

# Explicit $H^1$ -Estimate for the solution of the Lamé system with mixed boundary conditions.

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## Abstract

In this paper we consider Lamé system of equations on a polygonal convex domain with mixed boundary conditions of Dirichlet-Neumann type. An explicit  $L^2$  norm estimate for the gradient of the solution of this problem is established. This leads to an explicit bound of the  $H^1$  norm of this solution. Note that the obtained upper-bound is not optimal.

**keywords:** Lamé system; Korn's inequality; Poincare's inequality; inequality of trace; explicit estimates.

**AMS subject classification:** 35J57, 74B05

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## 1 Introduction

The static equilibrium of a deformable structure occupying a domain  $\Omega$  subset of  $\mathbb{R}^2$  is governed by the Lamé linear elasto-static system of equations, see [5]. In this paper, we restrict the study to a convex domain  $\Omega$  whose boundary has a polygonal shape that posses  $m + 1$  edges with  $m \geq 2$ . We denote  $\Gamma = \cup \Gamma_i$  its boundary and  $d(\Omega)$  its diameter. This system is given by

$$\begin{cases} Lu = f & \text{a.e in } \Omega, \\ \sigma \cdot \vec{n}_i = g_i & \text{on } (\Gamma - \Gamma_0) \cap \Gamma_i, \quad 1 \leq i \leq m \\ u = 0 & \text{on } \Gamma_0. \end{cases} \quad (1)$$

We assume that condition  $(H_2)$  of Theorem 2.3 stated in the paper [9] is satisfied by  $\Gamma$ . This condition is formulated in (5) below. The vector function  $u = (u^1, u^2)$  satisfying the system (8) describes a displacement in

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the plane. In this model, we impose a Dirichlet homogeneous condition on  $\Gamma_0$  and a Neumann condition on the rest of the boundary. The equality on the boundary is understood in the sense of the trace. We denote  $L$  the Lamé operator defined by:

$$Lu := -\operatorname{div} \sigma(u) = -\operatorname{div}[2\mu\varepsilon(u) + \lambda \operatorname{Tr} \varepsilon(u) Id] \quad (2)$$

The data functions  $f$  and  $g$  at the right hand side satisfy  $f \in [L^2(\Omega)]^2$  and  $g \in [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]^2$ . The vector  $\vec{n}_i$  represents the outside normal to  $\Gamma_i$ . We write  $\mu$  and  $\lambda$  the Lamé's coefficients. We place ourselves in the isotropic framework, the deformation tensor  $\varepsilon$  is defined by

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^t u), \quad (3)$$

The weak form of problem (1) is (see [2], [5]):

find  $u \in V; \forall v \in V$

$$\int_{\Omega} 2\mu\varepsilon(u)\varepsilon(v) + \lambda \operatorname{div} u \operatorname{div} v \, dx = \int_{\Omega} f v \, dx + \int_{\Gamma - \Gamma_0} g v \, d\sigma(x) \quad (4)$$

where

$$V = \{v \in [H^1(\Omega)]^2; \quad v = 0 \quad \text{on} \quad \Gamma_0\}$$

The problem of existence and uniqueness in  $V$  of the solution of (4) is classic, (see [2]).

If we denote  $\theta$  the interior angle between  $\Gamma_j$  and  $\Gamma_k$  such that  $\bar{\Gamma}_j \cap \bar{\Gamma}_k \neq \emptyset$  and if we denote  $\gamma$  the interior angle between the Neumann part of the boundary  $\Gamma_N$  and the Dirichlet part of the boundary  $\Gamma_D$  such that  $\Gamma_N \cap \Gamma_D \neq \emptyset$ , then we impose

$$\theta \leq 2\pi, \quad \gamma \leq \pi. \quad (5)$$

The reason behind this assumption on the boundary is to get a better regularity of the solution of the weak problem (4). Precisely, in that case we have, following [9],  $u \in [H^{\frac{3}{2}+\alpha}(\Omega)]^2$  for some positive  $\alpha$ , which implies in particular, using the appropriate Sobolev embedding, see [1], that  $u \in [C^{0, \frac{1}{2}+\alpha}(\bar{\Omega})]^2$  i.e.  $u$  is  $(\frac{1}{2} + \alpha)$ -holder continuous. Let us denote

$$\|\varepsilon(u)\|_{0,\Omega} := \left( \int_{\Omega} \varepsilon(u)\varepsilon(u) \, dx \right)^{\frac{1}{2}}, \quad \|\nabla u\|_{0,\Omega} := \left( \int_{\Omega} |\nabla u^1|^2 + |\nabla u^2|^2 \, dx \right)^{\frac{1}{2}}.$$

By using the second Korn inequality, see [7], the trace and the Poincaré's inequalities, one easily gets from (4) the following estimate

$$\|\nabla u\|_{0,\Omega} \leq \frac{1}{c_k} \frac{1}{2\mu} (c_p \|f\|_{0,\Omega} + c_{p,t} \|g\|_{\frac{1}{2}, \Gamma - \Gamma_0}), \quad (6)$$

where  $c_{p,t}$  is a constant that depends of Poincaré constant and the constant of trace inequality.  $c_k$  is the constant of the Korn's inequality. Note that the value of the constant  $c_k$  and  $c_{p,t}$  appearing in (6) are unknown and can not be explicitly lower-bounded in the general case. We propose to determine explicitly these constants. The main result of this work is stated in the following theorem:

**Theorem 1.** *The unique weak solution  $u$  of (4) on the polygonal domain  $\Omega$  admits the explicit upper bound*

$$\frac{1}{\mu} [2\sqrt{3}(c_p \|f\|_{0,\Omega} + c_t \|g\|_{\frac{1}{2},\Gamma-\Gamma_0}) + 1] \quad (7)$$

where  $c_p := d(\Omega)$ ,  $c_{tr} := 2\sqrt{d(\Omega)}$  and  $d(\Omega)$  represent the diameter of  $\Omega$ .

The estimate (7) is similar to (6), the constants that are present are the same. Before demonstrating this theorem, it is useful to go through some remarks and results. Denote  $x_i$ , for  $1 \leq i \leq m$ , the vertex of the polygon that connects  $\Gamma_{i-1}$  with  $\Gamma_i$  and  $x_0$  the one that connects  $\Gamma_m$  to  $\Gamma_0$ . Define the auxiliary function  $u_\epsilon \in H^1(\Omega)$  as the unique solution to the following Dirichlet problem

$$\begin{cases} Lu_\epsilon = f & \text{a.e in } \Omega, \\ u_\epsilon = u_\epsilon^d & \text{on } \Gamma. \end{cases} \quad (8)$$

Where  $u_\epsilon^d$  is the trace of the function

$$\phi_\epsilon(x)u(x) \quad (9)$$

on the boundary  $\Gamma$ ; if  $\epsilon < \frac{|\Gamma_i|}{2} \forall i, 0 \leq i \leq m$  then  $\phi_\epsilon$  is defined by

$$\begin{cases} \phi_\epsilon(x) = 0, & \|x - x_i\| \leq \epsilon^2, \quad 0 \leq i \leq m; \\ \phi_\epsilon(x) = \exp\left[-\frac{\epsilon^{\frac{1}{2}}(\epsilon - \|x - x_i\|)}{\|x - x_i\| - \epsilon^2}\right], & \epsilon^2 < \|x - x_i\| < \epsilon, \quad 0 \leq i \leq m; \\ \phi_\epsilon(x) = 1, & \epsilon \leq \|x - x_i\|, \quad 0 \leq i \leq m, \end{cases}$$

let us denote

$$D_{i,\epsilon} := \{x \in \mathbb{R}^2 \text{ such that } \|x - x_i\| < \epsilon^2\}.$$

We easily see that  $\phi_\epsilon \in C^0(\overline{\Omega})$ , consequently, there will be no jump when passing to the distributional derivative and thus  $\nabla u_\epsilon \in L^2(\Omega)$  i.e.  $u_\epsilon \in H^1(\Omega)$ . It is shown, using Lebesgue's dominated convergence theorem for instance, that  $\|\phi_\epsilon - 1\|_{0,\Gamma_i} \rightarrow 0$  i.e. we have convergence in  $L^2$  along the edge  $\Gamma_i$ . The functions  $\phi_\epsilon$  are identically zero on a small neighborhood of the respective vertices of the polygon.

