# Explicit $H^1$-Estimate for the solution of the Lamé system with mixed boundary conditions.

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**Abstract**

In this paper we consider Lamé system of equations on a polygonal convex domain with mixed boundary conditions of Dirichlet-Neumann type. An explicit $L^2$ norm estimate for the gradient of the solution of this problem is established. This leads to an explicit bound of the $H^1$ norm of this solution. Note that the obtained upper-bound is not optimal.

**keywords:** Lamé system; Korn's inequality; Poincare's inequality; inequality of trace; explicit estimates.

**AMS subject classification:** 35J57, 74B05
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## 1 Introduction

The static equilibrium of a deformable structure occupying a domain $\Omega$ subset of $\mathbb{R}^2$ is governed by the Lamé linear elasto-static system of equations, see [5]. In this paper, we restrict the study to a convex domain $\Omega$ whose boundary has a polygonal shape that posses $m+1$ edges with $m \geq 2$. We denote $\Gamma = \cup\Gamma_i$ its boundary and $d(\Omega)$ its diameter. This system is given by

$$\left\{\begin{array}{c} Lu = f \quad \text{a.e} \quad \text{in} \quad \Omega, \\ \sigma \cdot \overrightarrow{n_i} = g_i \quad \text{on} \quad (\Gamma - \Gamma_0) \cap \Gamma_i, \qquad 1 \leq i \leq m \\ u = 0 \quad \text{on} \quad \Gamma_0. \end{array}\right. \tag{1}$$

We assume that condition $(H_2)$ of Theorem 2.3 stated in the paper [9] is satisfied by $\Gamma$. This condition is formulated in (5) below. The vector function $u = (u^1, u^2)$ satisfying the system (8) describes a displacement in

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the plane. In this model, we impose a Dirichlet homogeneous condition on $\Gamma_0$ and a Neumann condition on the rest of the boundary. The equality on the boundary is understood in the sense of the trace. We denote $L$ the Lamé operator defined by:

$$Lu := -\operatorname{div}\sigma(u) = -\operatorname{div}[2\mu\varepsilon(u) + \lambda \operatorname{Tr}\varepsilon(u)Id] \tag{2}$$

The data functions $f$ and $g$ at the right hand side satisfy $f \in [L^2(\Omega)]^2$ and $g \in [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]^2$. The vector $\overrightarrow{n_i}$ represents the outside normal to $\Gamma_i$. We write $\mu$ and $\lambda$ the Lamé's coefficients. We place ourselves in the isotropic framework, the deformation tensor $\varepsilon$ is defined by

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^t u), \tag{3}$$

The weak form of problem (1) is (see [2], [5]):
find $u \in V$; $\forall v \in V$

$$\int_\Omega 2\mu\varepsilon(u)\varepsilon(v) + \lambda \operatorname{div} u \ \operatorname{div} v \ dx = \int_\Omega fv \ dx + \int_{\Gamma-\Gamma_0} gv \ d\sigma(x) \tag{4}$$

where

$$V = \{v \in [H^1(\Omega)]^2; \quad v = 0 \quad on \quad \Gamma_0\}$$

The problem of existence and uniqueness in $V$ of the solution of (4) is classic, (see [2]).

If we denote $\theta$ the interior angle between $\Gamma_j$ and $\Gamma_k$ such that $\overline{\Gamma}_j \cap \overline{\Gamma}_k \neq \emptyset$ and if we denote $\gamma$ the interior angle between the Neumann part of the boundary $\Gamma_N$ and the Dirichlet part of the boundary $\Gamma_D$ such that $\Gamma_N \cap \Gamma_D \neq \emptyset$, then we impose

$$\theta \leq 2\pi, \qquad \gamma \leq \pi. \tag{5}$$

The reason behind this assumption on the boundary is to get a better regularity of the solution of the weak problem (4). Precisely, in that case we have, following [9], $u \in [H^{\frac{3}{2}+\alpha}(\Omega)]^2$ for some positive $\alpha$, which implies in particular, using the appropriate Sobolev embedding, see [1], that $u \in [C^{0,\frac{1}{2}+\alpha}(\overline{\Omega})]^2$ i.e. $u$ is $(\frac{1}{2} + \alpha)-$holder continuous. Let us denote

$$||\varepsilon(u)||_{0,\Omega} := (\int_\Omega \varepsilon(u)\varepsilon(u) \ dx)^{\frac{1}{2}}, \qquad ||\nabla u||_{0,\Omega} := (\int_\Omega |\nabla u^1|^2 + |\nabla u^2|^2 \ dx)^{\frac{1}{2}}.$$

By using the second Korn inequality, see [7], the trace and the Poincaré's inequalities, one easily gets from (4) the following estimate

$$||\nabla u||_{0,\Omega} \leq \frac{1}{c_k}\frac{1}{2\mu}(c_p||f||_{0,\Omega} + c_{p,t}||g||_{\frac{1}{2},\Gamma-\Gamma_0}), \tag{6}$$

where $c_{p,t}$ is a constant that depends of Poincaré constant and the constant of trace inequality. $c_k$ is the constant of the Korn's inequality. Note that the value of the constant $c_k$ and $c_{p,t}$ appearing in (6) are unknown and can not be explicitly lower-bounded in the general case. We propose to determine explicitly these constants. The main result of this work is stated in the following theorem:

**Theorem 1.** *The unique weak solution $u$ of (4) on the polygonal domain $\Omega$ admits the explicit upper bound*

$$||\nabla u||_{0,\Omega} \leq \frac{3}{\mu}(1 + c_p||f||_{0,\Omega} + c_{tr}||g||_{\frac{1}{2},\Gamma-\Gamma_0}) \tag{7}$$

*where $c_p := d(\Omega)$, $c_{tr} := 2\sqrt{d(\Omega)}$ and $d(\Omega)$ represent the diameter of $\Omega$.*

The estimate (7) is similar to (6), the constants that are present are the same. Before demonstrating this theorem, it is useful to go through some remarks and results. Denote $x_i$, for $1 \leq i \leq m$, the vertex of the polygon that connects $\Gamma_{i-1}$ with $\Gamma_i$ and $x_0$ the one that connects $\Gamma_m$ to $\Gamma_0$. Define the auxiliary function $u_\epsilon \in H^1(\Omega)$ as the unique solution to the following Dirichlet problem

$$\begin{cases} Lu_\epsilon = f \quad \text{a.e} \quad \text{in} \quad \Omega, \\ u_\epsilon = u_\epsilon^d \quad \text{on} \quad \Gamma. \end{cases} \tag{8}$$

Where $u_\epsilon^d$ is the trace of the function

$$\phi_\epsilon(x)u(x) \tag{9}$$

on the boundary $\Gamma$; if $\epsilon < \frac{|\Gamma_i|}{2}$ $\forall i, 0 \leq i \leq m$ then $\phi_\epsilon$ is defined by

$$\begin{cases} \phi_\epsilon(x) & = 0, \qquad ||x - x_i|| \leq \epsilon^2, \quad 0 \leq i \leq m; \\ \phi_\epsilon(x) & = \exp[-\frac{\epsilon^{\frac{1}{2}}(\epsilon - ||x-x_i||)}{||x-x_i|| - \epsilon^2}], \qquad \epsilon^2 < ||x - x_i|| < \epsilon, \quad 0 \leq i \leq m; \\ \phi_\epsilon(x) & = 1, \qquad \epsilon \leq ||x - x_i||, \quad 0 \leq i \leq m, \end{cases}$$

let us denote

$$D_{i,\epsilon} := \{x \in \mathbb{R}^2 \quad \text{such that} \quad ||x - x_i|| < \epsilon^2\}.$$

We easily see that $\phi_\epsilon \in C^0(\overline{\Omega})$, consequently, there will be no jump when passing to the distributionnal derivative and thus $\nabla u_\epsilon \in L^2(\Omega)$ i.e. $u_\epsilon \in H^1(\Omega)$. It is shown, using Lebesgue's dominated convergence theorem for instance, that $||\phi_\epsilon - 1||_{0,\Gamma_i} \to 0$ i.e. we have convergence in $L^2$ along the edge $\Gamma_i$. The functions $\phi_\epsilon$ are identically zero on a small neighborhood of the respective vertices of the polygon.

In the sequel, we denote $u_\epsilon$ the vector-valued function $u_\epsilon = (u_\epsilon^1, u_\epsilon^2)$.

