



Explicit H^1 -Estimate for the solution of the Lamé system with mixed boundary conditions

Djamel Ait-Akli, Abdelkader Merakeb

► To cite this version:

Djamel Ait-Akli, Abdelkader Merakeb. Explicit H^1 -Estimate for the solution of the Lamé system with mixed boundary conditions. 2019. hal-01992510v3

HAL Id: hal-01992510

<https://hal.science/hal-01992510v3>

Preprint submitted on 16 May 2019 (v3), last revised 14 Feb 2020 (v13)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Explicit H^1 -Estimate for the solution of the Lamé system with mixed boundary conditions.

Djamel AIT AKLI^{a,*}, Abdelkader Merakeb^a

^a*L2CSP, U.M.M.T.O, 15000 Tizi-Ouzou, ALGERIA*

Abstract

In this paper we consider Lamé system of equations on a polygonal domain with mixed boundary conditions of Dirichlet-Neumann type. An explicit L^2 -estimate for the gradient of the solution of this problem is established. This leads to an explicit bound of the H^1 norm of this solution. Note that the obtained explicit upper bound is not optimal.

Keywords: Lamé system; Korn's inequality; Poincaré inequality; inequality of trace; explicit estimates.

2010 MSC: 35J57; 74B05

1. Introduction

The static equilibrium of a deformable structure occupying a domain Ω subset of \mathbb{R}^2 is governed by the Lamé linear elasto-static system of equations, see [1]. In this paper, we restrict the study to a domain Ω whose boundary has a polygonal shape that posses $m + 1$ edges with $m \geq 2$. We denote by $\Gamma = \cup \Gamma_i := \partial\Omega$ its boundary and by $d(\Omega)$ its diameter. This system is given by

$$\left\{ \begin{array}{ll} Lu = f & \text{a.e in } \Omega, \\ \sigma \cdot \vec{n}_i = g_i & \text{on } (\Gamma - \Gamma_0) \cap \Gamma_i, \quad 1 \leq i \leq m \\ u = 0 & \text{on } \Gamma_0. \end{array} \right. \quad (1)$$

*Corresponding author

Email addresses: d.aitakli@yahoo.com (Djamel AIT AKLI), merakeb_kader@yahoo.fr (Abdelkader Merakeb)

We assume that condition (H_2) of Theorem 2.3 stated in the paper [2] is satisfied by Γ . This condition is formulated in (5) below. The vector function $u = (u^1, u^2)$ satisfying the system (1) describes a displacement in the plane. In this model, we impose a Dirichlet homogeneous condition on Γ_0 and a Neumann condition on the rest of the boundary. The equality on the boundary is understood in the sense of the trace. We denote L the Lamé operator defined by:

$$Lu := -\operatorname{div} \sigma(u) = -\operatorname{div}[2\mu\varepsilon(u) + \lambda \operatorname{Tr} \varepsilon(u) Id] \quad (2)$$

The data functions f and g at the right hand side satisfy $f \in [L^2(\Omega)]^2$ and $g \in [H^{\frac{1}{2}}(\Gamma)]^2$. The vector \vec{n}_i represents the outside normal to Γ_i . We write μ and λ the Lamé's coefficients. We place ourselves in the isotropic framework, the deformation tensor ε is defined by

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^t u), \quad (3)$$

The weak form of problem (1) is (see [1], [3]):

find $u \in V; \quad \forall v \in V$

$$\int_{\Omega} 2\mu\varepsilon(u)\varepsilon(v) + \lambda \operatorname{div} u \operatorname{div} v \, dx = \int_{\Omega} f v \, dx + \int_{\Gamma-\Gamma_0} g v \, d\sigma(x) \quad (4)$$

where

$$V = \{v \in [H^1(\Omega)]^2; \quad v = 0 \quad \text{on} \quad \Gamma_0\}$$

The problem of existence and uniqueness in V of the solution of (4) is classic, (see [3]).

If we denote θ the interior angle between Γ_j and Γ_k such that $\bar{\Gamma}_j \cap \bar{\Gamma}_k \neq \emptyset$ and if we denote γ the interior angle between the Neumann part of the boundary Γ_N and the Dirichlet part of the boundary Γ_D such that $\Gamma_N \cap \Gamma_D \neq \emptyset$, then we impose

$$\theta \leq 2\pi, \quad \gamma \leq \pi. \quad (5)$$

The reason behind this assumption on the boundary is to get a better regularity of the solution of the weak problem (4). Precisely, in that case we have, following

[2], $u \in [H^{\frac{3}{2}+\epsilon}(\Omega)]^2$, which implies in particular, using the appropriate Sobolev embedding, see [4], that $u \in [C^{0,\frac{1}{2}+\epsilon}(\overline{\Omega})]^2$ i.e. u is $(\frac{1}{2} + \epsilon)$ -holder continuous. Let us denote

$$\|\varepsilon(u)\|_{0,\Omega} := (\int_{\Omega} \varepsilon(u) \varepsilon(u) \, dx)^{\frac{1}{2}}, \quad \|\nabla u\|_{0,\Omega} := (\int_{\Omega} |\nabla u^1|^2 + |\nabla u^2|^2 \, dx)^{\frac{1}{2}}.$$

