Explicit H^1-Estimate for the solution of the Lamé system with mixed boundary conditions.

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Abstract

In this paper we consider Lamé system of equations on a polygonal domain with mixed boundary conditions of Dirichlet-Neumann type. An explicit L^2 norm estimate for the gradient of the solution of this problem is established. This leads to an explicit bound of the H^1 norm of this solution. Note that the obtained upper-bound is not optimal.

keywords: Lamé system; Korn's inequality; Poincare's inequality; inequality of trace; explicit estimates.

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1 Introduction

The static equilibrium of a deformable structure occupying a domain Ω subset of $\mathbb{R}^2$ is governed by the Lamé linear elasto-static system of equations, see [5]. In this paper, we restrict the study to a convex domain Ω whose boundary has a polygonal shape that posses $m+1$ edges with $m \geq 2$. We denote $\Gamma = \cup\Gamma_i$ its boundary and $d(\Omega)$ its diameter. This system is given by

$$\begin{cases} Lu = f \quad \text{a.e} \quad \text{in} \quad \Omega, \\ \sigma \cdot \overrightarrow{n_i} = g_i \quad \text{on} \quad (\Gamma - \Gamma_0) \cap \Gamma_i, \qquad 1 \leq i \leq m \\ u = 0 \quad \text{on} \quad \Gamma_0. \end{cases} \tag{1}$$

We assume that condition (H_2) of Theorem 2.3 stated in the paper [9] is satisfied by Γ. This condition is formulated in (5) below. The vector function $u = (u^1, u^2)$ satisfying the system (8) describes a displacement in the plane. In this model, we impose a Dirichlet homogeneous condition on

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Γ_0 and a Neumann condition on the rest of the boundary. The equality on the boundary is understood in the sense of the trace. We denote L the Lamé operator defined by:

$$Lu := -\operatorname{div}\sigma(u) = -\operatorname{div}[2\mu\varepsilon(u) + \lambda \operatorname{Tr}\varepsilon(u)Id] \tag{2}$$

The data functions f and g at the right hand side satisfy $f \in [L^2(\Omega)]^2$ and $g \in [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]^2$. The vector $\overrightarrow{n_i}$ represents the outside normal to Γ_i. We write μ and λ the Lamé's coefficients. We place ourselves in the isotropic framework, the deformation tensor ε is defined by

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^t u), \tag{3}$$

The weak form of problem (1) is (see [2], [5]):
find $u \in V$; $\forall v \in V$

$$\int_\Omega 2\mu\varepsilon(u)\varepsilon(v) + \lambda \operatorname{div} u \ \operatorname{div} v \ dx = \int_\Omega fv \ dx + \int_{\Gamma-\Gamma_0} gv \ d\sigma(x) \tag{4}$$

where

$$V = \{v \in [H^1(\Omega)]^2; \quad v = 0 \quad on \quad \Gamma_0\}$$

The problem of existence and uniqueness in V of the solution of (4) is classic, (see [2]).

If we denote θ the interior angle between Γ_j and Γ_k such that $\overline{\Gamma}_j \cap \overline{\Gamma}_k \neq \emptyset$ and if we denote γ the interior angle between the Neumann part of the boundary Γ_N and the Dirichlet part of the boundary Γ_D such that $\Gamma_N \cap \Gamma_D \neq \emptyset$, then we impose

$$\theta \leq 2\pi, \qquad \gamma \leq \pi. \tag{5}$$

The reason behind this assumption on the boundary is to get a better regularity of the solution of the weak problem (4). Precisely, in that case we have, following [9], $u \in [H^{\frac{3}{2}+\alpha}(\Omega)]^2$ for some positive α, which implies in particular, using the appropriate Sobolev embedding, see [1], that $u \in [C^{0,\frac{1}{2}+\alpha}(\overline{\Omega})]^2$ i.e. u is $(\frac{1}{2}+\alpha)-$holder continuous. Let us denote

$$||\varepsilon(u)||_{0,\Omega} := (\int_\Omega \varepsilon(u)\varepsilon(u) \ dx)^{\frac{1}{2}}, \qquad ||\nabla u||_{0,\Omega} := (\int_\Omega |\nabla u^1|^2 + |\nabla u^2|^2 \ dx)^{\frac{1}{2}}.$$

By using the second Korn inequality, see [7], the trace and the Poincaré's inequalities, one easily gets from (4) the following estimate

$$||\nabla u||_{0,\Omega} \leq \frac{1}{c_k}\frac{1}{2\mu}(c_p||f||_{0,\Omega} + c_{p,t}||g||_{\frac{1}{2},\Gamma-\Gamma_0}), \tag{6}$$

where $c_{p,t}$ is a constant that depends of Poincaré constant and the constant of trace inequality. c_k is the constant of the Korn's inequality. Note

that the value of the constant c_k and $c_{p,t}$ appearing in (6) are unknown and can not be explicitly lower-bounded in the general case. We propose to determine explicitly these constants. The main result of this work is stated in the following theorem:

Theorem 1. *The unique weak solution u of (4) on the polygonal domain Ω admits the explicit upper bound*

$$||\nabla u||_{0,\Omega} \leq \frac{2}{\mu}(c_p||f||_{0,\Omega} + c_{tr}||g||_{\frac{1}{2},\Gamma-\Gamma_0}) \tag{7}$$

where $c_p := d(\Omega)$, $c_{tr} := 2\sqrt{d(\Omega)}$ and $d(\Omega)$ represent the diameter of Ω.

The estimate (7) is similar to (6), the constants that are present are the same. Before demonstrating this theorem, it is useful to go through some remarks and results. Denote x_i, for $1 \leq i \leq m$, the vertex of the polygon that connects Γ_{i-1} with Γ_i and x_0 the one that connects Γ_m to Γ_0. Define the auxiliary function $u_\epsilon \in H^1(\Omega)$ as the unique solution to the following Dirichlet problem

$$\begin{cases} Lu_\epsilon = f \quad \text{a.e} \quad \text{in} \quad \Omega, \\ u_\epsilon = u_\epsilon^d \quad \text{on} \quad \Gamma. \end{cases} \tag{8}$$

Where u_ϵ^d is the trace of the function

$$\phi_\epsilon(x)u(x) \tag{9}$$

on the boundary Γ; if $\epsilon < \frac{|\Gamma_i|}{2}$ $\forall i, 0 \leq i \leq m$ then ϕ_ϵ is defined by

$$\begin{cases} \phi_\epsilon(x) & = 0, \qquad ||x - x_i|| \leq \epsilon^2, \quad 0 \leq i \leq m; \\ \phi_\epsilon(x) & = \exp[-\frac{\epsilon^{\frac{1}{2}}(\epsilon - ||x-x_i||)}{||x-x_i|| - \epsilon^2}], \qquad \epsilon^2 < ||x - x_i|| < \epsilon, \quad 0 \leq i \leq m; \\ \phi_\epsilon(x) & = 1, \qquad \epsilon \leq ||x - x_i||, \quad 0 \leq i \leq m, \end{cases}$$

let us denote

$$D_{i,\epsilon} := \{x \in \mathbb{R}^2 \quad \text{such that} \quad ||x - x_i|| < \epsilon^2\}.$$

We easily see that $\phi_\epsilon \in C^0(\overline{\Omega})$, consequently, there will be no jump when passing to the distributionnal derivative and thus $\nabla u_\epsilon \in L^2(\Omega)$ i.e. $u_\epsilon \in H^1(\Omega)$. It is shown, using Lebesgue's dominated convergence theorem for instance, that $||\phi_\epsilon - 1||_{0,\Gamma_i} \to 0$ i.e. we have convergence in L^2 along the edge Γ_i. The functions ϕ_ϵ are identically zero on a small neighborhood of the respective vertices of the polygon.