## 2 Weak problem for $u_\epsilon$ and an approximation result

First of all, we construct the weak problem verified by the approximating function $u_\epsilon$. With the approximating displacement $u_\epsilon \in V$ is associated the approximating stress tensor

$$\sigma_\epsilon := 2\mu\varepsilon(u_\epsilon) + \lambda \operatorname{Tr}\varepsilon(u_\epsilon) I, \tag{10}$$

since $Lu_\epsilon = \operatorname{div}\sigma_\epsilon = f$, then $\sigma_\epsilon \in [H(\operatorname{div})(\Omega)]^{2\times 2}$. For a fixed $\epsilon$, by density of the regular functions in the space $H(\operatorname{div})(\Omega)$, there exists $\sigma_\epsilon^n \in [C^\infty(\overline{\Omega})]^{2\times 2}$ such that $\sigma_\epsilon^n \to \sigma_\epsilon$ in $[H(\operatorname{div})(\Omega)]^{2\times 2}$. This means

$$||\sigma_\epsilon^n - \sigma_\epsilon||_{\operatorname{div},\Omega} := ||\operatorname{div}\sigma_\epsilon^n - \operatorname{div}\sigma_\epsilon||_{0,\Omega} + ||\sigma_\epsilon^n - \sigma_\epsilon||_{0,\Omega} \to 0 \tag{11}$$

when $n \to \infty$. We put $\operatorname{div}\sigma_\epsilon^n = f^n$, then integrating by part against a test function $v \in [C^\infty(\overline{\Omega})]^2 \cap V$ yields the following

$$\int_\Omega \sigma_\epsilon^n \nabla v = \int_\Omega f^n v + \int_\Gamma \sigma_\epsilon^n \cdot \overrightarrow{n} v \mathrm{d}\sigma.$$

Passing to the limit in $n$ using (11), we find $\forall v \in [C^\infty(\overline{\Omega})]^2 \cap V$

$$\int_\Omega \sigma_\epsilon \nabla v = \int_\Omega fv + \ < \sigma_\epsilon \cdot \overrightarrow{n}, v >_{[H^{\frac{1}{2}}]'(\Gamma-\Gamma_0)\times[H^{\frac{1}{2}}](\Gamma-\Gamma_0)},$$

where $\sigma_\epsilon \cdot \overrightarrow{n} =: g_\epsilon \in [H^{\frac{1}{2}}(\Gamma-\Gamma_0)]'$ is the image of the normal component $\sigma_\epsilon$ by the trace operator on $\Gamma$. Since, following the main result in [4], $[C^\infty(\overline{\Omega})]^2 \cap V$ is a dense subset of $V \subset H^1(\Omega)$, then, according to the definition (3) and the expression (10), the function $u_\epsilon$ satisfy

$\forall v \in V$,

$$\int_\Omega 2\mu\varepsilon(u_\epsilon)\varepsilon(v) + \int_\Omega \lambda \operatorname{div} u_\epsilon \operatorname{div} v = \int_\Omega fv + \ < g_\epsilon, v >_{[H^{\frac{1}{2}}]'(\Gamma-\Gamma_0)\times H^{\frac{1}{2}}(\Gamma-\Gamma_0)}; \tag{12}$$

this is the weak problem satisfied by the approximating function $u_\epsilon$.

Let us recall, (see [8]), that the $H^{\frac{1}{2}}-$norm in one dimension on $\Gamma_i$ is defined by:

$$||u||_{\frac{1}{2},\Gamma_i} := (||u||_{0,\Gamma_i}^2 + \int_{\Gamma_i}\int_{\Gamma_i} \frac{|u(x)-u(y)|^2}{||x-y||^2} dxdy)^{\frac{1}{2}}.$$

**Remark 2.** *For any sufficiently small $\epsilon > 0$, it is possible to overlap $\Omega$ with a collection of open sets $(W_j^\epsilon)_j$ such that for any $j$, $W_j^\epsilon \cap \Gamma$ is either empty or equals one of the following subsets: for some $0 \le i \le m-1$*

$$1) \quad \Gamma_i^{1,\epsilon} := \{x \in \Gamma_i; \quad 0 < ||x - x_i|| < 2\epsilon\};$$

$$2)\quad \Gamma_i^{2,\epsilon} := \{x \in \Gamma_i;\quad 0 < ||x - x_{i+1}|| < 2\epsilon\};$$

$$3)\quad \Gamma_i^{3,\epsilon} := \{x \in \Gamma_i;\quad ||x - x_i|| > \frac{3}{2}\epsilon \quad and \quad ||x - x_{i+1}|| > \frac{3}{2}\epsilon\};$$

$$4)\quad \Gamma_i^{4,\epsilon} = \{x \in \Gamma_i \cup \Gamma_{i+1};\quad ||x - x_{i+1}|| < \epsilon^2\};$$

*and for $i = m$*

$$1)\quad \Gamma_m^{1,\epsilon} := \{x \in \Gamma_m;\quad 0 < ||x - x_m|| < 2\epsilon\};$$

$$2)\quad \Gamma_m^{2,\epsilon} := \{x \in \Gamma_m;\quad 0 < ||x - x_0|| < 2\epsilon\};$$

$$3)\quad \Gamma_m^{3,\epsilon} := \{x \in \Gamma_m;\quad ||x - x_m|| > \frac{3}{2}\epsilon \quad and \quad ||x - x_0|| > \frac{3}{2}\epsilon\};$$

$$4)\quad \Gamma_m^{4,\epsilon} = \{x \in \Gamma_m \cup \Gamma_0;\quad ||x - x_0|| < \epsilon^2\}$$

*let $(\vartheta_j^\epsilon)_j$, with supp $\vartheta_j^\epsilon \subset W_j^\epsilon$, a $C^1$-partition of unity with respect to this overlap; since $\vartheta_j^\epsilon \in H^{\frac{1}{2}}(\Gamma)$ then*

$$\begin{aligned}
||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma} &= ||(\phi_\epsilon - 1)\sum_j \vartheta_j^\epsilon||_{\frac{1}{2},\Gamma} \\
&\leq \sum_{i=0}^{m} ||(\phi_\epsilon - 1) \sum_{j, supp\vartheta_j^\epsilon \subset \Gamma_i} \vartheta_j^\epsilon||_{\frac{1}{2},\Gamma_i} + \sum_{i=0}^{m} ||(\phi_\epsilon - 1)||_{\frac{1}{2},\Gamma_i^{4,\epsilon}} \\
&\leq \sum_{i=0}^{m} ||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i} + (m+1)||(\phi_\epsilon - 1)||_{\frac{1}{2},\Gamma_0^{4,\epsilon}} \\
&= \sum_{i=0}^{m} ||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i} + (m+1)||(\phi_\epsilon - 1)||_{0,\Gamma_0^{4,\epsilon}} \qquad (13)
\end{aligned}$$

*So, using the definition and symmetry of $\phi_\epsilon$, we get for all $0 \leq i \leq m$*

$$\begin{aligned}
||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i} &= ||(\phi_\epsilon - 1) \sum_{j, supp\vartheta_j^\epsilon \subset \Gamma_i} \vartheta_j^\epsilon||_{\frac{1}{2},\Gamma_i} \\
&\leq ||(\phi_\epsilon - 1) \sum_{j, supp\vartheta_j^\epsilon \subset \Gamma_i^{1,\epsilon}} \vartheta_j^\epsilon||_{\frac{1}{2},\Gamma_i} + ||(\phi_\epsilon - 1) \sum_{j, supp\vartheta_j^\epsilon \subset \Gamma_i^{2,\epsilon}} \vartheta_j^\epsilon||_{\frac{1}{2},\Gamma_i} \\
&+ ||(\phi_\epsilon - 1) \sum_{j, supp\vartheta_j^\epsilon \subset \Gamma_i^{3,\epsilon}} \vartheta_j^\epsilon||_{\frac{1}{2},\Gamma_i} + 2||(\phi_\epsilon - 1)||_{\frac{1}{2},\Gamma_i^{4,\epsilon}} \\
&\leq \sum_{j=1}^{3} ||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i^{j,\epsilon}} + 2||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i^{4,\epsilon}} \\
&= 2||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i^{1,\epsilon}} + 2||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i^{4,\epsilon}}
\end{aligned}$$

*thus we have*

$$\begin{aligned}
||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i} \leq 2||\phi_\epsilon - 1||_{0,\Gamma_i^{1,\epsilon}} + 2(\int_{\Gamma_i^{1,\epsilon}} \int_{\Gamma_i^{1,\epsilon}} \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{||x - y||^2} dxdy)^{\frac{1}{2}} \\
+ 2||(\phi_\epsilon - 1)||_{0,\Gamma_i^{4,\epsilon}}. \qquad (14)
\end{aligned}$$

**Lemma 3.** *The functions $\phi_\epsilon$ admit the following limit for all $0 \leq i \leq m$*

$$||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i} \to 0 \qquad as \quad \epsilon \to 0$$

*Proof.* If we choose the vertex point $x_i$ as the origin of the $\mathbb{R}^2$-orthonormal coordinate system and $\Gamma_i$ supported by the positive half $x-$axis then the abscisses of $x \in \Gamma_i^{1,\epsilon} \equiv ]0, 2\epsilon[$ verify

$$||x - x_i|| = |x| = x.$$

the $H^{\frac{1}{2}}$-semi-norm on $\Gamma_i$ writes by using the definition of $\phi_\epsilon$

$$|\phi_\epsilon - 1|^2_{\frac{1}{2},\Gamma_i} := \int_{\Gamma_i}\int_{\Gamma_i} \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{||x-y||^2} dxdy \leq 2\int_0^\epsilon \int_0^\epsilon \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{|x-y|^2} dxdy. \tag{15}$$

