# Analysis and optimization of a simple crop irrigation model

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**Abstract**

We propose a new simplified crop irrigation model and study the optimal control which consists in maximizing the biomass production at harvesting time, under a constraint on the total amount of water used. Under water scarcity, we show that the optimal strategy could have a singular arc and therefore can be better than a simple bang-bang control as commonly used. The gain is illustrated on numerical simulations. This result is a promising first step towards the application of control theory to the problem of optimal irrigation scheduling.
**Key-words.** Crop irrigation, water management, optimal control, state constraint.

## 1 Model description and assumptions

We consider a simplified dynamical model of crop irrigation, inspired from [6], where $S(t)$ and $B(t)$ stand respectively for the relative soil humidity (a number between 0 and 1) and the crop biomass at time $t$ belonging to an interval $[0, T]$ representing the crop growth season:

$$\dot{S} = k_1(-\varphi(t)K_S(S) - (1-\varphi(t))K_R(S) + k_2 u(t)) \quad (1)$$
$$\dot{B} = k_3\varphi(t)K_S(S) \quad (2)$$

with the initial condition (at the sowing date 0)

$$S(0) = 1 \quad (3)$$
$$B(0) = 0 \quad (4)$$

and $T$ being the harvesting date. The control variable $u(t) = F(t)/F_{max} \in [0, 1]$ is the ratio of the input water flow rate $F(t)$ at time $t$ over the maximal flow $F_{max}$ that the irrigation device allows.

On an agronomic point of view, Eq. (1) represents the variation of a vertically averaged soil moisture as influenced by three fluxes: crop transpiration, crop evaporation, and crop irrigation. Unlike [6], we use here the simplified hypothesis made in [1]: transpiration and evaporation can be partitioned using a variable $\varphi(t)$ representing the crop radiation use efficiency and independent

of water stress. Both transpiration and evaporation fluxes are regulated by soil moisture as in [6] using two functions $K_S$ and $K_R$ (see Assumption 1 and Fig. 1 below). Eq.(2) determines the amount of biomass produced per time unit. It is simply related to the transpiration flux using the water use efficiency principle [10, 6]. Note also that the proposed model does not consider rainfall inputs and might be better associated to greenhouse grown crops.

**Assumption 1.** *$K_S(\cdot)$ and $K_R(\cdot)$ are continuous piecewise linear functions from $[0,1]$ to $[0,1]$, that take values $0$ at $S=0$ and $1$ at $S=1$, with*

$$S_w := \sup\{S \in [0,1] \ s.t \ K_S(S) = 0\} > S_h := \sup\{S \in [0,1] \ s.t \ K_R(S) = 0\}$$

*Moreover, there exists a threshold $S^\star \in (0,1)$ such that*

$$\begin{cases} K_S(S) = 1, \ S \in [S^\star, 1]; & K_S(S) < 1, \ S \in [0, S^\star) \\ 0 < K_R(S^\star) < 1 \end{cases} \tag{5}$$

*and $K_S(\cdot)$, $K_R(\cdot)$ are increasing when not equal to $0$ or $1$.*

The thresholds $S_w$ represents the plant wilting point, which is usually higher than the hydroscopic point denoted by $S_h$. We shall say that $S$ is a *corner point* of $K_S$, resp. $K_R$ when the function is non differentiable at $S$ (therefore $S^\star$ is necessarily a corner point of $K_S$). A typical instance of functions $K_S(\cdot)$ , $K_R(\cdot)$ are given by the following expressions (see Fig. 1)

$$K_S(S) = \begin{cases} 0 & S \in [0, S_w] \\ \dfrac{S - S_w}{S^\star - S_w} & S \in [S_w, S^\star] \\ 1 & S \in [S^\star, 1] \end{cases} \qquad K_R(S) = \begin{cases} 0 & S \in [0, S_h] \\ \dfrac{S - S_w}{1 - S_w} & S \in [S_h, 1] \end{cases}$$

with $0 < S_h < S_w < S^\star < 1$.

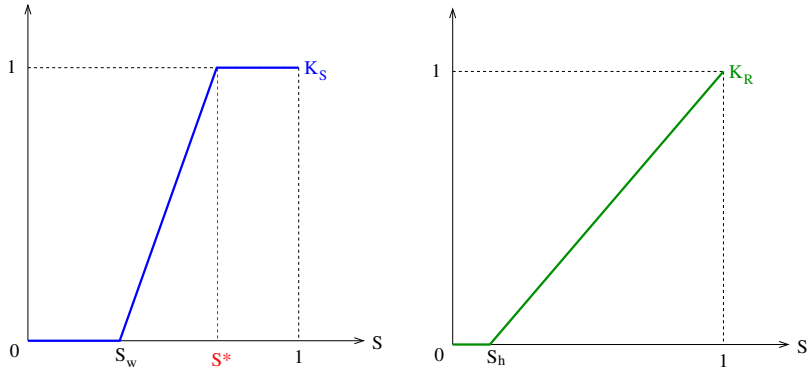


Figure 1: Graphs of the functions $K_S$ and $K_R$

**Assumption 2.** *$\varphi(\cdot)$ is a $C^1$ increasing function from $[0,T]$ to $[0,1]$ with $\varphi(0) = 0$ and $\varphi(T) = 1$.*

**Assumption 3.** *$k_1$, $k_2$, $k_3$ are positive parameters with*

$$k_2 \geq 1$$

Under these assumptions, one has straightforwardly the following property.

**Lemma 1.** *Any solution $S(\cdot)$ of (1) verifies $S(t) > S_h$ for any $t \geq 0$.*

*Proof.* □

The condition $k_2 > 1$ is a *controllability* assumption, in the sense that it allows the variable $S$ to stay equal to 1 with the constant control $u = 1/k_2$. However, the dynamics is naturally subject to the state constraint

$$S(t) \leq 1, \quad t \in [0, T] \tag{6}$$

We shall say that a control function $u(\cdot)$ is *admissible* when (6) is satisfied.