In the sequel, we denote u_ϵ the vector-valued function $u_\epsilon = (u_\epsilon^1, u_\epsilon^2)$.

2 Weak problem for u_ϵ and an approximation result

First of all, we construct the weak problem verified by the approximating function u_ϵ. With the approximating displacement $u_\epsilon \in V$ is associated the approximating stress tensor

$$\sigma_\epsilon := 2\mu\varepsilon(u_\epsilon) + \lambda \operatorname{Tr}\varepsilon(u_\epsilon) I, \tag{10}$$

since $Lu_\epsilon = \operatorname{div}\sigma_\epsilon = f$, then $\sigma_\epsilon \in [H(\operatorname{div})(\Omega)]^{2\times 2}$. For a fixed ϵ, by density of the regular functions in the space $H(\operatorname{div})(\Omega)$, there exists $\sigma_\epsilon^n \in [C^\infty(\overline{\Omega})]^{2\times 2}$ such that $\sigma_\epsilon^n \to \sigma_\epsilon$ in $[H(\operatorname{div})(\Omega)]^{2\times 2}$. This means

$$||\sigma_\epsilon^n - \sigma_\epsilon||_{\operatorname{div},\Omega} := ||\operatorname{div}\sigma_\epsilon^n - \operatorname{div}\sigma_\epsilon||_{0,\Omega} + ||\sigma_\epsilon^n - \sigma_\epsilon||_{0,\Omega} \to 0 \tag{11}$$

when $n \to \infty$. We put $\operatorname{div}\sigma_\epsilon^n = f^n$, then integrating by part against a test function $v \in [C^\infty(\overline{\Omega})]^2 \cap V$ yields the following

$$\int_\Omega \sigma_\epsilon^n \nabla v = \int_\Omega f^n v + \int_\Gamma \sigma_\epsilon^n \cdot \overrightarrow{n} v \mathrm{d}\sigma.$$

Passing to the limit in n using (11), we find $\forall v \in [C^\infty(\overline{\Omega})]^2 \cap V$

$$\int_\Omega \sigma_\epsilon \nabla v = \int_\Omega fv + \ < \sigma_\epsilon \cdot \overrightarrow{n}, v >_{[H^{\frac{1}{2}}]'(\Gamma-\Gamma_0)\times[H^{\frac{1}{2}}](\Gamma-\Gamma_0)},$$

where $\sigma_\epsilon \cdot \overrightarrow{n} =: g_\epsilon \in [H^{\frac{1}{2}}(\Gamma-\Gamma_0)]'$ is the image of the normal component σ_ϵ by the trace operator on Γ. Since, following the main result in [4], $[C^\infty(\overline{\Omega})]^2 \cap V$ is a dense subset of $V \subset H^1(\Omega)$, then, according to the definition (3) and the expression (10), the function u_ϵ satisfy

$\forall v \in V$,

$$\int_\Omega 2\mu\varepsilon(u_\epsilon)\varepsilon(v) + \int_\Omega \lambda \operatorname{div} u_\epsilon \operatorname{div} v = \int_\Omega fv + \ < g_\epsilon, v >_{[H^{\frac{1}{2}}]'(\Gamma-\Gamma_0)\times H^{\frac{1}{2}}(\Gamma-\Gamma_0)}; \tag{12}$$

this is the weak problem satisfied by the approximating function u_ϵ.

Let us recall, (see [8]), that the $H^{\frac{1}{2}}-$norm in one dimension on Γ_i is defined by:

$$||u||_{\frac{1}{2},\Gamma_i} := (||u||^2_{0,\Gamma_i} + \int_{\Gamma_i}\int_{\Gamma_i} \frac{|u(x)-u(y)|^2}{||x-y||^2} dxdy)^{\frac{1}{2}}.$$

Remark 2. *For any sufficiently small $\epsilon > 0$, it is possible to overlap Ω with a collection of open sets $(W_j^\epsilon)_j$ such that for any j, $W_j^\epsilon \cap \Gamma$ is either empty or equals one of the following subsets: for some $0 \le i \le m-1$*

$$1) \quad \Gamma_i^{1,\epsilon} := \{x \in \Gamma_i; \quad 0 < ||x - x_i|| < 2\epsilon\};$$

$$2)\quad \Gamma_i^{2,\epsilon} := \{x \in \Gamma_i;\quad 0 < ||x - x_{i+1}|| < 2\epsilon\};$$

$$3)\quad \Gamma_i^{3,\epsilon} := \{x \in \Gamma_i;\quad ||x - x_i|| > \frac{3}{2}\epsilon \quad and \quad ||x - x_{i+1}|| > \frac{3}{2}\epsilon\};$$

$$4)\quad \Gamma_i^{4,\epsilon} = \{x \in \Gamma_i \cup \Gamma_{i+1};\quad ||x - x_{i+1}|| < \epsilon^2\};$$

and for $i = m$

$$1)\quad \Gamma_m^{1,\epsilon} := \{x \in \Gamma_m;\quad 0 < ||x - x_m|| < 2\epsilon\};$$

$$2)\quad \Gamma_m^{2,\epsilon} := \{x \in \Gamma_m;\quad 0 < ||x - x_0|| < 2\epsilon\};$$

$$3)\quad \Gamma_m^{3,\epsilon} := \{x \in \Gamma_m;\quad ||x - x_m|| > \frac{3}{2}\epsilon \quad and \quad ||x - x_0|| > \frac{3}{2}\epsilon\};$$

$$4)\quad \Gamma_m^{4,\epsilon} = \{x \in \Gamma_m \cup \Gamma_0;\quad ||x - x_0|| < \epsilon^2\}$$

let $(\vartheta_j^\epsilon)_j$, with supp $\vartheta_j^\epsilon \subset W_j^\epsilon$, a C^1-partition of unity with respect to this overlap; since $\vartheta_j^\epsilon \in H^{\frac{1}{2}}(\Gamma)$ then

$$\begin{aligned}
||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma} &= ||(\phi_\epsilon - 1)\sum_j \vartheta_j^\epsilon||_{\frac{1}{2},\Gamma} \\
&\leq \sum_{i=0}^m ||(\phi_\epsilon - 1)\sum_{j,supp\vartheta_j^\epsilon \subset \Gamma_i} \vartheta_j^\epsilon||_{\frac{1}{2},\Gamma_i} + \sum_{i=0}^m ||(\phi_\epsilon - 1)||_{\frac{1}{2},\Gamma_i^{4,\epsilon}} \\
&\leq \sum_{i=0}^m ||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i} + (m+1)||(\phi_\epsilon - 1)||_{\frac{1}{2},\Gamma_0^{4,\epsilon}} \\
&= \sum_{i=0}^m ||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i} + (m+1)||(\phi_\epsilon - 1)||_{0,\Gamma_0^{4,\epsilon}} \qquad (13)
\end{aligned}$$

