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# Coprime sampling with embedded random delays

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## Abstract

Spectrum sensing over a broad frequency band plays an important role in cognitive radio, but it also leads to high sampling rate when the bandwidth is large. The recently proposed coprime sampling scheme has been recognized as an attractive mechanism because it allows to significantly reduce the sampling rate. However in some cases, classical high resolution spectrum estimation methods fail when coprime sampling data are used. This situation has never been considered in the open literature under the framework of practical coprime sampling. In this paper, firstly, the conditions of this phenomenon are specified. Then, a new coprime sampling scheme based on embedded random delays is proposed to prevent this phenomenon. Simulation results show the effectiveness of the proposed coprime sampling scheme.

*Keywords:*

Spectrum sensing, coprime sampling, frequency estimation.

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## 1. INTRODUCTION

Due to the strong development of wireless techniques, the demand for frequency spectrum leads to an increasing research interest in radio frequency spectrum analysis. To optimally utilize the spectrum opportunities, a promising solution is using cognitive radio [1], which allows secondary users to access the licensed spectrum bands when the primary user is absent. To that end, the frequency bands preassigned to the primary users need to be sensed before sharing the unoccupied frequency bands. Many techniques have been developed to detect dynamic frequencies, including energy detection [2], match filtering [3], and cyclostationary feature detection [4], etc. For most of these techniques, the sampling rate is an important issue closely related to hardware implementation as well as cost. In practice, the above methods are limited by the Nyquist sampling rate, which causes high complexity for hardware when the bandwidth is large.

To mitigate Nyquist rate sampling burden on hardware, many sub-Nyquist rate sensing methods have been developed [5], [6]. Recently, coprime sampling [7], [8] has attracted growing interest. For this technique, two samplers are adopted to sample a signal constituted of a sum of sinusoidal signals at sub-Nyquist

sampling rates with sampling intervals  $MT$  and  $NT$ , respectively (referred as the classical coprime scheme in this paper), where  $M$  and  $N$  are two coprime integers and  $T$  is the Nyquist sampling period. By using the concept of coarray [9], two uniformly sample sets generate a virtual data set and higher degrees of freedom (DOFs) can be achieved. Qin *et al.* [10] generalized the classical coprime scheme and exploited overlapped sampling data to further increase the DOFs provided by the classical coprime scheme. However, the above mentioned techniques are based on the hypothesis of uncorrelated sinusoidal components, which is not always satisfied for a finite number of data samples, even if the different signals are statistically independent. For some specific conditions, the above mentioned techniques completely fail. To the best of our knowledge, this problem has never been clearly discussed in the open literature concerning the high resolution spectral analysis methods based on coprime sampling.

In this paper, we deal with the problem of estimating frequencies in a signal composed of a sum of independent sinusoidal components based on coprime sampling principle. Considering the commonly used signal model of classical coprime sampling scheme, our main contributions are as follows. Firstly, the **sources covariance matrix diagonal property loss phenomenon** under finite samples for some particular situations is discovered, and we show that the Multiple Signal Classification (MUSIC) method [11] applied to coprime sampling fails for such situations. Next, we design an original mechanism to fix this problem. This is achieved by organizing the collected samples in blocks and by introducing random delays between the successive blocks during the sampling process at both samplers such that the phase of the different sinusoidal components becomes artificially random from one block to another, which allows to restore the **diagonal property** of sources covariance matrix. Finally, from a finite number of randomly delayed blocks, the signal covariance matrix is estimated and the classical coprime-MUSIC method [8] is applied.

The rest of this paper is organized as follows. The classical coprime sampling model and some related high resolution spectral analysis techniques are reviewed in Section 2. The **diagonal property loss phenomenon** is described in Section 3 and the proposed embedded random delay mechanism is presented in Section 4. Section 5 shows some simulation results and conclusions are drawn in Section 6.

## 2. PROBLEM FORMULATION

Consider the following signal composed of  $D$  sinusoidal components buried in an additive noise

$$x(t) = \sum_{i=1}^D A_i e^{j(2\pi f_i t + \phi_i)} + \omega(t) \quad (1)$$

where  $A_i$  is the amplitude,  $f_i$  is the frequency of the  $i$ -th sinusoidal component,  $\phi_i$  is the corresponding phase assumed to be uniformly distributed in range  $[0, 2\pi]$  and independent from each other, and  $\omega(t)$  is a zero mean additive white Gaussian noise, independent from the  $D$  sinusoidal components.