The set of admissible controls $u(\cdot)$ are measurable functions taking value in $[0, 1]$, such that the solution of (1),(3) verifies the constraint (6). To each such control function, we associate the total water delivered on the time interval $[0, T]$, and the biomass production at the harvesting date $T$, given respectively by

$$Q[u(\cdot)] := F_{max} \int_0^T u(t)\, dt, \quad B_T[u(\cdot)] := B(T)$$

Remark that this constraint can be tackled replacing equation (1) by

$$\dot{S} = k_1 \Big( -\varphi(t) K_S(S) - (1 - \varphi(t)) K_R(S) + k_2 \chi(S, u(t)) \Big) \tag{7}$$

where the function $\chi(\cdot)$ is defined as follows

$$\chi(S, u) := \begin{cases} \min(1/k_2, u) & \text{if } S = 1 \\ u & \text{if } S < 1 \end{cases} \tag{8}$$

The total water delivered on the time interval $[0, T]$ is given by

$$Q[u(\cdot)] := F_{max} \int_0^T u(t)\, dt$$

# 2 Study of solutions above the $S^\star$ threshold

Let us introduce the following notations and definitions.

## 2.1 Some notations and definitions

i. For any $t_0 \in [0, T]$ and $S_0 \in [0, 1]$, we denote by $S_{t_0,S_0,0}(\cdot)$, resp. $S_{t_0,S_0,1}(\cdot)$, the solution of the differential equation (7) over $[0, T]$ with $S(t_0) = S_0$ and the constant control $u = 0$, resp. $u = 1$.

   Then, for any solution $S(\cdot)$ of (7) with $S(t_0) = S_0$ and an admissible control function $u(\cdot)$, one has clearly

$$S_{t_0,S_0,0}(t) \leq S(t) \leq S_{t_0,S_0,1}(t), \quad t \in [t_0, T] \tag{9}$$

Notice that for $t_0 < T$ and $S_0 > 0$ and, the map $t \mapsto S_{t_0,S_0,0}(t)$ is decreasing over $[t_0, T)$. Similarly, for $t_0 > 0$ and $S_0 > 0$, the map $t \mapsto S_{t_0,S_0,1}(t)$ is increasing over $[0, t_0)$.

ii. For any $t_0 \in [0, T)$ and $S_0 \in [0, 1]$, we denote by $\mathcal{A}_{t_0,S_0}(t)$ the attainable set at time $t \in (t_0, T]$ for the variable $S$, that is the set of values of $S(t)$, where $S(\cdot)$ is solution of (7) for an admissible control function $u(\cdot)$. From the continuous dependency of the solution of (7) with respect to $u$ and property (9), the attainable set is the interval:

$$\mathcal{A}_{t_0,S_0}(t) = \big[S_{t_0,S_0,0}(t), S_{t_0,S_0,1}(t)\big] \tag{10}$$

iii. For $t_0 \in [0, T)$ and $S_0 \in (S^\star, 1]$, we define

$$t^+(t_0, S_0) = \begin{cases} T \quad \text{if } S_{t_0,S_0,0}(t) > S^\star,\ t \in [t_0, T] \\ \inf\{t > t_0\,;\, S_{t_0,S_0,0}(t) = S^\star\} \quad \text{otherwise} \end{cases} \tag{11}$$

Similarly, for any $(t_0, S_0) \in (0, T] \times (S^\star, 1]$, we define

$$t^-(t_0, S_0) = \begin{cases} 0 \quad \text{if } S_{t_0,S_0,1}(t) > S^\star,\ \forall t \in [0, t_0] \\ \sup\{t < t_0\,;\, S_{t_0,S_0,1}(t) = S^\star\} \quad \text{otherwise} \end{cases} \tag{12}$$

iv. Take $(t_1, S_1) \in [0, T) \times [S^\star, 1]$ and $(t_2, S_2) \in (t_1, T] \times [S^\star, 1]$ such that $S_2 \in \mathcal{A}_{t_1,S_1}(t_2)$. If $t^+(t_1, S_1) > t^-(t_2, S_2)$, we consider the function

$$I(t) := S_{t_1,S_1,0}(t) - S_{t_2,S_2,1}(t), \quad t \in [t_1, t_2] \tag{13}$$

From point i., $I(\cdot)$ is decreasing on $[t_1, t_2]$. As $(t_2, S_2)$ is attainable from $(t_1, S_1)$, there exists an admissible solution $S(\cdot)$ such that $S(t_1) = S_1$ and $S(t_2) = S_2$. One has then $I(t_1) \geq 0$. Otherwise, one has $S_{t_2,S_2,1}(t_1) - S_{t_1,S_1,1}(t_1) = S_{t_2,S_2,1}(t_1) - S(t_1) > 0$, and as (9) gives $S_{t_2,S_2,1}(t_2) - S_{t_1,S_1,1}(t_2) = S(t_2) - S_{t_1,S_1,1}(t_2) < 0$, there exists $t_c \in (t_1, t_2)$ such that $S_{t_2,S_2,1}(t_c) = S_{t_1,S_1,1}(t_c)$, which contradicts the uniqueness of the solution of (1) with $u = 1$. Similarly, one has $I(t_2) \leq 0$. By the intermediate value theorem, we deduce that there exists an unique $\bar{t}(t_1, S_1, t_2, S_2) \in [t_1, t_2]$ such that $I(\bar{t}(t_1, S_1, t_2, S_2)) = 0$, and that one has

$$S_{t_1,S_1,0}(\bar{t}(t_1, S_1, t_2, S_2)) = S_{t_2,S_2,1}(\bar{t}(t_1, S_1, t_2, S_2)) > S^\star \tag{14}$$

We define now the particular MRAP controls.