So, using the definition and symmetry of ϕ_ϵ, we get for all $0 \leq i \leq m$

$$\begin{aligned}
||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i} &= ||(\phi_\epsilon - 1)\sum_{j,supp\vartheta_j^\epsilon \subset \Gamma_i} \vartheta_j^\epsilon||_{\frac{1}{2},\Gamma_i} \\
&\leq ||(\phi_\epsilon - 1)\sum_{j,supp\vartheta_j^\epsilon \subset \Gamma_i^{1,\epsilon}} \vartheta_j^\epsilon||_{\frac{1}{2},\Gamma_i} + ||(\phi_\epsilon - 1)\sum_{j,supp\vartheta_j^\epsilon \subset \Gamma_i^{2,\epsilon}} \vartheta_j^\epsilon||_{\frac{1}{2},\Gamma_i} \\
&+ ||(\phi_\epsilon - 1)\sum_{j,supp\vartheta_j^\epsilon \subset \Gamma_i^{3,\epsilon}} \vartheta_j^\epsilon||_{\frac{1}{2},\Gamma_i} + 2||(\phi_\epsilon - 1)||_{\frac{1}{2},\Gamma_i^{4,\epsilon}} \\
&\leq \sum_{j=1}^3 ||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i^{j,\epsilon}} + 2||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i^{4,\epsilon}} \\
&= 2||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i^{1,\epsilon}} + 2||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i^{4,\epsilon}}
\end{aligned}$$

thus we have

$$\begin{aligned}
||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i} \leq 2||\phi_\epsilon - 1||_{0,\Gamma_i^{1,\epsilon}} + 2(\int_{\Gamma_i^{1,\epsilon}}\int_{\Gamma_i^{1,\epsilon}} \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{||x-y||^2} dxdy)^{\frac{1}{2}} \\
+ 2||(\phi_\epsilon - 1)||_{0,\Gamma_i^{4,\epsilon}}. \qquad (14)
\end{aligned}$$

Lemma 3. *The functions ϕ_ϵ admit the following limit for all $0 \leq i \leq m$*

$$||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma_i} \to 0 \qquad as \quad \epsilon \to 0$$

Proof. If we choose the vertex point x_i as the origin of the $\mathbb{R}^2$-orthonormal coordinate system and Γ_i supported by the positive half $x-$axis then the abscisses of $x \in \Gamma_i^{1,\epsilon} \equiv]0, 2\epsilon[$ verify

$$||x - x_i|| = |x| = x.$$

the $H^{\frac{1}{2}}$-semi-norm on Γ_i writes by using the definition of ϕ_ϵ

$$|\phi_\epsilon - 1|^2_{\frac{1}{2},\Gamma_i} := \int_{\Gamma_i}\int_{\Gamma_i} \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{||x-y||^2} dxdy \leq 2\int_0^\epsilon \int_0^\epsilon \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{|x-y|^2} dxdy. \tag{15}$$

Consider the decomposition of (15) into four partial double integrals

$$1) \int_0^{\epsilon^2}\int_0^{\epsilon^2} \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{|x-y|^2} dxdy = 0,$$

this is obvious.

$$2) \int_{\epsilon^2}^{\epsilon}\int_{\epsilon^2}^{\epsilon} \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{|x-y|^2} dxdy \leq \int_{\epsilon^2}^{\epsilon}\int_{\epsilon^2}^{\epsilon} \frac{|\exp[\ -\frac{\epsilon^{\frac{1}{2}}(\epsilon-x)}{(x-\epsilon^2)}\ ] - \exp[\ -\frac{\epsilon^{\frac{1}{2}}(\epsilon-y)}{(y-\epsilon^2)}\ ]|^2}{|x-y|^2} dxdy$$

The function $F(x) := \exp[\ -\frac{\epsilon^{\frac{1}{2}}(\epsilon-x)}{(x-\epsilon^2)}\ ]$ is $C^1(]\epsilon^2, \epsilon[)$ and thus lipschitz. We have, using the fact that $x \to F'(x)$ is increasing on $[\epsilon^2, \frac{\epsilon}{2}]$, that

$$|F'(x)| \leq \frac{\epsilon^{\frac{1}{2}}(1-\epsilon)}{\epsilon(\frac{1}{2}-\epsilon)^2} \exp(\frac{-\epsilon^{\frac{1}{2}}}{\frac{1}{2}-\epsilon}) =: L_1,$$

$\forall x \in [\epsilon^2, \frac{\epsilon}{2}]$. On the other hand

$$|F'(x)| \leq \frac{\epsilon^{\frac{1}{2}}}{\epsilon(1-\epsilon)} \exp(\frac{2}{1-\epsilon}) =: L_2,$$

$\forall x \in [\frac{\epsilon}{2}, \epsilon]$. Therefore we conclude that

$$|F'(x)| \leq L := \max(L_1, L_2) \leq L_1 + L_2$$

for all $x \in [\epsilon^2, \epsilon]$. This yields

$$
\begin{aligned}
&(\int_{\epsilon^2}^{\epsilon}\int_{\epsilon^2}^{\epsilon} \frac{|\phi_\epsilon(x)-\phi_\epsilon(y)|^2}{|x-y|^2}dxdy)^{\frac{1}{2}} \\
&\leq (\int_{\epsilon^2}^{\epsilon}\int_{\epsilon^2}^{\epsilon} \frac{|\frac{\epsilon^{\frac{1}{2}}}{\epsilon(1-\epsilon)}\exp(\frac{2}{1-\epsilon}) + \frac{\epsilon^{\frac{1}{2}}(1-\epsilon)}{\epsilon(\frac{1}{2}-\epsilon)^2}\exp(\frac{-\epsilon^{\frac{1}{2}}}{\frac{1}{2}-\epsilon})|^2|x-y|^2}{|x-y|^2}dxdy)^{\frac{1}{2}} \\
&\leq (\int_{\epsilon^2}^{\epsilon}\int_{\epsilon^2}^{\epsilon} |\frac{\epsilon^{\frac{1}{2}}}{\epsilon(1-\epsilon)}\exp(\frac{2}{1-\epsilon})|^2dxdy)^{\frac{1}{2}} + (\int_{\epsilon^2}^{\epsilon}\int_{\epsilon^2}^{\epsilon} |\frac{\epsilon^{\frac{1}{2}}(1-\epsilon)}{\epsilon(\frac{1}{2}-\epsilon)^2}\exp(\frac{-\epsilon^{\frac{1}{2}}}{\frac{1}{2}-\epsilon})|^2dxdy)^{\frac{1}{2}} \\
&\to 0 \quad as \quad \epsilon \to 0
\end{aligned}
$$

