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The Geometric Theorem

(Paris Album No. 1—Spring 2018)

Adrien Deloro* and Joshua Wiscons†

15th January 2019

Solomon saith, *There is no new thing upon the earth.* So that as Plato had an imagination, *That all knowledge was but remembrance;* so Solomon giveth his sentence, *That all novelty is but oblivion.* Whereby you may see that the river of Lethe runneth as well above ground as below.

§ 1. Introduction — § 2. Theorem A — § 3. The B-Sides

1 Introduction

The presence or lack of involutions may be regarded as the main dividing line in the study of abstract groups of finite Morley rank, in great part since, to-date, the most successful prism of analysis was borrowed from finite group theory. We do not wish to dwell on the topic as our methods are of a different nature. The present short note is a revival of and contribution to the geometry of involutions in groups of finite Morley rank.

1.1 A theorem and conjecture

In what follows, we use “ranked group” interchangeably with “group of finite Morley rank”; for the present work there is surprisingly little to know on these objects and § 1.3 will help orient the reader. We shall prove the result below and some of its consequences. A group G is U_2^\perp (this stands for: no 2-unipotent subgroups) if it contains no infinite elementary abelian 2-group. A subgroup $C < G$ is TI (this stands for: trivial intersections) if: $(\forall g) (C \neq C^g \rightarrow C \cap C^g = 1)$.

Theorem A (“The Geometric Theorem”). *Let G be a connected, U_2^\perp , ranked group with involutions. Suppose that G has a definable, connected subgroup $C < G$ which is TI and almost self-normalising.*

Then $\bigcup_{g \in G} C^g$ does not contain all strongly real elements of G .

Theorem A is a revisitation of the classical analysis by Nesin of specific $SO_3(\mathbb{R})$ -ish configurations [Nes89]. It is however new and strictly more general, arguably thanks to modern torsion tools and perhaps a more efficient geometrisation. Its name refers to the fact that such a configuration naturally defines a G -invariant point-line-plane incidence structure that is, in fact, a three-dimensional projective space. And this has nothing to do with “Bachmann’s Theorem”.

The residual configuration is perfectly sound, at least as sound as $PGL_2(\mathbb{K})$, and we boldly pose the following.

Conjecture (“ A_1 -Conjecture”). *Let G be a connected, U_2^\perp , ranked group with involutions. Suppose that G has a definable, connected subgroup $C < G$ which is TI and almost self-normalising. Suppose that $N = N_G(C)$ is not strongly embedded. Then $G \simeq PGL_2(\mathbb{K})$.*

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However tempting (and faithful to the spirit of the Cherlin-Zilber Conjecture), it would be immodest to drop the assumption on N : in the strongly embedded case the authors have no contradiction in sight for now. But they intend to return to the A_1 -conjecture in the near future. One could even try to work in a broader setting than finite Morley rank. As a matter of fact the conjecture should identify the root system of an algebraic group, not an abstract group. The anisotropic real form of $\mathrm{PGL}_2(\mathbb{C})$, namely $\mathrm{SO}_3(\mathbb{R})$, satisfies all assumptions of Theorem A except the model-theoretic aspect of uncountable categoricity, which suggests working in the presence of a dimension function generalising both Morley rank and o -minimal dimension, as in [DW18] for instance. But for the moment we shall be content with Theorem A and its consequences.

1.2 Corollaries

The proof of Theorem A is in § 2. Here we briefly list its consequences, which range from classical to new. Corollaries A.1 and A.2 are immediately derived; Theorem B and its Corollaries B.1 and B.2 will be proved in § 3.

Corollary A.1. *Let G be a connected, U_2^\perp , ranked group. Suppose that G has a definable, connected subgroup $C < G$ whose conjugates partition G . Then G has no involutions.*

Corollary A.1 (which is new as well) was our starting point; then we realised that it was enough to capture all generic and strongly real elements to force genuine covering. Its setting is *not* the same as Jaligot’s “Full Frobenius groups” [Jal01], where malnormality (viz. $(\forall g) (g \notin C \rightarrow C \cap C^g = 1)$) was assumed, and absence of involutions proved after Delahan and Nesin. It is a trivial matter to show that if in Corollary A.1 one supposes in addition that C is soluble or contains no unipotent torsion, then it is self-normalising, hence malnormal. Such configurations have been revived by Wagner [Wag18] as a generalisation of the ill-named “bad groups”, of which more will be said below.

Modulo their non-trivial but standard local analysis, one immediately retrieves Corollary A.2, a celebrated classic. (Announced in rank 3 by Cherlin [Che79, §5.2, Theorem 1] with a flawed proof, the result has an interesting story. In an unpublished preprint [Bor84] Borovik foresaw the importance of geometrisation; the geometric line was simultaneously followed by Borovik and Poizat in a Soviet-published, much delayed article [BP91, Theorem 5.4], and by Poizat in his self-published, fast-track, book [Poi87, § 3.8]; Nesin’s approach in rank 3 [Nes89] was different and independent, and to be later generalised by Corredor.)