Similarly to [8] and [10], two sub-Nyquist samplers operating at sampling intervals  $MT$  and  $NT$  respectively are utilized to sample the noise contaminated signal, with  $M$  and  $N$  two coprime numbers and  $\frac{1}{T} = 2f_{max}$  the Nyquist rate ( $f_i < f_{max}$ ). The collected samples are organized in blocks and the two data subsets associated to the  $l$ -th ( $l \geq 0$ ) block can be expressed as

$$x_M[Nl + n] = \sum_{i=1}^D A_i e^{j(\pi q_i M(Nl+n) + \phi_i)} + \omega(M(Nl + n)) \quad (2)$$

$$x_N[Ml + m] = \sum_{i=1}^D A_i e^{j(\pi q_i N(Ml+m) + \phi_i)} + \omega(N(Ml + m)) \quad (3)$$

where  $q_i = 2f_i T = \frac{f_i}{f_{max}}$  is the normalized frequency with  $q_i \in (-1, 1)$ , and  $1 \leq m \leq M - 1, 0 \leq n \leq N - 1$ .

The sampling signal vectors of the  $l$ -th block can be constructed with the above data

$$\mathbf{y}_M[l] = [x_M[Nl], x_M[Nl + 1], \dots, x_M[Nl + N - 1]]^T \quad (4)$$

$$\mathbf{y}_N[l] = [x_N[Ml + 1], \dots, x_N[Ml + M - 1]]^T \quad (5)$$

Concatenating  $\mathbf{y}_M[l]$  and  $\mathbf{y}_N[l]$  leads to the following observed data vector

$$\mathbf{y}[l] = [\mathbf{y}_M^T[l], \mathbf{y}_N^T[l]]^T = \sum_{i=1}^D \mathbf{a}(q_i) A_i e^{j\phi_i} e^{j\pi q_i M N l} + \mathbf{w}[l] \quad (6)$$

where  $\mathbf{a}(q_i) = [[1 \dots e^{j q_i \pi M(N-1)}] [e^{j q_i \pi N} \dots e^{j q_i \pi N(M-1)}]]^T$  and  $\mathbf{w}[l]$  is the corresponding noise vector. The covariance matrix of the observed data vector is given by

$$\mathbf{R}_y = E[\mathbf{y}[l] \mathbf{y}^H[l]] = \sum_{i=1}^D A_i^2 \mathbf{a}(q_i) \mathbf{a}^H(q_i) + \sigma_n^2 \mathbf{I} = \mathbf{A} \mathbf{R}_s \mathbf{A}^H + \sigma_n^2 \mathbf{I} \quad (7)$$

where  $\mathbf{A} = [\mathbf{a}(q_1) \quad \mathbf{a}(q_2) \quad \dots \quad \mathbf{a}(q_D)]$ ,  $\mathbf{R}_s = \text{diag}(A_1^2, A_2^2, \dots, A_D^2)$ ,  $\sigma_n^2$  is the noise power and  $\mathbf{I}$  is a  $(N + M - 1) \times (N + M - 1)$  identity matrix.

Based on the theoretical covariance matrix (7), many techniques have been proposed to estimate the frequencies [8], [12] and references therein. Most of them require to operate the vectorization of  $\mathbf{R}_y$ , select and rearrange the elements in the above obtained vector [13] to obtain constant lag difference between the

successive terms (equivalent to virtual uniform Nyquist rate sampling), and process a spatial smoothing to achieve a rank restored covariance matrix [14] such that a subspace based method can be exploited to achieve a high resolution performance. Among these methods, MUSIC algorithm is the mostly preferred and it is also adopted in this paper. In most situations, excellent performance has been obtained in terms of detectable number of frequencies and resolution power [8],[10]. But we have discovered that in some particular situations, depending on the values of  $M, N$  and  $q_i$ , the above mentioned methods totally fail. In the following section, we will specify the conditions under which these situations happen.