$$
\begin{aligned}
3)\ &\int_0^{\epsilon^2}\int_{\epsilon^2}^{\epsilon} \frac{|\phi_\epsilon(x)-\phi_\epsilon(y)|^2}{|x-y|^2}dxdy \\
&\leq \int_0^{\epsilon^2}\int_{\epsilon^2}^{\frac{\epsilon}{2}} \frac{1}{|x-\epsilon^2|^2}\exp^2[\ -\frac{\epsilon^{\frac{1}{2}}(\epsilon-x)}{(x-\epsilon^2)}]dxdy + \int_0^{\epsilon^2}\int_{\frac{\epsilon}{2}}^{\epsilon^2} \frac{1}{|x-\epsilon^2|^2}\exp^2[\ -\frac{\epsilon^{\frac{1}{2}}(\epsilon-x)}{(x-\epsilon^2)}]dxdy \\
&\leq \epsilon^2\int_{\epsilon^2}^{\frac{\epsilon}{2}} \frac{1}{|x-\epsilon^2|^2}\exp^2[\ -\frac{\epsilon^{\frac{1}{2}+1}(1-\epsilon)}{2(x-\epsilon^2)}]dx + \epsilon^2\int_{\frac{\epsilon}{2}}^{\epsilon} \frac{1}{|\frac{\epsilon}{2}-\epsilon^2|^2}\exp^2[\ \frac{\epsilon^{\frac{1}{2}}(2\epsilon)}{(\frac{\epsilon}{2}-\epsilon^2)}]dx \\
&\leq \frac{\epsilon^{\frac{1}{2}}\exp(-\frac{\epsilon^{\frac{3}{2}}(1-\epsilon)}{x})}{1-\epsilon}]_{x=0}^{x=\epsilon} + \frac{1}{(\frac{1}{2}-\epsilon)^2}\frac{\epsilon}{2}\exp^2[\ \frac{2\epsilon^{\frac{1}{2}}}{(\frac{1}{2}-\epsilon)}\ ] \\
&\to 0 \quad as \quad \epsilon \to 0,
\end{aligned}
$$

here we used the properties of the exponential function and elementary majorizations.

$$
4)\ \int_{\epsilon^2}^{\epsilon}\int_0^{\epsilon^2} \frac{|\phi_\epsilon(x)-\phi_\epsilon(y)|^2}{|x-y|^2}dxdy = \int_{\epsilon^2}^{\epsilon}\int_0^{\epsilon^2} \frac{|\exp[\ -\frac{\epsilon^{\frac{1}{2}}(y-\epsilon)}{(y-\epsilon^2)}\ ]|^2}{|x-y|^2}dxdy \to 0,
$$

proceed in the same way as for 3).

Combining these integrals on one hand and using the facts: $||\phi_\epsilon - 1||_{0,\Gamma} \to 0$ and $2||(\phi_\epsilon - 1)||_{0,\Gamma_i^{4,\epsilon}} \to 0$ for all $0 \leq i \leq m$ on the other hand yield, using (14), the result of lemma 3. Consequently, using (13), it yields also

$$||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma} \to 0.$$

□

Since u is $(\frac{1}{2}+\alpha)-$holder continuous and thus uniformly continuous on Ω, the result of lemma (3) implies

$$||u_\epsilon - u||_{\frac{1}{2},\Gamma} \leq ||u(\phi_\epsilon - 1)||_{\frac{1}{2},\Gamma} \leq ||u||_{\infty,\Gamma}||\phi_\epsilon - 1||_{\frac{1}{2},\Gamma} \to 0.$$

One proves the following approximation lemma:

Lemma 4. *The function u_ϵ defined by (8) and the distribution g_ϵ defined in problem (12) satisfy the following limits*

$$a)\ ||\nabla u_\epsilon - \nabla u||_{0,\Omega} \to 0, \quad b)\ ||g_\epsilon - g||_{[H^{\frac{1}{2}}]',\Gamma-\Gamma_0} \to 0$$

as $\epsilon \to 0$

Proof. a) Consider the linear operator G that associate, for the fixed $f \in L^2(\Omega)$, to each $u^d \in H^{\frac{1}{2}}(\Gamma)$ the corresponding unique solution u of problem (8).

$$\begin{aligned} G : (H^{\frac{1}{2}}(\Gamma), ||.||_{\frac{1}{2},\Gamma}) &\to A \subset V, \\ u^d := u_{|\Gamma} &\to K(u^d) = u. \end{aligned}$$

Where $(A, ||.||_{H^1(\Omega)})$ denote the range of $H^{\frac{1}{2}}(\Gamma)$ under G. The inverse operator G^{-1} identifies with the trace operator which is obviously well defined and bijective for $u \in A$. Using the trace inequality on Γ, $\forall u \in A$, there exists $c > 0$

$$||u||_{\frac{1}{2},\Gamma} \leq c||\nabla u||_{0,\Omega}$$

this implies the continuity of the linear bijective operator G^{-1}. By the Banach isomorphism theorem the operator G is continuous, this means that there exists $c_{-1} > 0$ such that for all $u \in A$ we have

$$||\nabla u||_{0,\Omega} \leq c_{-1}||u||_{\frac{1}{2},\Gamma}$$

thus

$$||\nabla u - \nabla u_\epsilon||_{0,\Omega} \leq c_{-1}||u - u_\epsilon||_{\frac{1}{2},\Gamma} \to 0, \tag{16}$$

as a consequence of lemma (3), this proves a).

b) We make the same reasoning as for a). Given $g \in [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]'$, let $w \in V$ be the unique solution of

$$\int_\Omega 2\mu\varepsilon(w)\varepsilon(v) + \lambda \operatorname{div} w \ \operatorname{div} v \ dx =< g, v >_{[H^{\frac{1}{2}}(\Gamma-\Gamma_0)]',H^{\frac{1}{2}}(\Gamma-\Gamma_0)} \quad ,\forall v \in V. \tag{17}$$

choosing $v = w$, there exist $c' > 0$ such that

$$||\nabla w||_{0,\Omega} \leq c'||g||_{[H^{\frac{1}{2}}(\Gamma-\Gamma_0)]'}. \tag{18}$$

Let K be the operator that associate to each data $g \in [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]'$ the solution function w of the corresponding problem (17):

$$\begin{aligned} K : [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]' &\to B \subset V \\ g &\to K(g) = w. \end{aligned}$$

Where $(B, ||\ ||_{H^1(\Omega)})$ denote the range of $[H^{\frac{1}{2}}]'(\Gamma - \Gamma_0)$ under K. Following existence and uniqueness result for problem (17), K is well defined, furthermore it is linear and invertible. An equivalent formulation of (18) is: there exists a constant $c' > 0$ such that $\forall g \in [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]'$, we have

$$||K(g)||_{H^1} \leq c'||g||_{[H^{\frac{1}{2}}]'(\Gamma-\Gamma_0)}$$

i.e. K is continuous. Then, according to Banach's isomorphism theorem, we deduce that $\exists c'_{-1} > 0$ such that

$$||g||_{[H^{\frac{1}{2}}(\Gamma-\Gamma_0)]'} \leq c'_{-1}||\nabla w||_{0,\Omega}. \tag{19}$$

Rewriting (4) with $g \in H^{\frac{1}{2}}(\Gamma - \Gamma_0) \equiv D \subset [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]'$ then substracting (4) and (12) member-to-member, one find that $u - u_\epsilon$ satisfy: $\forall v \in V$,

$$\int_\Omega 2\mu(\varepsilon(u_\epsilon)-\varepsilon(u))\varepsilon(v)+\lambda \operatorname{div}(u_\epsilon-u)\ \operatorname{div} v\, dx =< g_\epsilon-g, v >_{H^{-\frac{1}{2}}(\Gamma-\Gamma_0),H^{\frac{1}{2}}(\Gamma-\Gamma_0)} . \tag{20}$$

