s,Y) \subset F(t,X) + \varepsilon B \quad \forall (s,Y) \in B((t,X),\delta).
\]

3. MAIN RESULTS

3.1 Filippov’s Theorem

The following theorem extends the celebrated Filippov’s theorem from Filippov (1967), to differential inclusions of type (1).

**Theorem 1.** Let \( \lambda > 0 \) and \( \delta_0 \geq 0 \) and assume (A1), (A2). Let \( y \in C([-\tau,0],\mathbb{R}^n) \) be such that \( y(\cdot) \) is absolutely continuous on \([t_0,T]\). Set \( R = \max_{t \in [t_0-\tau,T]} \| y(t) \| \).
where feasible trajectories (see Bettiol et al. (2010) for a counter
F-equivalent, in the case of continuous set-valued map.

Theorem 2. Let \(\lambda > 3.2\) neighboring feasible trajectories theorems
delay. The proof of Theorem 2 uses standard relaxation arguments.
which verifies Theorem 1. The following theorem establishes the possibility of approximating any relaxed \(F\)-trajectory by an \(F\)-trajectory starting from the initial condition.
Theorem 2. Let \(y(\cdot)\) be a relaxed \(F\)-trajectory. Assume (A1), (A2) and (A3). Then for every \(\delta > 0\) there exists an \(F\)-trajectory \(x(\cdot)\) satisfying \(x_0 = y_0\) and \(\sup_{t \in [0, T]} \|x(t) - y(t)\| \leq \delta\).
The proof of Theorem 2 uses standard relaxation arguments which are adapted to differential inclusions with delay.

3.2 Neighboring feasible trajectories theorems

Let \(\lambda > 0\). Consider the following inward pointing condition:

\[
\begin{aligned}
\forall t \in [0, T], \forall x \in \partial K, \forall y \in x + \tau \lambda B, \\
\forall v \in F(t, x, y) \text{ such that } \max_{n \in N^*_K(x)} (n, v) \geq 0,
\end{aligned}
\]

\[
(\text{IPCL}^\lambda)
\]

This condition is an extension of the well known inward pointing conditions to delay differential inclusions. It is equivalent, in the case of continuous set-valued map \(F\) and smooth boundary \(\partial K\), to the following:

\[
\begin{aligned}
\forall t \in [0, T], \forall x \in \partial K, \forall y \in x + \tau \lambda B, \\
\exists w \in F(t, x, y) \text{ satisfying } (n_x, w) < 0,
\end{aligned}
\]

where \(n_x\) denotes the unit outward normal to \(K\) at \(x\). Note that, in the case of non-smooth boundary, the condition \(\max_{n \in N^*_K(x)} (n, w) < 0\) is not sufficient in itself to construct feasible trajectories (see Bettiol et al. (2010) for a counter example). The (IPCL) condition can be interpreted as illustrated by Figure 1: Let \(t \in [0, T]\) and \(x\) be on the boundary of \(K\). A velocity at \(x\) is a vector which depends particularly on its history. This history, denoted by \(y\), lies in a ball of center \(x\) and radius \(\tau \lambda\) (where \(\tau\) is the constant delay and \(\lambda\) depends on \(F\)). This condition requires, for every velocity \(v\) pointing to the exterior of \(K\), the existence of a vector \(w\) in the lower limit of \(F(s, z, \xi)\) when \((s, z, \xi) \to (t, x, y)\) such that \(v - w\) points into the interior of \(K\).

The following theorem shows the existence of a feasible \(F\)-trajectory and provides an estimate of the distance (in the norm of uniform convergence) of this trajectory from a specified \(F\)-trajectory.

Theorem 3. Assume (A1)-(A3). Let \(\tau > 0\), \(r_0 > 0\) and \(\lambda_0 > 0\) and suppose that, for

\[
\lambda = \max\{\lambda_0, (1 + 1 + \lambda_0 \tau + r_0) e^{\tau \sigma} \sigma\},
\]

assumptions (A4) and (IPCL) hold true. Then there exists a constant \(C > 0\) such that for any \(t_0 \in [0, T]\) and every \(F\)-trajectory \(\hat{x}(\cdot)\) on \([t_0 - \tau, T]\) with \(\lambda_0\)-Lipschitz \(\hat{x}_{t_0}\) and \(\hat{x}(t_0) \in K \cap r_0 B\), and for any \(\varepsilon_0 > 0\), we can find a feasible \(F\)-trajectory \(x(\cdot)\) on \([t_0 - \tau, T]\) satisfying \(x_{t_0} = \hat{x}_{t_0}, x((t_0, T]) \subset \text{Int} K\) and

\[
\|x(t) - \hat{x}(t)\| \leq C \left( \max_{t \in [t_0, T]} d_K(\hat{x}(t)) + \varepsilon_0 \right).
\]