### 3. DIAGONAL PROPERTY LOSS PHENOMENON

In practice, we can only obtain a finite number of samples of a particular realization, for which the sinusoidal component parameters  $q_i, A_i$  and  $\phi_i$  ( $i = 1, 2, \dots, D$ ) are constant. For convenience and without loss of generality, we consider the noise-free situation. The covariance matrix (7) is then estimated, over  $L$  blocks of samples, by

$$\begin{aligned}\hat{\mathbf{R}}_{\mathbf{y}} &= \frac{1}{L} \sum_{l=0}^{L-1} \mathbf{y}[l] \mathbf{y}^H[l] \\ &= \mathbf{A} \hat{\mathbf{R}}_{\mathbf{s}} \mathbf{A}^H\end{aligned}\quad (8)$$

where  $\hat{\mathbf{R}}_{\mathbf{s}}$  is no longer a diagonal matrix, **but it is a matrix whose**  $\{i, k\}$ -th elements can be expressed as

$$\hat{\mathbf{R}}_{\mathbf{s}}(i, k) = \frac{A_i A_k e^{j(\phi_i - \phi_k)}}{L} \sum_{l=0}^{L-1} e^{j\pi(q_i - q_k)MNl} \quad (9)$$

where,  $A_i, A_k, \phi_i, \phi_k$ , with  $i, k = 1, 2, \dots, D$ , are constant for a given realization.

From the hypothesis presented in the previous section,  $\mathbf{R}_{\mathbf{s}}$  is diagonal. However it can be observed that if there exists a pair of normalized frequencies verifying

$$q_i - q_k = \frac{2b}{MN} \quad (10)$$

where  $b$  is an integer, then the term in equation (9) turns to be equal to  $A_i A_k e^{j(\phi_i - \phi_k)}$ , which is independent of  $L$ . It is obvious that matrix  $\hat{\mathbf{R}}_{\mathbf{s}}$  **will not be diagonal even for a big value of  $L$** . In fact, in this situation, this matrix becomes rank deficient. For any pair of distinct frequencies  $q_i$  and  $q_k$  satisfying this condition, **the diagonal property of  $\hat{\mathbf{R}}_{\mathbf{s}}$  can no more be hold. This phenomenon** has never been reported in the framework of practical coprime sampling. In this situation which seems to happen quite frequently, the estimated

covariance matrix (8) obtained from finite samples does not exhibit the same properties as the theoretical matrix (7).

For co-array MUSIC method, matrix  $\hat{\mathbf{R}}_{\mathbf{y}}$  is vectorized and spatial smoothing is applied to construct a new covariance matrix of the virtual signal. The vectorization of  $\hat{\mathbf{R}}_{\mathbf{y}}$  can be given as

$$\text{vec}(\hat{\mathbf{R}}_{\mathbf{y}}) = \sum_{i=1}^D A_i^2 \mathbf{a}^*(q_i) \otimes \mathbf{a}(q_i) + \sum_{h=1}^D \sum_{k=1, h \neq k}^D \xi_{hk} A_h A_k e^{j\pi\phi_{hk}} \mathbf{a}^*(q_k) \otimes \mathbf{a}(q_h) \quad (11)$$

where  $\phi_{hk} = \phi_h - \phi_k$ ,  $\xi_{hk} = 1$  only when condition (10) is met and 0 otherwise. Then the virtual signal vector can be written as

$$\mathbf{x}_v = \mathbf{F} \text{vec}(\hat{\mathbf{R}}_{\mathbf{y}}) = \mathbf{F}\mathbf{B}\mathbf{p} + \mathbf{F}\mathbf{B}'\mathbf{p}' \quad (12)$$