Applying (19) to $w = u_\epsilon - u$ we get:

$$||g_\epsilon - g||_{[H^{\frac{1}{2}}(\Gamma-\Gamma_0)]'} \leq c'_{-1}||\nabla u_\epsilon - \nabla u||_{0,\Omega}.$$

Considering (16), we infer b). □

Remark 5. *As a consequence of the previous lemma we have:* $\forall v \in V$ *and* $\forall \delta > 0$ *there exists* $\epsilon_0 > 0$ *such that* $\forall 0 < \epsilon < \epsilon_0$

$$\begin{aligned} | \int_{\Gamma-\Gamma_0} gvd\sigma(x) - < g_\epsilon, v >_{[H^{\frac{1}{2}}]'(\Gamma-\Gamma_0),H^{\frac{1}{2}}(\Gamma-\Gamma_0)} | &= | < g - g_\epsilon, v >_{[H^{\frac{1}{2}}]'(\Gamma-\Gamma_0),H^{\frac{1}{2}}(\Gamma-\Gamma_0)} | \\ &\leq ||g - g_\epsilon||_{[H^{\frac{1}{2}}]'(\Gamma-\Gamma_0)}\ ||v||_V \\ &\leq \frac{\delta}{\sqrt{m}}. \end{aligned}$$

We take now the idea of decomposition: consider the range of u by a sobolev extension operator

$$P : H^1(\Omega) \to H^1(\mathbb{R}^2)$$

which is always possible for the lipschitz domain Ω, this function is still denoted u. Given $\epsilon > 0$, there exists $(\Omega_i^\epsilon)_{i=1,m}$ an overlap of Ω and a C^1-partition of unity $(\varphi_i^\epsilon)_i$ with respect to this overlap such that if $M_i^\epsilon = \text{supp } \varphi_i^\epsilon \subset \Omega_i^\epsilon$ the following two conditions are satisfied:

(i) $c1$:
$1)\forall i = \overline{1, m-1}$: ∂M_i^ϵ intersects Γ in exactly two points p_i and p_{i+1}
$p_i \in D_{i,\epsilon} \cap \Gamma_{i-1}, p_{i+1} \in D_{i+1,\epsilon} \cap \Gamma_{i+1}$ such that $(\text{supp } \varphi_i^\epsilon u_\epsilon) \cap \Gamma \subset \Gamma_i$,

2) $for \quad i = 0 \quad \partial M_0^\epsilon$ intersects Γ in exactly two points q_0 and q_m
$q_m \in D_{0,\epsilon} \cap \Gamma_m, q_1 \in D_{1,\epsilon} \cap \Gamma_1$ such that $(\text{supp } \varphi_0^\epsilon u_\epsilon) \cap \Gamma \subset \Gamma_0$,

3) $for \quad i = m \quad \partial M_m^\epsilon$ intersects Γ in exactly two points r_{m-1} and r_0
$r_{m-1} \in D_{0,\epsilon} \cap \Gamma_{m-1}, r_0 \in D_{1,\epsilon} \cap \Gamma_0$ such that $(\text{supp } \varphi_m^\epsilon u_\epsilon) \cap \Gamma \subset \Gamma_m$,

(ii) $c2: \quad \forall\ i \neq j: \quad 2(\mu+\lambda) \max_{k=i,j} ||\nabla(\varphi_k^\epsilon u_\epsilon)||_{0,\Omega_i^\epsilon \cap \Omega_j^\epsilon} \leq \frac{\epsilon}{m\sqrt{m}}$.

Put $u_{\epsilon,i} := \varphi_i^\epsilon u_\epsilon$. Since we are looking for explicit estimates, we should use Poincaré, trace and Korn's inequalities relatively to suitable geometrical configuration i.e. for which they are explicitly formulated. The configuration that best fits our polygonal domain Ω is the half-plane $\mathbb{R}^{2+}$ containing the domain Ω for Korn's inequality, the square S_d with edge's length equal to $d(\Omega)$ for the other two inequalities. Thus we determine theses constants thanks to results available for this type of domains. All this suggests to extend by zero the functions $u_{\epsilon,i}$ outside Ω. The definition of the functions $u_{\epsilon,i}$ is adapted to make such an extension.

3 Technical tools

We introduce some useful lemmas, which will play essential roles in proving theorem (1).

3.1 Extension of the functions $u_{\epsilon,i}$

Given $\epsilon > 0$, we consider for $i,\ 1 \leq i \leq m$, the extension by zero of $u_{\epsilon,i}$ from Ω to the half-plane $\mathbb{R}^{2+}$ containing Ω such that $\Gamma_i \subset \partial\mathbb{R}^{2+}$. The extended function is defined by

$$\begin{cases} \tilde{u}_{\epsilon,i} = u_{\epsilon,i}, & a.e. \quad x \in \overline{M_i^\epsilon \cap \Omega}, \\ \tilde{u}_{\epsilon,i} = 0, & x \in \mathbb{R}^{2+} - \overline{M_i^\epsilon \cap \Omega}. \end{cases} \tag{21}$$

We have obviously the following

$$||\partial_{x_i} \tilde{u}_{\epsilon,i}||_{0,\mathbb{R}^{2+}} = ||\partial_{x_i} \tilde{u}_{\epsilon,i}||_{0,M_i^\epsilon \cap \Omega} = ||\partial_{x_i} u_{\epsilon,i}||_{0,M_i^\epsilon \cap \Omega}. \tag{22}$$

The inequalities are established for the extended H^1 regular functions defined on a square containing Ω.

3.2 Explicit constant in the Poincaré inequality

We show in the following lemma that the function $u_{\epsilon,i} \in V_i$ satisfy the Poincaré inequality for which we determine explicitly the constant.

Lemma 6. *For all i, $0 \leq i \leq m$, the function $u_{\epsilon,i}$ satisfy:*

$$||u_{\epsilon,i}||_{0,\Omega} \leq d(\Omega)||\nabla u_{\epsilon,i}||_{0,\Omega}, \tag{23}$$

the constant $d(\Omega)$ means the diameter of Ω.

Proof. We establish Poincaré inequality for one of the two components $u^l_{\epsilon,i}, l = 1, 2$, the same estimate hold with the other. Note *abcd* the square S_d such that $a = (a_1, a_2), b = (b_1, b_2), c = (c_1, c_2)$ et $d = (d_1, d_2)$ and such that $\Gamma_i \subset S_i := [c, d]$; so $\tilde{u}^l_{\epsilon,i} = 0$ on $\partial S_d - \Gamma_i$.