Corollary A.2 (in rank 3, Nesin [Nes89, Main Theorem (2)]; in general, Borovik-Poizat [BP91, Theorem 5.4] or Corredor [Cor89, Theorem 1(5)]). *Let G be a simple ranked group whose proper, definable, connected subgroups are all nilpotent. Then G has no involutions.*

Parenthetically let us observe that if only the definable, connected, *soluble* subgroups are supposed to be nilpotent, then this is open. Recall that the *Borel subgroups* are the definable, connected, soluble subgroups which are maximal as such.

Conjecture. *Let G be a simple ranked group whose Borel subgroups are all nilpotent. Then G has no involutions.*

Such a configuration is sometimes referred to as a “bad group”; the intriguing terminology has varied over the years, so one could consider refraining from using it. Also notice that what really matters in these tight configurations is the impossibility to apply the Schur-Zilber method inside Borel subgroups. One might therefore include such hypothetical objects in the wider class of “asomic groups”, i.e. groups not interpreting an infinite field—always bearing in mind that other methods for constructing a field exist.

What prevents us at present from proving the last conjecture is essentially that as opposed to the algebraic world, Borel subgroups need no longer be conjugate in definable, connected subgroups of G . This contrasts with the following result derived from Theorem A with only marginal extra work and in which *hereditarily conjugate* means that every definable connected subgroup of G enjoys Borel conjugacy. (We did our best to avoid the assumption, and could not.)

Theorem B (generalises Borovik-Burldges [BB08, Theorem 3.8]). *Let G be a connected, U_2^\perp , ranked group in which all Borel subgroups are nilpotent and hereditarily conjugate. Then the (possibly trivial) 2-torsion of G is central.*

As opposed to Borovik and Burldges, we however make no assumptions on p -unipotence. From there we easily obtain Borovik and Burldges' linear theorem, based on the preliminary analysis by Poizat [Poi01] (and later Mustafin [Mus04]).

Corollary B.1 (Borovik-Burldges, [BB08, Main Theorem]). *Let \mathbb{K} be a ranked field of characteristic 0 and $G \leq \mathrm{GL}_n(\mathbb{K})$ be a simple, definable, non Zariski-closed subgroup. Then G has no involutions.*

We also found something we were not looking for.

Corollary B.2. *Let G be a connected ranked group in which all Borel subgroups are good tori. Then the (possibly trivial) 2-torsion of G is central.*

Remark. The authors learnt shortly before submitting that Altmel, Berkman, and Wagner had also obtained a very special case of Corollary B.2, when the Prüfer 2-rank is 1 [ABW18, Theorem 8]. Their methods are of course geometric but seem to 1. focus on the plane of involutions instead of a 3-dimensional geometry and 2. use Bachmann's Theorem.

1.3 A self-guide to the present paper

We close this introduction by reminding our reader what (s)he will need to deem familiar in order to follow us.

- Basic group-theoretic definitions such as a strongly real element [BN94, § 10.1] and a strongly embedded subgroup [BN94, § 10.5].
- Some rough notion of what a group of finite Morley rank is: in particular the additivity property [BN94, § 4.1.2], enabling basic rank computations; also the notion of connectedness [BN94, § 5.2] and why generic subsets of a connected group must intersect. Something *almost* holds if it does up to taking connected components.
- Very interestingly, classical and fundamental results such as the Chevalley-Zilber generation lemma ("indecomposability theorem" [BN94, Theorem 5.26]), the Schur-Zilber linearisation lemma ("field theorem" [BN94, Theorem 9.1]), and the analysis of groups of small Morley rank *play no role at all*. We would be glad to take it as an indication that our result goes beyond the mere theory of groups of finite Morley rank. But on the other hand we shall invoke Macintyre's theorem [BN94, Theorem 8.1] that an infinite field with a rank function is algebraically closed: something not true in other model-theoretic settings.

What really matters is torsion; torsion-lifting [BN94, Exercise 11 on p. 98], basic divisibility properties [BN94, Exercises 9 and 13 on p. 82], but also the following.

- Recall that a U_2^\perp group is one not containing an infinite elementary abelian 2-group. In the ranked case, Sylow 2-subgroups (which are as always conjugate [BN94, Theorem 10.11]) then become toral-by-finite [BN94, Corollary 6.22], i.e. finite extensions of $(\mathbb{Z}/2^\infty\mathbb{Z})^d$, where $\mathbb{Z}/2^\infty\mathbb{Z}$ is the Prüfer quasi-cyclic 2-group. In a connected, U_2^\perp , ranked group *with* involutions, the integer d known as the Prüfer 2-rank is non-zero [BBC07].
- Torality principles from [BC09] are key features: in particular, in a connected, U_2^\perp ranked group, every 2-element lies in a 2-torus.
- It is essential to know what a good or decent torus [Che05] is and, in particular, the generic behaviour of the Cartan subgroups $C_G(T)$ and their conjugacy, and also their connectedness [AB08, Theorem 1].