Consider the decomposition of (15) into four partial double integrals

$$1) \int_0^{\epsilon^2}\int_0^{\epsilon^2} \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{|x-y|^2} dxdy = 0,$$

this is obvious.

$$2) \int_{\epsilon^2}^{\epsilon}\int_{\epsilon^2}^{\epsilon} \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{|x-y|^2} dxdy \leq \int_{\epsilon^2}^{\epsilon}\int_{\epsilon^2}^{\epsilon} \frac{|\exp[\ -\frac{\epsilon^{\frac{1}{2}}(\epsilon-x)}{(x-\epsilon^2)}\ ] - \exp[\ -\frac{\epsilon^{\frac{1}{2}}(\epsilon-y)}{(y-\epsilon^2)}\ ]|^2}{|x-y|^2} dxdy$$

The function $F(x) := \exp[\ -\frac{\epsilon^{\frac{1}{2}}(\epsilon-x)}{(x-\epsilon^2)}\ ]$ is $C^1(]\epsilon^2, \epsilon[)$ and thus lipschitz. We have, using the fact that $x \to F'(x)$ is increasing on $[\epsilon^2, \frac{\epsilon}{2}]$, that

$$|F'(x)| \leq \frac{\epsilon^{\frac{1}{2}}(1-\epsilon)}{\epsilon(\frac{1}{2}-\epsilon)^2} \exp(\frac{-\epsilon^{\frac{1}{2}}}{\frac{1}{2}-\epsilon}) =: L_1,$$

$\forall x \in [\epsilon^2, \frac{\epsilon}{2}]$. On the other hand

$$|F'(x)| \leq \frac{\epsilon^{\frac{1}{2}}}{\epsilon(1-\epsilon)} \exp(\frac{2}{1-\epsilon}) =: L_2,$$

$\forall x \in [\frac{\epsilon}{2}, \epsilon]$. Therefore we conclude that

$$|F'(x)| \leq L := \max(L_1, L_2) \leq L_1 + L_2$$

for all $x \in [\epsilon^2, \epsilon]$. This yields

$$
\begin{aligned}
&(\int_{\epsilon^2}^{\epsilon}\int_{\epsilon^2}^{\epsilon} \frac{|\phi_\epsilon(x)-\phi_\epsilon(y)|^2}{|x-y|^2}dxdy)^{\frac{1}{2}} \\
&\leq (\int_{\epsilon^2}^{\epsilon}\int_{\epsilon^2}^{\epsilon} \frac{|\frac{\epsilon^{\frac{1}{2}}}{\epsilon(1-\epsilon)}\exp(\frac{2}{1-\epsilon}) + \frac{\epsilon^{\frac{1}{2}}(1-\epsilon)}{\epsilon(\frac{1}{2}-\epsilon)^2}\exp(\frac{-\epsilon^{\frac{1}{2}}}{\frac{1}{2}-\epsilon})|^2|x-y|^2}{|x-y|^2}dxdy)^{\frac{1}{2}} \\
&\leq (\int_{\epsilon^2}^{\epsilon}\int_{\epsilon^2}^{\epsilon} |\frac{\epsilon^{\frac{1}{2}}}{\epsilon(1-\epsilon)}\exp(\frac{2}{1-\epsilon})|^2dxdy)^{\frac{1}{2}} + (\int_{\epsilon^2}^{\epsilon}\int_{\epsilon^2}^{\epsilon} |\frac{\epsilon^{\frac{1}{2}}(1-\epsilon)}{\epsilon(\frac{1}{2}-\epsilon)^2}\exp(\frac{-\epsilon^{\frac{1}{2}}}{\frac{1}{2}-\epsilon})|^2dxdy)^{\frac{1}{2}} \\
&\to 0 \quad as \quad \epsilon \to 0
\end{aligned}
$$

$$
\begin{aligned}
3)\ &\int_0^{\epsilon^2}\int_{\epsilon^2}^{\epsilon} \frac{|\phi_\epsilon(x)-\phi_\epsilon(y)|^2}{|x-y|^2}dxdy \\
&\leq \int_0^{\epsilon^2}\int_{\epsilon^2}^{\frac{\epsilon}{2}} \frac{1}{|x-\epsilon^2|^2}\exp^2[\ -\frac{\epsilon^{\frac{1}{2}}(\epsilon-x)}{(x-\epsilon^2)}]dxdy + \int_0^{\epsilon^2}\int_{\frac{\epsilon}{2}}^{\epsilon^2} \frac{1}{|x-\epsilon^2|^2}\exp^2[\ -\frac{\epsilon^{\frac{1}{2}}(\epsilon-x)}{(x-\epsilon^2)}]dxdy \\
&\leq \epsilon^2\int_{\epsilon^2}^{\frac{\epsilon}{2}} \frac{1}{|x-\epsilon^2|^2}\exp^2[\ -\frac{\epsilon^{\frac{1}{2}+1}(1-\epsilon)}{2(x-\epsilon^2)}]dx + \epsilon^2\int_{\frac{\epsilon}{2}}^{\epsilon} \frac{1}{|\frac{\epsilon}{2}-\epsilon^2|^2}\exp^2[\ \frac{\epsilon^{\frac{1}{2}}(2\epsilon)}{(\frac{\epsilon}{2}-\epsilon^2)}]dx \\
&\leq \frac{\epsilon^{\frac{1}{2}}\exp(-\frac{\epsilon^{\frac{3}{2}}(1-\epsilon)}{x})}{1-\epsilon}]_{x=0}^{x=\epsilon} + \frac{1}{(\frac{1}{2}-\epsilon)^2}\frac{\epsilon}{2}\exp^2[\ \frac{2\epsilon^{\frac{1}{2}}}{(\frac{1}{2}-\epsilon)}\ ] \\
&\to 0 \quad as \quad \epsilon \to 0,
\end{aligned}
$$

here we used the properties of the exponential function and elementary majorizations.

$$
4)\ \int_{\epsilon^2}^{\epsilon}\int_0^{\epsilon^2} \frac{|\phi_\epsilon(x)-\phi_\epsilon(y)|^2}{|x-y|^2}dxdy = \int_{\epsilon^2}^{\epsilon}\int_0^{\epsilon^2} \frac{|\exp[\ -\frac{\epsilon^{\frac{1}{2}}(y-\epsilon)}{(y-\epsilon^2)}\ ]|^2}{|x-y|^2}dxdy \to 0,
$$

proceed in the same way as for 3).

Combining these integrals on one hand and using the facts: $||\phi_\epsilon - 1||_{0,\Gamma} \to 0$ and $2||(\phi_\epsilon - 1)||_{0,\Gamma_i^{4,\epsilon}} \to 0$ for all $0 \leq i \leq m$ on the other hand yield, using (14), the result of lemma 3. Consequently, using (13), it yields also

$$||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma} \to 0.$$

□

Since $u$ is $(\frac{1}{2}+\alpha)-$holder continuous and thus uniformly continuous on $\Omega$, the result of lemma (3) implies

$$||u_\epsilon - u||_{\frac{1}{2},\Gamma} \leq ||u(\phi_\epsilon - 1)||_{\frac{1}{2},\Gamma} \leq ||u||_{\infty,\Gamma}||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma} \to 0.$$

One proves the following approximation lemma:

**Lemma 4.** *The function $u_\epsilon$ defined by (8) and the distribution $g_\epsilon$ defined in problem (12) satisfy the following limits*

$$a)\ ||\nabla u_\epsilon - \nabla u||_{0,\Omega} \to 0, \quad b)\ ||g_\epsilon - g||_{[H^{\frac{1}{2}}]',\Gamma-\Gamma_0} \to 0$$

*as* $\epsilon \to 0$

*Proof.* $a$) Consider the linear operator $G$ that associate, for the fixed $f \in L^2(\Omega)$, to each $u^d \in H^{\frac{1}{2}}(\Gamma)$ the corresponding unique solution $u$ of problem (8).

$$\begin{aligned} G : (H^{\frac{1}{2}}(\Gamma), ||.||_{\frac{1}{2},\Gamma}) &\to A \subset V, \\ u^d := u_{|\Gamma} &\to K(u^d) = u. \end{aligned}$$

Where $(A, ||.||_{H^1(\Omega)})$ denote the range of $H^{\frac{1}{2}}(\Gamma)$ under $G$. The inverse operator $G^{-1}$ identifies with the trace operator which is obviously well defined and bijective for $u \in A$. Using the trace inequality on $\Gamma$, $\forall u \in A$, there exists $c > 0$

$$||u||_{\frac{1}{2},\Gamma} \leq c||\nabla u||_{0,\Omega}$$

this implies the continuity of the linear bijective operator $G^{-1}$. By the Banach isomorphism theorem the operator $G$ is continuous, this means that there exists $c_{-1} > 0$ such that for all $u \in A$ we have

$$||\nabla u||_{0,\Omega} \leq c_{-1}||u||_{\frac{1}{2},\Gamma}$$

thus

$$||\nabla u - \nabla u_\epsilon||_{0,\Omega} \leq c_{-1}||u - u_\epsilon||_{\frac{1}{2},\Gamma} \to 0, \tag{16}$$

as a consequence of lemma (3), this proves $a$).