By using the second Korn inequality, see [5], the trace and the Poincaré's inequalities, one easily get from (4) the following estimate

$$\|\nabla u\|_{0,\Omega} \leq \frac{1}{c_k} \frac{1}{2\mu} (c_p \|f\|_{0,\Omega} + c_{p,t} \|g\|_{\frac{1}{2},\Gamma}), \quad (6)$$

where $c_{p,t}$ is a constant that depends of Poincaré constant and the constant of trace inequality. c_k is the constant of the Korn's inequality. Note that the value of the constant c_k and $c_{p,t}$ appearing in (6) are unknown and can not be explicitly lower-bounded in the general case. We propose to determine explicitly these constants. The main result of this work is stated in the following theorem:

Theorem 1.1. *The unique weak solution u of (4) on the polygonal domain Ω admits the explicit upper bound*

$$\|\nabla u\|_{0,\Omega} \leq \frac{3}{\mu} (c_p \|f\|_{0,\Omega} + c_{tr} \|g\|_{\frac{1}{2},\Gamma}) \quad (7)$$

where $c_p := d(\Omega)$, $c_{tr} := 2\sqrt{d(\Omega)}$ and $d(\Omega)$ represent the diameter of Ω .

The estimate (7) is similar to (6), the constants that are present are the same. Before demonstrating this theorem, it is useful to go through some remarks and results. Consider the auxiliary function $u_\epsilon \in V$ defined by:

$$u_\epsilon(x) = \phi_\epsilon(x) u(x) \quad (8)$$

with:

$$\begin{cases} \phi_\epsilon(x) &= 0, & \|x - x_i\| \leq \frac{\sqrt{\epsilon}}{2} \\ \phi_\epsilon(x) &= 1 - \exp\left[\frac{(\|x - x_i\|^2 - \frac{\epsilon}{2^2})^2}{\|x - x_i\|^2 - \epsilon}\right], & \frac{\sqrt{\epsilon}}{2} < \|x - x_i\| < \sqrt{\epsilon} \\ \phi_\epsilon(x) &= 1, & \sqrt{\epsilon} \leq \|x - x_i\| \end{cases}$$

the functions $\phi_\epsilon \in C^2(\mathbb{R}^2)$ are identically zero on a sufficiently small neighborhood of the vertices.

In the sequel, we denote by u_ϵ the vector-valued function $u_\epsilon = (u_\epsilon^1, u_\epsilon^2)$.

2. Weak problem for u_ϵ and an approximation result

First of all, we construct the weak problem verified by the approximating function u_ϵ . With the approximating displacement $u_\epsilon \in V$ is associated the approximating stress tensor

$$\sigma_\epsilon := 2\mu\varepsilon(u_\epsilon) + \lambda \operatorname{Tr} \varepsilon(u_\epsilon) I, \quad (9)$$

since $Lu_\epsilon = \operatorname{div} \sigma_\epsilon = f_\epsilon$, then $\sigma_\epsilon \in [H(\operatorname{div})(\Omega)]^{2 \times 2}$. For a fixed ϵ , by density of the regular functions in the space $H(\operatorname{div})(\Omega)$, there exists $\sigma_\epsilon^n \in [C^\infty(\overline{\Omega})]^{2 \times 2}$ such that $\sigma_\epsilon^n \rightarrow \sigma_\epsilon$ in $[H(\operatorname{div})(\Omega)]^{2 \times 2}$. This means

$$\|\sigma_\epsilon^n - \sigma_\epsilon\|_{\operatorname{div}, \Omega} := \|\operatorname{div} \sigma_\epsilon^n - \operatorname{div} \sigma_\epsilon\|_{0, \Omega} + \|\sigma_\epsilon^n - \sigma_\epsilon\|_{0, \Omega} \rightarrow 0 \quad (10)$$

when $n \rightarrow \infty$. We put $\operatorname{div} \sigma_\epsilon^n = f_\epsilon^n$, then integrating by part against a test function $v \in [C^\infty(\overline{\Omega})]^2$ yields the following

$$\int_{\Omega} \sigma_\epsilon^n \nabla v = \int_{\Omega} f_\epsilon^n v + \int_{\Gamma} \sigma_\epsilon^n \cdot \vec{n} v d\sigma.$$

Passing to the limit in n using (10), we find $\forall v \in C^\infty(\overline{\Omega})$

$$\int_{\Omega} \sigma_\epsilon \nabla v = \int_{\Omega} f_\epsilon v + \langle \sigma_\epsilon \cdot \vec{n}, v \rangle_{[H^{\frac{1}{2}}]'\!(\Gamma) \times [H^{\frac{1}{2}}](\Gamma)},$$

where $\sigma_\epsilon \cdot \vec{n} =: g_\epsilon \in [H^{\frac{1}{2}}(\Omega)]'$ is the image of the normal component σ_ϵ by the trace operator on Γ . Since $C^\infty(\overline{\Omega})$ is a dense subset of $V \subset H^1(\Omega)$, then, according to the definition (3) and the expression (9), the function u_ϵ satisfy

$$\forall v \in V,$$

$$\int_{\Omega} 2\mu\varepsilon(u_\epsilon)\varepsilon(v) + \int_{\Omega} \lambda \operatorname{div} u_\epsilon \operatorname{div} v = \int_{\Omega} f_\epsilon v + \langle g_\epsilon, v \rangle_{[H^{\frac{1}{2}}]'\!(\Gamma) \times H^{\frac{1}{2}}(\Gamma)}. \quad (11)$$

One proves the following approximation lemma:

Lemma 2.1. *The function u_ϵ defined by (8) satisfy the following limits*

$$a) \|\nabla u_\epsilon - \nabla u\|_{0, \Omega} \rightarrow 0, \quad b) \|f_\epsilon - f\|_{0, \Omega} \rightarrow 0, \quad c) \|g_\epsilon - g\|_{[H^{\frac{1}{2}}]'\!(\Gamma)} \rightarrow 0$$

Proof. a) It is easy to see that $\phi_\epsilon \in C^1(\overline{B(x_i, \mathbb{R}^2)})$. It is shown using Lebesgue's dominated convergence theorem that $\|\phi_\epsilon - 1\|_{p, \Omega} \rightarrow 0$ for all $p > 1$, which implies that $\|\phi_\epsilon - 1\|_{\infty, \Omega} \rightarrow 0$. On the other hand

$$\begin{aligned} \|\nabla u_\epsilon - \nabla u\|_{0, \Omega} &= \|u \nabla \phi_\epsilon + \phi_\epsilon \nabla u - \nabla u\|_{0, \Omega} \\ &\leq \|u \nabla \phi_\epsilon\|_{0, \Omega} + \|(\phi_\epsilon - 1) \nabla u\|_{0, \Omega}. \end{aligned}$$

Using the fact that u is $(\frac{1}{2} + \epsilon)$ -holder continuous we have

$$\|\nabla u_\epsilon - \nabla u\|_{0, \Omega} \leq \|u\|_\infty \|\nabla \phi_\epsilon\|_{0, \Omega} + \|(\phi_\epsilon - 1)\|_{\infty, \Omega} \|\nabla u\|_{0, \Omega}.$$

One time again, the application of the dominated convergence theorem implies that

$$\|\nabla \phi_\epsilon\|_{0, \Omega} \rightarrow 0,$$

therefore

$$\|\nabla u_\epsilon - \nabla u\|_{0, \Omega} \rightarrow 0. \quad (12)$$

We prove the two remaining points b) and c) simultaneously. Rewriting (4) with $g \in H^{\frac{1}{2}}(\Gamma) \equiv D \subset [H^{\frac{1}{2}}(\Gamma)]'$ and choosing $v = u$ we obtain

$$\|\nabla u\|_{0, \Omega} \leq c(\|f\|_{0, \Omega} + \|g\|_{[H^{\frac{1}{2}}(\Gamma)]'}). \quad (13)$$

Let K be the operator that associate to each data pair $(f, g) \in L^2(\Omega) \times [H^{\frac{1}{2}}(\Gamma)]'$ the solution function u of the corresponding problem (11):

$$\begin{aligned} K : L^2(\Omega) \times [H^{\frac{1}{2}}(\Gamma)]' &\rightarrow H^1(\Omega) \\ (f, g) &\rightarrow K(f, g) = u. \end{aligned}$$

Following existence and uniqueness result for problem (11), K is well defined. An equivalent formulation of (13) is: there exist a constant $c > 0$ such that $\forall (f, g) \in L^2(\Omega) \times [H^{\frac{1}{2}}(\Gamma)]'$, we have

$$\|K(f, g)\|_{H^1} \leq c\|(f, g)\|_{L^2 \times [H^{\frac{1}{2}}]'} = c(\|f\|_{L^2(\Omega)} + \|g\|_{[H^{\frac{1}{2}}]'}),$$

i.e. K is continuous, furthermore it is linear and bijective. Then, according to Banach's isomorphism theorem, we deduce that $\exists c_{-1} > 0$ such that

$$\|f\|_{0,\Omega} + \|g\|_{[H^{\frac{1}{2}}(\Gamma)]'} \leq c_{-1} \|\nabla u\|_{0,\Omega}. \quad (14)$$

Subtracting (4) and (11) member-to-member, one find that $u - u_\epsilon$ satisfy: $\forall v \in V$,

$$\int_{\Omega} 2\mu(\varepsilon(u_\epsilon) - \varepsilon(u))\varepsilon(v) + \lambda \operatorname{div}(u_\epsilon - u) \operatorname{div} v \, dx = \langle g_\epsilon - g, v \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)}. \quad (15)$$

Applying (14) to $u_\epsilon - u$ we get:

$$\|f_\epsilon - f\|_{0,\Omega} + \|g_\epsilon - g\|_{[H^{\frac{1}{2}}(\Gamma)]'} \leq c_{-1} \|\nabla u_\epsilon - \nabla u\|_{0,\Omega}.$$

Considering (12), we infer b) and c). □

We take now the idea of decomposition: we consider the image of u by an extension operator

$$P : H^1(\Omega) \rightarrow H^1(\mathbb{R}^2)$$

which is always possible for the lipschitz domain Ω . Given $\epsilon > 0$; there exists $(\Omega_i^\epsilon)_{i=1,m}$ an overlap of Ω and a C^1 partition of unity $(\varphi_i^\epsilon)_i$ with respect to this overlap such that if $M_i^\epsilon = \operatorname{supp} \varphi_i^\epsilon \subset \Omega_i^\epsilon$ the following two conditions are satisfied:

(i) $c1 : \forall i = \overline{1, m} : \Gamma \cap \partial M_i \in D_\epsilon$ in such a way that $(\operatorname{supp} \varphi_i^\epsilon u_\epsilon) \cap \Gamma \subset \Gamma_i$,

(ii) $c2 : \forall i \neq j : 2(\mu + \lambda) \max_{k=i,j} \|\nabla(\varphi_k^\epsilon u_\epsilon)\|_{0,\Omega^\epsilon \cap \Omega_j^\epsilon} \leq \epsilon$,

where

$$D_\epsilon := \{x \in \mathbb{R}^2 \text{ such that } \|x - x_i\| < \epsilon\}.$$

Put $u_{\epsilon,i} := \varphi_i^\epsilon u_\epsilon$. Since we are looking for explicit estimates, we should use these inequalities relatively to a geometrical configuration for which they are explicitly formulated. The configuration that best fits our polygonal domain Ω

is the half-plane \mathbb{R}^{2+} containing the domain Ω for Korn's inequality, the square S_d with edge's length equal to $d(\Omega)$ for the other two inequalities. Thus we determine these constants thanks to results available for this type of domains. All this suggests to extend by zero the functions $u_{\epsilon,i}$ outside Ω . The definition of the functions $u_{\epsilon,i}$ is adapted to make such an extension.