Sketch of proof: The proof of this theorem is done in two main steps. Firstly, we prove the existence of solutions which satisfy the state constraint only on a subinterval of \([t_0, T]\). We show the existence of positive constants \(\delta\) and \(c\) for which, for every \(t \in [0, T]\) and every \(F\)-trajectory \(\hat{x}(\cdot)\) on \([t - \tau, T]\) and any \(\varepsilon > 0\), we can find an \(F\)-trajectory satisfying \(\|x(t) - \hat{x}(t)\| \leq C \left( \max_{t \in [t_0, T]} d_K(\hat{x}(t)) + \varepsilon \right)\). Theorem 3 together with Theorem 2 imply that under the inward pointing condition (IPCL), the set of \(F\)-trajectories lying in the interior of the constraint set \(K\),
for \( t \in (t_0, T) \) and starting at \( \tilde{x}_{t_0} \), is dense in the set of feasible relaxed \( F \)-trajectories. This results from the following corollary:

**Corollary 4.** Under all the assumptions of Theorem 3, for any feasible relaxed \( F \)-trajectory \( \hat{x}(\cdot) \) with \( \lambda_2 \)-Lipschitz \( \hat{x}_{t_0} \) and \( \hat{x}(t_0) \in K \cap \mathbb{R}^n \), and any \( \delta > 0 \), there exists a feasible \( F \)-trajectory \( x(\cdot) \) such that \( x_{t_0} = x_{t_0} = \hat{x}_{t_0} \) and \( x(t_0, T) \in \text{Int} K \) and \( \|x_t - \hat{x}_t\| < \delta \) for all \( t \in [t_0, T] \).

**Sketch of proof:** The proof of this corollary is illustrated by Figure 2. Let \( \delta > 0 \). Starting from a feasible relaxed trajectory \( \hat{x} \), thanks to Theorem 2, we can find an \( F \)-trajectory \( x(\cdot) \) (which may violate the constraint set) such that \( \|x_t - \hat{x}_t\| < \delta/2 \) for all \( t \in [t_0, T] \). For the neighboring trajectory \( \tilde{x} \), thanks to Theorem 3, we can find a feasible \( F \)-trajectory \( x(\cdot) \), such that \( \|x_t - \tilde{x}_t\| < \delta/2 \) for all \( t \in [t_0, T] \). Then, \( \|x_t - \tilde{x}_t\| < \delta \), for all \( t \in [t_0, T] \).

Now, assume the relaxed inward pointing condition given by (\( IPC_{\text{rel}} \)). The following theorem is related to Theorem 3, however neither one is contained in another.

**Theorem 5.** Assume (A1)–(A3). Let \( \tau > 0 \), \( r_0 > 0 \) and \( \lambda_0 > 0 \) and suppose that, for \( \lambda \) given by (6), assumptions (A4) and (\( IPC_{\text{rel}} \)) hold true. Then there exists a constant \( C \) such that for any \( t_0 \in [0, T] \) and every relaxed \( F \)-trajectory \( \hat{x}(\cdot) \) on \([t_0, T] \) with \( \lambda_0 \)-Lipschitz \( \hat{x}_{t_0} \) and \( \hat{x}(t_0) \in K \cap r_0 B \), and for any \( \varepsilon > 0 \), we can find a relaxed feasible \( F \)-trajectory \( x(\cdot) \) on \([t_0, T] \) satisfying \( x_{t_0} = \hat{x}_{t_0} \), \( x(t_0, T) \in \text{Int} K \) and

\[
\|x_t - \hat{x}_t\| \leq C \left( \max_{t \in [t_0, T]} d_K(x(t)) + \varepsilon_0 \right).
\]

Theorem 5 and the constructive argument of (Betti et al., 2012, Proof of Lemma 5.2) imply the following Corollary:

**Corollary 6.** Under all the assumptions of Theorem 5, for any relaxed feasible \( F \)-trajectory \( \hat{x}(\cdot) \) with \( \lambda_2 \)-Lipschitz \( \hat{x}_{t_0} \) and \( \hat{x}(t_0) \in K \cap r_0 B \), and any \( \delta > 0 \), there exists a feasible \( F \)-trajectory \( x(\cdot) \) such that \( x_{t_0} = \hat{x}_{t_0} \), \( x(t_0, T) \in \text{Int} K \) and \( \|x_t - \hat{x}_t\| < \delta \) for all \( t \in [t_0, T] \).