Since $\tilde{u}^l_{\epsilon,i}$ is absolutely continuous on the lines parallel to the coordinate axis, then applying the fundamental theorem of calculus to $\tilde{u}^l_{\epsilon,i}$ on

S_d for $l = 1, 2$, we have for all $(x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$

$$\tilde{u}^l_{\epsilon,i}(x_1, x_2) = \int_{a_1}^{x_1} \partial_{x_1} \tilde{u}^l_{\epsilon,i}(s, x_2)ds + \tilde{u}^l_{\epsilon,i}(a_1, x_2).$$

Since $(a_1, x_2) \in \partial S_d - \Gamma_i$, then $\forall (x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$

$$\tilde{u}^l_{\epsilon,i}(x_1, x_2) = \int_{a_1}^{x_1} \partial_{x_1} \tilde{u}^l_{\epsilon,i}(s, x_2)ds.$$

Using Cauchy-Schwarz inequality $\forall (x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$

$$|\tilde{u}^l_{\epsilon,i}(x_1, x_2)| \leq |x_1 - a|^{\frac{1}{2}} (\int_{a_1}^{x_1} |\partial_{x_1} \tilde{u}^l_{\epsilon,i}(s, x_2)|^2 ds)^{\frac{1}{2}}.$$

Taking the square of the two hand sides of this inequality and using the fact $|x_1 - a| \leq d(\Omega)$: $\forall (x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$ yields

$$|\tilde{u}^l_{\epsilon,i}(x_1, x_2)|^2 \leq |x_1 - a| \int_{a_1}^{x_1} |\partial_{x_1} \tilde{u}^l_{\epsilon,i}(s, x_2)|^2 ds \leq d(\Omega) \int_{a_1}^{d_1} |\partial_{x_1} \tilde{u}^l_{\epsilon,i}(s, x_2)|^2 ds.$$

Integrating on S_d with respect to the variables x_1 and x_2:

$$\begin{aligned} ||\tilde{u}^l_{\epsilon,i}||^2_{0,S_d} = \int_{a_2}^{b_2} \int_{a_1}^{d_1} |\tilde{u}^l_{\epsilon,i}(x_1, x_2)|^2 dx_1 dx_2 &\leq d(\Omega) \int_{a_2}^{b_2} \int_{a_1}^{d_1} \int_{a_1}^{d_1} |\partial_x \tilde{u}^l_{\epsilon,i}(s, x_2)|^2 ds dx_2 dx_1 \\ &\leq d^2(\Omega) \int \int_{S_d} |\partial_x \tilde{u}^l_{\epsilon,i}(s, x_2)|^2 ds dx_2. \end{aligned}$$

According to definition 21 and by considering (22) we get

$$||u^l_{\epsilon,i}||^2_{0,\Omega} \leq d^2(\Omega)||\nabla u^l_{\epsilon,i}||^2_{0,\Omega}.$$

We infer that

$$||u_{\epsilon,i}||^2_{0,\Omega} = ||u^1_{\epsilon,i}||^2_{0,\Omega} + ||u^2_{\epsilon,i}||^2_{0,\Omega} \leq d^2(\Omega)(||\nabla u^1_{\epsilon,i}||^2_{0,\Omega} + ||\nabla u^2_{\epsilon,i}||^2_{0,\Omega}) = d^2(\Omega)||\nabla u_{\epsilon,i}||^2_{0,\Omega}.$$

□

3.3 Explicit constant in the trace inequality

Using mainly the inequality of Poincaré stated in lemma (6), one establishes a trace inequality on Γ_i for the function $u_{\epsilon,i}$ with an explicit constant.

Lemma 7. *For all i, the functions $u_{\epsilon,i}$ satisfy:*

$$||u_{\epsilon,i}||_{0,\Gamma_i} \leq c_{tr}||\nabla u_{\epsilon,i}||_{0,\Omega} \tag{24}$$

where $c_{tr} := 2\sqrt{d(\Omega)}$ is the trace constant.

Proof. Let $\tilde{u}_{\epsilon,i}$ be defined on S_d such that $\Gamma_i \subset \partial S_d$. We establish trace inequality for one of the two components $u^l_{\epsilon,i}, l = 1, 2$, the same estimate hold with he other. Applying the inequality of trace on the boundary of a prallelogram (see lemma 4.2 in [3]) for $\tilde{u}^l_{\epsilon,i}$ on S_i, yields

$$||\tilde{u}^l_{\epsilon,i}||^2_{0,\Gamma_i} \leq ||\tilde{u}^l_{\epsilon,i}||^2_{0,S_i} \leq 2\frac{|S|}{|S_d|}||\tilde{u}^l_{\epsilon,i}||^2_{0,S_d} + 2\frac{|S_d|}{|S|}||\nabla\tilde{u}^l_{\epsilon,i}||^2_{0,S_d}.$$

Using estimate (6) we find

$$||\tilde{u}^l_{\epsilon,i}||^2_{0,\Gamma_i} \leq 2\frac{|S|}{|S_d|}d^2(\Omega)||\nabla\tilde{u}^l_{\epsilon,i}||^2_{0,S_d} + 2\frac{|S_d|}{|S|}||\nabla\tilde{u}^l_{\epsilon,i}||^2_{0,S_d},$$

hence by simplifying

$$||\tilde{u}^l_{\epsilon,i}||^2_{0,\Gamma_i} \leq 4d(\Omega)||\nabla\tilde{u}^l_{\epsilon,i}||^2_{0,S_d}.$$

Using 21 defining $\tilde{u}^l_{\epsilon,i}$ and (22) we have

$$||u^l_{\epsilon,i}||^2_{0,\Gamma_i} \leq 4d(\Omega)||\nabla u^l_{\epsilon,i}||^2_{0,\Omega}.$$

Summing over $l = 1, 2$ we get

$$||u_{\epsilon,i}||^2_{0,\Gamma_i} \leq 4d(\Omega)||\nabla u_{\epsilon,i}||^2_{0,\Omega}.$$

□

We need also the following elementary technical lemma:

Lemma 8. *Assume $v \in H^1(\Omega)$, then*

$$||\varepsilon(u)||_{0,\Omega} \leq ||\nabla u||_{0,\Omega} \quad , \qquad ||\operatorname{div} u||_{0,\Omega} \leq \sqrt{2}||\nabla u||_{0,\Omega}. \tag{25}$$

Proof. On one side

$$\begin{aligned}
\int_\Omega |\varepsilon(u)|^2_2\, dx = \frac{1}{4}\sum_{i,j=1}^{2}\int_\Omega (\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i})^2 dx &= \frac{1}{2}\sum_{i,j=1}^{2}\int_\Omega |\frac{\partial u^i}{\partial x_j}|^2 + (\frac{\partial u^i}{\partial x_j}\frac{\partial u^j}{\partial x_i})dx \\
&\leq \frac{1}{2}\int_\Omega (|\nabla u|^2_2 + \frac{1}{2}\sum_{i,j=1}^{2}(|\frac{\partial u^i}{\partial x_j}|^2 + |\frac{\partial u^j}{\partial x_i}|^2))dx \\
&\leq \frac{1}{2}\int_\Omega (|\nabla u|^2_2 + \frac{1}{2}(|\nabla u|^2_2 + |\nabla u|^2_2))dx \\
&\leq \int_\Omega |\nabla u|^2_2 dx = ||\nabla u||^2_{L^2(\Omega)^2}.
\end{aligned}$$

Therefore $(\int_\Omega |\varepsilon(u)|_2^2\ dx)^{\frac{1}{2}} \leq ||\nabla u||_{0,\Omega}$. On the other side

$$\begin{aligned}\int_\Omega \operatorname{div}^2 u\ dx = \int_\Omega |\partial_{x_1}u^1 + \partial_{x_2}u^2|^2 dx &\leq \int_\Omega (|\partial_{x_1}u^1| + |\partial_{x_2}u^2|)^2 dx \\ &\leq 2\int_\Omega |\partial_{x_1}u^1|^2 + |\partial_{x_2}u^2|^2 dx \\ &\leq 2\int_\Omega |\nabla u^1|_2^2 + |\nabla u^2|_2^2 dx = 2||\nabla u||^2_{L^2(\Omega)^2}.\end{aligned}$$

Therefore $(\int_\Omega \operatorname{div}^2 \mathrm{u}\, dx)^{\frac{1}{2}} \leq \sqrt{2}||\nabla u||_{0,\Omega}$.