**Definition 1.** *For any $(t_1, S_1) \in [0, T) \times [S^\star, 1]$ and $(t_2, S_2) \in (t_1, T] \times [S^\star, 1]$ such that $S_2 \in \mathcal{A}_{t_1,S_1}(t_2)$, we associate the* most rapid path approach to $S^\star$ *(MRAP in short) control, defined as follows.*

*i) If $t_-(t_2, S_2) \geq t_+(t_1, S_1)$:*

$$\tilde{u}(t) := \begin{cases} 0 & t \in [t_1, t^+(t_1, S_1)) \\ \tilde{u}_{S^\star}(t) & t \in [t_+(t_1, S_1), t^-(t_2, S_2)] \\ 1 & t \in (t_-(t_2, S_2), t_2] \end{cases} \tag{15}$$

*with the singular control*

$$\tilde{u}_{S^\star}(t) := \frac{\varphi(t) + (1 - \varphi(t))K_R(S^\star)}{k_2} \tag{16}$$

*Notice that under Assumption 3, this control is admissible as one has*

$$\tilde{u}_{S^\star}(t) < 1, \quad t \in [t_+(t_1, S_1), t^-(t_2, S_2)] \tag{17}$$

*ii) If* $t^-(t_2, S_2) < t^+(t_1, S_1)$*:*

$$\tilde{u}(t) := \begin{cases} 0 & t \in [t_1, \bar{t}(t_1, S_1, t_2, S_2)) \\ 1 & t \in (\bar{t}(t_1, S_1, t_2, S_2), t_2] \end{cases} \tag{18}$$

Clearly, the solution $\tilde{S}(\cdot)$ of (1) with $\tilde{S}(t_1) = S_1$ and control $\tilde{u}(\cdot)$ satisfies $\tilde{S}(t_2) = S_2$ and

$$S(t) \geq \tilde{S}(t) \text{ for any } t \text{ such that } S(t) \geq S^\star \tag{19}$$

(see Fig. 2 and 3).

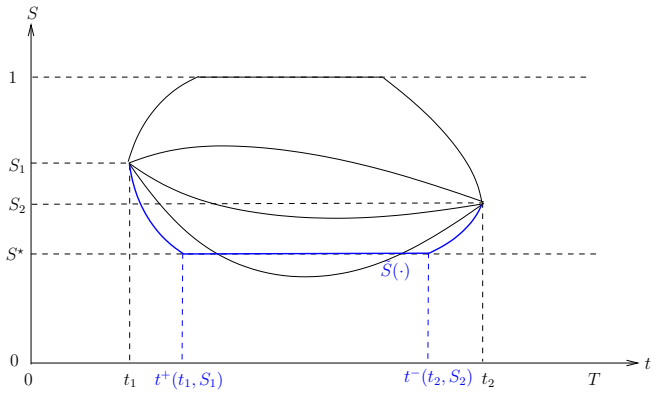


Figure 2: The MRAP trajectory $\tilde{S}(\cdot)$ compared to other possible trajectories $S(\cdot)$ when $t_-(t_2, S_2) > t_+(t_1, S_1)$

## 2.2 Properties of MRAP controls

The MRAP controls have been considered with the use of Green's theorem in several optimal control problems in the plane linear w.r.t. to the control [5, 4, 9, 3]. Here, we shall consider it as a comparison tool for the water quantity $Q[u(\cdot]$, that will play in important role in the next sections when a constraint on $Q[u(\cdot]$ is considered.

**Proposition 1.** *Let* $S(\cdot)$ *be a solution of* (1) *on* $[t_1, t_2]$ *(with* $0 \leq t_1 < t_2 \leq T$*) for an admissible control* $u(\cdot)$ *such that* $S(t) \geq S^\star$ *for any* $t \in [t_1, t_2]$*. Denote* $S_1 = S(t_1)$ *and* $S_2 = S(t_2)$*. Then, the solution* $\tilde{S}(\cdot)$ *of* (1) *on* $[t_1, t_2]$ *with* $\tilde{S}(t_1) = S_1$ *and the MRAP control* $\tilde{u}(\cdot)$ *defined in* (15) *or* (18)*, satisfies the following properties*

$$\tilde{S}(t_2) = S_2 \tag{20}$$

$$\tilde{S}(t) \leq S(t), \quad t \in [t_1, t_2] \tag{21}$$

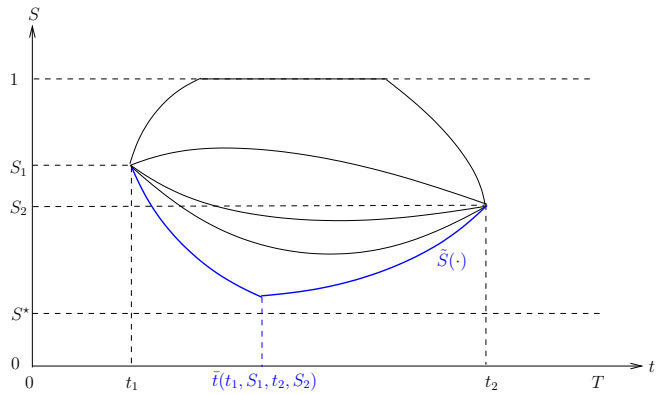


Figure 3: The MRAP trajectory $\tilde{S}(\cdot)$ compared to other possible trajectories $S(\cdot)$ when $t_-(t_2, S_2) < t_+(t_1, S_1)$

$$\int_{t_1}^{t_2} \tilde{u}(t)\, dt \leq \int_{t_1}^{t_2} u(t)\, dt \tag{22}$$

*Moreover, the last inequality is strict when $S(\cdot)$ and $\tilde{S}(\cdot)$ are not identical.*

*Proof.* Properties (20) and (21) follow directly from the construction of the MRAP control $\tilde{u}(\cdot)$ and (19).

Define the closed curve $\mathcal{C}$ in the $(t, S)$ plane, parameterized by $\theta \in [0, 2(t_2 - t_1)]$ as follows

$$(t, S)(\theta) = \begin{cases} (t_1 + \theta, \tilde{S}(t_1 + \theta)) & \theta \in [0, t_2 - t_1) \\ (2t_2 - t_1 - \theta, S(2t_2 - t_1 - \theta)) & \theta \in [t_2 - t_1, 2(t_2 - t_1)] \end{cases} \tag{23}$$