where  $\mathbf{F}$  is the selection matrix [10],  $\mathbf{B} = [\mathbf{a}^*(q_1) \otimes \mathbf{a}(q_1), \dots, \mathbf{a}^*(q_D) \otimes \mathbf{a}(q_D)]$ ,  $\mathbf{p} = [A_1^2, \dots, A_D^2]^T$  and  $\mathbf{B}' = [\mathbf{a}^*(q_1) \otimes \mathbf{a}(q_2), \dots, \mathbf{a}^*(q_1) \otimes \mathbf{a}(q_D), \dots, \mathbf{a}^*(q_D) \otimes \mathbf{a}(q_1), \dots, \mathbf{a}^*(q_D) \otimes \mathbf{a}(q_{D-1})]$ ,  $\mathbf{p}' = [\xi_{21} A_2 A_1 e^{j\phi_{21}}, \dots, \xi_{D1} A_D A_1 e^{j\phi_{D1}}, \dots, \xi_{1D} A_1 A_D e^{j\phi_{1D}}, \dots, \xi_{(D-1)D} A_{D-1} A_D e^{j\phi_{(D-1)D}}]^T$ . Here,  $h \neq k$  for each  $\mathbf{a}^*(q_k) \otimes \mathbf{a}(q_h)$  in  $\mathbf{B}'$  and  $\xi_{hk} A_h A_k e^{j\phi_{hk}}$  in  $\mathbf{p}'$ . We can write  $\mathbf{F}\mathbf{B}\mathbf{p}$  as

$$\mathbf{F}\mathbf{B}\mathbf{p} = \left[ \sum_{i=1}^D A_i^2 e^{j\pi q_i (1-(M+N))}, \sum_{i=1}^D A_i^2 e^{j\pi q_i (2-(M+N))}, \dots, \sum_{i=1}^D A_i^2 e^{j\pi q_i (M+N-1)} \right]^T = \mathbf{A}_v \mathbf{p} \quad (13)$$

where  $\mathbf{A}_v = [\mathbf{d}(q_1), \dots, \mathbf{d}(q_D)]$  with  $\mathbf{d}(q_i) = [e^{j\pi q_i (1-(M+N))}, e^{j\pi q_i (2-(M+N))}, \dots, e^{j\pi q_i (M+N-1)}]^T$ . Before applying the co-array MUSIC, the spatial smoothing is applied to  $\mathbf{x}_v$ . It can be seen that if the diagonal property of  $\hat{\mathbf{R}}_{\mathbf{s}}$  holds, the virtual signal vector  $\mathbf{x}_v$  can be written as  $\mathbf{A}_v \mathbf{p}$ . If the diagonal property loss condition is met, vector  $\mathbf{F}\mathbf{B}'\mathbf{p}'$  consisting of some cross terms between different frequencies will cause some problems to the co-array MUSIC. Due to the cross terms of different frequencies, the virtual signal vector  $\mathbf{x}_v$  can no longer be written in the form  $\mathbf{A}_v \mathbf{p}$ , which is the basis of all spatial smoothing based high resolution techniques. As a consequence, the co-array MUSIC fails in this case.

#### 4. COPRIME SAMPLING WITH EMBEDDED RANDOM DELAYS

In this section, we propose a technique to overcome the above mentioned **diagonal property loss phenomenon**. The main idea is to introduce randomness in the sampling process to artificially keep the uncorrelation between the sampled sinusoidal components. This is achieved by introducing random delays such that the phase of different sinusoidal components becomes random because of the different frequencies.

After acquiring the first block of data, a discrete random delay is introduced before starting each new block at both samplers as illustrated in Figure 1. It should be noticed that the DOFs is  $M + N - 1$  in this case because we embed the random delays after each block. The proposed scheme can be easily generalized to increase the DOFs by embedding the random delays after every  $B$  blocks. For instance, if the random delays are embedded after the 2-nd, 4-th, 6-th,...blocks, data from two blocks can be jointly used to construct the covariance matrix and the DOFs can be increased to  $MN + M + N - 1$ . Indeed, the DOFs can be further increased to  $(B - 1)MN + M + N - 1$  [10] if the random delays are embedded after the  $B$ -th,  $2B$ -th,  $3B$ -th,...blocks. In this paper, our main concern is to show the diagonal property loss problem and give a way to fix it. Without loss of generality, we have embedded the random delays after each block in this paper.