□

4 Proof of Theorem 1

To prove the estimate of theorem (1), we choose as a test function in (12) the compactly supported functions $u_{\epsilon,i}$ and use Korn's inequality, Poincaré's and trace's inequalities to explicitly upper bound $||\nabla u_{\epsilon,i}||_{0,\Omega}$. This leads, thanks to the approximation lemma (4), to explicitly upper bound $||\nabla u||_{0,\Omega}$.

Proof. Let ϵ_0 be as defined in remark (5) and fix $\epsilon < \epsilon_0$.
Step (i) At first, we establish an upper bound estimate for $||\varepsilon(u_{\epsilon,i})||_{0,\Omega}$. We recall that u_ϵ verifies:

$$\forall v \in V, \qquad \int_\Omega 2\mu\varepsilon(u_\epsilon)\varepsilon(v) + \lambda \operatorname{div} u_\epsilon \operatorname{div} v\ dx = \int_\Omega fv\ dx + < g_\epsilon, v >_{[H^{\frac{1}{2}}]'(\Gamma-\Gamma_0), H^{\frac{1}{2}}(\Gamma-\Gamma_0)},$$

this function is expressed $u_\epsilon = \sum_{i=1}^m u_{\epsilon,i} = \sum_i u_\epsilon \varphi_i$. Choose $v = u_{\epsilon,i}$, this gives

$$\begin{aligned}\int_{M_i^\epsilon} 2\mu\varepsilon^2(u_{\epsilon,i}) + \lambda \operatorname{div}^2 u_{\epsilon,i}\ dx = &-\sum_{j,j\neq i} 2\int_{M_i^\epsilon \cap M_j^\epsilon} \mu\varepsilon(u_{\epsilon,j})\varepsilon(u_{\epsilon,i}) + \lambda \operatorname{div} u_{\epsilon,j} \operatorname{div} u_{\epsilon,i}\ dx \\ &+ \int_{M_i^\epsilon} fu_{\epsilon,i}dx + < g_\epsilon, u_{\epsilon,i} >_{[H^{\frac{1}{2}}]'(\Gamma-\Gamma_0), H^{\frac{1}{2}}(\Gamma-\Gamma_0)}.\end{aligned}$$

Using remark (5) and since the sequence $||u_{\epsilon,i}||_V$ is uniformly bounded for all i, we get

$$\begin{aligned}\int_{M_i^\epsilon} 2\mu\varepsilon^2(u_{\epsilon,i}) + \lambda \operatorname{div}^2 u_{\epsilon,i}\ dx = &-\sum_{j,j\neq i} 2\int_{M_i^\epsilon \cap M_j^\epsilon} \mu\varepsilon(u_{\epsilon,j})\varepsilon(u_{\epsilon,i}) + \lambda \operatorname{div} u_{\epsilon,j} \operatorname{div} u_{\epsilon,i}\ dx \\ &+ \int_{M_i^\epsilon} fu_{\epsilon,i}dx + \int_{\Gamma_i} gu_{\epsilon,i}d\sigma(x) + \frac{\delta}{\sqrt{m}}.\end{aligned}$$

Applying Cauchy-Schwarz inequality yields

$$\begin{aligned}\int_{M_i^\epsilon} 2\mu\varepsilon^2(u_{\epsilon,i}) + \lambda \operatorname{div}^2 u_{\epsilon,i}\ dx \leq & \sum_{j,j\neq i} 2\mu||\varepsilon(u_{\epsilon,j})||_{0,M_i^\epsilon\cap M_j^\epsilon}||\varepsilon(u_{\epsilon,i})||_{0,M_i^\epsilon\cap M_j^\epsilon} \\ & + 2\lambda||\operatorname{div} u_{\epsilon,j}||_{0,M_i^\epsilon\cap M_j^\epsilon}||\operatorname{div} u_{\epsilon,i}||_{0,M_i^\epsilon\cap M_j^\epsilon} \\ & + ||f||_{0,M_i^\epsilon}||u_{\epsilon,i}||_{0,M_i^\epsilon} + ||g||_{0,\Gamma_i}||u_{\epsilon,i}||_{0,\Gamma_i} + \frac{\delta}{\sqrt{m}}\end{aligned}$$

Lemma (8) allows us to write the following

$$\begin{aligned}\int_{M_i^\epsilon} 2\mu\varepsilon^2(u_{\epsilon,i}) + \lambda \operatorname{div}^2 u_{\epsilon,i}\ dx \leq & \sum_{j,j\neq i} 2\mu||\nabla u_{\epsilon,j}||_{0,M_i^\epsilon\cap M_j^\epsilon}||\nabla u_{\epsilon,i}||_{0,M_i^\epsilon\cap M_j^\epsilon} \\ & + 2\lambda||\nabla u_{\epsilon,j}||_{0,M_i^\epsilon\cap M_j^\epsilon}||\nabla u_{\epsilon,i}||_{0,M_i^\epsilon\cap M_j^\epsilon} \\ & + ||f||_{0,M_i^\epsilon}||u_{\epsilon,i}||_{0,\Omega} + ||g||_{0,\Gamma_i}||u_{\epsilon,i}||_{0,\Gamma} + \frac{\delta}{\sqrt{m}}.\end{aligned}$$

Using (23) and (24) we get

$$\begin{aligned}\int_{M_i^\epsilon} 2\mu\varepsilon^2(u_{\epsilon,i}) + \lambda \operatorname{div}^2 u_{\epsilon,i}\ dx \leq & \sum_{j,j\neq i} (2\mu+2\lambda)||\nabla u_{\epsilon,j}||_{0,M_i^\epsilon\cap M_j^\epsilon}||\nabla u_{\epsilon,i}||_{0,M_i^\epsilon\cap M_j^\epsilon} \\ & + c_p||f||_{0,M_i^\epsilon}||\nabla u_{\epsilon,i}||_{0,\Omega} + c_{tr}||g||_{0,\Gamma_i}||\nabla u_{\epsilon,i}||_{0,\Omega} + \frac{\delta}{\sqrt{m}}\end{aligned}$$

The condition $c2$ of the definition of the partition of unity gives

$$\begin{aligned}\int_{M_i^\epsilon} 2\mu\varepsilon^2(u_{\epsilon,i}) + \lambda \operatorname{div}^2 u_{\epsilon,i}\ dx \leq & \frac{\epsilon}{\sqrt{m}}||\nabla u_{\epsilon,i}||_{0,M_i^\epsilon} + c_p||f||_{0,M_i^\epsilon}||\nabla u_{\epsilon,i}||_{0,M_i^\epsilon} \\ & + c_{tr}||g||_{0,\Gamma_i}||\nabla u_{\epsilon,i}||_{0,M_i^\epsilon} + \frac{\delta}{\sqrt{m}}.\end{aligned} \tag{26}$$