On the other hand, from equation (1), one has

$$\Delta := \int_{t_1}^{t_2} \tilde{u}(t)\, dt - \int_{t_1}^{t_2} u(t)\, dt = \oint_{\mathcal{C}} \frac{1}{k_2}\big(\varphi(t) + (1 - \varphi(t)K_r(S)\big)dt + \frac{1}{k_1} dS \tag{24}$$

which is of the form

$$\Delta = \oint_{\mathcal{C}} P(t, S)dt + G(t, S)dS \tag{25}$$

From property (21), the curve $\mathcal{C}$ is anticlockwise oriented (see Fig. 4). Then, by Green's Theorem, one has

$$\Delta = \iint_D \left( \frac{\partial Q}{\partial t} - \frac{\partial P}{\partial S} \right) dtdS \tag{26}$$

where $\mathcal{D}$ is the domain enclosed by the curve $\mathcal{C}$. We obtain here

$$\Delta = \iint_D -\varphi(t)K_S'(S) - (1 - \varphi(t))K_r'(S)\, dtdS \tag{27}$$

From Assumption 1, we have $\Delta \leq 0$ and $\Delta < 0$ when $\mathcal{D}$ is of non-empty interior, that is when the solution $\tilde{S}(\cdot)$ does not coincide with $S(\cdot)$. □

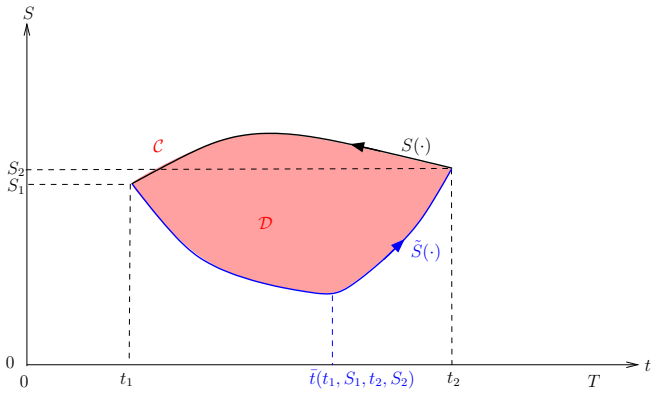


Figure 4: The domain $\mathcal{D}$ is delimited by the closed curve $\mathcal{C}$ composed of the graphs of $\tilde{S}(\cdot)$ and $S(\cdot)$ anticlockwise oriented

### 2.3 About water consumption and biomass production

Let us define the number

$$\bar{B} := k_3 \int_0^T \varphi(t)\, dt \tag{28}$$

which is the highest biomass production at time $T$, and the $S$-profile generated by the null control (i.e. without any irrigation).

$$\underline{S}(t) := S_{0,1,0}(t), \quad t \in [0, T] \tag{29}$$

One can straightforwardly check that the following lemma is verified.

**Lemma 2.** *If $\underline{S}(T) \geq S^\star$, then the control $u(t) = 0$, $t \in [0, T]$ ensures the maximal biomass production $\bar{B}$ at time $T$.*

When $\underline{S}(T) < S^\star$, let us consider the MRAP control for $t_1 = 0$, $S_1 = 1$, $t_2 = T$, $S_2 = S^\star$, that we denote by $\tilde{u}_{0T}(\cdot)$. One has then the following property, as a straightforward consequence of Proposition 1.

**Corollary 1.** *Assume $\underline{S}(T) < S^\star$. The control $\tilde{u}_{0T}(\cdot)$ has the least water consumption $Q[\tilde{u}_{0T}(\cdot)]$ among all admissible control $u(\cdot)$ giving the same biomass production $\bar{B}$ at time $T$.*

## 3 The optimal control problem

Given a maximal quantity of available water $\bar{Q} > 0$, we consider the problem of maximizing the biomass production

$$B^\star(\bar{Q}) := \sup_{u(\cdot)} B(T) \tag{30}$$

among measurable functions $u(\cdot)$ taking values in $[0, 1]$, under the state constraint (6) and the integral constraint on the control

$$Q[u(\cdot)] \leq \bar{Q} \tag{31}$$

where $B(\cdot)$ is solution of (1)-(2). One can easily check, with the usual arguments of the theory of optimal control (see for instance [12], that such a problem admits an optimal solution. We shall say from now that a control $u(\cdot)$ is *admissible* when constraints (6) and (31) are both satisfied. From Lemma 2, we know that when the condition $\underline{S}(T) \geq S^\star$ is fulfilled, the null control is optimal whatever is $\bar{Q}$. When this condition is not satisfied, one obtains straightforwardly from Corollary 1 the following properties.

**Lemma 3.** *Assume $\underline{S}(T) < S^\star$. One has the following properties.*

*i. When $\bar{Q} \geq Q[\tilde{u}_{OT}(\cdot)]$, $\tilde{u}_{OT}(\cdot)$ is optimal and one has $B^\star(\bar{Q}) = \bar{B}$.*

*ii. When $\bar{Q} < Q[\tilde{u}_{OT}(\cdot)]$, any admissible trajectory is such that $B(T) < \bar{B}$. Moreover, the constraint (6) is not active on any optimal trajectory.*

**Remark 1.** *When $\bar{Q} > Q[\tilde{u}_{OT}(\cdot)]$, the optimal strategy is not unique: any control that guarantees $S(t) \geq S^\star$ at any $t \in [0, T]$ yields $B(T) = \bar{B}$.*

## 3.1 First results

In the following, we consider non-trivial cases for which the null control or the control $\tilde{u}_{OT}(\cdot)$ are not optimal.

**Assumption 4.**

$$\underline{S}(T) < S^\star \quad \textit{and} \quad 0 < \bar{Q} < Q[\tilde{u}_{OT}(\cdot)]$$

Under this assumption, one can define the time

$$\underline{t} := \inf\{t \in [0, T] \text{ s.t. } \underline{S}(t) < S^\star\} < T \tag{32}$$

One has the following first result, concerning the behavior of optimal solutions with respect to $S = S^\star$.