In this paper, discrete random delay is considered but continuous random delay could have been chosen too. For practical convenience of implementation, we consider the introduced random delays to be multiple of the Nyquist sampling period  $T$  with a discrete uniform distribution. The delay embedded at the front of the  $p$ -th block is denoted as  $t_p T$ , where  $t_p$  is a random integer ruled by the discrete uniform distribution  $U[0, \alpha - 1]$  ( $\alpha \geq 2$ ). It means that  $t_p$  randomly takes one integer value in set  $[0, \alpha - 1]$  with probability  $\frac{1}{\alpha}$ . At the  $l$ -th block, the total accumulated delay is

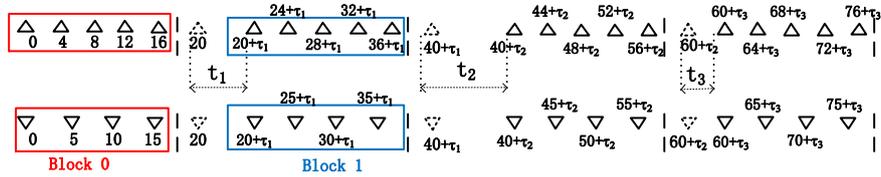


Figure 1: Random delay between coprime sampling blocks,  $M=4$ ,  $N=5$

$$\tau_l T = \sum_{p=1}^l t_p T \quad (14)$$

Then the new concatenated samples block vector in equation (6) can be modified as

$$\tilde{\mathbf{y}}[l] = \sum_{i=1}^D \mathbf{a}(q_i) A_i e^{j\phi_i} e^{j\pi q_i (MNI + \tau_l)} + \tilde{\mathbf{w}}[l] \quad (15)$$

Its covariance matrix can then be estimated over the  $L$  obtained blocks. Similarly to (9), the  $\{i, k\}$ -th

element in  $\tilde{\mathbf{R}}_{\mathbf{s}}$  can be written as

$$\tilde{\mathbf{R}}_{\mathbf{s}}(i, k) = \frac{A_i A_k e^{j(\phi_i - \phi_k)}}{L} \sum_{l=0}^{L-1} e^{j\pi(q_i - q_k)(MNl + \tau_l)} \quad (16)$$

Since  $A_i, A_k, \phi_i, \phi_k$  are constant for a particular realization, we consider only the summation item in  $\tilde{\mathbf{R}}_{\mathbf{s}}(i, k)$ . It should be noticed that even for a set of received samples, only one realization of the random delays is drawn. It is impossible to derive a closed-form expression of the summation term in equation (16). However it makes sense to observe the statistical mean of this term, which is given by

$$E \left[ \frac{1}{L} \sum_{l=0}^{L-1} e^{j\pi(q_i - q_k)(MNl + \tau_l)} \right] = \frac{1}{L} \sum_{l=0}^{L-1} \left( \frac{e^{j\pi(q_i - q_k)(MN + \frac{\alpha-1}{2})} \sin\left(\frac{\pi(q_i - q_k)\alpha}{2}\right)}{\alpha \sin\left(\frac{\pi(q_i - q_k)}{2}\right)} \right)^l \quad (17)$$

To better understand how the embedded delays affect the non-diagonal terms and fix the diagonal property loss problem, we use equation (17) to approximately show the impact of the introduced delays. When  $L$  increases, the summation term in equation (16) tends to take a value close to its statistical mean. Observing equation (17), it comes that the value of the summation item is given as the sum of the first  $L$  terms of a geometric series for which the modulus of the common ratio is less than or equal to one (equal to one only when  $|q_i - q_k|$  is even). Since  $\alpha \geq 2$  and  $q_i \neq q_k$  with  $|q_i - q_k| < 2$  by definition, the modulus of the common ratio will be always less than one. Hence, as  $L$  is chosen large enough, the non diagonal elements of the estimated signal correlation matrix  $\tilde{\mathbf{R}}_{\mathbf{s}}$  will be very small for any  $\alpha$  and normalized frequencies. In fact, the non diagonal elements tend to zero as  $L$  goes to infinity. Therefore, even in the diagonal property loss condition defined in (10), two distinct signal components ( $q_i \neq q_k$ ) will never be linearly correlated, which means that no diagonal property loss will occur in the proposed scheme. The coprime subspace based methods can then be applied for frequencies estimation even under the diagonal property loss condition.

It seems that parameter  $\alpha$  could be chosen to optimize the performance of frequencies estimation, because the smaller the amplitude of (17) is, the better is the performance of the subspace based techniques. However, for a given number of blocks  $L$ , it is not possible to optimally choose  $\alpha$  to obtain a lowest value of (17) because the normalized frequencies are not known initially.