$b$) We make the same reasoning as for $a$). Given $g \in [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]'$, let $w \in V$ be the unique solution of

$$\int_\Omega 2\mu\varepsilon(w)\varepsilon(v) + \lambda \operatorname{div} w \ \operatorname{div} v \ dx =< g, v >_{[H^{\frac{1}{2}}(\Gamma-\Gamma_0)]',H^{\frac{1}{2}}(\Gamma-\Gamma_0)} \quad ,\forall v \in V. \tag{17}$$

choosing $v = w$, there exist $c' > 0$ such that

$$||\nabla w||_{0,\Omega} \leq c'||g||_{[H^{\frac{1}{2}}(\Gamma-\Gamma_0)]'}. \tag{18}$$

Let $K$ be the operator that associate to each data $g \in [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]'$ the solution function $w$ of the corresponding problem (17):

$$\begin{aligned} K : [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]' &\to D \subset V \\ g &\to K(g) = w. \end{aligned}$$

Where $(D, ||\ ||_{H^1(\Omega)})$ denote the range of $[H^{\frac{1}{2}}]'(\Gamma - \Gamma_0)$ under $K$. Following existence and uniqueness result for problem (17), $K$ is well defined, furthermore it is linear and invertible. An equivalent formulation of (18) is: there exists a constant $c' > 0$ such that $\forall g \in [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]'$, we have

$$||K(g)||_{H^1} \leq c'||g||_{[H^{\frac{1}{2}}]'(\Gamma-\Gamma_0)}$$

i.e. $K$ is continuous. Then, according to Banach's isomorphism theorem, we deduce that $\exists c'_{-1} > 0$ such that

$$||g||_{[H^{\frac{1}{2}}(\Gamma-\Gamma_0)]'} \leq c'_{-1}||\nabla w||_{0,\Omega}. \tag{19}$$

Rewriting (4) with $g \in H^{\frac{1}{2}}(\Gamma - \Gamma_0) \equiv D \subset [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]'$ then substracting (4) and (12) member-to-member, one find that $u - u_\epsilon$ satisfy: $\forall v \in V$,

$$\int_\Omega 2\mu(\varepsilon(u_\epsilon)-\varepsilon(u))\varepsilon(v)+\lambda \operatorname{div}(u_\epsilon-u)\ \operatorname{div} v\, dx =< g_\epsilon-g, v >_{H^{-\frac{1}{2}}(\Gamma-\Gamma_0),H^{\frac{1}{2}}(\Gamma-\Gamma_0)}. \tag{20}$$

Applying (19) to $w = u_\epsilon - u$ we get:

$$||g_\epsilon - g||_{[H^{\frac{1}{2}}(\Gamma-\Gamma_0)]'} \leq c'_{-1}||\nabla u_\epsilon - \nabla u||_{0,\Omega}.$$

Considering (16), we infer $b$). □

**Remark 5.** *As a consequence of the previous lemma we have: For all $v \in V$, there exists $\epsilon_0 > 0$ such that $\forall 0 < \epsilon < \epsilon_0$*

$$\begin{aligned} |\int_{\Gamma-\Gamma_0} gvd\sigma(x) - < g_\epsilon, v >_{[H^{\frac{1}{2}}]'(\Gamma-\Gamma_0),H^{\frac{1}{2}}(\Gamma-\Gamma_0)}| &= |< g - g_\epsilon, v >_{[H^{\frac{1}{2}}]'(\Gamma-\Gamma_0),H^{\frac{1}{2}}(\Gamma-\Gamma_0)}| \\ &\leq ||g - g_\epsilon||_{[H^{\frac{1}{2}}]'(\Gamma-\Gamma_0)}\ ||v||_{\frac{1}{2},\Gamma} \\ &\leq \frac{1}{8(m+1)^2}||\nabla v||_{0,\Omega} \end{aligned}$$

*and such that $||\nabla u_\epsilon - \nabla u||_{0,\Omega} \leq \frac{1}{2}$.*

For the rest of the paper, we fix $\epsilon$, $0 < \epsilon < \epsilon_0$.

**Remark 6.** *We take now the idea of decomposition. Fix an open cover $(\Omega_i)_{i\in\overline{\{0,m\}}} \supset \Omega$ of the convex domain $\Omega$. This cover is choosen such that for all $i$, $\Omega_i \cap \Gamma = \Gamma_i$ and $\Omega_i$ intersects $\Gamma$ at exactly two points. These two points must belong respectively to $D_{i,\epsilon}$ and $D_{i+1,\epsilon}$ where the successive indices correspond respectively to those of the extremities $x_i$ and $x_{i+1}$ of $\Gamma_i$. There exists, $(\psi_i)_i \in C^\infty(\Omega)$, a $C^\infty-$partition of unity with respect to that cover. Put $u_{\epsilon,i} := \psi_i u_\epsilon$. The definition of the cover is required to satisfy the following condition: for all $i \neq j$,*

$$\int_{\Omega_i\cap\Omega_j} |f|^2 dx \leq \frac{1}{4(m+1)^4}.$$

*Thus*

$$u_\epsilon = \sum_{i=0}^{m} \psi_i u_\epsilon = \sum_{i=0}^{m} u_{\epsilon,i}$$

*Since we are looking for explicit estimates, we should use Poincaré, trace and Korn's inequalities relatively to suitable geometric configurations i.e. for which they are explicitly formulated. The configuration that best fits our polygonal convex domain $\Omega$ is the half-plane $\mathbb{R}^{2+}$ containing the convex polygonal domain $\Omega$ for Korn's inequality, the square $S_d$ with edge's length equal to $d(\Omega)$ for the other two inequalities. Thus we determine theses constants thanks to results available for this type of domains. All this suggests to extend by zero the functions $u_{\epsilon,i}$ outside the convex domain $\Omega$. The definition of the functions $u_{\epsilon,i}$ is adapted to make such an extension.*

# 3 Technical tools

We introduce some useful lemmas, which will play essential roles in proving theorem (1).

## 3.1 Extension of the functions $u_{\epsilon,i}$

We consider for $i,\ 1 \leq i \leq m$, the extension by zero of $u_{\epsilon,i}$ from the convex domain $\Omega$ to the half-plane $\mathbb{R}^{2+}$ containing $\Omega$ such that $\Gamma_i \subset \partial\mathbb{R}^{2+}$. The extended function is defined by

$$\begin{cases} \tilde{u}_{\epsilon,i} = u_{\epsilon,i}, & a.e. \quad x \in \overline{\Omega_i \cap \Omega}, \\ \tilde{u}_{\epsilon,i} = 0, & x \in \mathbb{R}^{2+} - \overline{\Omega_i \cap \Omega}. \end{cases} \tag{21}$$

We have, obviously, the following

$$||\partial_{x_i} \tilde{u}_{\epsilon,i}||_{0,\mathbb{R}^{2+}} = ||\partial_{x_i} \tilde{u}_{\epsilon,i}||_{0,\Omega_i \cap \Omega} = ||\partial_{x_i} u_{\epsilon,i}||_{0,\Omega_i \cap \Omega}. \tag{22}$$

The inequalities are established for the extended $H^1$ regular functions defined on a square containing the convex polygonal domain $\Omega$.

## 3.2 Explicit constant in the Poincaré inequality

We show in the following lemma that the function $u_{\epsilon,i} \in V_i$ satisfy the Poincaré inequality for which we determine explicitly the constant.

**Lemma 7.** *For all $i$, $0 \leq i \leq m$, the function $u_{\epsilon,i}$ satisfy:*

$$||u_{\epsilon,i}||_{0,\Omega} \leq d(\Omega)||\nabla u_{\epsilon,i}||_{0,\Omega}, \tag{23}$$

*the constant $d(\Omega)$ means the diameter of $\Omega$.*

*Proof.* We establish Poincaré inequality for one of the two components $u^l_{\epsilon,i}, l = 1, 2$, the same estimate hold with the other. Note *abcd* the square $S_d$ such that $a = (a_1, a_2), b = (b_1, b_2), c = (c_1, c_2)$ et $d = (d_1, d_2)$ and such that $\Gamma_i \subset S_i := [c, d]$; so $\tilde{u}^l_{\epsilon,i} = 0$ on $\partial S_d - \Gamma_i$.