If such tools, and a few more to be cited in due time, were available in the \mathcal{o} -minimal context, Proposition 1 might carry over to there.

2 Theorem A

Theorem A. *Let G be a connected, U_2^\perp , ranked group with involutions. Suppose that G has a definable, connected subgroup $C < G$ which is TI and almost self-normalising.*

Then $\bigcup_{g \in G} C^g$ does not contain all strongly real elements.

The present section is devoted to proving Theorem A; there is nothing to say on how to derive Corollaries A.1 and A.2. A brief outline of the proof itself is as follows, in the same notation as the statement.

- The main alternative (Proposition 1 of § 2.1) is highly restrictive. If C is TI and self-normalising, then either $N_G(C)$ is strongly embedded, or G shares many features of groups of type A_1 over \mathbb{C} or \mathbb{R} ; which covers the structure of the Sylow 2-subgroup, the generic distribution of involutions, and even the generic nature of G .
- The assumption that $\bigcup_G C^g$ contains all strongly real elements actually rules out strong embedding, so Proposition 1 has much to say. But as opposed to $\mathrm{PGL}_2(\mathbb{K})$, the configuration is now unviable: this is proved in § 2.2, Proposition 2. To find a contradiction we introduce a projective geometry; this is remarkably accelerated by going 3-dimensional with a built-in polarity, a strategy which must be compared with Nesin's work.
- The final argument uses Borel's fixed point theorem from algebraic geometry. We insist that the famous theorem by Bachmann [BN94, §8.3], however beautiful, belongs to metric geometry: methodologically speaking, it can therefore be no part of a ranked study, where fields are algebraically closed.

2.1 The main alternative

Notation. *Throughout this subsection and the next, G will be a connected, U_2^\perp , ranked group with involutions, and $C < G$ a definable, connected, TI and almost self-normalising subgroup.*

We let $N = N_G(C)$ and $I \subseteq G$ be the set of involutions; $I \cdot I$ is then the set of strongly real elements of G .

Proposition 1 (A French Nocturne). *Let G be a connected, U_2^\perp , ranked group with involutions. Suppose that G has a definable, connected subgroup $C < G$ which is TI and almost self-normalising.*

Then the following are equivalent:

- (i) $N = N_G(C)$ is not strongly embedded;
- (ii) N/C has even order;
- (iii) the Sylow 2-subgroup is as in $\mathrm{PGL}_2(\mathbb{C})$ and for $i \in I$ (the set of involutions), $C_G^\circ(i)$ is a conjugate of C ;
- (iv) $N = C \rtimes \mathbb{Z}/2\mathbb{Z}$ (inversion action);
- (v) C consists of elements which are strongly real in G (alt.: $C \subseteq I \cdot I$);
- (vi) the generic element of G is strongly real (alt.: $I \cdot I$ is generic in G);
- (vii) every strongly real element generic as such is in $\bigcup_G C^g$ (alt.: $\mathrm{rk}(I \cdot I \setminus \bigcup_G C^g) < \mathrm{rk}(I \cdot I)$);
- (viii) for $i \in I$, $C_G(i)$ is not connected;
- (ix) if i, j is a generic pair of involutions, there is $k \in I$ commuting with both.

Remarks.

- The reader may like us muse upon the fact that the first four conditions and (viii) provide *structural* descriptions, whereas the other four yield *generic* information. Having such an equivalence is strong praise for generic methods in the study of groups of finite Morley rank.

- All are satisfied in $\mathrm{PGL}_2(\mathbb{K})$. Algebraically, $\mathrm{SO}_3(\mathbb{R})$ has the properties, but its dimension function does not agree with the Morley rank. The A_1 -conjecture is precisely that Proposition 1 characterises $\mathrm{PGL}_2(\mathbb{K})$ among groups of finite Morley rank.
- In $\mathrm{PGL}_2(\mathbb{K})$ and $\mathrm{SO}_3(\mathbb{R})$ it also happens that $I \cdot I = G$, i.e. *every* element is strongly real. This does not seem to follow from Proposition 1, although it would be a consequence of the A_1 -conjecture. And the latter does not look noticeably easier under the extra assumption that $I \cdot I = G$.
- Knowing the mere structure of the Sylow 2-subgroup in (iii) does not seem to be enough. But [DJ16] provides a thorough analysis of $C_G^\circ(i)$ in N_G° -groups, and we now say a (very optional) word about the CiBoes.
In CiBo_1 , the (to our current knowledge, possibly TI) centraliser $C_G(i) = C_G^\circ(i)$ is strongly embedded. CiBo_3 is irrelevant here: neither the centraliser of an involution nor that of a maximal 2-torus is TI. Under the assumption that its connected centralisers $C_G^\circ(i)$ are genuinely TI, CiBo_2 is a model of Proposition 1, making the A_1 -conjecture non-trivial.
- Proposition 1 easily implies that $I \cdot C$ is generic in G . The converse may fail, as shown by the possibly TI configuration of type CiBo_1 . To our surprise, genericity of $I \cdot C$ does not seem to be a relevant property.
- Non-connectedness of $C_G(i)$ implies $C_G^\circ(i) = C$, but here again the converse may fail. And there are no claims whatsoever on whether $N = C$ or not in the strongly embedded case.