The proofs of Theorem 5 and Corollary 6 are the adaptation of what is done before to the case of the relaxed inward pointing condition (\( IPC_{\text{rel}} \)).

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4. APPLICATIONS OF THE RELAXATION THEOREMS IN OPTIMAL CONTROL

The relaxation theorems obtained in this paper allow to show that the value function of the original optimal control problem coincides with the value function of the relaxed one. This is described, briefly, by the following.

Let \( g : \mathbb{R}^n \to \mathbb{R} \) be a \( \lambda_1 \)-Lipschitz function. Suppose that the assumptions of Theorem 5 are satisfied. Let \( \mathcal{S}_{[x_0, T]}(x_0) \) be the set of all solutions to (1), (3), where \( x_{t_0} = x_0 \) and \( x_0 \in \mathcal{X}_\lambda \), with \( \mathcal{X}_\lambda := \{ \psi \in C([-\tau, 0], \mathbb{R}^n) : \psi \text{ is } \lambda \text{-Lipschitz, } \psi(0) \in K \} \).

Consider the Mayer optimal control problem

\[
\min \{ g(x(T)) : x(\cdot) \in \mathcal{S}_{[0, T]}(x_0) \}.
\]

The value function, associated to problem (9),

\[
V : [0, T] \times \mathcal{X} \to \mathbb{R} \cup \{+\infty\},
\]

is defined by

\[
V(t_0, y_0) = \inf \{ g(x(T)) : x(\cdot) \in \mathcal{S}_{[t_0, T]}(y_0) \}
\]

with the convention that \( V(t_0, y_0) = +\infty \) if \( \mathcal{S}_{[t_0, T]}(y_0) = \emptyset \).

Thanks to Corollary 6, one can prove that \( V \) is equal to the value function of the relaxed Mayer problem, and thus any optimal solution to the Mayer problem is also optimal for the relaxed Mayer problem. Indeed, let us denote by \( \Gamma \) the value function of the relaxed Mayer problem. Fix \( (t_0, y_0) \in [0, T] \times \mathcal{X} \). We have clearly \( \nabla V(t_0, y_0) \leq V(t_0, y_0) \). On other hand, for every \( \varepsilon > 0 \),

\[
\nabla V(t_0, y_0) \leq \frac{\partial g(x(T))}{\partial x} + \lambda_1 \left. \frac{\partial^2 g(x(T))}{\partial x^2} \right|_{x_0} + \varepsilon,
\]

where \( \frac{\partial g(x(T))}{\partial x} \) is a relaxed feasible trajectory verifying \( \nabla V(t_0, y_0) \geq \frac{\partial g(x(T))}{\partial x} - \varepsilon/2 \) and \( x(\cdot) \) an associated feasible trajectory satisfying (thanks to Corollary 6)

\[
\frac{\partial g(x(T))}{\partial x} \leq \varepsilon/2 \lambda_1, \quad \forall t \in [0, T].
\]

Being true for arbitrarily small \( \varepsilon \), inequality (11) implies that \( \nabla V(t_0, y_0) = V(t_0, y_0) \).

In addition, Theorem 2 together with Theorem 5, allow to prove that \( V \) is Lipschitz on \( \mathcal{X} \). This latter property allows to characterize the optimal solutions of the Mayer problem by means of the relaxed differential inclusion (see (Frankowska and Mazzola, 2013, Theorem 5.3), for more details in the case of differential inclusions without delay).