Since $\tilde{u}^l_{\epsilon,i}$ is absolutely continuous on the lines parallel to the coordinate axis, then applying the fundamental theorem of calculus to $\tilde{u}^l_{\epsilon,i}$ on
$S_d$ for $l = 1, 2$ , we have for all $(x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$

$$\tilde{u}^l_{\epsilon,i}(x_1, x_2) = \int_{a_1}^{x_1} \partial_{x_1} \tilde{u}^l_{\epsilon,i}(s, x_2)ds + \tilde{u}^l_{\epsilon,i}(a_1, x_2).$$

Since $(a_1, x_2) \in \partial S_d - \Gamma_i$, then $\forall (x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$

$$\tilde{u}^l_{\epsilon,i}(x_1, x_2) = \int_{a_1}^{x_1} \partial_{x_1} \tilde{u}^l_{\epsilon,i}(s, x_2)ds.$$

Using Cauchy-Schwarz inequality $\forall (x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$

$$|\tilde{u}^l_{\epsilon,i}(x_1, x_2)| \leq |x_1 - a|^{\frac{1}{2}} \left(\int_{a_1}^{x_1} |\partial_{x_1} \tilde{u}^l_{\epsilon,i}(s, x_2)|^2 ds\right)^{\frac{1}{2}}.$$

Taking the square of the two hand sides of this inequality and using the fact $|x_1 - a| \leq d(\Omega)$: $\forall (x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$ yields

$$|\tilde{u}^l_{\epsilon,i}(x_1, x_2)|^2 \leq |x_1 - a| \int_{a_1}^{x_1} |\partial_{x_1} \tilde{u}^l_{\epsilon,i}(s, x_2)|^2 ds \leq d(\Omega) \int_{a_1}^{d_1} |\partial_{x_1} \tilde{u}^l_{\epsilon,i}(s, x_2)|^2 ds.$$

Integrating on $S_d$ with respect to the variables $x_1$ and $x_2$:

$$\begin{aligned} ||\tilde{u}^l_{\epsilon,i}||^2_{0,S_d} = \int_{a_2}^{b_2} \int_{a_1}^{d_1} |\tilde{u}^l_{\epsilon,i}(x_1, x_2)|^2 dx_1 dx_2 &\leq d(\Omega) \int_{a_2}^{b_2} \int_{a_1}^{d_1} \int_{a_1}^{d_1} |\partial_x \tilde{u}^l_{\epsilon,i}(s, x_2)|^2 ds dx_2 dx_1 \\ &\leq d^2(\Omega) \int \int_{S_d} |\partial_x \tilde{u}^l_{\epsilon,i}(s, x_2)|^2 ds dx_2. \end{aligned}$$

According to definition 21 and by considering (22) we get

$$||u^l_{\epsilon,i}||^2_{0,\Omega} \leq d^2(\Omega) ||\nabla u^l_{\epsilon,i}||^2_{0,\Omega}.$$

We infer that

$$||u_{\epsilon,i}||^2_{0,\Omega} = ||u^1_{\epsilon,i}||^2_{0,\Omega} + ||u^2_{\epsilon,i}||^2_{0,\Omega} \leq d^2(\Omega)(||\nabla u^1_{\epsilon,i}||^2_{0,\Omega} + ||\nabla u^2_{\epsilon,i}||^2_{0,\Omega}) = d^2(\Omega) ||\nabla u_{\epsilon,i}||^2_{0,\Omega}.$$

□

### 3.3 Explicit constant in the trace inequality

Using mainly the inequality of Poincaré stated in lemma (7), one establishes a trace inequality on $\Gamma_i$ for the function $u_{\epsilon,i}$ with an explicit constant.

**Lemma 8.** *For all $i$, the functions $u_{\epsilon,i}$ satisfy:*

$$||u_{\epsilon,i}||_{0,\Gamma_i} \leq c_{tr}||\nabla u_{\epsilon,i}||_{0,\Omega} \tag{24}$$

*where $c_{tr} := 2\sqrt{d(\Omega)}$ is the trace constant.*

*Proof.* Let $\tilde{u}_{\epsilon,i}$ be defined on $S_d$ such that $\Gamma_i \subset \partial S_d$. We establish trace inequality for one of the two components $u^l_{\epsilon,i}, l = 1, 2$, the same estimate hold with he other. Applying the inequality of trace on the boundary of a prallelogram (see lemma 4.2 in [3]) for $\tilde{u}^l_{\epsilon,i}$ on $S_i$, yields

$$||\tilde{u}^l_{\epsilon,i}||^2_{0,\Gamma_i} \leq ||\tilde{u}^l_{\epsilon,i}||^2_{0,S_i} \leq 2\frac{|S|}{|S_d|}||\tilde{u}^l_{\epsilon,i}||^2_{0,S_d} + 2\frac{|S_d|}{|S|}||\nabla\tilde{u}^l_{\epsilon,i}||^2_{0,S_d}.$$

Using estimate (7) we find

$$||\tilde{u}^l_{\epsilon,i}||^2_{0,\Gamma_i} \leq 2\frac{|S|}{|S_d|}d^2(\Omega)||\nabla\tilde{u}^l_{\epsilon,i}||^2_{0,S_d} + 2\frac{|S_d|}{|S|}||\nabla\tilde{u}^l_{\epsilon,i}||^2_{0,S_d},$$

hence by simplifying

$$||\tilde{u}^l_{\epsilon,i}||^2_{0,\Gamma_i} \leq 4d(\Omega)||\nabla\tilde{u}^l_{\epsilon,i}||^2_{0,S_d}.$$

Using 21 defining $\tilde{u}^l_{\epsilon,i}$ and (22) we have

$$||u^l_{\epsilon,i}||^2_{0,\Gamma_i} \leq 4d(\Omega)||\nabla u^l_{\epsilon,i}||^2_{0,\Omega}.$$

Summing over $l = 1, 2$ we get

$$||u_{\epsilon,i}||^2_{0,\Gamma_i} \leq 4d(\Omega)||\nabla u_{\epsilon,i}||^2_{0,\Omega}.$$

□

To prove estimate (7), we construct auxiliary approximating functions $u^{\beta,L}_{\epsilon,i}$ that are solution of a problem that is similar to (12). These auxiliary functions enjoy an orthogonality property, this allows us to deduce easily and explicitly, using Korn's inequality, Poincaré's and trace's inequalities, an upper bound $||\nabla u_{\epsilon,i}||_{0,\Omega}$ and therefore prove the main theorem.

# 4 Proof of theorem 1

The space $V \subset H^1$ has a countable dense basis then it is a separable Hilbert spaces. So, it admits a countable orthonormal Hilbert basis $\{e_\tau^l\}_{\tau\in\mathbb{N}}$ resp. $\{e_\tau^{l'}\}_{\tau\in\mathbb{N}}$ with respect to the inner product defined over $V$,

$$< u, v >_l := \int_\Omega \varepsilon(u)\varepsilon(v) + \text{divu divvdx} \quad \text{resp.} \quad < \text{u, v} >_{l'} := \int_\Omega \nabla \text{u} \nabla \text{vdx}$$

these are indeed inner products on $V$, one can see it by considering the Poincaré's and Korn's inequalities.

**Remark 9.** *Consider $V$ as a Hilbert space for the inner product $<,>_l$. Following the definition of an orthonormal basis of a Hilbert space we have: for all $i$, $0 \leq i \leq m$, $\forall \beta > 0$ there exists a finite linear combination*

$$\sum_{\tau=1}^{\tau_i} \alpha_\tau^{i,\beta} e_{\tau,i}^l =: u_{\epsilon,i}^\beta$$

*of elements $\{e_{\tau,i}^l\}_{\tau\in I^i} =: B_i^l \subset \{e_\tau^l\}_{\tau\in\mathbb{N}}$, with $I^i$ a finite subset of $\mathbb{N}$ with $\tau_i$ elements, such that*

$$||u_{\epsilon,i} - \sum_{\tau=1}^{\tau_i} \alpha_\tau^{i,\beta} e_{\tau,i}^l||_{1,\Omega} \leq \beta. \tag{25}$$

*for all $i$, $0 \leq i \leq m$*

The same remark holds for the case where the inner product is

$$< u, v >_{l'} := \int_\Omega \nabla u \nabla v dx.$$

in this case, the finite subset corresponding to $B_i^l$ in the previous remark is denoted $B_i^{l'}$.

Let $u_{\epsilon,i}$ be as defined in remark (6). Recall that $\sum_i u_{\epsilon,i} \in V$ satisfy

$$\int_\Omega \varepsilon(\sum_i u_{\epsilon,i})\varepsilon(v) + (\text{div} \sum_{\text{i}} \text{u}_{\epsilon,\text{i}})(\text{divv})\text{dx} = \int_\Omega \text{fvdx} + \int_{\Gamma-\Gamma_0} \text{gvd}\sigma \tag{26}$$

for all $v \in V$. One should remark that the functions $u_{\epsilon,i}$ are not necessarily two by two orthogonal; in which case, this would yield immediatly the desired result. So, the idea is to approximate the functions $u_{\epsilon,i}$ on one hand by functions $\overline{u}_{\epsilon,i}^{\beta,l}$ that are orthogonal with respect to a new basis $\overline{B}_i^l$ built from elements of $B_i^l$ and on the other hand by $\overline{u}_{\epsilon,i}^{\beta,l'}$ that are orthogonal with respect to another new basis $\overline{B}_i^{l'}$ built from elements of $B_i^{l'}$. This construction is detailed below. These approximating functions solve a similar problem as (26).