**Proposition 2.** *Having $u(t) = 0$ for $t \in [0, \underline{t}]$ is optimal. Moreover, any optimal solution $u(\cdot)$ is such that $S(t) \leq S^\star$ for any $t \in [\underline{t}, T]$, and $Q[u(\cdot)] = \bar{Q}$.*

*Proof.* Notice first that any solution $S(\cdot)$, $B(\cdot)$ is such that the set

$$E := \{t \in [0, T] \text{ s.t. } S(t) < S^\star\}$$

is non-empty, otherwise one would have $B(T) = \bar{B}$, which is excluded by Lemma 3.ii under Assumption 4. Let $t^\star := \inf E < T$. By continuity of $S(\cdot)$, one has necessarily $S(t^\star) = S^\star$ and by Proposition 1 one has

$$\int_0^{t^\star} \tilde{u}_{OT}(t)\, dt \leq \int_0^{t^\star} u(t)\, dt \tag{33}$$

Notice that one has $\tilde{u}_{OT}(t) = \tilde{u}_{S^\star}(t)$ for $t \in [t^\star, T]$. From Assumption 4, the inequality

$$\int_0^{T} u(t)\, dt < \int_0^{T} \tilde{u}_{OT}(t)\, dt \tag{34}$$

is fulfilled. Consequently, (33) and (34) give the inequality

$$\int_{t^\star}^{T} u(t)\, dt < \int_{t^\star}^{T} \tilde{u}_{S^\star}(t)\, dt$$

where $\tilde{u}_{S^\star}(t) < 1$ for $t \in [t^\star, T]$ (cf property (17)). Therefore, the set

$$E_1 := \{t \in [t^\star, T] \text{ s.t. } u(t) < 1\}$$

is necessarily of non-null measure. Moreover, the set $E \cap E_1$ is also of non-null measure (otherwise one would have $u(t) = 1$ for a.e. $t \in E$ that would imply that $S(\cdot)$ is increasing on $E$, which contradicts $S(t^\star) = S^\star$).

If $t^\star > \underline{t}$, inequality (33) is strict (by Proposition 1), and one can consider a control $v(\cdot)$ such that

$$\begin{array}{ll} v(t) = \tilde{u}_{OT}(t), & t \in [0, t^\star], \\ v(t) = u(t), & t \in [t^\star, T] \setminus (E \cap E_1), \\ v(t) \in [u(t), 1], & t \in E \cap E_1 \end{array}$$

with

$$0 < \int_{E \cap E_1} \big(v(t) - u(t)\big)\, dt \leq \int_0^{t^\star} \big(u(t) - \tilde{u}_{0T}(t)\big)\, dt$$

Then, one has

$$Q[v(\cdot)] \leq Q[u(\cdot)] \leq \bar{Q}$$

which guarantees that $v(\cdot)$ is admissible. Its associated solution $S_v(\cdot)$, $B_v(\cdot)$ satisfies then $S_v(t) \geq S(t)$ for any $t \in [0, T]$ with

$$\int_{E \cap E_1} S_v(t)\, dt > \int_{E \cap E_1} S(t)\, dt$$

As $S(t) < S^\star$ for $t \in E \cap E_1$, one obtains under Assumption 1 the inequality

$$\int_{E \cap E_1} \varphi(t) K_S(S_v(t))\, dt > \int_{E \cap E_1} \varphi(t) K_S(S(t))\, dt \tag{35}$$

which yields

$$B_v(T) = \int_0^T \varphi(t) K_S(S_v(t))\, dt > \int_0^T \varphi(t) K_S(S(t)) = B(T) \tag{36}$$

We conclude that an optimal solution has to verify $t^\star = \underline{t}$ , that is such that

$$S(t) = \underline{S}(t), \quad t \in [0, \underline{t}]$$

or equivalently that having $u(t) = 0$ for $t \in [0, \underline{t}]$ is optimal.

Consider now a solution $S(\cdot)$, $B(\cdot)$ with an admissible control $u(\cdot)$ that is null on $[0, \underline{t}]$ and such that the set

$$F := \{t \in [\underline{t}, T] \text{ s.t. } S(t) > S^\star\}$$

is non empty. From Proposition 1, one has

$$\int_F \tilde{u}_{OT}(t)\, dt < \int_F u(t)\, dt$$

Let us consider an admissible control $v(\cdot)$ such that

$$\begin{array}{ll} v(t) = \tilde{u}_{[0,T]}(t), & t \in F, \\ v(t) = u(t), & t \in [0,T] \setminus (F \cup (E \cap E_1)), \\ v(t) \in [u(t), 1], & t \in E \cap E_1 \end{array}$$

with

$$0 < \int_{E \cap E_1} \big(v(t) - u(t)\big)\, dt \leq \int_F \big(u(t) - \tilde{u}_{0T}(t)\big)\, dt$$

Its solution $S_v(\cdot)$, $B_v(\cdot)$ satisfies $S_v(t) = S^\star$ for $t \in F$ and $S_v(t) \geq S^\star$ for $t \in [0,T] \setminus F$ with

$$\int_{E \cap E_1} S_v(t)\, dt > \int_{E \cap E_1} S(t)\, dt$$

As before, we obtain inequalities (35), (36). We conclude that an optimal solution has to verify $F = \emptyset$, that is such that $S(t) \leq S^\star$ for $t \in [\underline{t}, T]$.

Finally, consider an admissible control $u(\cdot)$ that is null on $[0, \underline{t}]$ with $S(t) \leq S^\star$ for $t \in [\underline{t}, T]$ and $Q[u(\cdot)] < \bar{Q}$. As previously, one can consider another admissible control $v(\cdot)$ such that:

$$\begin{array}{ll} v(t) = u(t), & t \in [0,T] \setminus (E \cap E_1), \\ v(t) \in [u(t), 1], & t \in E \cap E_1 \end{array}$$

with

$$0 < F_{max} \int_{E \cap E_1} \big(v(t) - u(t)\big)\, dt \leq \bar{Q} - Q[u(\cdot)]$$

Its solution $S_v(\cdot)$, $B_v(\cdot)$ satisfies $S_v(t) \geq S(t)$ for $t \in [0,T]$ with

$$\int_{E \cap E_1} S_v(t)\, dt > \int_{E \cap E_1} S(t)\, dt$$

One obtains again inequality (36), which shows that the control $u(\cdot)$ cannot be optimal. Therefore, an optimal control $u(\cdot)$ has to satisfy $Q[u(\cdot)] = \bar{Q}$. □