3. Technical tools

We introduce some useful lemmas, which will play essential roles in proving theorem (1.1).

3.1. Extension of the functions $u_{\epsilon,i}$

Given $\epsilon > 0$, we consider for i , $1 \leq i \leq m$, the extension by zero of $u_{\epsilon,i}$ from Ω to the half-plane \mathbb{R}^{2+} containing Ω such that $\Gamma_i \subset \partial\mathbb{R}^{2+}$. The extended function is defined by

$$\begin{cases} \tilde{u}_{\epsilon,i} = u_{\epsilon,i}, & a.e. \quad x \in \overline{M_i^\epsilon \cap \Omega}, \\ \tilde{u}_{\epsilon,i} = 0, & x \in \mathbb{R}^{2+} - \overline{M_i^\epsilon \cap \Omega}. \end{cases} \quad (16)$$

We have obviously the following

$$\|\partial_{x_i} \tilde{u}_{\epsilon,i}\|_{0,\mathbb{R}^{2+}} = \|\partial_{x_i} \tilde{u}_{\epsilon,i}\|_{0,M_i^\epsilon \cap \Omega} = \|\partial_{x_i} u_{\epsilon,i}\|_{0,M_i^\epsilon \cap \Omega}. \quad (17)$$

The inequalities are established for the extended H^1 regular functions defined on a square containing Ω .

3.2. Explicit constant in the Poincaré inequality

We show in the following lemma that the function $u_{\epsilon,i} \in V_i$ satisfy the Poincaré inequality for which we determine explicitly the constant.

Lemma 3.1. *For all i , $1 < i < m$, the function $u_{\epsilon,i}$ satisfy:*

$$\|u_{\epsilon,i}\|_{0,\Omega} \leq d(\Omega) \|\nabla u_{\epsilon,i}\|_{0,\Omega}, \quad (18)$$

the constant $d(\Omega)$ means the diameter of Ω .

Proof. We establish Poincaré inequality for one of the two components $u_{\epsilon,i}^l, l = 1, 2$, the same estimate hold with the other. Note $abcd$ the square S_d such that $a = (a_1, a_2), b = (b_1, b_2), c = (c_1, c_2)$ et $d = (d_1, d_2)$ and such that $\Gamma_i \subset S_i := [c, d]$; so $\tilde{u}_{\epsilon,i}^l = 0$ on $\partial S_d - \Gamma_i$.

Since $\tilde{u}_{\epsilon,i}^l$ is absolutely continuous on the lines parallel to the coordinate axis, then applying the fundamental theorem of calculus to $\tilde{u}_{\epsilon,i}^l$ on

S_d for $l = 1, 2$, we have for all $(x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$

$$\tilde{u}_{\epsilon,i}^l(x_1, x_2) = \int_{a_1}^{x_1} \partial_{x_1} \tilde{u}_{\epsilon,i}^l(s, x_2) ds + \tilde{u}_{\epsilon,i}^l(a_1, x_2).$$

Since $(a_1, x_2) \in \partial S_d - \Gamma_i$, then $\forall (x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$

$$\tilde{u}_{\epsilon,i}^l(x_1, x_2) = \int_{a_1}^{x_1} \partial_{x_1} \tilde{u}_{\epsilon,i}^l(s, x_2) ds.$$

Using Cauchy-Schwarz inequality $\forall (x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$

$$|\tilde{u}_{\epsilon,i}^l(x_1, x_2)| \leq |x_1 - a|^{1/2} \left(\int_{a_1}^{x_1} |\partial_{x_1} \tilde{u}_{\epsilon,i}^l(s, x_2)|^2 ds \right)^{1/2}.$$

Taking the square of the two hand sides of this inequality and using the fact $|x_1 - a| \leq d(\Omega)$: $\forall (x_1, x_2) \in [a_1, d_1] \times [a_2, b_2]$ yields

$$|\tilde{u}_{\epsilon,i}^l(x_1, x_2)|^2 \leq |x_1 - a| \int_{a_1}^{x_1} |\partial_{x_1} \tilde{u}_{\epsilon,i}^l(s, x_2)|^2 ds \leq d(\Omega) \int_{a_1}^{d_1} |\partial_{x_1} \tilde{u}_{\epsilon,i}^l(s, x_2)|^2 ds.$$