In the sequel, we denote  $u_\epsilon$  the vector-valued function  $u_\epsilon = (u_\epsilon^1, u_\epsilon^2)$ .

## 2 Weak problem for $u_\epsilon$ and approximation results

First of all, we construct the weak problem verified by the approximating function  $u_\epsilon$ . With the approximating displacement  $u_\epsilon \in V$  is associated the approximating stress tensor

$$\sigma_\epsilon := 2\mu\varepsilon(u_\epsilon) + \lambda \operatorname{Tr} \varepsilon(u_\epsilon)I, \quad (10)$$

since  $Lu_\epsilon = \operatorname{div} \sigma_\epsilon = f$ , then  $\sigma_\epsilon \in [H(\operatorname{div})(\Omega)]^{2 \times 2}$ . For a fixed  $\epsilon$ , by density of the regular functions in the space  $H(\operatorname{div})(\Omega)$ , there exists  $\sigma_\epsilon^n \in [C^\infty(\overline{\Omega})]^{2 \times 2}$  such that  $\sigma_\epsilon^n \rightarrow \sigma_\epsilon$  in  $[H(\operatorname{div})(\Omega)]^{2 \times 2}$ . This means

$$\|\sigma_\epsilon^n - \sigma_\epsilon\|_{\operatorname{div}, \Omega} := \|\operatorname{div} \sigma_\epsilon^n - \operatorname{div} \sigma_\epsilon\|_{0, \Omega} + \|\sigma_\epsilon^n - \sigma_\epsilon\|_{0, \Omega} \rightarrow 0 \quad (11)$$

when  $n \rightarrow \infty$ . We put  $\operatorname{div} \sigma_\epsilon^n = f^n$ , then integrating by part against a test function  $v \in [C^\infty(\overline{\Omega})]^2 \cap V$  yields the following

$$\int_{\Omega} \sigma_\epsilon^n \nabla v = \int_{\Omega} f^n v + \int_{\Gamma} \sigma_\epsilon^n \cdot \vec{n} v d\sigma.$$

Passing to the limit in  $n$  using (11), we find  $\forall v \in [C^\infty(\overline{\Omega})]^2 \cap V$

$$\int_{\Omega} \sigma_\epsilon \nabla v = \int_{\Omega} f v + \langle \sigma_\epsilon \cdot \vec{n}, v \rangle_{[H^{\frac{1}{2}}]^\prime(\Gamma - \Gamma_0) \times [H^{\frac{1}{2}}]^\prime(\Gamma - \Gamma_0)},$$

where  $\sigma_\epsilon \cdot \vec{n} =: g_\epsilon \in [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]^\prime$  is the image of the normal component  $\sigma_\epsilon$  by the trace operator on  $\Gamma$ . Since, following the main result in [4],  $[C^\infty(\overline{\Omega})]^2 \cap V$  is a dense subset of  $V \subset H^1(\Omega)$ , then, according to the definition (3) and the expression (10), the function  $u_\epsilon$  satisfy

$\forall v \in V$ ,

$$\int_{\Omega} 2\mu\varepsilon(u_\epsilon)\varepsilon(v) + \int_{\Omega} \lambda \operatorname{div} u_\epsilon \operatorname{div} v = \int_{\Omega} f v + \langle g_\epsilon, v \rangle_{[H^{\frac{1}{2}}]^\prime(\Gamma - \Gamma_0) \times [H^{\frac{1}{2}}]^\prime(\Gamma - \Gamma_0)}; \quad (12)$$

this is the weak problem satisfied by the approximating function  $u_\epsilon$ .

Let us recall, (see [8]), that the  $H^{\frac{1}{2}}$ -norm in one dimension on  $\Gamma_i$  is defined by:

$$\|u\|_{\frac{1}{2}, \Gamma_i} := (\|u\|_{0, \Gamma_i}^2 + \int_{\Gamma_i} \int_{\Gamma_i} \frac{|u(x) - u(y)|^2}{\|x - y\|^2} dx dy)^{\frac{1}{2}}.$$

**Remark 2.** For any sufficiently small  $\epsilon > 0$ , it is possible to overlap  $\Omega$  with a collection of open sets  $(W_j^\epsilon)_j$  such that for any  $j$ ,  $W_j^\epsilon \cap \Gamma$  is either empty or equals one of the following subsets: for some  $0 \leq i \leq m - 1$

$$1) \quad \Gamma_i^{1, \epsilon} := \{x \in \Gamma_i; \quad 0 < \|x - x_i\| < 2\epsilon\};$$

- 2)  $\Gamma_i^{2,\epsilon} := \{x \in \Gamma_i; \quad 0 < \|x - x_{i+1}\| < 2\epsilon\};$   
 3)  $\Gamma_i^{3,\epsilon} := \{x \in \Gamma_i; \quad \|x - x_i\| > \frac{3}{2}\epsilon \quad \text{and} \quad \|x - x_{i+1}\| > \frac{3}{2}\epsilon\};$   
 4)  $\Gamma_i^{4,\epsilon} = \{x \in \Gamma_i \cup \Gamma_{i+1}; \quad \|x - x_{i+1}\| < \epsilon^2\};$

and for  $i = m$

- 1)  $\Gamma_m^{1,\epsilon} := \{x \in \Gamma_m; \quad 0 < \|x - x_m\| < 2\epsilon\};$   
 2)  $\Gamma_m^{2,\epsilon} := \{x \in \Gamma_m; \quad 0 < \|x - x_0\| < 2\epsilon\};$   
 3)  $\Gamma_m^{3,\epsilon} := \{x \in \Gamma_m; \quad \|x - x_m\| > \frac{3}{2}\epsilon \quad \text{and} \quad \|x - x_0\| > \frac{3}{2}\epsilon\};$   
 4)  $\Gamma_m^{4,\epsilon} = \{x \in \Gamma_m \cup \Gamma_0; \quad \|x - x_0\| < \epsilon^2\}$

let  $(\vartheta_j^\epsilon)_j$ , with  $\text{supp } \vartheta_j^\epsilon \subset W_j^\epsilon$ , a  $C^1$ -partition of unity with respect to this overlap; since  $\vartheta_j^\epsilon \in H^{\frac{1}{2}}(\Gamma)$  then

$$\begin{aligned}
 \|\phi_\epsilon - 1\|_{\frac{1}{2},\Gamma} &= \|(\phi_\epsilon - 1) \sum_j \vartheta_j^\epsilon\|_{\frac{1}{2},\Gamma} \\
 &\leq \sum_{i=0}^m \|(\phi_\epsilon - 1) \sum_{j, \text{supp } \vartheta_j^\epsilon \subset \Gamma_i} \vartheta_j^\epsilon\|_{\frac{1}{2},\Gamma_i} + \sum_{i=0}^m \|(\phi_\epsilon - 1)\|_{\frac{1}{2},\Gamma_i^{4,\epsilon}} \\
 &\leq \sum_{i=0}^m \|\phi_\epsilon - 1\|_{\frac{1}{2},\Gamma_i} + (m+1) \|(\phi_\epsilon - 1)\|_{\frac{1}{2},\Gamma_0^{4,\epsilon}} \\
 &= \sum_{i=0}^m \|\phi_\epsilon - 1\|_{\frac{1}{2},\Gamma_i} + (m+1) \|(\phi_\epsilon - 1)\|_{0,\Gamma_0^{4,\epsilon}} \tag{13}
 \end{aligned}$$