## 5. NUMERICAL SIMULATION

In this section, in order to illustrate the above highlighted problem, firstly we provide the MUSIC spectrum under the diagonal property loss condition with the classical coprime sampling. Then, the MUSIC

spectrum obtained from the proposed embedded random delay sampling is displayed to show the benefit brought by the new proposed coprime sampling scheme. Finally, RMSE performance is given to show that the proposed sampling scheme does not affect the estimation performance when there is no **diagonal property loss**.

### 5.1. MUSIC spectrum in the **diagonal property loss condition**

In order to illustrate the **diagonal property loss phenomenon**, let's consider the coprime integers  $M = 4$  and  $N = 5$ , and  $D = 7$  sinusoidal components with unit amplitude. The signal-to-noise ratio (SNR) is set to  $20dB$ . Consider  $L = 1000$  blocks of samples. The 7 normalized frequencies in this example are selected such that there exist exactly two pairs among them verifying the **diagonal property loss condition** (10). The normalized frequencies are  $q_1 = -0.40, q_2 = -0.34, q_3 = -0.17, q_4 = 0.23, q_5 = 0.39, q_6 = 0.56, q_7 = 0.88$ , and the condition is met with  $q_4 - q_3$  and  $q_6 - q_2$ .

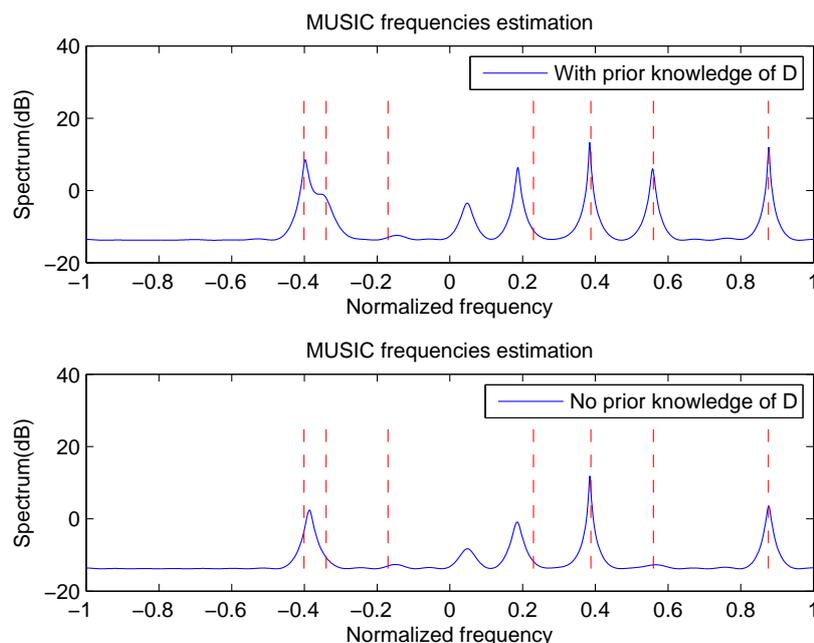


Figure 2: MUSIC spectrum with coprime sampling in the **diagonal property loss condition**, 7 sinusoidal components,  $M = 4$  and  $N = 5$ .

The coprime sampling MUSIC algorithm [8] is then applied in two scenarios: 1) the number of components  $D = 7$  is known and the dimension of the signal subspace dimension is set to 7 when performing MUSIC algorithm; 2) No prior knowledge of the number of components is assumed, and the minimum description length criteria (MDL) [15] is used to determine the signal subspace dimension. In this situation, the subspace signal dimension is found to be equal to 5. Fig. 2 shows the estimated MUSIC spectrum under **diagonal**

**property loss phenomenon.** The vertical dotted lines refer to the true position of frequencies. It can be observed that the frequencies are not correctly estimated in both scenarios. Phantom peaks appear at wrong frequency position and some true frequencies can not be detected.

Fig. 3 depicts the estimated MUSIC spectrum with the proposed embedded random delays sampling method. The same frequencies setting as before is considered and  $\alpha$  is set to 6. It can be observed that the frequencies are correctly estimated even with two pairs of frequencies verifying condition (10).