Integrating on S_d with respect to the variables x_1 and x_2 :

$$\begin{aligned} \|\tilde{u}_{\epsilon,i}^l\|_{0,S_d}^2 &= \int_{a_2}^{b_2} \int_{a_1}^{d_1} |\tilde{u}_{\epsilon,i}^l(x_1, x_2)|^2 dx_1 dx_2 \leq d(\Omega) \int_{a_2}^{b_2} \int_{a_1}^{d_1} \int_{a_1}^{d_1} |\partial_{x_1} \tilde{u}_{\epsilon,i}^l(s, x_2)|^2 ds dx_2 dx_1 \\ &\leq d^2(\Omega) \int \int_{S_d} |\partial_x \tilde{u}_{\epsilon,i}^l(s, x_2)|^2 ds dx_2. \end{aligned}$$

According to definition 16 and by considering (17) we get

$$\|u_{\epsilon,i}^l\|_{0,\Omega}^2 \leq d^2(\Omega) \|\nabla u_{\epsilon,i}^l\|_{0,\Omega}^2.$$

We infer that

$$\|u_{\epsilon,i}\|_{0,\Omega}^2 = \|u_{\epsilon,i}^1\|_{0,\Omega}^2 + \|u_{\epsilon,i}^2\|_{0,\Omega}^2 \leq d^2(\Omega) (\|\nabla u_{\epsilon,i}^1\|_{0,\Omega}^2 + \|\nabla u_{\epsilon,i}^2\|_{0,\Omega}^2) = d^2(\Omega) \|\nabla u_{\epsilon,i}\|_{0,\Omega}^2.$$

□

3.3. Explicit constant in the trace inequality

Using mainly the inequality of Poincaré stated in lemma (3.1), one establishes a trace inequality on Γ_i for the function $u_{\epsilon,i}$ with an explicit constant.

Lemma 3.2. *For all i , the functions $u_{\epsilon,i}$ satisfy:*

$$\|u_{\epsilon,i}\|_{0,\Gamma_i} \leq c_{tr} \|\nabla u_{\epsilon,i}\|_{0,\Omega} \quad (19)$$

where $c_{tr} := 2\sqrt{d(\Omega)}$ is the trace constant.

Proof. Let $\tilde{u}_{\epsilon,i}$ be defined on S_d such that $\Gamma_i \subset \partial S_d$. We establish trace inequality for one of the two components $u_{\epsilon,i}^l$, $l = 1, 2$, the same estimate hold with the other. Applying the inequality of trace on the boundary of a parallelogram (see lemma 4.2 in [6]) for $\tilde{u}_{\epsilon,i}^l$ on S_i , yields

$$\|\tilde{u}_{\epsilon,i}^l\|_{0,\Gamma_i}^2 \leq \|\tilde{u}_{\epsilon,i}^l\|_{0,S_i}^2 \leq 2 \frac{|S|}{|S_d|} \|\tilde{u}_{\epsilon,i}^l\|_{0,S_d}^2 + 2 \frac{|S_d|}{|S|} \|\nabla \tilde{u}_{\epsilon,i}^l\|_{0,S_d}^2.$$

Using estimate (3.1) we find

$$\|\tilde{u}_{\epsilon,i}^l\|_{0,\Gamma_i}^2 \leq 2 \frac{|S|}{|S_d|} d^2(\Omega) \|\nabla \tilde{u}_{\epsilon,i}^l\|_{0,S_d}^2 + 2 \frac{|S_d|}{|S|} \|\nabla \tilde{u}_{\epsilon,i}^l\|_{0,S_d}^2,$$

hence by simplifying

$$\|\tilde{u}_{\epsilon,i}^l\|_{0,\Gamma_i}^2 \leq 4d(\Omega) \|\nabla \tilde{u}_{\epsilon,i}^l\|_{0,S_d}^2.$$

Using 16 defining $\tilde{u}_{\epsilon,i}^l$ and (17) we have

$$\|u_{\epsilon,i}^l\|_{0,\Gamma_i}^2 \leq 4d(\Omega) \|\nabla u_{\epsilon,i}^l\|_{0,\Omega}^2.$$

Summing over $l = 1, 2$ we get

$$\|u_{\epsilon,i}\|_{0,\Gamma_i}^2 \leq 4d(\Omega) \|\nabla u_{\epsilon,i}\|_{0,\Omega}^2.$$

□

We need also the following elementary technical lemma:

Lemma 3.3. *Assume $v \in H^1(\Omega)$, then*

$$\|\varepsilon(u)\|_{0,\Omega} \leq \|\nabla u\|_{0,\Omega} \quad , \quad \|\operatorname{div} u\|_{0,\Omega} \leq \sqrt{2} \|\nabla u\|_{0,\Omega}. \quad (20)$$

Proof. On one side

$$\begin{aligned}
\int_{\Omega} |\varepsilon(u)|_2^2 dx &= \frac{1}{4} \sum_{i,j=1}^2 \int_{\Omega} \left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i} \right)^2 dx = \frac{1}{2} \sum_{i,j=1}^2 \int_{\Omega} \left| \frac{\partial u^i}{\partial x_j} \right|^2 + \left(\frac{\partial u^i}{\partial x_j} \frac{\partial u^j}{\partial x_i} \right) dx \\
&\leq \frac{1}{2} \int_{\Omega} (|\nabla u|_2^2 + \frac{1}{2} \sum_{i,j=1}^2 (|\frac{\partial u^i}{\partial x_j}|^2 + |\frac{\partial u^j}{\partial x_i}|^2)) dx \\
&\leq \frac{1}{2} \int_{\Omega} (|\nabla u|_2^2 + \frac{1}{2} (|\nabla u|_2^2 + |\nabla u|_2^2)) dx \\
&\leq \int_{\Omega} |\nabla u|_2^2 dx = \|\nabla u\|_{L^2(\Omega)^2}^2.
\end{aligned}$$