## 3.2 Application of the Maximum Principle

Notice that one can write equivalently the optimization problem (1)-(2)-(30) as an (non-autonomous) scalar optimal control problem

$$\max_{u(\cdot)} \int_0^T \varphi(t) K_S(S(t))\, dt \tag{37}$$

where $S(\cdot)$ is solution of (1), under constraints (6) and (31), or equivalently as a optimal control in the plane for the dynamics

$$\dot{S} = k_1\Big( - \varphi(t) K_S(S) - (1 - \varphi(t)) K_R(S) + k_2 u(t)\Big), \quad S(0) = 1 \tag{38}$$

$$\dot{V} = u(t), \quad V(0) = 0 \tag{39}$$

with the target

$$V(T) \leq \bar{V} := \frac{\bar{Q}}{F_{max}} \tag{40}$$

and the criterion (37). Moreover, we know from Lemma 2 that under Assumption 4, we do not have to consider the state constraint (6) for the optimal solution. Let us write the Hamiltonian associated to this optimal control problem:

$$H(t,S,\lambda_S,\lambda_V,u) := \lambda_S k_1\Big(k_2 u - (\varphi(t)K_S(S) + (1-\varphi(t))K_R(S))\Big) + \lambda_V u + \lambda_0\varphi(t)K_S(S) \tag{41}$$

and its adjoint equations:

$$\dot{\lambda}_S \in \varphi(t)\big(\lambda_S k_1 - \lambda_0\big)\partial_C K_S(S(t)) + (1-\varphi(t))\lambda_S k_1 \partial_C K_R(S(t)) \tag{42}$$

$$\dot{\lambda}_V = 0 \tag{43}$$

where $\partial_C K_S$, $\partial_C K_R$ denote the Clark generalized gradient of the Lipschitz maps $K_S$, $K_R$. Therefore, $\lambda_V$ is constant.

The Maximum Principle of Pontryagin [7] ensures that for any optimal solution $S(\cdot)$, $V(\cdot)$, $u(\cdot)$, there exists an adjoint vector $\lambda(\cdot) = (\lambda_S(\cdot), \lambda_V(\cdot))$ solution of the adjoint system (42)-(43) and a scalar $\lambda_0$ equal to 0 or 1 such that

$$\lambda_0 + |\lambda_S(t)| + |\lambda_V(t)| \neq 0, \quad t \in [0,T] \tag{44}$$

which satisfy the transversality condition

$$\lambda_S(T) = 0 \tag{45}$$

(as $S(T)$ is free and by Proposition 2 one has necessarily $V(T) = \bar{V}$), along with the maximization condition

$$H(t,S(t),\lambda_S(t),\lambda_V(t),u(t)) = \max_{v\in[0,1]} H(t,S(t),\lambda_S(t),\lambda_V(t),v), \quad \text{a.e. } t \in [0,T] \tag{46}$$

Defining the *switching function*

$$\phi(t) := \lambda_S(t)k_1k_2 + \lambda_V \tag{47}$$

the maximization (46) gives, for a.e. $t \in [0,T]$

$$u(t) = \begin{cases} 1 & \text{if } \phi(t) > 0 \\ ? & \text{if } \phi(t) = 0 \\ 0 & \text{if } \phi(t) < 0 \end{cases} \tag{48}$$

We first show that that an optimal solution cannot be abnormal.

**Lemma 4.** *For any optimal solution, one has* $\lambda_0 = 1$.

*Proof.* If $\lambda_0 = 0$, the only solution of (42) for the terminal condition (45) is $\lambda_S(t) = 0$ for $t \in [0,T]$. Moreover, the constant value of $\lambda_V$ has to be negative to fulfill the conditions (45) and (44). This implies that the $\phi(t)$ is negative for any $t \in [0,T]$ and by (48), one has $u(t) = 0$ for a.e. $t \in [0,T]$ i.e. $\underline{S}(\cdot)$ is the optimal trajectory. Let $\underline{t} \in [0,T]$ be such that $\underline{S}(\underline{t}) = S^\star$. Then the control $v(\cdot)$ defined by

$$\bar{v}(t) = \begin{cases} 0 & t \in [0,\underline{t}) \\ \dfrac{\bar{Q}}{T-\underline{t}} & t \in [\underline{t},T] \end{cases}$$

is admissible, and its associated solution $S_v(\cdot)$ verifies

$$S_v(t) = \underline{S}(t),\ t \in [0, \underline{t}), \quad S_v(t) > \underline{S}(t),\ t \in [\underline{t}, T]$$

which implies the inequality

$$\int_0^T \varphi(t) K_S(S_v(t))\, dt > \int_0^T \varphi(t) K_S(\underline{S}(t))$$

and thus is a contradiction with the optimality of $\underline{S}(\cdot)$. □

We prove now sign properties of the adjoint variables, that will play a fundamental role in the following.

**Proposition 3.** *For any optimal solution, one has $\lambda_S(T) > 0$ for $t \in [0, T)$. Moreover, one has $\lambda_V < 0$.*

*Proof.* Let us consider the set

$$E := \{t \in [0, T) \text{ s.t. } \lambda_S(t) < 0\}$$

and assume by contradiction that $E$ is non-empty. As $K_S$, $K_R$ are non decreasing functions, the elements of $\partial_C K_S$, $\partial_C K_R$ are all non-positive, and one obtains that $\lambda_S$ is non-increasing on $E$. Therefore, one has $\sup E = T$ and $\lambda_S(T) < 0$, which is a contradiction with the transversality condition (45).

If the constant $\lambda_V$ is non negative, then one has $\phi(t) > 0$ for any $t \in [0, T)$ which implies by (48) that one has $u(t) = 1$ for almost any $t \in [0, T]$. Then one obtains

$$V(T) > \int_0^T \tilde{u}_{OT}(t)\, dt = \frac{Q[\tilde{u}_{OT}(\cdot)]}{F_{max}} > \bar{V}$$

which is in contradiction with the target condition (40). □

Proposition 3 and the transversality condition (45) imply that one has $\phi(T) < 0$, which yields straightforwardly the following properties.

**Corollary 2.** *There exists $\bar{t} < T$ such that $u(t) = 0$ for $t \in [\bar{t}, T]$ is optimal. Moreover, one has $S(T) < S^\star$.*

Let us now study the possibilities of singular arcs (we recall that a singular is a piece of an optimal trajectory such that the switching function $\phi$ remains equal to zero).