So, using the definition and symmetry of  $\phi_\epsilon$ , we get for all  $0 \leq i \leq m$

$$\begin{aligned}
 \|\phi_\epsilon - 1\|_{\frac{1}{2},\Gamma_i} &= \|(\phi_\epsilon - 1) \sum_{j, \text{supp } \vartheta_j^\epsilon \subset \Gamma_i} \vartheta_j^\epsilon\|_{\frac{1}{2},\Gamma_i} \\
 &\leq \|(\phi_\epsilon - 1) \sum_{j, \text{supp } \vartheta_j^\epsilon \subset \Gamma_i^{1,\epsilon}} \vartheta_j^\epsilon\|_{\frac{1}{2},\Gamma_i} + \|(\phi_\epsilon - 1) \sum_{j, \text{supp } \vartheta_j^\epsilon \subset \Gamma_i^{2,\epsilon}} \vartheta_j^\epsilon\|_{\frac{1}{2},\Gamma_i} \\
 &\quad + \|(\phi_\epsilon - 1) \sum_{j, \text{supp } \vartheta_j^\epsilon \subset \Gamma_i^{3,\epsilon}} \vartheta_j^\epsilon\|_{\frac{1}{2},\Gamma_i} + 2 \|(\phi_\epsilon - 1)\|_{\frac{1}{2},\Gamma_i^{4,\epsilon}} \\
 &\leq \sum_{j=1}^3 \|\phi_\epsilon - 1\|_{\frac{1}{2},\Gamma_i^{j,\epsilon}} + 2 \|\phi_\epsilon - 1\|_{\frac{1}{2},\Gamma_i^{4,\epsilon}} \\
 &= 2 \|\phi_\epsilon - 1\|_{\frac{1}{2},\Gamma_i^{1,\epsilon}} + 2 \|\phi_\epsilon - 1\|_{\frac{1}{2},\Gamma_i^{4,\epsilon}}
 \end{aligned}$$

thus we have

$$\begin{aligned}
 \|\phi_\epsilon - 1\|_{\frac{1}{2},\Gamma_i} &\leq 2 \|\phi_\epsilon - 1\|_{0,\Gamma_i^{1,\epsilon}} + 2 \left( \int_{\Gamma_i^{1,\epsilon}} \int_{\Gamma_i^{1,\epsilon}} \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{\|x - y\|^2} dx dy \right)^{\frac{1}{2}} \\
 &\quad + 2 \|(\phi_\epsilon - 1)\|_{0,\Gamma_i^{4,\epsilon}}. \tag{14}
 \end{aligned}$$

**Lemma 3.** *The functions  $\phi_\epsilon$  admit the following limit for all  $0 \leq i \leq m$*

$$\|\phi_\epsilon - 1\|_{\frac{1}{2}, \Gamma_i} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

*Proof.* If we choose the vertex point  $x_i$  as the origin of the  $\mathbb{R}^2$ -orthonormal coordinate system and  $\Gamma_i$  supported by the positive half  $x$ -axis then the abscissas of  $x \in \Gamma_i^{1, \epsilon} \equiv ]0, 2\epsilon[$  verify

$$\|x - x_i\| = |x| = x.$$

the  $H^{\frac{1}{2}}$ -semi-norm on  $\Gamma_i$  writes by using the definition of  $\phi_\epsilon$

$$|\phi_\epsilon - 1|_{\frac{1}{2}, \Gamma_i}^2 := \int_{\Gamma_i} \int_{\Gamma_i} \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{|x - y|^2} dx dy \leq 2 \int_0^\epsilon \int_0^\epsilon \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{|x - y|^2} dx dy. \quad (15)$$

Consider the decomposition of (15) into four partial double integrals

$$1) \int_0^{\epsilon^2} \int_0^{\epsilon^2} \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{|x - y|^2} dx dy = 0,$$

this is obvious.

$$2) \int_{\epsilon^2}^\epsilon \int_{\epsilon^2}^\epsilon \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{|x - y|^2} dx dy \leq \int_{\epsilon^2}^\epsilon \int_{\epsilon^2}^\epsilon \frac{|\exp[-\frac{\epsilon^{\frac{1}{2}}(\epsilon-x)}{(x-\epsilon^2)}] - \exp[-\frac{\epsilon^{\frac{1}{2}}(\epsilon-y)}{(y-\epsilon^2)}]|^2}{|x - y|^2} dx dy$$

The function  $F(x) := \exp[-\frac{\epsilon^{\frac{1}{2}}(\epsilon-x)}{(x-\epsilon^2)}]$  is  $C^1(] \epsilon^2, \epsilon[)$  and thus lipschitz. We have, using the fact that  $x \rightarrow F'(x)$  is increasing on  $[\epsilon^2, \frac{\epsilon}{2}]$ , that

$$|F'(x)| \leq \frac{\epsilon^{\frac{1}{2}}(1-\epsilon)}{\epsilon(\frac{1}{2}-\epsilon)^2} \exp(\frac{-\epsilon^{\frac{1}{2}}}{\frac{1}{2}-\epsilon}) =: L_1,$$

$\forall x \in [\epsilon^2, \frac{\epsilon}{2}]$ . On the other hand

$$|F'(x)| \leq \frac{\epsilon^{\frac{1}{2}}}{\epsilon(1-\epsilon)} \exp(\frac{2}{1-\epsilon}) =: L_2,$$

$\forall x \in [\frac{\epsilon}{2}, \epsilon]$ . Therefore we conclude that

$$|F'(x)| \leq L := \max(L_1, L_2) \leq L_1 + L_2$$

for all  $x \in [\epsilon^2, \epsilon]$ . This yields

$$\begin{aligned} & \left( \int_{\epsilon^2}^{\epsilon} \int_{\epsilon^2}^{\epsilon} \frac{|\phi_{\epsilon}(x) - \phi_{\epsilon}(y)|^2}{|x - y|^2} dx dy \right)^{\frac{1}{2}} \\ & \leq \left( \int_{\epsilon^2}^{\epsilon} \int_{\epsilon^2}^{\epsilon} \frac{\left| \frac{\epsilon^{\frac{1}{2}}}{\epsilon(1-\epsilon)} \exp\left(\frac{2}{1-\epsilon}\right) + \frac{\epsilon^{\frac{1}{2}(1-\epsilon)}}{\epsilon(\frac{1}{2}-\epsilon)^2} \exp\left(\frac{-\frac{\epsilon^{\frac{1}{2}}}{\frac{1}{2}-\epsilon}}\right) \right|^2 |x - y|^2}{|x - y|^2} dx dy \right)^{\frac{1}{2}} \\ & \leq \left( \int_{\epsilon^2}^{\epsilon} \int_{\epsilon^2}^{\epsilon} \left| \frac{\epsilon^{\frac{1}{2}}}{\epsilon(1-\epsilon)} \exp\left(\frac{2}{1-\epsilon}\right) \right|^2 dx dy \right)^{\frac{1}{2}} + \left( \int_{\epsilon^2}^{\epsilon} \int_{\epsilon^2}^{\epsilon} \left| \frac{\epsilon^{\frac{1}{2}(1-\epsilon)}}{\epsilon(\frac{1}{2}-\epsilon)^2} \exp\left(\frac{-\frac{\epsilon^{\frac{1}{2}}}{\frac{1}{2}-\epsilon}}\right) \right|^2 dx dy \right)^{\frac{1}{2}} \\ & \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \end{aligned}$$