Therefore $(\int_{\Omega} |\varepsilon(u)|_2^2 dx)^{\frac{1}{2}} \leq \|\nabla u\|_{0,\Omega}$. On the other side

$$\begin{aligned}
\int_{\Omega} \operatorname{div}^2 u dx &= \int_{\Omega} |\partial_{x_1} u^1 + \partial_{x_2} u^2|^2 dx \leq \int_{\Omega} (|\partial_{x_1} u^1| + |\partial_{x_2} u^2|)^2 dx \\
&\leq 2 \int_{\Omega} |\partial_{x_1} u^1|^2 + |\partial_{x_2} u^2|^2 dx \\
&\leq 2 \int_{\Omega} |\nabla u^1|_2^2 + |\nabla u^2|_2^2 dx = 2 \|\nabla u\|_{L^2(\Omega)^2}^2.
\end{aligned}$$

Therefore $(\int_{\Omega} \operatorname{div}^2 u dx)^{\frac{1}{2}} \leq \sqrt{2} \|\nabla u\|_{0,\Omega}$.

□

4. Proof of Theorem 1.1

To prove the estimate of theorem (1.1), we choose as a test function in (11) the compactly supported functions $u_{\epsilon,i}$ and use Korn's inequality, Poincaré's and trace's inequalities to explicitly upper bound $\|\nabla u_{\epsilon,i}\|_{0,\Omega}$. This leads, thanks to the approximation lemma (2.1), to explicitly upper bound $\|\nabla u\|_{0,\Omega}$.

Proof. Fix $\epsilon > 0$.

Step (i) At first, we establish an upper bound estimate for $\|\varepsilon(u_{\epsilon,i})\|_{0,\Omega}$. We recall that u_{ϵ} verifies:

$$\forall v \in V, \quad \int_{\Omega} 2\mu \varepsilon(u_{\epsilon}) \varepsilon(v) + \lambda \operatorname{div} u_{\epsilon} \operatorname{div} v dx = \int_{\Omega} f_{\epsilon} v dx + \langle g_{\epsilon}, v \rangle_{[H^{\frac{1}{2}}]'\!(\Gamma), H^{\frac{1}{2}}(\Gamma)},$$

this function is expressed $u_\epsilon = \sum_i u_{\epsilon,i} = \sum_i u_\epsilon \varphi_i$. Choose $v = u_{\epsilon,i}$, this gives

$$\begin{aligned} \int_{M_i^\epsilon} 2\mu \varepsilon^2(u_{\epsilon,i}) + \lambda \operatorname{div}^2 u_{\epsilon,i} \, dx &= - \sum_{j,j \neq i} 2 \int_{M_i^\epsilon \cap M_j^\epsilon} \mu \varepsilon(u_{\epsilon,j}) \varepsilon(u_{\epsilon,i}) + \lambda \operatorname{div} u_{\epsilon,j} \operatorname{div} u_{\epsilon,i} \, dx \\ &\quad + \int_{M_i^\epsilon} f_\epsilon u_{\epsilon,i} dx + \langle g_\epsilon, u_{\epsilon,i} \rangle_{[H^{\frac{1}{2}}]^\prime(\Gamma), H^{\frac{1}{2}}(\Gamma)}. \end{aligned}$$

Applying Cauchy-Schwarz inequality yields

$$\begin{aligned} \int_{M_i^\epsilon} 2\mu \varepsilon^2(u_{\epsilon,i}) + \lambda \operatorname{div}^2 u_{\epsilon,i} \, dx &\leq \sum_{j,j \neq i} 2\mu \|\varepsilon(u_{\epsilon,j})\|_{0,M_i^\epsilon \cap M_j^\epsilon} \|\varepsilon(u_{\epsilon,i})\|_{0,M_i^\epsilon \cap M_j^\epsilon} \\ &\quad + 2\lambda \|\operatorname{div} u_{\epsilon,j}\|_{0,M_i^\epsilon \cap M_j^\epsilon} \|\operatorname{div} u_{\epsilon,i}\|_{0,M_i^\epsilon \cap M_j^\epsilon} \\ &\quad + \|f_\epsilon\|_{0,M_i^\epsilon} \|u_{\epsilon,i}\|_{0,M_i^\epsilon} + \|g_\epsilon\|_{[H^{\frac{1}{2}}]^\prime, \Gamma} \|u_{\epsilon,i}\|_{\frac{1}{2}, \Gamma_i} \end{aligned}$$

Lemma (3.3) allows us to write the following

$$\begin{aligned} \int_{M_i^\epsilon} 2\mu \varepsilon^2(u_{\epsilon,i}) + \lambda \operatorname{div}^2 u_{\epsilon,i} \, dx &\leq \sum_{j,j \neq i} 2\mu \|\nabla u_{\epsilon,j}\|_{0,M_i^\epsilon \cap M_j^\epsilon} \|\nabla u_{\epsilon,i}\|_{0,M_i^\epsilon \cap M_j^\epsilon} \\ &\quad + 2\lambda \|\nabla u_{\epsilon,j}\|_{0,M_i^\epsilon \cap M_j^\epsilon} \|\nabla u_{\epsilon,i}\|_{0,M_i^\epsilon \cap M_j^\epsilon} \\ &\quad + \|f_\epsilon\|_{0,M_i^\epsilon} \|u_{\epsilon,i}\|_{0,\Omega} + \|g_\epsilon\|_{[H^{\frac{1}{2}}]^\prime, \Gamma} \|u_{\epsilon,i}\|_{\frac{1}{2}, \Gamma}. \end{aligned}$$