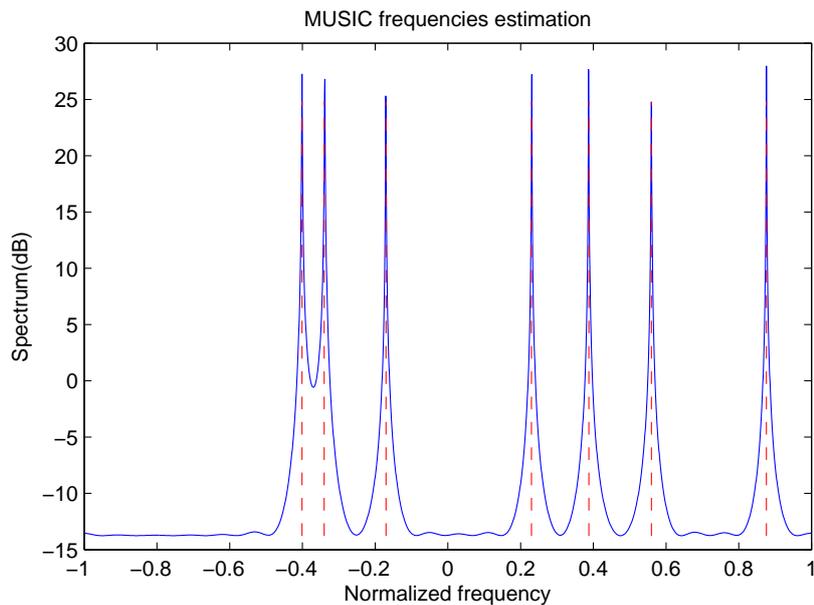


Figure 3: MUSIC spectrum using embedded random delays coprime sampling, 7 sinusoidal components,  $M = 4$  and  $N = 5$ .

### 5.2. Estimation performance

In the following, the proposed embedded random delays scheme and the classical coprime scheme are compared. The performance is assessed in terms of RMSE, defined as

$$RMSE = \sqrt{\frac{1}{DU} \sum_{i=1}^D \sum_{u=1}^U (\hat{q}_i(u) - q_i)^2} \quad (18)$$

where  $\hat{q}_i(u)$  is the estimate of the normalized frequency  $q_i$  in the  $u$ -th estimation trial,  $u = 1, 2, \dots, U$ . In the following simulations, the following parameters are chosen,  $U = 1000, M = 4, N = 5$

As explained above, the classical co-array MUSIC fails when condition (10) is met. We first show the performance of the classical co-array MUSIC algorithm when the difference between some frequencies is

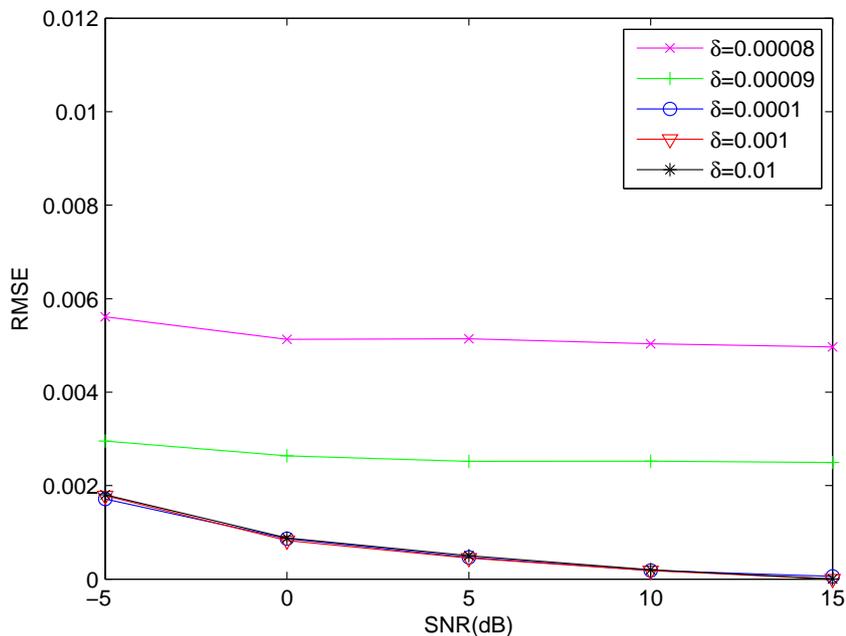


Figure 4: RMSE performance when the difference between some frequencies is close to  $2b/(MN)$ ,  $b = 4$ ,  $M = 4$ ,  $N = 5$ .