$$\begin{aligned} 3) & \int_0^{\epsilon^2} \int_{\epsilon^2}^{\epsilon} \frac{|\phi_{\epsilon}(x) - \phi_{\epsilon}(y)|^2}{|x - y|^2} dx dy \\ & \leq \int_0^{\epsilon^2} \int_{\epsilon^2}^{\frac{\epsilon}{2}} \frac{1}{|x - \epsilon^2|^2} \exp^2\left[-\frac{\epsilon^{\frac{1}{2}}(\epsilon - x)}{(x - \epsilon^2)}\right] dx dy + \int_0^{\epsilon^2} \int_{\frac{\epsilon}{2}}^{\epsilon} \frac{1}{|x - \epsilon^2|^2} \exp^2\left[-\frac{\epsilon^{\frac{1}{2}}(\epsilon - x)}{(x - \epsilon^2)}\right] dx dy \\ & \leq \epsilon^2 \int_{\epsilon^2}^{\frac{\epsilon}{2}} \frac{1}{|x - \epsilon^2|^2} \exp^2\left[-\frac{\epsilon^{\frac{1}{2}+1}(1-\epsilon)}{2(x - \epsilon^2)}\right] dx + \epsilon^2 \int_{\frac{\epsilon}{2}}^{\epsilon} \frac{1}{|\frac{\epsilon}{2} - \epsilon^2|^2} \exp^2\left[\frac{\epsilon^{\frac{1}{2}}(2\epsilon)}{(\frac{\epsilon}{2} - \epsilon^2)}\right] dx \\ & \leq \frac{\epsilon^{\frac{1}{2}} \exp\left(-\frac{\epsilon^{\frac{3}{2}}(1-\epsilon)}{x}\right)}{1-\epsilon} \Big|_{x=0}^{x=\frac{\epsilon}{2}} + \frac{1}{(\frac{1}{2}-\epsilon)^2} \frac{\epsilon}{2} \exp^2\left[\frac{2\epsilon^{\frac{1}{2}}}{(\frac{1}{2}-\epsilon)}\right] \\ & \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0, \end{aligned}$$

here we used the properties of the exponential function and elementary majorizations.

$$4) \int_{\epsilon^2}^{\epsilon} \int_0^{\epsilon^2} \frac{|\phi_{\epsilon}(x) - \phi_{\epsilon}(y)|^2}{|x - y|^2} dx dy = \int_{\epsilon^2}^{\epsilon} \int_0^{\epsilon^2} \frac{|\exp\left[-\frac{\epsilon^{\frac{1}{2}}(y-\epsilon)}{(y-\epsilon^2)}\right]|^2}{|x - y|^2} dx dy \rightarrow 0,$$

proceed in the same way as for 3).

Combining these integrals on one hand and using the facts:  $\|\phi_{\epsilon} - 1\|_{0,\Gamma} \rightarrow 0$  and  $2\|(\phi_{\epsilon} - 1)\|_{0,\Gamma_i^{4,\epsilon}} \rightarrow 0$  for all  $0 \leq i \leq m$  on the other hand yield, using (14), the result of lemma 3. Consequently, using (13), it yields also

$$\|\phi_{\epsilon} - 1\|_{\frac{1}{2},\Gamma} \rightarrow 0.$$

□

Since  $u$  is  $(\frac{1}{2} + \alpha)$ -holder continuous and thus uniformly continuous on  $\Omega$ , the result of lemma (3) implies

$$\|u_{\epsilon} - u\|_{\frac{1}{2},\Gamma} \leq \|u(\phi_{\epsilon} - 1)\|_{\frac{1}{2},\Gamma} \leq \|u\|_{\infty,\Gamma} \|\phi_{\epsilon} - 1\|_{\frac{1}{2},\Gamma} \rightarrow 0.$$

One proves the following approximation lemma:

**Lemma 4.** *The function  $u_\epsilon$  defined by (8) and the distribution  $g_\epsilon$  defined in problem (12) satisfy the following limits*

$$a) \|\nabla u_\epsilon - \nabla u\|_{0,\Omega} \rightarrow 0, \quad b) \|g_\epsilon - g\|_{[H^{\frac{1}{2}}]', \Gamma - \Gamma_0} \rightarrow 0$$

as  $\epsilon \rightarrow 0$

*Proof.* a) Consider the linear operator  $G$  that associate, for the fixed  $f \in L^2(\Omega)$ , to each  $u^d \in H^{\frac{1}{2}}(\Gamma)$  the corresponding unique solution  $u$  of problem (8).

$$G : (H^{\frac{1}{2}}(\Gamma), \|\cdot\|_{\frac{1}{2},\Gamma}) \rightarrow A \subset V, \\ u^d := u|_\Gamma \rightarrow K(u^d) = u.$$

Where  $(A, \|\cdot\|_{H^1(\Omega)})$  denote the range of  $H^{\frac{1}{2}}(\Gamma)$  under  $G$ . The inverse operator  $G^{-1}$  identifies with the trace operator which is obviously well defined and bijective for  $u \in A$ . Using the trace inequality on  $\Gamma$ ,  $\forall u \in A$ , there exists  $c > 0$

$$\|u\|_{\frac{1}{2},\Gamma} \leq c \|\nabla u\|_{0,\Omega}$$

this implies the continuity of the linear bijective operator  $G^{-1}$ . By the Banach isomorphism theorem the operator  $G$  is continuous, this means that there exists  $c_{-1} > 0$  such that for all  $u \in A$  we have

$$\|\nabla u\|_{0,\Omega} \leq c_{-1} \|u\|_{\frac{1}{2},\Gamma}$$

thus

$$\|\nabla u - \nabla u_\epsilon\|_{0,\Omega} \leq c_{-1} \|u - u_\epsilon\|_{\frac{1}{2},\Gamma} \rightarrow 0, \quad (16)$$

as a consequence of lemma (3), this proves a).

b) We make the same reasoning as for a). Given  $g \in [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]'$ , let  $w \in V$  be the unique solution of

$$\int_{\Omega} 2\mu \varepsilon(w) \varepsilon(v) + \lambda \operatorname{div} w \operatorname{div} v \, dx = \langle g, v \rangle_{[H^{\frac{1}{2}}(\Gamma - \Gamma_0)]', H^{\frac{1}{2}}(\Gamma - \Gamma_0)} \quad , \quad (17)$$

for all  $v \in V$ . Choosing  $v = w$ , there exist  $c' > 0$  such that

$$\|\nabla w\|_{0,\Omega} \leq c' \|g\|_{[H^{\frac{1}{2}}(\Gamma - \Gamma_0)]'}. \quad (18)$$

Let  $K$  be the operator that associate to each data  $g \in [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]'$  the solution function  $w$  of the corresponding problem (17):

$$K : [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]' \rightarrow D \subset V \\ g \rightarrow K(g) = w.$$



Where  $(D, \|\cdot\|_{H^1(\Omega)})$  denote the range of  $[H^{\frac{1}{2}}]'(\Gamma - \Gamma_0)$  under  $K$ . Following existence and uniqueness result for problem (17),  $K$  is well defined, furthermore it is linear and invertible. An equivalent formulation of (18) is: there exists a constant  $c' > 0$  such that  $\forall g \in [H^{\frac{1}{2}}]'(\Gamma - \Gamma_0)'$ , we have

$$\|K(g)\|_{H^1} \leq c' \|g\|_{[H^{\frac{1}{2}}]'(\Gamma - \Gamma_0)'}.$$

i.e.  $K$  is continuous. Then, according to Banach's isomorphism theorem, we deduce that  $\exists c'_{-1} > 0$  such that

$$\|g\|_{[H^{\frac{1}{2}}]'(\Gamma - \Gamma_0)'} \leq c'_{-1} \|\nabla w\|_{0,\Omega}. \quad (19)$$