Using (18) and (19) we get

$$\begin{aligned} \int_{M_i^\epsilon} 2\mu \varepsilon^2(u_{\epsilon,i}) + \lambda \operatorname{div}^2 u_{\epsilon,i} \, dx &\leq \sum_{j,j \neq i} (2\mu + 2\lambda) \|\nabla u_{\epsilon,j}\|_{0,M_i^\epsilon \cap M_j^\epsilon} \|\nabla u_{\epsilon,i}\|_{0,M_i^\epsilon \cap M_j^\epsilon} \\ &\quad + c_p \|f_\epsilon\|_{0,M_i^\epsilon} \|\nabla u_{\epsilon,i}\|_{0,\Omega} + c_{tr} \|g_\epsilon\|_{[H^{\frac{1}{2}}]^\prime, \Gamma} \|\nabla u_{\epsilon,i}\|_{0,\Omega} \end{aligned}$$

The condition c_2 of the definition of the partition of unity gives

$$\begin{aligned} \int_{M_i^\epsilon} 2\mu \varepsilon^2(u_{\epsilon,i}) + \lambda \operatorname{div}^2 u_{\epsilon,i} \, dx &\leq \frac{\epsilon}{2} \|\nabla u_{\epsilon,i}\|_{0,M_i^\epsilon} + c_p \|f_\epsilon\|_{0,M_i^\epsilon} \|\nabla u_{\epsilon,i}\|_{0,M_i^\epsilon} \\ &\quad + c_{tr} \|g_\epsilon\|_{[H^{\frac{1}{2}}]^\prime, \Gamma} \|\nabla u_{\epsilon,i}\|_{0,M_i^\epsilon}. \end{aligned} \tag{21}$$

Estimate (21) becomes

$$2\mu \|\varepsilon(u_{\epsilon,i})\|_{0,M_i^\epsilon}^2 \leq \|\nabla u_{\epsilon,i}\|_{0,M_i^\epsilon} (\epsilon + c_p \|f_\epsilon\|_{0,M_i^\epsilon} + c_{tr} \|g_\epsilon\|_{[H^{\frac{1}{2}}]^\prime, \Gamma}). \tag{22}$$

Step (ii) We give a lower-bound for $\|\varepsilon(u_{\epsilon,i})\|_{0,M_i^\epsilon}$ in term of $\|\nabla u\|_{0,M_i^\epsilon}$ for all $i, 1 \leq i \leq m$. Since the deformation is a linear application with respect to the first derivatives of $u_{\epsilon,i}$, then with the same notation as in (16) and by using (17) we have

$$2\mu\|\varepsilon(\tilde{u}_{\epsilon,i})\|_{0,\mathbb{R}^{2+}}^2 = 2\mu\|\varepsilon(\tilde{u}_{\epsilon,i})\|_{0,\Omega}^2 = 2\mu\|\varepsilon(u_{\epsilon,i})\|_{0,M_i^\epsilon}^2. \quad (23)$$

Applying the estimate in corollary 1.2.2 of [7] to $\tilde{u}_{\epsilon,i}$ gives

$$\frac{1}{2} \times 2\mu\|\nabla \tilde{u}_{\epsilon,i}\|_{0,\mathbb{R}^{2+}}^2 \leq 2\mu\|\varepsilon(\tilde{u}_{\epsilon,i})\|_{0,\mathbb{R}^{2+}}^2.$$

Hence, by using (23), $\forall i, 1 \leq i \leq m+1$, we get

$$\mu\|\nabla u_{\epsilon,i}\|_{0,\Omega}^2 = \mu\|\nabla \tilde{u}_{\epsilon,i}\|_{0,\mathbb{R}^{2+}}^2 \leq 2\mu\|\varepsilon(\tilde{u}_{\epsilon,i})\|_{0,\mathbb{R}^{2+}}^2 = 2\mu\|\varepsilon(u_{\epsilon,i})\|_{0,\Omega}^2. \quad (24)$$

Combining (22) and (24) we get for all i

$$\mu\|\nabla u_{\epsilon,i}\|_{0,M_i^\epsilon}^2 \leq \int_{M_i^\epsilon} 2\mu\varepsilon^2(u_{\epsilon,i}) + \lambda \operatorname{div}^2 u_{\epsilon,i} \, dx \leq \|\nabla u_{\epsilon,i}\|_{0,M_i^\epsilon} (\epsilon + c_p\|f_\epsilon\|_{0,M_i^\epsilon} + c_{tr}\|g_\epsilon\|_{[H^{\frac{1}{2}}]',\Gamma}),$$

i.e.

$$\mu\|\nabla u_{\epsilon,i}\|_{0,M_i^\epsilon} \leq \epsilon + c_p\|f_\epsilon\|_{0,M_i^\epsilon} + c_{tr}\|g_\epsilon\|_{[H^{\frac{1}{2}}]',\Gamma}. \quad (25)$$

Taking the square of the two hand sides of (25) and using Young inequality

$$\mu^2\|\nabla u_{\epsilon,i}\|_{0,M_i^\epsilon}^2 \leq 3(\epsilon^2 + c_p^2\|f_\epsilon\|_{0,M_i^\epsilon}^2 + c_{tr}^2\|g_\epsilon\|_{[H^{\frac{1}{2}}]',\Gamma}^2).$$