close to  $2b/(MN)$ . In Figure 4, we consider 2 frequencies, which are  $q'_1 = -0.17$  and  $q'_2 = 0.23 + \delta$  such that  $q'_2 - q'_1 = 0.4 + \delta$ , where  $\delta$  is a small offset  $b = 4$ . We set  $\delta$  to several values for comparison and no prior knowledge of  $D$  is assumed in this case. It can be observed from Figure 4 that when  $\delta$  varies from 0.01 to 0.0001, the classical co-array MUSIC achieves similar performance. When  $\delta$  continues to decrease, RMSE increases dramatically. This indicates that when the difference between some frequencies tends to be close to  $2b/(MN)$ , the non-diagonal terms in  $\hat{\mathbf{R}}_s$  become non-negligible. The diagonal property loss problem becomes significant.

The classical coprime scheme and the proposed scheme are compared in Figure 5. For simplicity, signal with only two sinusoidal components  $(q_1, q_2)$  is considered. Because the classical coprime mechanism fails under the **diagonal property loss condition** while our proposed method can still robustly perform, we arbitrarily choose frequencies which do not satisfy the **diagonal property loss condition** for comparison. Without loss of generality,  $q_1$  is chosen to be  $-0.84$  and  $q_2$  is randomly chosen in each estimation trial. Also, we consider different values of  $\alpha$  to compare how the embedded delay distributions affect the performance, namely  $\alpha = 6, 16, 31$ .

It can be observed from Fig. 5 that the proposed scheme can obtain similar RMSE performance as that of the classical coprime scheme. Moreover, with different values of  $\alpha$ , the performance does not significantly

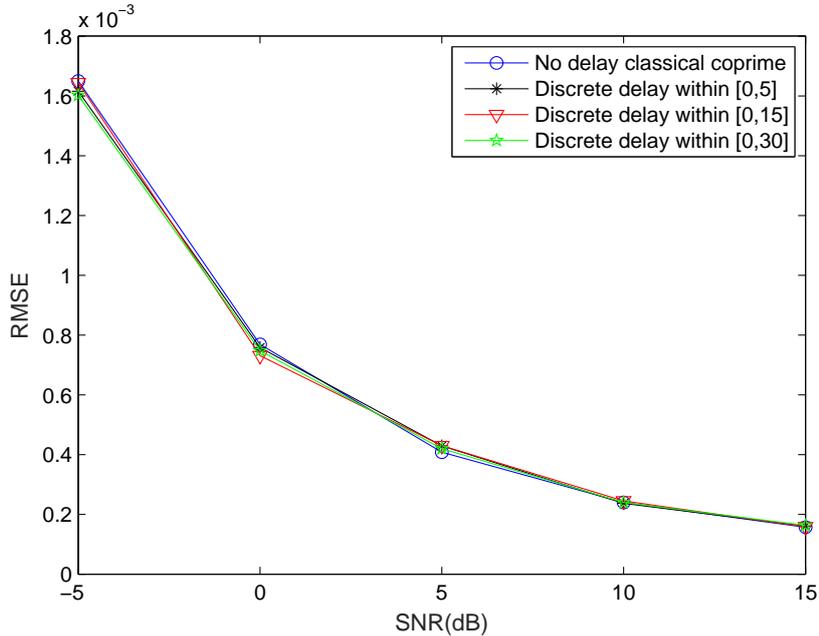


Figure 5: RMSE performance with classical coprime sampling and embedded random delay coprime sampling,  $M = 4, N = 5$ .

change, which means that the embedded delays do not significantly affect the performance of coprime sampling process while being able to fix the problem of the **diagonal property loss**.

## 6. CONCLUSION

In this paper, we have reviewed the classical coprime sampling mechanism and shown that the classical coprime subspace based methods suffer from **diagonal property loss phenomenon** in specific practical conditions. The closed-form expression of the condition is given and a new approach is proposed to prevent the **diagonal property loss**. We artificially introduce the randomness into sinusoid components by embedding random delays in the coprime sampling process to keep uncorrelation between different sinusoidal components. Our proposed scheme fixes the **diagonal property loss problem** in coprime subspace based methods without sacrificing the performance. Future works can be conducted to optimize the random distribution of the embedded delays.

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