Let $\overline{\{B_i^l\}}$ and $\overline{\{B_{i+1}^l\}}$ be the finite dimensional subspaces of $V$ spanned by the sets $B_i^l$ and $B_{i+1}^l$ corresponding to indices $i$ and $i+1$ as defined in remark (9). Let us explain the procedure of construction of the approximating functions within these two subspaces. Suppose that $B_i^l \cap B_{i+1}^l = \{e_{*,i+1}^l\} \neq \emptyset$. For simplicity of presentation we supposed that the intersection contains only one element, the same reasoning works equally for the case where the intersection contains more than one element. Let us modify $Le_{*,i+1}^l =: f_{*,i+1}^l \in H^{-1}(\Omega)$ into a distribution $\overline{f}_{*,i+1}^l \in H^{-1}(\Omega)$ such that, if $\overline{e}_{*,i+1}^l \in V$ is the solution function of

$$\begin{cases} L\overline{e}_{*,i+1}^l(x) = \overline{f}_{*,i+1}^l & \text{a.e} \quad \text{in} \quad \Omega, \\ \overline{e}_{*,i+1}^l = e_{*,i+1}^l & \text{on} \quad \Gamma. \end{cases}$$

then, it does satisfy the two following hypothesis

$$h1) \quad x \to L(\overline{e}_{*,i+1}^l)(x) \quad \textit{and} \quad x \to L(e_{\tau,i}^l(x))$$

are linearly independent for all $0 \leq \tau \leq \tau_i$, consequently, the same hold for $\overline{e}_{*,i+1}^l$ and $e_{\tau,i}^l$ for all $0 \leq \tau \leq \tau_i$.

$$(h2) \quad ||\nabla \overline{e}_{*,i+1}^l - \nabla e_{*,i+1}^l||_{0,\Omega} \leq C(||\overline{e}_{*,i+1}^l - e_{*,i+1}^l||_{\frac{1}{2},\Gamma} + ||L\overline{e}_{*,i+1}^l - Le_{*,i+1}^l||_{-1,\Omega}) \leq \frac{1}{(m+1)^2}.$$

This last hypothesis can be realized by writing

$$||L\overline{e}_{*,i}^l - Le_{*,i}^l||_{-1,\Omega} \leq ||\overline{h}_n - h_n||_{-1,\Omega} + ||h_n - Le_{*,i}^l||_{-1,\Omega}$$

where, by density, $h_n \in L^2(\Omega)$ can be taken so that $||h_n - f_{*,i+1}^l||_{-1,\Omega} \leq \frac{1}{2(m+1)^2}$; posing $\overline{f}_{*,i+1}^l := \overline{h}_n$ with $\overline{h}_n = h_n$ on $\Omega$ except on a sufficiently small region of $\Omega$ where it is equal to zero such that $||\overline{h}_n - h_n||_{-1,\Omega} \leq \frac{1}{2(m+1)^2}$.

Consider the new set $\overline{B}_{i+1}^l$ that consists of the same elements as $B_{i+1}^l$ except for $e_{*,i+1}^l$ which is replaced by $\overline{e}_{*,i+1}^l$. Since $\overline{\{\overline{B}_{i+1}^l\}}$ is of finite dimension and its elements are, following the construction of $e_{*,i+1}^l$, linearly independent then, by a Gram schmidt process, we build from these elements a sequence of orthogonal functions that forms an orthogonal basis for the set spanned by $\overline{B}_{i+1}^l$. The resulting set is of the same dimension as the former. Thus, the set $\overline{B}_i^l \cup B_{i+1}$ contains $\tau_i + \tau_{i+1}$ orthogonal elements. Repeatring the same procedure with $\overline{B}_i^l \cup B_{i+1}$ and $B_{i+2}$, for all $0 \leq i \leq m$, as described previously, we construct a collection of finite subsets $\{\overline{B}_i^l\}_{i=\overline{0,m}}$ that contains only orthogonal elements and assumption $(h2)$ is met for all $i$ within these basis sets.

There exists coefficients $(\alpha_{\tau,i}^{\beta,l})_\tau$ such that $u_{\epsilon,i}^\beta$ writes in the basis $B_i^l$

$$u_{\epsilon,i}^{\beta,l} := \sum_{\tau=1}^{\tau_i} \alpha_{\tau,i}^{\beta,l} e_{\tau,i}^l \in V,$$

then define $\overline{u}_{\epsilon,i}^{\beta,l}$ written in the basis $\overline{B}_i^l$ by

$$\overline{u}_{\epsilon,i}^{\beta,l} := \alpha_{*,i+1}^{\beta,l}\overline{e}_{*,i+1}^{l} + \sum_{\tau=2}^{\tau_i} \alpha_{\tau,i}^{\beta,l} e_{\tau,i}^{l} \in V,$$

this is the auxiliary function for which we have replaced the common base functions $e_{*,i+1}^{l}$, appearing in its expression as a linear combination and belonging to the intersection, by the modified ones. One can easily see that $\overline{u}_{\epsilon,i}^{\beta,l}$ satisfy hypothesis $(h2)$. We are now ready to present a proof of the main result.

*Proof.* Since $L(\overline{u}_{\epsilon,i}^{\beta,l}) \in H^{-1}(\Omega)$ for all $i$, we can not write an adequate weak formulation similar to (12) for which $\sum_i \overline{u}_{\epsilon,i}^{\beta,l} \in V$ would be a solution. To overcome this obstacle, we consider, following the main result of ([4]), an approximating smooth function $\overline{u}_{\epsilon,i,n}^{\beta,l} \in C^\infty(\bar{\Omega}) \cap V$ such that

$$||\overline{u}_{\epsilon,i,n}^{\beta,L} - \overline{u}_{\epsilon,i}^{\beta,l}||_{1,\Omega} \to 0 \quad \text{as} \quad n \to \infty. \tag{27}$$

There exists $f_n^{\beta,l} \in L^2(\Omega)$ and $g_{\epsilon,n}^{\beta,l} \in [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]'$ such that the function $\sum_i u_{\epsilon,i,n}^{\beta,l} \in V$ is a solution of

$$\int_\Omega \varepsilon(\sum_i \overline{u}_{\epsilon,i,n}^{\beta,l})\varepsilon(v) + \mathrm{div}(\sum_{\mathrm{i}} \overline{\mathrm{u}}_{\epsilon,\mathrm{i},\mathrm{n}}^{\beta,\mathrm{l}})\mathrm{divvdx} = \int_\Omega \mathrm{f}_\mathrm{n}^{\beta,\mathrm{l}}\mathrm{vdx} + < \mathrm{g}_{\epsilon,\mathrm{n}}^{\beta,\mathrm{l}}, \mathrm{v} > \tag{28}$$

for all $v \in V$. By an appropriate choice of the linear combination in (25), we can make the following estimate as small as we want

$$||f - f_n^{\beta,l}||_{0,\Omega} + ||g_\epsilon - g_{\epsilon,n}^{\beta,l}||_{[H^{\frac{1}{2}}]',\Gamma-\Gamma_0}, \tag{29}$$

indeed, by an argument similar to that used in lemma (4), there exists $C > 0$ such that

$$\begin{aligned}
&||f - f_n^{\beta,l}||_{0,\Omega} + ||g_\epsilon - g_{\epsilon,n}^{\beta,l}||_{[H^{\frac{1}{2}}]',\Gamma-\Gamma_0} \\
&\leq C||u_\epsilon - \overline{u}_{\epsilon,i,n}^{\beta,l}||_{1,\Omega} \\
&= C||\nabla u_\epsilon - \nabla \overline{u}_{\epsilon,i,n}^{\beta,l}||_{0,\Omega} \\
&= C||\nabla \sum_i u_{\epsilon,i} - \nabla \sum_i \overline{u}_{\epsilon,i,n}^{\beta,l}||_{0,\Omega} \\
&\leq C \sum_i ||\nabla u_{\epsilon,i} - \nabla \overline{u}_{\epsilon,i,n}^{\beta,l}||_{0,\Omega} \\
&\leq C \sum_i ||\nabla u_{\epsilon,i} - \nabla \overline{u}_{\epsilon,i}^{\beta,l}||_{0,\Omega} + C \sum_i ||\nabla \overline{u}_{\epsilon,i}^{\beta,l} - \nabla \overline{u}_{\epsilon,i,n}^{\beta,l}||_{0,\Omega} \\
&\leq C \sum_i ||\nabla u_{\epsilon,i} - \nabla u_{\epsilon,i}^{\beta,l}||_{0,\Omega} + C||\nabla u_{\epsilon,i}^{\beta,l} - \nabla \overline{u}_{\epsilon,i}^{\beta,l}|| + C \sum_i ||\nabla \overline{u}_{\epsilon,i}^{\beta,l} - \nabla \overline{u}_{\epsilon,i,n}^{\beta,l}||_{0,\Omega},
\end{aligned}$$

to conclude (29), we use (25) with $\beta$ sufficiently small for the first term, hypothesis (h2) for the second term of the last inequality and (27) for the third term.