**Lemma 5.** *A singular arc cannot occur at locus where $S$ is not a corner point of $K_S$ or $K_R$.*

*Proof.* A singular arc occurs when the switching function $\phi$ is equal to zero on a time interval $I$ of non-null measure. This amounts to have $\lambda_S$ constant equal to $\lambda_S^\star := -\lambda_v/(k_1 k_2) > 0$ on a such an interval. If $K_S$ and $K_R$ are differentiable at $S(t_1)$ where $t_1 \in I$, one has then, from (42)

$$\varphi(t)\big(\lambda_S^\star k_1 - 1\big) K_S'(S(t_1)) + (1 - \varphi(t)) \lambda_S^\star k_1 K_R'(S(t_1)) = 0 \tag{49}$$

for any $t \in I$ in a neighborhood of $t_1$. From Lemma 1, one has $K_R(S(t_1)) > 0$ and thus $K_R'(S(t_1)) > 0$ (by Assumption 1). Differentiating the left member of (49) w.r.t. $t$ at $t_1$ yields

$$\varphi'(t_1)\Big(\big(\lambda_S^\star k_1 - 1\big) K_S'(S(t_1)) - \lambda_S^\star k_1 K_R'(S(t_1))\Big) > 0$$

which shows that (49) cannot be satisfied on a neighborhood of $t_1$. □

### 3.3 An optimal synthesis

We consider the "saturated one slot" (SOS) feedback control, as follows.

**Definition 2.** *For $t_S \in [\underline{t}, T-\bar{V}]$, we define the time-varying feedback control:*

$$\psi_{t_S}^{SOS}(t,S,V) := \begin{cases} 0 & \text{if } t < t_S \text{ or } V = \bar{V}, \\ 1 & \text{if } t \geq t_S \text{ and } S < S^\star \text{ and } V < \bar{V}, \\ \tilde{u}_{S^\star}(t) & \text{if } t \geq t_S \text{ and } S = S^\star \text{ and } V < \bar{V} \end{cases} \tag{50}$$

This strategy consists in irrigating crops at once, from a time $t_S$ larger or equal to $\underline{t}$ (accordingly to Proposition 2). It consists in delivering water at the maximal flow rate ($u = 1$) as long as the humidity rate $S$ is below $S^\star$, or maintaining $S = S^\star$ (with the singular control $\tilde{u}_{S^\star}(\cdot)$, accordingly to Proposition 2). Moreover, from Proposition 2 and Corollary 2, an optimal solution has to use all the water quantity $\bar{V}$ before the final time $T$. Therefore, the very last possible time $t_S$ is necessarily less than $T - \bar{V}$.

One has then the following result about the optimality of the SOS strategy.

**Proposition 4.** *Under Assumption 4, if for any $t \in (\underline{t}, T-\bar{V})$, $\underline{S}(t)$ is not a corner point of $K_S$ or $K_R$, then the SOS strategy is optimal, i.e. there exists $t_S \in [\underline{t}, T-\bar{V}]$ such that the feedback control* (50) *is optimal.*

*Proof.* Notice first that an optimal solution $S(\cdot)$ satisfies $S(t) \geq \underline{S}(t)$ for any $t \in [0,T]$ with $S(t) \leq S^\star$ for $t \in [\underline{t}, T]$ (cf Proposition 2). From Lemma 5 and Assumption 1, we conclude that the only possible singular arc is when $S(\cdot)$ remains equal to $S^\star$.

As $u(t) = 0$ is optimal for $t \in [0, \underline{t}]$ (Proposition 2), the switching function $\phi(\cdot)$ has to be non-positive on $[0, \underline{t}]$, or equivalently one should have $\lambda_S(t) \leq \lambda_{S^\star} = -\lambda_v/(k_1 k_2$ for $t \in [0, \underline{t}]$. On the interval $[0, \underline{t}]$, $S(\cdot)$ is thus decreasing with $S(\underline{t}) = S^\star$. Let $S_R := \sup\{S \in [0,1] \text{ s.t. } K_R(S) < 1\}$, which satisfies $S_R > S^\star$ by Assumption 1. Define then $t_R := \inf\{t \in [0, \underline{t}] \text{ s.t. } S(t_R) < S_R\}$. By hypothesis, $K_R$ and $K_S$ are respectively increasing and constant on $[S^\star, S_R]$. From (42), we deduce that $\lambda_S(\cdot)$ is increasing on $[t_R, \underline{t}]$ and thus one has $\lambda_S(t) < \lambda_{S^\star}$ for $t \in [t_R, \underline{t})$. Consider then the set

$$C := \{t \in [\underline{t}, T] \text{ s.t. } \lambda_S(t) \geq \lambda_{S^\star}\}$$

which is of non empty interior (otherwise $u(t) = 0$ would be optimal for any $t \in [\underline{t}, T]$, which is not possible by Proposition 2). Notice that one has necessarily $\sup C < T$ because the only possible singular arc is $S = S^\star$ and one should have $S(T) < S^\star$ (Corollary 2). Moreover, one can check that the property

$$S(t) = S^\star \text{ with } t \in int\, C \Rightarrow \lambda_S(t) = \lambda_{S^\star} \tag{51}$$

is fulfilled, because $\lambda_S(t) > \lambda_{S^\star}$ would imply $u(\tau) = 1$ for a.e. $\tau$ in a neighborhood of $t$ in $C$, violating $S(t) \leq S^\star$ for any $t \in [\underline{t}, T]$.