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5. EXAMPLE: LANDFILL WASTE MANAGEMENT

Here, we present an example which concerns the management of landfill waste. A landfill is a controlled site for the disposal of waste materials where the solid waste is disposed and treated. The treatment is decomposed in two parts: “biological treatment” of the solubilized substrates by means of a microbial biodegradation and “physical treatment” of the unsolubilized substrates through the recirculation of the landfill leachate. In fact, landfill leachate is the liquid that drains from waste during the landfill operation; knowing its highest acidity, the re-circulation of this waste-water improve the system mixing and then the solubilization of the solid material. A simplified model of
this complex system is proposed in Rapaport et al. (2016). This is given by
t
\[
\begin{align*}
\dot{x}_1 &= -f(x_1, u) \\
\dot{x}_2 &= f(x_1, u) - g(x_2)x_3, \\
\dot{x}_3 &= g(x_2)x_3, 
\end{align*}
\]

where \(x_1\) and \(x_2\) denote the unsolubilized and solubilized substrates concentrations, \(x_3\) is the concentration of the biomass that degrades the solubilized substrate and \(g\) is the biomass growth rate function. The function \(f\) describes the transfer kinetics of \(x_1\) to \(x_2\), which is parametrized by \(u\), the leachate re-circulation flow; it is supposed to be positive, equal to zero at \(x_1 = 0 \ (u = 0, \ \text{respectively})\) and increasing with respect to \(x_1 \ (u, \ \text{respectively})\). The function \(g\) describes the specific microbial growth rate, it is positive, equal to zero at \(u\) and increasing with respect to \(x_1\) (\(u, \ \text{respectively}\)).

In general, \(F\) can be written in the form of (1), where the set-valued function \(g\) describes the specific microbial growth rate, it is positive, equal to zero at \(u\) and increasing with respect to \(x_1\) (\(u, \ \text{respectively}\)).

\[
g(x_2) = \frac{\bar{g} x_2}{k_1 + x_2 + x_2^2/k_2},
\]

where \(\bar{g}, k_1\) and \(k_2\) are positive parameters. The objective of this paper is to derive relaxations of feasible trajectories of a delay differential inclusion. This result is refined in Corollary 4 by considering the relaxed inward pointing condition \(IPC^3\).

In this paper we deal with an interesting problem concerning the existence of solutions for delay differential inclusions, in presence of state constraints. We present a relaxation result stating that the set of trajectories lying in the interior of the constraint is dense in the set of constrained trajectories of the corresponding convexified inclusion. The proof of this result is given in several steps. First, in Theorem 1, we extend the celebrated Filippov’s theorem to differential inclusions with constant delay. Then, thanks to this extension, we give, in Theorem 2, a relaxation result without state constraint. Then, we generalize this latter result to the case where the state variable is constrained to the closure of an open subset of \(\mathbb{R}^n\). Under the new inward pointing condition \(IPC^3\), we show, in Corollary 4, that each feasible trajectory of a delay differential inclusion can be approximated (in the norm of uniform convergence) by trajectories lying in the interior of the constraint set \(K\) and starting at the same initial condition. We also estimate, in Theorem 3, the distance between feasible and non-feasible trajectories, using the magnitude of the constraints violation. This result is refined in Corollary 6 by considering the relaxed inward pointing condition \(IPC^3\). Thanks to our relaxation theorems, we show that the value function of an optimal control problem coincides with the value function of its relaxed one. An example illustrating the applicability of our result is also discussed.

6. CONCLUSION

In this paper we deal with an interesting problem concerning the existence of solutions for delay differential inclusions, in presence of state constraints. We present a relaxation result stating that the set of trajectories lying in the interior of the constraint is dense in the set of constrained trajectories of the corresponding convexified inclusion. The proof of this result is given in several steps. First, in Theorem 1, we extend the celebrated Filippov’s theorem to differential inclusions with constant delay. Then, thanks to this extension, we give, in Theorem 2, a relaxation result without state constraint. Then, we generalize this latter result to the case where the state variable is constrained to the closure of an open subset of \(\mathbb{R}^n\). Under the new inward pointing condition \(IPC^3\), we show, in Corollary 4, that each feasible trajectory of a delay differential inclusion can be approximated (in the norm of uniform convergence) by trajectories lying in the interior of the constraint set \(K\) and starting at the same initial condition. We also estimate, in Theorem 3, the distance between feasible and non-feasible trajectories, using the magnitude of the constraints violation. This result is refined in Corollary 6 by considering the relaxed inward pointing condition \(IPC^3\). Thanks to our relaxation theorems, we show that the value function of an optimal control problem coincides with the value function of its relaxed one. An example illustrating the applicability of our result is also discussed.

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