Rewriting (4) with  $g \in H^{\frac{1}{2}}(\Gamma - \Gamma_0) \equiv D \subset [H^{\frac{1}{2}}]'(\Gamma - \Gamma_0)'$  then subtracting (4) and (12) member-to-member, one find that  $u - u_\epsilon$  satisfy:  $\forall v \in V$ ,

$$\int_{\Omega} 2\mu(\varepsilon(u_\epsilon) - \varepsilon(u))\varepsilon(v) + \lambda \operatorname{div}(u_\epsilon - u) \operatorname{div} v \, dx = \langle g_\epsilon - g, v \rangle_{H^{-\frac{1}{2}}(\Gamma - \Gamma_0), H^{\frac{1}{2}}(\Gamma - \Gamma_0)}. \quad (20)$$

Applying (19) to  $w = u_\epsilon - u$  we get:

$$\|g_\epsilon - g\|_{[H^{\frac{1}{2}}]'(\Gamma - \Gamma_0)'} \leq c'_{-1} \|\nabla u_\epsilon - \nabla u\|_{0,\Omega}.$$

Considering (16), we infer  $b$ ).  $\square$

**Remark 5.** As a consequence of the previous lemma, there exists  $\epsilon_0 > 0$  such that  $\forall \epsilon, 0 < \epsilon < \epsilon_0$  we have

$$\|g - g_\epsilon\|_{[H^{\frac{1}{2}}]'(\Gamma - \Gamma_0)'} \leq \frac{1}{2} \quad \text{and} \quad \|\nabla u_\epsilon - \nabla u\|_{0,\Omega} \leq \frac{1}{2}.$$

Actually, We can do this, since by assumption  $g \in H^{\frac{1}{2}}(\Gamma - \Gamma_0)$ , by using the continuity of the embedding

$$I : H^{\frac{1}{2}}(\Gamma - \Gamma_0) \rightarrow L^2(\Gamma - \Gamma_0) \rightarrow [H^{\frac{1}{2}}]'(\Gamma - \Gamma_0),$$

hence

$$\begin{aligned} \|g_\epsilon\|_{[H^{\frac{1}{2}}]'(\Gamma - \Gamma_0)'} &\leq \|g\|_{[H^{\frac{1}{2}}]'(\Gamma - \Gamma_0)'} + \frac{1}{2} \leq \|g\|_{0,\Gamma - \Gamma_0} + \frac{1}{2} \\ &\leq \|g\|_{\frac{1}{2},\Gamma - \Gamma_0} + \frac{1}{2}. \end{aligned}$$

Let  $\epsilon_0$  be such as defined in remark (5). For the rest of the paper, we fix  $\epsilon, 0 < \epsilon < \epsilon_0$ .

**Remark 6.** Let  $(\Omega_i)_{i=1,m}$  be a collection of  $m$  sets included in the polygonal domain  $\Omega$  such that:

- 1) For all  $i, 1 \leq i \leq m$ , we assume that  $\Omega_i \cap \Gamma = \Gamma_i$ .
- 2) we require that  $\forall i, 1 \leq i \leq m, \Omega_i \cap \Omega_{i+1} \subset D_{i,\epsilon}$  and  $\Omega_i \cap \Omega_j = \emptyset$  for all  $j \notin \{i-1, i+1\}$ .

i.e. only sets  $(\Omega_i)_i$  with consecutive indices do intersect and the intersection set is included in  $D_{i,\epsilon}$ , where  $i$  is the common extremity between  $\Gamma_i$  and  $\Gamma_{i+1}$ . Let us consider a particular decomposition of the function  $u_\epsilon$ . For this, define

$$f_1 := \begin{cases} f & \text{in } \cup_{i=1}^m \Omega_i \\ 0 & \text{in } \Omega - \cup_{i=1}^m \Omega_i, \end{cases}$$

then; there exists  $f_2 \in L^2(\Omega)$ , and  $\lambda_1, \lambda_2 \in \mathbb{R}$  such that

$$f = \lambda_1 f_1 + \lambda_2 f_2 \quad \text{and} \quad \|f\|_{0,\Omega}^2 = \|f_1\|_{0,\Omega}^2 + \|f_2\|_{0,\Omega}^2$$

i.e.  $f_1$  and  $f_2$  are orthogonal.

Let  $w_1$  and  $w_2$  be, respectively, the solutions of

$$\begin{cases} L(w_1) = \lambda_1 f_1 & \text{in } \Omega \\ \sigma(w_1) \cdot \vec{n} = g' & \text{on } \Gamma - \Gamma_0 \\ w_1 = 0 & \text{on } \Gamma_0 \end{cases} \quad \begin{cases} L(w_2) = \lambda_2 f_2 & \text{in } \Omega \\ w_2 = 0 & \text{on } \Gamma, \end{cases}$$

where  $g'$  is such that  $u_\epsilon = w_1 + w_2$ .

**Remark 7.** One should note that, with this definition of  $w^1$  and  $w_2$ , one has

$$\sigma(w_2) \cdot \vec{n} = 0 \quad \text{on } \Gamma.$$

Indeed, by density, there exists  $w_2^n \in C_0^\infty(\Omega)$  such that

$$\|0 - \sigma(w_2) \cdot \vec{n}\|_{0,\Gamma} = \|\sigma(w_2^n) \cdot \vec{n} - \sigma(w_2) \cdot \vec{n}\|_{0,\Gamma} \leq \|\nabla w_2^n - \nabla w_2\|_{\frac{1}{2},\Omega} \leq \|w_2^n - w_2\|_{\frac{3}{2},\Omega} \rightarrow 0,$$

thus,  $\sigma(w_2) \cdot \vec{n} = 0$  on  $\Gamma$ . Consequently

$$\sigma(w_1) \cdot \vec{n} = g_\epsilon \quad \text{on } \Gamma - \Gamma_0.$$

It is easily seen, from the previous remark, that

$$u_\epsilon = w_1 + w_2 \tag{21}$$

and that the functions  $w_1, w_2$  satisfy respectively

$$2\mu \int_{\Omega} \varepsilon(w_1) \varepsilon(v) + \lambda \int_{\Omega} \operatorname{div} w_1 \operatorname{div} v \, dx = \int_{\Omega} \lambda_1 f_1 v \, dx + \langle g_\epsilon, v \rangle_{[H^{\frac{1}{2}}]^\prime(\Gamma - \Gamma_0), H^{\frac{1}{2}}(\Gamma - \Gamma_0)} \tag{22}$$

for all  $v \in V$ .

$$2\mu \int_{\Omega} \varepsilon(w_2)\varepsilon(v) + \lambda \int_{\Omega} \operatorname{div} w_2 \operatorname{div} v \, dx = \int_{\Omega} \lambda_2 f_2 v \, dx \quad (23)$$

for all  $v \in H_0^1(\Omega)$ . Let  $(\Omega_i)_i$  be such as defined in remark (6). We state an approximation property of the function  $w_1$ . Define, for each  $i$ ,  $1 \leq i \leq m$ , the function  $u_{\epsilon,i}$  to be the solution of

$$\begin{cases} L(u_{\epsilon,i}) &= \lambda_1 f_1 = \lambda_1 f & \Omega_i \\ \sigma(u_{\epsilon,i}) \cdot \vec{n} &= g_{\epsilon} & \Gamma_i \\ u_{\epsilon,i} &= 0 & \partial\Omega_i - \Gamma_i. \end{cases}$$

If we denote

$$H_{\Gamma_i}^1(\Omega_i) := \{v \in H^1(\Omega_i); v = 0 \text{ on } \partial\Omega_i - \Gamma_i\}$$

There exists, following the principal result proved in [4], a sequence  $u_{\epsilon,i}^n \in C^\infty(\overline{\Omega_i}) \cap H_{\Gamma_i}^1(\Omega_i)$  such that

$$\|\nabla u_{\epsilon,i} - \nabla u_{\epsilon,i}^n\|_{0,\Omega_i} \leq \epsilon.$$