Let us show now that $C$ is a connected set. If not, there exists $t_m \in [\underline{t}, T] \setminus C$ inbetween two consecutive connected components of $C$ that is a local minimum of $\lambda_S(\cdot)$ with $\lambda_S(t_m) < \lambda_{S^\star}$. One has then $S(t_m) < S^\star$ and by hypothesis, $K_S$ and $K_R$ are differentiable at $S(t_m)$. From (42), one can write

$$\dot{\lambda}_S(t_m) = \varphi(t_m)(\lambda_S(t_m)k_1 - 1)K'_{Sm} + (1-\varphi(t_m))\lambda_S(t_m)k_1 K'_{Rm} = 0 \tag{52}$$

where $K'_{Sm}$, $K'_{Rm}$ denote the derivatives of $K_s$, $K_R$ at $S(t_m)$,

$$\ddot{\lambda}_S(t_m) = \varphi'(t_m)\big((\lambda_S(t_m)k_1 - 1)K'_{Sm} - \lambda_S(t_m)k_1 K'_{Rm}\big) \tag{53}$$

(using the fact that $K_S$, $K_R$ are linear around $S(t_m)$). By Lemma 1, one has $K_R(t_m) > 0$ and under Assumption 1, one has then $K'_{Rm} > 0$. From (52), one obtains $\lambda_S(t_m)k_1 - 1 < 0$ and then from (53) with Assumption 2, $\ddot{\lambda}_S(t_m) < 0$, which contradicts that $\lambda_S(\cdot)$ has a local minimum at $t_m$.

Finally, we obtained an optimal control defined for almost any $t \in (0, T]$ as follows.

1. $u(t) = 0$ is optimal for $t \notin C$,
2. $u(t) = 1$ is optimal for $t \in interior C$ with $S(t) < S^\star$, as a singular arc can occur only at $S = S^\star$,
3. the singular control $u(t) = \tilde{u}_{S^\star}(t)$ is optimal for $t \in int\, C$ when $S(t) = S^\star$, because of property (51) and the only way to have a neighborhood of $t$ belonging to $C$ with $S \leq S^\star$ is to stay on the singular arc $S = S^\star$.

This proves, along with the fact that $C$ is connected, the optimality of the feedback (50).

□

# 4 Numerical illustration

We present in Fig. 5 the simulations performed with irrigation strategies SOS and OS and with inputs data given in Table 1. The "One Shot" (OS) strategy consists in delivering water at maximum flow rate during a single irrigation period at a triggering time $t_S$:

**Definition 3.** *For $t_S \in [\underline{t}, T - \bar{V}]$, we define the open-loop control :*

$$u^{OS}_{t_S}(t) := \begin{cases} 0 & \text{if } t < t_S \text{ or } t > \min(t_S + \bar{Q}/F_{max}, T)) \\ 1 & \text{if } t \in [t_S, \min(t_S + \bar{Q}/F_{max}, T)) \end{cases}$$

It represents a class of widely used irrigation strategies, typically when drip irrigation is not available. For illustrative purposes only, we have considered dimensionless parameters (by normalizing the units) and function $\varphi$ in the family of $t \mapsto (t/T)^\alpha$ ($\alpha > 0$).

| $T$ | $k_1$ | $k_2$ | $k_3$ | $S^\star$ | $S_w$ | $S_h$ | $F_{max}$ | $\bar{Q}$ | $\alpha$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2.1 | 5 | 1 | 0.7 | 0.4 | 0.2 | 1.2 | 0.1 | 4 |

Table 1: Normalized parameters used for the simulations

The optimal OS strategy was obtained for $t_S = 0.697$ and produced a biomass $B(T) = 0.152$. The corresponding humidity dynamics is plotted in Fig. 5c. It can be seen that some value of $S$ are above $S^\star$. It can be therefore

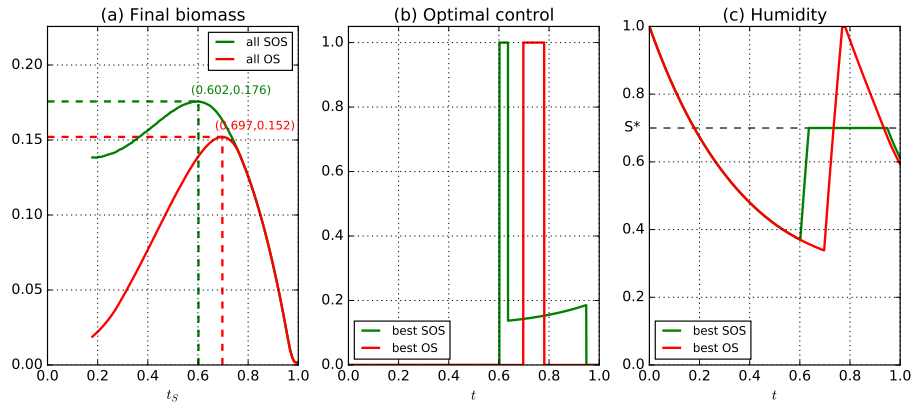


Figure 5: Comparison of SO and $SOS$ controls strategies on one typical example. Model parameters used are given in Table 1.

concluded from the application of of Proposition 4 that an OS strategy cannot be optimal. This is further illustrated by applying the SOS strategies for the same inputs data. We find that the best SOS strategy gives a final biomass $B(T) = 0.176$ which is 15% higher than what gives the best OS strategy. The associated control is a bang-singular-bang (see Fig. 5b).

Notice that the SOS strategy requires more knowledge or online measurements than the OS control for its real application (as the expression of the singular control (16) needs the function $\varphi(\cdot)$ and the values $S^\star$, $k_2$ and $K_R(S^\star)$). Moreover it change gradually the input flow rate during the singular arc phase. This is why it can be considered as a more *sophisticated* strategy.

# 5 Conclusion

We have introduced a simple crop irrigation model in order to study optimal irrigation scheduling using a mathematical analysis. We have shown, using a comparison tool, that the state constraint of this model is never activated for the optimal control problem solutions. Moreover we have shown that, under water scarcity, an optimal trajectory has to reach as fast as possible the domain for which the relative humidity is below or equal to the threshold of maximal crop transpiration, and then do not leave this domain until the harvesting time. However, due to water scarcity, it has to be below the threshold at some stage. We have then compared two control strategies: the one-shot (OS), commonly used in practice and a more sophisticated one, the saturated one-shot (SOS), that could exhibit a singular arc. We have shown numerically the superiority of this last strategy. We conjecture that the SOS strategy is indeed an optimal control for this model. This would be a promising result since SOS irrigation schemes are not so intuitive controls and because they can be also tested on more detailed simulation models. This shall be the matter of a future work.

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