The same estimate (29) holds with $\overline{u}_{\epsilon,i,n}^{\beta,l'}$ in place of $\overline{u}_{\epsilon,i,n}^{\beta,l}$. Using the estimate (29), we get also

$$|\int_\Omega f_n^{\beta,l}(\overline{u}_{\epsilon,i,n}^{\beta,l}-u_{\epsilon,i})dx \leq ||f_n^{\beta,l}||_{0,\Omega}\,||\overline{u}_{\epsilon,i,n}^{\beta,l}-u_{\epsilon,i}|| \leq (||f||_{0,\Omega}+\frac{1}{m})||\overline{u}_{\epsilon,i,n}^{\beta,l}-u_{\epsilon,i}||_{1,\Omega} \leq \frac{1}{32(m+1)^3}$$

and,

$$\begin{aligned} < g_{\epsilon,n}^{\beta,l}, \overline{u}_{\epsilon,i,n}^{\beta,l} - u_{\epsilon,i} > &\leq ||g_{\epsilon,n}^{\beta,l}||_{[H^{\frac{1}{2}}]',\Gamma-\Gamma_0}\;||\overline{u}_{\epsilon,i,n}^{\beta,l} - u_{\epsilon,i}|| \\ &\leq (||g_\epsilon||_{[H^{\frac{1}{2}}]',\Gamma-\Gamma_0} + \frac{1}{m})||\overline{u}_{\epsilon,i,n}^{\beta,l} - u_{\epsilon,i}||_{1,\Omega} \\ &\leq \frac{1}{32(m+1)^3}. \end{aligned}$$

Choosing $v = \overline{u}_{\epsilon,i,n}^{\beta,l}$ in (28), we obtain

$$\begin{aligned} \int_\Omega \varepsilon(\sum_i \overline{u}_{\epsilon,i,n}^{\beta,l})\varepsilon(\overline{u}_{\epsilon,i,n}^{\beta,l}) + \mathrm{div}\sum_{\mathrm{i}} \overline{\mathrm{u}}_{\epsilon,\mathrm{i},\mathrm{n}}^{\beta,\mathrm{l}}\mathrm{div}\overline{\mathrm{u}}_{\epsilon,\mathrm{i},\mathrm{n}}^{\beta,\mathrm{l}}\mathrm{dx} &= \int_\Omega f_n^{\beta,l}\overline{u}_{\epsilon,i,n}^{\beta,l} + < g_{\epsilon,n}^{\beta,l}, \overline{u}_{\epsilon,i,n}^{\beta,l} > \\ &\leq \int_\Omega f_n^{\beta,L}u_{\epsilon,i} + < g_{\epsilon,n}^{\beta,l}, u_{\epsilon,i} > + \frac{1}{16(m+1)^3} \\ &\leq \int_{\Omega_i} f u_{\epsilon,i} + < g_\epsilon, u_{\epsilon,i} > + \frac{1}{8(m+1)^3}. \end{aligned}$$

Using remark (5) and the cauchy-schwarz inequality, we have

$$\begin{aligned} &2\mu\int_\Omega \varepsilon(\sum_i \overline{u}_{\epsilon,i,n}^{\beta,l})\varepsilon(\overline{u}_{\epsilon,i,n}^{\beta,l}) + (\mathrm{div}\sum_{\mathrm{i}} \overline{\mathrm{u}}_{\epsilon,\mathrm{i},\mathrm{n}}^{\beta,\mathrm{l}})(\mathrm{div}\overline{\mathrm{u}}_{\epsilon,\mathrm{i},\mathrm{n}}^{\beta,\mathrm{l}})\mathrm{dx} \\ &\leq \int_{\Omega_i} f u_{\epsilon,i}dx + \int_{\Gamma-\Gamma_0} g u_{\epsilon,i}d\sigma + \frac{1}{8(m+1)^2}||\nabla u_{\epsilon,i}||_{0,\Omega_i} + \frac{1}{8(m+1)^3} \\ &\leq ||f||_{0,\Omega_i}\;||u_{\epsilon,i}||_{0,\Omega_i} + ||g||_{0,\Gamma_i}\;||u_{\epsilon,i}||_{0,\Gamma_i} + \frac{1}{8(m+1)^2}||\nabla u_{\epsilon,i}||_{0,\Omega_i} + \frac{1}{8(m+1)^3}. \end{aligned}$$

Letting $n \to \infty$

$$\begin{aligned} 2\mu\int_\Omega \varepsilon(\sum_i \overline{u}_{\epsilon,i}^{\beta,l})\varepsilon(\overline{u}_{\epsilon,i}^{\beta,l}) + \mathrm{div}\sum_{\mathrm{i}} \overline{\mathrm{u}}_{\epsilon,\mathrm{i}}^{\beta,\mathrm{l}}\mathrm{div}\overline{\mathrm{u}}_{\epsilon,\mathrm{i}}^{\beta,\mathrm{l}}\mathrm{dx} &\leq ||f||_{0,\Omega_i}\;||u_{\epsilon,i}||_{0,\Omega_i} + ||g||_{0,\Gamma_i}\;||u_{\epsilon,i}||_{0,\Gamma_i} \\ &\quad + \frac{1}{8(m+1)^2}||\nabla u_{\epsilon,i}||_{0,\Omega_i} + \frac{1}{8(m+1)^3}. \end{aligned}$$

Now, either

$$||\nabla u_{\epsilon,i}||_{0,\Omega_i} \leq \frac{1}{m+1} \quad (*)$$

then, we have done with the explicit upper bound for the functoin $u_{\epsilon,i}$. Or

$$||\nabla u_{\epsilon,i}||_{0,\Omega_i} \geq \frac{1}{m+1} \quad (**),$$

in this case, using (23) and (24) yield

$$\begin{aligned} 2\mu \int_\Omega \varepsilon(\sum_i \overline{u}^{\beta,l}_{\epsilon,i,n})\varepsilon(\overline{u}^{\beta,l}_{\epsilon,i,n}) + (\mathrm{div} \sum_{\mathrm{i}} \overline{\mathrm{u}}^{\beta,\mathrm{l}}_{\epsilon,\mathrm{i},\mathrm{n}})(\mathrm{div}\overline{\mathrm{u}}^{\beta,\mathrm{l}}_{\epsilon,\mathrm{i},\mathrm{n}})\mathrm{dx} \\ \leq ||\nabla u_{\epsilon,i}||_{0,\Omega_i}(c_p||f||_{0,\Omega_i} + c_t||g||_{0,\Gamma_i} + \frac{1}{4(m+1)^2}). \end{aligned}$$

Using the orthogonality of the functions $\overline{u}^{\beta}_{\epsilon,i}$ withe respect to $<,>_L$,

$$||\varepsilon(\overline{u}^{\beta,l}_{\epsilon,i})||^2 \leq ||\nabla u_{\epsilon,i}||_{0,\Omega_i}(c_p||f||_{0,\Omega_i} + c_t||g||_{0,\Gamma_i} + \frac{1}{4(m+1)^2}).$$

Again, since $\epsilon(u_{\epsilon,i})$ is linear with respect to the component of $\nabla u_{\epsilon,i}$ then, using $(h2)$ , we can have the following approximation

$$||\varepsilon(u_{\epsilon,i})||_{0,\Omega} \leq ||\varepsilon(\overline{u}^{\beta,l}_{\epsilon,i})|| + \frac{1}{4(m+1)^3},$$

it yields in addition to $(**)$

$$\begin{aligned} 2\mu||\varepsilon(u_{\epsilon,i})||^2_{0,\Omega} &\leq \frac{1}{4(m+1)^3} + ||\nabla u_{\epsilon,i}||_{0,\Omega_i}(c_p||f||_{0,\Omega_i} + c_t||g||_{0,\Gamma_i} + \frac{1}{4(m+1)^2}) \\ &\leq ||\nabla u_{\epsilon,i}||_{0,\Omega_i}(c_p||f||_{0,\Omega_i} + c_t||g||_{0,\Gamma_i} + \frac{1}{2(m+1)^2}), \end{aligned} \tag{30}$$

for all $i$. On the other hand we give a lower-bound for $||\varepsilon(u_{\epsilon,i})||_{0,\Omega_i}$ in term of $||\nabla u_{\epsilon,i}||_{0,\Omega_i}$ for all $i$, $1 \leq i \leq m$. Since the deformation is a linear application with respect to the first derivatives of $u_{\epsilon,i}$, then with the same notation as in (21) and by using (22) we have

$$2\mu||\varepsilon(\tilde{u}_{\epsilon,i})||^2_{0,\mathbb{R}^{2+}} = 2\mu||\varepsilon(\tilde{u}_{\epsilon,i})||^2_{0,\Omega} = 2\mu||\varepsilon(u_{\epsilon,i})||^2_{0,\Omega_i}. \tag{31}$$