Denote

$$f_{\epsilon,i}^n := L(u_{\epsilon,i}^n) \quad g_{\epsilon,i}^n := \sigma(u_{\epsilon,i}^n) \cdot \vec{n},$$

using the same argument as in the proof of lemma (4), we infer that  $\exists C > 0$  such that

$$\|\lambda_1 f_1 - f_{\epsilon,i}^n\|_{0,\Omega_i} + \|g_{\epsilon,i}^n - g_{\epsilon,i}\|_{[H^{\frac{1}{2}}]^\prime(\Gamma_i)} \leq C \|\nabla u_{\epsilon,i} - \nabla u_{\epsilon,i}^n\|_{0,\Omega} \quad (24)$$

then, one can easily see that we can choose  $n$  and  $\epsilon$  such that

$$\frac{c_k}{c_p} \|f_{\epsilon,i}^n - \lambda_1 f_1\|_{0,\Omega_i} \leq \frac{1}{2} \quad \text{and} \quad \frac{c_k}{c_{p,t}} \|g_{\epsilon,i}^n - g_{\epsilon,i}\|_{[H^{\frac{1}{2}}]^\prime(\Gamma_i)} \leq \frac{1}{2}. \quad (25)$$

Since we are looking for explicit estimates, we should use Poincaré, trace and Korn's inequalities relatively to suitable geometric configurations i.e. for which they are explicitly formulated. The configuration that best fits our polygonal convex domain  $\Omega$  is the half-plane  $\mathbb{R}^{2+}$  containing the convex polygonal domain  $\Omega$  for Korn's inequality, the square  $S_d$  with edge's length equal to  $d(\Omega)$  for the other two inequalities. Thus we determine these constants thanks to results available for this type of domains. All this suggests to extend by zero the functions  $u_{\epsilon,i}$  outside the convex domain  $\Omega$ . The definition of the functions  $u_{\epsilon,i}$  is adapted to make such an extension.

### 3 Technical tools

We introduce some useful lemmas, which will play essential roles in proving theorem (1).

#### 3.1 Extension of the functions $u_{\epsilon,i}^n$

We consider for  $i$ ,  $1 \leq i \leq m$ , the extension by zero of  $u_{\epsilon,i}^n$  from the convex domain  $\Omega$  to the half-plane  $\mathbb{R}^{2+}$  containing  $\Omega$  such that  $\Gamma_i \subset \partial\mathbb{R}^{2+}$ . The extended function is defined by

$$\tilde{u}_{\epsilon,i}^n = \begin{cases} u_{\epsilon,i}^n, & a.e. \quad x \in \overline{\Omega_i \cap \Omega}, \\ 0, & x \in \mathbb{R}^{2+} - \overline{\Omega_i \cap \Omega}. \end{cases} \quad (26)$$

We have, obviously, the following

$$\|\partial_{x_i} \tilde{u}_{\epsilon,i}^n\|_{0,\mathbb{R}^{2+}} = \|\partial_{x_i} \tilde{u}_{\epsilon,i}^n\|_{0,\Omega_i \cap \Omega} = \|\partial_{x_i} u_{\epsilon,i}^n\|_{0,\Omega_i \cap \Omega}. \quad (27)$$

Then

$$w_1^n := \sum_i \tilde{u}_{\epsilon,i}^n \in V$$

satisfies

$$2\mu \int_{\Omega} \varepsilon(w_1^n) \varepsilon(v) dx + \lambda \int_{\Omega} \operatorname{div} w_1^n \operatorname{div} v dx = \int_{\Omega} L(w_1^n) v dx + \int_{\Gamma - \Gamma_0} g_{\epsilon,i}^n v d\sigma \quad (28)$$

for all  $v \in V$ . Notice also that

$$L(w_1^n)(x) = \sum_i f_{\epsilon,i}^n(x) \quad \forall x \in \Omega.$$

Subtracting (22) from (28), we obtain

$$\begin{aligned} 2\mu \int_{\Omega} \varepsilon(w_1^n - w_1) \varepsilon(v) + \lambda \int_{\Omega} \operatorname{div}(w_1^n - w_1) \operatorname{div} v dx &= \int_{\Omega} (\sum_i f_{\epsilon,i}^n - \lambda_1 f_1) v dx \\ &+ \langle g_{\epsilon,i}^n - g_{\epsilon}, v \rangle_{[H^{\frac{1}{2}}]'\!(\Gamma - \Gamma_0), H^{\frac{1}{2}}(\Gamma - \Gamma_0)}, \end{aligned}$$

for all  $v \in V$ .

Choose  $v = w_1^n - w_1$  then, using Cauchy-Schwarz on one hand, and Poincar, Korn, trace inequalities on the other hand we easily get

$$\begin{aligned} c_k \|\nabla w_1^n - \nabla w_1\|_{0,\Omega} &\leq c_p \left\| \sum_i f_{\epsilon,i}^n - \lambda_1 f_1 \right\|_{0,\Omega} + c_{p,t} \|g_{\epsilon,i}^n - g_{\epsilon}\|_{[H^{\frac{1}{2}}]'\!(\Gamma - \Gamma_0)} \\ &\leq c_p \sum_i \|f_{\epsilon,i}^n - \lambda_1 f_1\|_{0,\Omega_i} + c_{p,t} \sum_i \|g_{\epsilon,i}^n - g_{\epsilon}\|_{[H^{\frac{1}{2}}]'\!(\Gamma_i)} \end{aligned}$$

and thus, using the approximation result (25), there exists  $n$  such that

$$\|\nabla w_1^n - \nabla w_1\|_{0,\Omega} \leq \frac{1}{2}$$

The inequalities are established for the extended  $H^1$  regular functions defined on a square containing the convex polygonal domain  $\Omega$ .

### 3.2 Explicit constant in the Poincaré inequality

We show in the following lemma that the function  $u_{\epsilon,i}^n \in V_i$  satisfy the Poincaré inequality for which we determine explicitly the constant.

**Lemma 8.** *For all  $i$ ,  $0 \leq i \leq m$ , the function  $u_{\epsilon,i}^n$  satisfy:*

$$\|u_{\epsilon,i}^n\|_{0,\Omega} \leq d(\Omega) \|\nabla u_{\epsilon,i}^n\|_{0,\Omega}, \quad (29)$$

the constant  $d(\Omega)$  means the diameter of  $\Omega$ .

*Proof.* We establish Poincaré inequality for one of the two components  $u_{\epsilon,i}^{n,l}$ ,  $l = 1, 2$ , the same estimate hold with the other. Note  $abcd$  the square  $S_d$  such that  $a = (a_1, a_2)$ ,  $b = (b_1, b_2)$ ,  $c = (c_1, c_2)$  et  $d = (d_1, d_2)$  and such that  $\Gamma_i \subset S_i := [c, d]$ ; so  $\tilde{u}_{\epsilon,i}^{n,l} = 0$  on  $\partial S_d - \Gamma_i$ .