By summing over $i = \overline{1, m}$

$$\mu^2 \sum_i \|\nabla u_{\epsilon,i}\|_{0,\Omega}^2 \leq 3(m\epsilon^2 + c_p^2 \sum \|f_\epsilon\|_{0,M_i^\epsilon}^2 + c_{tr}^2 \sum \|g_\epsilon\|_{[H^{\frac{1}{2}}]',\Gamma}^2). \quad (26)$$

Applying the appropriate identity on the left hand side of (26)

$$\begin{aligned} \mu^2 \|\nabla \sum_i u_{\epsilon,i}\|_{0,\Omega}^2 &\leq 2\mu^2 \sum_{i \neq j} \int_{M_i \cap M_j} \nabla u_{\epsilon,i} \nabla u_{\epsilon,j} \, dx \\ &\quad + 3(m\epsilon^2 + c_p^2\|f_\epsilon\|_{0,\Omega}^2 + c_{tr}^2\|g_\epsilon\|_{[H^{\frac{1}{2}}]',\Gamma}^2). \end{aligned} \quad (27)$$

Cauchy-Schwarz inequality yields using condition c2

$$2\mu^2 \int_{M_i^\epsilon \cap M_j^\epsilon} \nabla u_{\epsilon,i} \nabla u_{\epsilon,j} \, dx \leq 2\mu^2 \|\nabla u_{\epsilon,i}\|_{M_i^\epsilon \cap M_j^\epsilon} \|\nabla u_{\epsilon,j}\|_{M_i^\epsilon \cap M_j^\epsilon} \leq \epsilon^2.$$

Inequality (27) becomes

$$\mu^2 \|\nabla u_\epsilon\|_{0,\Omega}^2 \leq \epsilon^2 + 3(m\epsilon^2 + c_p^2 \|f_\epsilon\|_{0,\Omega}^2 + c_{tr}^2 \|g_\epsilon\|_{[H^{\frac{1}{2}}]',\Gamma}^2),$$

taking the square root of the two hand sides,

$$\|\nabla u_\epsilon\|_{0,\Omega} \leq \frac{1}{\mu} (\epsilon \sqrt{3m+1} + 3c_p \|f_\epsilon\|_{0,\Omega} + 3c_{tr} \|g_\epsilon\|_{[H^{\frac{1}{2}}]',\Gamma}).$$

Letting $\epsilon \rightarrow 0$ and using the approximation lemma (2.1)

$$\|\nabla u\|_{0,\Omega} \leq \frac{1}{\mu} (3c_p \|f\|_{0,\Omega} + 3c_{tr} \|g\|_{[H^{\frac{1}{2}}]',\Gamma}). \quad (28)$$

Since $g \in L^2(\Gamma)$, by continuity of the injection

$$I : H^{\frac{1}{2}}(\Gamma) \equiv D \subset [H^{\frac{1}{2}}(\Gamma)]' \rightarrow [H^{\frac{1}{2}}(\Gamma)]',$$

the estimate in Theorem (1.1) is deduced immediately from (28). □

Finally, in order to get the explicit H^1 estimate of u_ϵ , and so that of u , we use the poincaré inequality (18) to bound $\|u_\epsilon\|_{0,\Omega}$ at one hand and the estimate (7) at the other hand.

Conclusion

The result of this lemma is still valid if the structure occupies a non-convex domain. Indeed, the convexity of the domain Ω does not affect since the essential tool used in the proof is the partition of the unity. In the point of view of numerical analysis, estimate of theorem (1.1) is interesting. Indeed, error estimates in finite element method of the type

$$\|u - u_h\|_{0,\Omega} \leq Ch \|\nabla u\|_{0,\Omega}$$

involve the quantity $\|\nabla u\|_{0,\Omega}$. Assuming that the constant C can be calculated, then it is possible to explicitly bound $\|\nabla u\|_{0,\Omega}$ which implies a better estimate of $\|u\|_{0,\Omega}$.

Another interesting fact in the estimation (7) that makes it effective is that it does not depend on the characteristic parameters of the polygonal domain Ω ,

namely, the edges's length, their number as well as the measures of the angles. The estimate is therefore indifferently applicable to all polygons. All this allows the possibility to generalize this result to a C^1 class domain.

References

- [1] G. Duvaut, J. L. Lions, Les inéquations en mécanique et en physique, Dunod, Paris, 1972.
- [2] S. Nicaise, About the lamé system in a polygonal or polyhedral domain and a coupled problem between the lame system and the plate equation. i: regularity of the solutions, Annali della scuola Normale superiore di Pisa, classe di scienze 4e série 19 (03) (1992) 327–361.
- [3] G. Allaire, Analyse numérique et optimisation. Une introduction ‘a la modélisation et à la simulation numérique, 2nd Edition, Editions de lécole polytechnique, 2004.
- [4] R. Adams, J. Fournier, Sobolev spaces, 2nd Edition, Academic Press, 1975.
- [5] C. Horgan, Korn’s inequalities and their applications in continuum mechanics, SIAM Review 37 (4) (1995) 491–511.
- [6] S. Sauter, C. Carstensen, A posteriori error analysis for elliptic pdes on domains with complicated structures, Numerische Mathematik 96 (4) (2004) 691–721.
- [7] C. Eck, J. Jarusek, M. Krbec, Unilateral contact problems: variational methods and existence theorems, CRC Pure and Applied Mathematics, Chapman and Hall, 2005.