Applying the estimate in corollary 1.2.2 of [6] to $\tilde{u}_{\epsilon,i}$ gives

$$\frac{1}{2} \times 2\mu||\nabla\tilde{u}_{\epsilon,i}||^2_{0,\mathbb{R}^{2+}} \leq 2\mu||\varepsilon(\tilde{u}_{\epsilon,i})||^2_{0,\mathbb{R}^{2+}}.$$

Hence, by using (31), $\forall i, 0 \leq i \leq m$, we get

$$\mu||\nabla u_{\epsilon,i}||^2_{0,\Omega} = \mu||\nabla\tilde{u}_{\epsilon,i}||^2_{0,\mathbb{R}^{2+}} \leq 2\mu||\varepsilon(\tilde{u}_{\epsilon,i})||^2_{0,\mathbb{R}^{2+}} = 2\mu||\varepsilon(u_{\epsilon,i})||^2_{0,\Omega}. \tag{32}$$

Combining (30) and (32) gives

$$||\nabla u_{\epsilon,i}||^2_{0,\Omega} \leq ||\varepsilon(u_{\epsilon,i})||^2_{0,\Omega} \leq ||\nabla u_{\epsilon,i}||_{0,\Omega_i}(c_p||f||_{0,\Omega_i} + ||g||c_t||\nabla u_{\epsilon,i}||_{0,\Omega} + \frac{1}{2(m+1)^2}),$$

thus

$$||\nabla u_{\epsilon,i}||_{0,\Omega} \leq c_p||f||_{0,\Omega_i} + c_t||g|| + \frac{1}{2(m+1)^2}.$$

Taking the square and using elementary majorizations

$$||\nabla u_{\epsilon,i}||^2_{0,\Omega} = ||\nabla u_{\epsilon,i}||^2_{0,\Omega_i} \leq 3(c_p||f||^2_{0,\Omega_i} + c_t||g||^2_{0,\Gamma_i} + \frac{1}{4(m+1)^4}),$$

summing over $i$ gives

$$\sum_i ||\nabla u_{\epsilon,i}||^2_{0,\Omega} = \sum_i ||\nabla u_{\epsilon,i}||^2_{0,\Omega_i} \leq 3(c_p \sum_i ||f||^2_{0,\Omega_i} + c_t \sum_i ||g||^2_{0,\Gamma_i} + \sum \frac{1}{4(m+1)^4}),$$

posing $\tilde{g}_i = g$ on $\Gamma_i$ and $\tilde{g}_i = 0$ on $\Gamma - \Gamma_i$, define also $\tilde{f}_i = f$ on $\Omega_i$ and $\tilde{f}_i = 0$ elsewhere, we get

$$\sum_i ||\nabla u_{\epsilon,i}||^2_{0,\Omega} \leq 3(c_p \sum_i ||\tilde{f}_i||^2_{0,\Omega_i} + c_t \sum_i ||\tilde{g}_i||^2_{0,\Gamma_i} + \frac{1}{4(m+1)^3}).$$

Using the fact: for all $i, j$, $i \neq j$

$$|\int_\Omega \tilde{f}_i\tilde{f}_j| = |\int_{\Omega_i\cap\Omega_j} \tilde{f}_i\tilde{f}_j| = \int_{\Omega_i\cap\Omega_j} |f|^2dx \leq \frac{1}{4(m+1)^4} \quad and \quad \int_\Gamma \tilde{g}_i\tilde{g}_j d\sigma = 0,$$

in addition to an appropriate identity, we obtain

$$\begin{aligned}\sum_i ||\nabla u_{\epsilon,i}||^2_{0,\Omega_i} &\leq 3(c_p \sum_i ||\tilde{f}_i||^2_{0,\Omega_i} + \sum \frac{1}{4(m+1)^4} + c_{tr}^2 \sum_i ||\tilde{g}_i||^2_{0,\Gamma_i} + \frac{1}{4(m+1)^3}) \\ &\leq 3(c_p||f||^2_{0,\Omega} + c_{tr}^2||g||^2_{0,\Gamma} + \frac{1}{2(m+1)^3}).\end{aligned}$$

Using approximation again

$$||\nabla \overline{u}^{\beta,l'}_{\epsilon,i}||_{0,\Omega} \leq ||\nabla(u_{\epsilon,i})||_{0,\Omega} + \frac{3}{2(m+1)^3},$$

yields

$$\sum_i ||\nabla \overline{u}^{\beta,l'}_{\epsilon,i}||^2_{0,\Omega} \leq \frac{3}{2(m+1)^3} + 3(c_p||f||^2_{0,\Omega} + c_{tr}^2||g||^2_{0,\Gamma} + \frac{1}{2(m+1)^3}).$$

Using the orthogonality of the functions $\overline{u}^{\beta,L'}_{\epsilon,i}$ withe respect to $<,>_{L'}$

$$||\sum_i \nabla \overline{u}^{\beta,l'}_{\epsilon,i}||^2_{0,\Omega} \leq 3(c_p||f||^2_{0,\Omega} + c_{tr}^2||g||^2_{0,\Gamma} + \frac{1}{(m+1)^3}),$$

thus

$$||\sum_i \nabla \overline{u}_{\epsilon,i}^{\beta,l'}||_{0,\Omega} \leq \sqrt{3}(c_p||f||_{0,\Omega} + c_{tr}||g||_{0,\Gamma} + \frac{1}{m+1}),$$

and the fact

$$||\nabla \sum_{i=0}^{m} u_{\epsilon,i} - \nabla \sum_{i=0}^{m} \overline{u}_{\epsilon,i}^{\beta,l'}||_{0,\Omega} \leq \sum_{i=0}^{m} ||u_{\epsilon,i} - \overline{u}_{\epsilon,i}^{\beta,l'}||_{1,\Omega} \leq \sum_{i=0}^{m} \frac{1}{(m+1)^3}$$

if we choose $\beta$ sufficiently small in estimate (25) and following hypothesis $(h2)$, it follows

$$\begin{aligned} ||\nabla u_\epsilon||_{0,\Omega}^2 \leq ||\nabla \sum_i u_{\epsilon,i}||_{0,\Omega}^2 &\leq \frac{1}{(m+1)^2} + ||\sum_i \nabla \overline{u}_{\epsilon,i}^{\beta,L'}||_{0,\Omega}^2 \\ &\leq \frac{1}{(m+1)^2} + \sqrt{3}(c_p||f||_{0,\Omega} + c_{tr}||g||_{0,\Gamma} + \frac{1}{m+1}), \end{aligned}$$

$$||\nabla u_\epsilon||_{0,\Omega} \leq \frac{\sqrt{3}}{\mu}(c_p||f||_{0,\Omega} + c_t||g||_{0,\Gamma-\Gamma_0} + \frac{2}{m+1}), \tag{33}$$

considering $(*)$, estimate (33) becomes

$$||\nabla u_\epsilon||_{0,\Omega} \leq \max[1, \quad \frac{\sqrt{3}}{\mu}(c_p||f||_{0,\Omega} + c_t||g||_{0,\Gamma-\Gamma_0} + 1)]. \tag{34}$$

Actually, to meet exactly (7), the estimate (34) should be written for $g \in H^{\frac{1}{2}}(\Gamma)$. We can do this, since by assumption $g \in H^{\frac{1}{2}}(\Gamma - \Gamma_0)$, by using the continuity of the injection

$$I : H^{\frac{1}{2}}(\Gamma - \Gamma_0) \to L^2(\Gamma - \Gamma_0),$$

the estimate in Theorem (1) for $u_\epsilon$ is deduced immediately from (33). We conclude the theorem for $u$ by applying lemma (4). □

Finally; in order to get the explicit $H^1$ estimate of $u_\epsilon$, and so that of $u$, we use the poincaré inequality (23) to bound $||u_\epsilon||_{0,\Omega}$ at one hand and the estimate (7) at the other hand.

## Conclusion

In the point of view of numerical analysis, estimate of theorem (1) is interesting. Indeed, error estimates in finite element method of the type

$$||u - u_h||_{0,\Omega} \leq Ch||\nabla u||_{0,\Omega}$$

involve the quantity $||\nabla u||_{0,\Omega}$. Assuming that the constant $C$ can be calculated, then it is possible to explicitly bound $||\nabla u||_{0,\Omega}$ which implies a better estimate of $||u||_{0,\Omega}$.

Another interesting feature of the estimation (7) that makes it effective is that it does not depend on the characteristic parameters of the polygonal domain $\Omega$, namely, the edges's length, their number as well as the measures of the angles. The estimate is therefore indifferently applicable to all polygons. All this allows the possibility to generalize this result, by approximation, to a $C^1$ class domain.

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