Estimate (26) becomes

$$2\mu||\varepsilon(u_{\epsilon,i})||^2_{0,M_i^\epsilon} \leq ||\nabla u_{\epsilon,i}||_{0,M_i^\epsilon}(\frac{\epsilon}{\sqrt{m}} + c_p||f||_{0,M_i^\epsilon} + c_{tr}||g||_{0,\Gamma_i} + \frac{\delta}{\sqrt{m}}). \tag{27}$$

Step (ii) We give a lower-bound for $||\varepsilon(u_{\epsilon,i})||_{0,M_i^\epsilon}$ in term of $||\nabla u||_{0,M_i^\epsilon}$ for all i, $1 \leq i \leq m$. Since the deformation is a linear application with respect to the first derivatives of $u_{\epsilon,i}$, then with the same notation as in (21) and by using (22) we have

$$2\mu||\varepsilon(\tilde{u}_{\epsilon,i})||^2_{0,\mathbb{R}^{2+}} = 2\mu||\varepsilon(\tilde{u}_{\epsilon,i})||^2_{0,\Omega} = 2\mu||\varepsilon(u_{\epsilon,i})||^2_{0,M_i^\epsilon}. \tag{28}$$

Applying the estimate in corollary 1.2.2 of [6] to $\tilde{u}_{\epsilon,i}$ gives

$$\frac{1}{2} \times 2\mu||\nabla\tilde{u}_{\epsilon,i}||^2_{0,\mathbb{R}^{2+}} \leq 2\mu||\varepsilon(\tilde{u}_{\epsilon,i})||^2_{0,\mathbb{R}^{2+}}.$$

Hence, by using (28), $\forall i, 1 \leq i \leq m+1$, we get

$$\mu||\nabla u_{\epsilon,i}||^2_{0,\Omega} = \mu||\nabla\tilde{u}_{\epsilon,i}||^2_{0,\mathbb{R}^{2+}} \leq 2\mu||\varepsilon(\tilde{u}_{\epsilon,i})||^2_{0,\mathbb{R}^{2+}} = 2\mu||\varepsilon(u_{\epsilon,i})||^2_{0,\Omega}. \quad (29)$$

Combining (27) and (29) we get for all i

$$\begin{aligned}\mu||\nabla u_{\epsilon,i}||^2_{0,M_i^\epsilon} &\leq \int_{M_i^\epsilon} 2\mu\varepsilon^2(u_{\epsilon,i}) + \lambda \operatorname{div}^2 u_{\epsilon,i}\, dx \\ &\leq ||\nabla u_{\epsilon,i}||_{0,M_i^\epsilon}(\frac{\epsilon}{\sqrt{m}} + c_p||f||_{0,M_i^\epsilon} + c_{tr}||g||_{0,\Gamma_i} + \frac{\delta}{\sqrt{m}}),\end{aligned}$$

i.e.

$$\mu||\nabla u_{\epsilon,i}||_{0,M_i^\epsilon} \leq \frac{\epsilon}{\sqrt{m}} + c_p||f||_{0,M_i^\epsilon} + c_{tr}||g||_{0,\Gamma_i} + \frac{\delta}{\sqrt{m}}. \quad (30)$$

Taking the square of the two hand sides of (30) and using Young inequality

$$\mu^2||\nabla u_{\epsilon,i}||^2_{0,M_i^\epsilon} \leq 4(\frac{\epsilon^2}{m} + c_p^2||f||^2_{0,M_i^\epsilon} + c_{tr}^2||g||^2_{0,\Gamma_i} + \frac{\delta^2}{m}).$$

By summing over $i = \overline{0,m}$

$$\mu^2 \sum_i ||\nabla u_{\epsilon,i}||^2_{0,\Omega} \leq 4(\epsilon^2 + c_p^2 \sum_i ||f||^2_{0,M_i^\epsilon} + c_{tr}^2 \sum_i ||g||^2_{0,\Gamma_i} + \delta^2). \quad (31)$$

Applying the appropriate identity on the left hand side of (31)

$$\begin{aligned}\mu^2||\nabla \sum_i u_{\epsilon,i}||^2_{0,\Omega} \leq 2\mu^2 \sum_{i\neq j} \int_{M_i\cap M_j} \nabla u_{\epsilon,i}\nabla u_{\epsilon,j} dx \\ + 4(\epsilon^2 + c_p^2||f||^2_{0,\Omega} + c_{tr}^2||g||^2_{0,\Gamma-\Gamma_0} + \delta^2).\end{aligned} \quad (32)$$

Cauchy-Schwarz inequality yields using condition $c2$

$$2\mu^2 \int_{M_i^\epsilon\cap M_j^\epsilon} \nabla u_{\epsilon,i}\nabla u_{\epsilon,j} dx \leq 2\mu^2||\nabla u_{\epsilon,i}||_{M_i^\epsilon\cap M_j^\epsilon}||\nabla u_{\epsilon,j}||_{M_i^\epsilon\cap M_j^\epsilon} \leq \epsilon^2.$$

Inequality (32) becomes

$$\mu^2||\nabla u_\epsilon||^2_{0,\Omega} \leq \epsilon^2 + 4(\epsilon^2 + c_p^2||f||^2_{0,\Omega} + c_{tr}^2||g||^2_{0,\Gamma-\Gamma_0} + \delta^2),$$

taking the square root of the two hand sides,

$$||\nabla u_\epsilon||_{0,\Omega} \leq \frac{1}{\mu}(3\epsilon + 2c_p||f||_{0,\Omega} + 2c_{tr}||g||_{0,\Gamma-\Gamma_0} + \delta).$$

Letting $\epsilon \to 0$ and using the approximation lemma (4), $\forall \delta > 0$

$$||\nabla u||_{0,\Omega} \leq \frac{1}{\mu}(2c_p||f||_{0,\Omega} + 2c_{tr}||g||_{0,\Gamma-\Gamma_0} + \delta). \tag{33}$$

Since $g \in H^{\frac{1}{2}}(\Gamma - \Gamma_0)$, by continuity of the injection

$$I : H^{\frac{1}{2}}(\Gamma - \Gamma_0) \to L^2(\Gamma - \Gamma_0),$$

the estimate in Theorem (1) is deduced immediately from (33).

□

Finally, in order to get the explicit H^1 estimate of u_ϵ, and so that of u, we use the poincaré inequality (23) to bound $||u_\epsilon||_{0,\Omega}$ at one hand and the estimate (7) at the other hand.

Conclusion

In the point of view of numerical analysis, estimate of theorem (1) is interesting. Indeed, error estimates in finite element method of the type

$$||u - u_h||_{0,\Omega} \leq Ch||\nabla u||_{0,\Omega}$$

involve the quantity $||\nabla u||_{0,\Omega}$. Assuming that the constant C can be calculated, then it is possible to explicitly bound $||\nabla u||_{0,\Omega}$ which implies a better estimate of $||u||_{0,\Omega}$.

Another interesting fact in the estimation (7) that makes it effective is that it does not depend on the characteristic parameters of the polygonal domain Ω, namely, the edges's length, their number as well as the measures of the angles. The estimate is therefore indifferently applicable to all polygons. All this allows the possibility to generalize this result to a C^1 class domain.

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