Since  $\tilde{u}_{\epsilon,i}^{n,l}$  is absolutely continuous on the lines parallel to the coordinate axis, then applying the fundamental theorem of calculus to  $\tilde{u}_{\epsilon,i}^{n,l}$  on  $S_d$  for  $l = 1, 2$ , we have for all  $(x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$

$$\tilde{u}_{\epsilon,i}^{n,l}(x_1, x_2) = \int_{a_1}^{x_1} \partial_{x_1} \tilde{u}_{\epsilon,i}^{n,l}(s, x_2) ds + \tilde{u}_{\epsilon,i}^{n,l}(a_1, x_2).$$

Since  $(a_1, x_2) \in \partial S_d - \Gamma_i$ , then  $\forall (x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$

$$\tilde{u}_{\epsilon,i}^{n,l}(x_1, x_2) = \int_{a_1}^{x_1} \partial_{x_1} \tilde{u}_{\epsilon,i}^{n,l}(s, x_2) ds.$$

Using Cauchy-Schwarz inequality  $\forall (x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$

$$|\tilde{u}_{\epsilon,i}^{n,l}(x_1, x_2)| \leq |x_1 - a|^{1/2} \left( \int_{a_1}^{x_1} |\partial_{x_1} \tilde{u}_{\epsilon,i}^{n,l}(s, x_2)|^2 ds \right)^{1/2}.$$

Taking the square of the two hand sides of this inequality and using the fact  $|x_1 - a| \leq d(\Omega)$ :  $\forall (x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$  yields

$$|\tilde{u}_{\epsilon,i}^{n,l}(x_1, x_2)|^2 \leq |x_1 - a| \int_{a_1}^{x_1} |\partial_{x_1} \tilde{u}_{\epsilon,i}^{n,l}(s, x_2)|^2 ds \leq d(\Omega) \int_{a_1}^{d_1} |\partial_{x_1} \tilde{u}_{\epsilon,i}^{n,l}(s, x_2)|^2 ds.$$

Integrating on  $S_d$  with respect to the variables  $x_1$  and  $x_2$ :

$$\begin{aligned} \|\tilde{u}_{\epsilon,i}^{n,l}\|_{0,S_d}^2 &= \int_{a_2}^{b_2} \int_{a_1}^{d_1} |\tilde{u}_{\epsilon,i}^{n,l}(x_1, x_2)|^2 dx_1 dx_2 \leq d(\Omega) \int_{a_2}^{b_2} \int_{a_1}^{d_1} \int_{a_1}^{d_1} |\partial_x \tilde{u}_{\epsilon,i}^{n,l}(s, x_2)|^2 ds dx_2 dx_1 \\ &\leq d^2(\Omega) \int \int_{S_d} |\partial_x \tilde{u}_{\epsilon,i}^{n,l}(s, x_2)|^2 ds dx_2. \end{aligned}$$

According to definition 26 and by considering (27) we get

$$\|u_{\epsilon,i}^{n,l}\|_{0,\Omega}^2 \leq d^2(\Omega) \|\nabla u_{\epsilon,i}^{n,l}\|_{0,\Omega}^2.$$

We infer that

$$\|u_{\epsilon,i}^n\|_{0,\Omega}^2 := \|u_{\epsilon,i}^{n,1}\|_{0,\Omega}^2 + \|u_{\epsilon,i}^{n,2}\|_{0,\Omega}^2 \leq d^2(\Omega) (\|\nabla u_{\epsilon,i}^{n,1}\|_{0,\Omega}^2 + \|\nabla u_{\epsilon,i}^{n,2}\|_{0,\Omega}^2) = d^2(\Omega) \|\nabla u_{\epsilon,i}^n\|_{0,\Omega}^2.$$

□

### 3.3 Explicit constant in the trace inequality

Using mainly the inequality of Poincaré stated in lemma (8), one establishes a trace inequality on  $\Gamma_i$  for the function  $u_{\epsilon,i}$  with an explicit constant.

**Lemma 9.** *For all  $i$ , the functions  $u_{\epsilon,i}$  satisfy:*

$$\|u_{\epsilon,i}^n\|_{0,\Gamma_i} \leq c_{tr} \|\nabla u_{\epsilon,i}^n\|_{0,\Omega} \quad (30)$$

where  $c_{tr} := 2\sqrt{d(\Omega)}$  is the trace constant.

*Proof.* Let  $\tilde{u}_{\epsilon,i}^n$  be defined on  $S_d$  such that  $\Gamma_i \subset \partial S_d$ . We establish trace inequality for one of the two components  $\tilde{u}_{\epsilon,i}^{n,l}$ ,  $l = 1, 2$ , the same estimate hold with the other. Applying the inequality of trace on the boundary of a parallelogram (see lemma 4.2 in [3]) for  $\tilde{u}_{\epsilon,i}^{n,l}$  on  $S_i$ , it yields

$$\|\tilde{u}_{\epsilon,i}^{n,l}\|_{0,\Gamma_i}^2 \leq \|\tilde{u}_{\epsilon,i}^{n,l}\|_{0,S_i}^2 \leq 2 \frac{|S|}{|S_d|} \|\tilde{u}_{\epsilon,i}^{n,l}\|_{0,S_d}^2 + 2 \frac{|S_d|}{|S|} \|\nabla \tilde{u}_{\epsilon,i}^{n,l}\|_{0,S_d}^2.$$

Using estimate (8) we find

$$\|\tilde{u}_{\epsilon,i}^{n,l}\|_{0,\Gamma_i}^2 \leq 2 \frac{|S|}{|S_d|} d^2(\Omega) \|\nabla \tilde{u}_{\epsilon,i}^{n,l}\|_{0,S_d}^2 + 2 \frac{|S_d|}{|S|} \|\nabla \tilde{u}_{\epsilon,i}^{n,l}\|_{0,S_d}^2,$$

hence by simplifying

$$\|\tilde{u}_{\epsilon,i}^{n,l}\|_{0,\Gamma_i}^2 \leq 4d(\Omega) \|\nabla \tilde{u}_{\epsilon,i}^{n,l}\|_{0,S_d}^2.$$

Using (26) defining  $\tilde{u}_{\epsilon,i}$  and (27) we have

$$\|u_{\epsilon,i}^{n,l}\|_{0,\Gamma_i}^2 \leq 4d(\Omega) \|\nabla u_{\epsilon,i}^{n,l}\|_{0,\Omega}^2.$$

Summing over  $l = 1, 2$  we get

$$\|u_{\epsilon,i}^n\|_{0,\Gamma_i}^2 \leq 4d(\Omega) \|\nabla u_{\epsilon,i}^n\|_{0,\Omega}^2.$$

□

To prove estimate (7), we construct auxiliary approximating functions  $u_{\epsilon,i}^{\beta,L}$  that are solution of a problem that is similar to (12). These auxiliary functions enjoy an orthogonality property, this allows us to deduce easily and explicitly, using Korn's inequality, Poincaré's and trace's inequalities, an upper bound  $\|\nabla u_{\epsilon,i}\|_{0,\Omega}$  and therefore prove the main theorem.

## 4 Proof of theorem 1

We are ready now to present a proof of the main theorem:

*Proof.* Let  $i$  be fixed. Choose  $v = u_{\epsilon,i}^n$  in (28) to get

$$2\mu \int_{\Omega_i} \varepsilon(w_1^n) \varepsilon(u_{\epsilon,i}^n) + \lambda \int_{\Omega_i} \operatorname{div} w_1^n \operatorname{div} u_{\epsilon,i}^n \, dx = \int_{\Omega_i} \sum_i f_{\epsilon,i}^n(x) u_{\epsilon,i}^n \, dx + \int_{\Gamma_i} g_{\epsilon,i}^n u_{\epsilon,i}^n \, d\sigma. \quad (31)$$

Considering assumption 2) in remark (6) then, applying Cauchy-Schwarz inequality to (31), we obtain

$$2\mu \|\varepsilon(u_{\epsilon,i}^n)\|_{0,\Omega_i}^2 \leq \|f_{\epsilon,i}^n\|_{0,\Omega_i} \|u_{\epsilon,i}^n\|_{0,\Omega_i} + \|g_{\epsilon,i}^n\|_{0,\Gamma_i} \|u_{\epsilon,i}^n\|_{0,\Gamma_i}. \quad (32)$$

Since the deformation is a linear application with respect to the first derivatives of  $u_{\epsilon,i}$ , then with the same notations as in (26) and by using (27) we have

$$2\mu \|\varepsilon(\tilde{u}_{\epsilon,i}^n)\|_{0,\mathbb{R}^{2+}}^2 = 2\mu \|\varepsilon(\tilde{u}_{\epsilon,i}^n)\|_{0,\Omega}^2 = 2\mu \|\varepsilon(u_{\epsilon,i}^n)\|_{0,\Omega_i}^2. \quad (33)$$

Applying the estimate in corollary 1.2.2 of [6] to  $\tilde{u}_{\epsilon,i}^n$  gives

$$\frac{1}{2} \times 2\mu \|\nabla \tilde{u}_{\epsilon,i}^n\|_{0,\mathbb{R}^{2+}}^2 \leq 2\mu \|\varepsilon(\tilde{u}_{\epsilon,i}^n)\|_{0,\mathbb{R}^{2+}}^2.$$

Hence, by using (33),  $\forall i, 0 \leq i \leq m$ , we get

$$\mu \|\nabla u_{\epsilon,i}^n\|_{0,\Omega}^2 = \mu \|\nabla \tilde{u}_{\epsilon,i}^n\|_{0,\mathbb{R}^{2+}}^2 \leq 2\mu \|\varepsilon(\tilde{u}_{\epsilon,i}^n)\|_{0,\mathbb{R}^{2+}}^2 = 2\mu \|\varepsilon(u_{\epsilon,i}^n)\|_{0,\Omega}^2. \quad (34)$$

Estimate (32) becomes

$$\mu \|\nabla(u_{\epsilon,i}^n)\|_{0,\Omega_i}^2 \leq \|f_{\epsilon,i}^n\|_{0,\Omega_i} \|u_{\epsilon,i}^n\|_{0,\Omega_i} + \|g_{\epsilon,i}^n\|_{0,\Gamma_i} \|u_{\epsilon,i}^n\|_{0,\Gamma_i}, \quad (35)$$

the estimates (8), (9) on the other hand yield

$$\mu \|\nabla u_{\epsilon,i}^n\|_{0,\Omega_i} \leq c_p \|f_{\epsilon,i}^n\|_{0,\Omega_i} + c_t \|g_{\epsilon,i}^n\|_{\Gamma_i},$$

by taking the square of the two hand sides

$$\mu^2 \|\nabla \tilde{u}_{\epsilon,i}^n\|_{0,\Omega}^2 = \mu^2 \|\nabla u_{\epsilon,i}^n\|_{0,\Omega_i}^2 \leq 2(c_p^2 \|f_{\epsilon,i}^n\|_{0,\Omega}^2 + c_t^2 \|g_{\epsilon,i}^n\|_{\Gamma_i}^2),$$

summing over  $i$ , it gives

$$\mu^2 \sum_i \|\nabla(\tilde{u}_{\epsilon,i}^n)\|_{0,\Omega}^2 \leq 2(c_p^2 \sum_i \|f_{\epsilon,i}^n\|_{0,\Omega}^2 + c_t^2 \sum_i \|g_{\epsilon,i}^n\|_{0,\Gamma_i}^2).$$

Using again the assumption 2) of remark (6), we obtain

$$\mu^2 \|\nabla w_1^n\|_{0,\Omega}^2 = \mu^2 \|\nabla(\sum_i \tilde{u}_{\epsilon,i}^n)\|_{0,\Omega}^2 \leq 2(c_p^2 \sum_i \|f_{\epsilon,i}^n\|_{0,\Omega}^2 + c_t^2 \|g_{\epsilon,i}^n\|_{0,\Gamma-\Gamma_0}^2).$$

Applying the approximation result (24) and remark (5), it yield

$$\mu^2 \|\nabla w_1\|_{0,\Omega}^2 \leq 2(c_p^2 \|\lambda_1 f_1\|_{0,\Omega}^2 + c_t^2 \|g\|_{\frac{1}{2},\Gamma-\Gamma_0}^2 + \frac{1}{2}) \quad (36)$$

On the other hand, choose  $v = w_2$  in (23)

$$\int_{\Omega} \varepsilon^2(w_2) dx = \int_{\Omega} \lambda_2 f_2 w_2 dx$$

using the Korn inequality in the case of Dirichlet homogeneous conditions for  $w_2$ , (see []), we have

$$\mu \|\nabla w_2\|_{0,\Omega} \leq 2\mu \|\varepsilon(w_2)\|_{0,\Omega} \leq \frac{d(\Omega)}{2} \|\lambda_2 f_2\|_{0,\Omega} = c_p \|\lambda_2 f_2\|_{0,\Omega}$$

thus

$$\mu^2 \|\nabla w_2\|_{0,\Omega}^2 \leq c_p^2 \|\lambda_2 f_2\|_{0,\Omega}^2 \quad (37)$$

Combining (36) and (37), we get

$$\mu^2 (\|\nabla w_1\|_{0,\Omega}^2 + \|\nabla w_2\|_{0,\Omega}^2) \leq 3c_p^2 (\|\lambda_1 f_1\|_{0,\Omega}^2 + \|\lambda_2 f_2\|_{0,\Omega}^2) + 2c_t^2 \|g\|_{\frac{1}{2},\Gamma-\Gamma_0}^2 + \frac{1}{2}$$

The orthogonality of the functions  $f_1$  and  $f_2$  yields

$$\begin{aligned} \mu^2 (\|\nabla w_1\|_{0,\Omega}^2 + \|\nabla w_2\|_{0,\Omega}^2) &\leq 3c_p^2 \|\lambda_1 f_1 + \lambda_2 f_2\|_{0,\Omega}^2 + 2c_t^2 \|g\|_{\frac{1}{2},\Gamma-\Gamma_0}^2 + \frac{1}{2} \\ &\leq 3c_p^2 \|f\|_{0,\Omega}^2 + 2c_t^2 \|g\|_{\frac{1}{2},\Gamma-\Gamma_0}^2 + \frac{1}{2}, \end{aligned}$$

thus, taking the square root

$$\mu \|\nabla w_1\|_{0,\Omega} \leq \sqrt{3}(c_p \|f\|_{0,\Omega} + c_t \|g\|_{\frac{1}{2},\Gamma-\Gamma_0}) + \frac{1}{2} \quad \text{and} \quad \mu \|\nabla w_2\|_{0,\Omega} \leq \sqrt{3}(c_p \|f\|_{0,\Omega} + c_t \|g\|_{\frac{1}{2},\Gamma-\Gamma_0}) + \frac{1}{2}$$

which implies, in virtue of (21)

$$\|\nabla u_{\epsilon}\|_{0,\Omega} \leq \|\nabla w_1\|_{0,\Omega} + \|\nabla w_2\|_{0,\Omega} \leq \frac{1}{\mu} [2\sqrt{3}(c_p \|f\|_{0,\Omega} + c_t \|g\|_{\frac{1}{2},\Gamma-\Gamma_0}) + 1]$$

We conclude the theorem for  $u$  by applying lemma (4).  $\square$

Finally; in order to get the explicit  $H^1$  estimate of  $u_{\epsilon}$ , and so that of  $u$ , we use the poincaré inequality (29) to bound  $\|u_{\epsilon}\|_{0,\Omega}$  at one hand and the estimate (7) at the other hand.



## Conclusion

In the point of view of numerical analysis, estimate of theorem (1) is interesting. Indeed, error estimates in finite element method of the type

$$\|u - u_h\|_{0,\Omega} \leq Ch \|\nabla u\|_{0,\Omega}$$

involve the quantity  $\|\nabla u\|_{0,\Omega}$ . Assuming that the constant  $C$  can be calculated, then it is possible to explicitly bound  $\|\nabla u\|_{0,\Omega}$  which implies a better estimate of  $\|u\|_{0,\Omega}$ .

Another interesting feature of the estimate (7) that makes it effective is that it does not depend on the characteristic parameters of the polygonal domain  $\Omega$ , namely, the edges's length, their number as well as the measures of the angles. The estimate is therefore indifferently applicable to all polygons. All this allows the possibility to generalize this result, by approximation, to a  $C^1$  class domain.

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