Proof. By an obvious rank computation, $\Gamma = \bigcup_{g \in G} C^g$ is generic in G . It follows that C contains the connected centraliser of each of its non-trivial elements: if $1 \neq x \in C$, then $C_G(x) \leq N_G(C)$ since C is TI, so $C_G^\circ(x) \leq N_G^\circ(C) = C$ by almost self-normalisation.

Now, C also contains a maximal decent torus of G : letting $T \leq G$ be one such, we know that $\bigcup_{g \in G} C_G(T)^g$ is generic [Che05]. Since so is Γ , up to conjugacy there is $1 \neq x \in C \cap C_G(T)$, whence $T \leq C_G^\circ(x) \leq C$. Hence by torality [BC09, Theorem 3], every 2-element is in a conjugate of C .

Since C is TI, an element $x \neq 1$ belongs to at most one conjugate of C , which we then denote by C_x . As we just saw C_x is defined (at least) for x a generic element of G or a 2-element. We stress that C_x does not stand for $C_G^\circ(x)$, but for the unique conjugate of C containing x ; in the strongly embedded case equality could fail and all we know is $C_G^\circ(x) \leq C_x$.

There will be two main blocks: (i)–(vii) and (viii)–(ix). Actually we treat (viii) somehow separately, since the current status of generic use of the Borovik-Cartan polar decomposition in odd type is not fully clarified. This is why we shall prove: [(i)–(vii)] \Leftrightarrow [(ix) \Leftrightarrow (viii)]. Of course (iv) \Rightarrow (viii), but we want to stress that equivalence of (i)–(vii) with (ix) does *not* rely on unpublished material. The proof begins.

(i) \Rightarrow (ii). Let $g \in G \setminus N$ such that $N \cap N^g$ contains an involution k . If N/C had odd order, then we would obtain $k \in C \cap C^g$, whence $C = C^g$ and $g \in N_G(C) = N$, a contradiction.

(ii) \Rightarrow (iii). Suppose that N/C has even order. Lifting torsion, let $\alpha \in N \setminus C$ be a 2-element with least possible order, so that $\alpha^2 \in C$. However $\alpha \in C_\alpha \neq C$, so $\alpha^2 \in C_\alpha \cap C = 1$ and α is an involution. Now $C_\alpha \neq C$ implies that α inverts C . If $i \in C$ is any involution then $C = C_G^\circ(i)$; moreover i normalises $C_\alpha \neq C$ so i inverts C_α , which proves that C contains a unique involution. Hence the Prüfer 2-rank is 1, and since $\alpha \notin C$ normalises C , the Sylow 2-subgroup of G is as in $\mathrm{PGL}_2(\mathbb{C})$ (for reference, [DJ10, Proposition 27]).

(iii) \Rightarrow (iv). Consider a Sylow 2-subgroup, isomorphic to $\mathbb{Z}/2^\infty\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, with central involution $i \in C$ and some non-central involution k . Every involution in C is toral in C and the Prüfer rank is 1, so i is the only involution in C . In particular $C \neq C_k$, so k inverts C . Hence $N \leq C_G(i) \leq N(C_G^\circ(i)) = N$, so by Steinberg's torsion theorem [Del09], N/C has exponent 2. By the structure of the Sylow 2-subgroup, $N = C \rtimes \langle k \rangle$.

(iv) \Rightarrow (v). Clear.

(v) \Rightarrow (vi). As we know, $\Gamma = \bigcup_G C^g$ is generic in G .

(vi) \Rightarrow (vii). A strongly real element generic as such is then just a generic element of G ; and as we know, Γ is generic in G .

(vii) \Rightarrow (i). Suppose (vii) and yet that N is strongly embedded. Let $S = I \cdot I$ be the set of strongly real elements and $\Sigma = S \cap \Gamma$. The assumption is that $\text{rk}(S \setminus \Sigma) < \text{rk } S$, which implies $\text{rk } \Sigma = \text{rk } S$ (the degree of S is a priori unknown; of course once S is proved generic, this will be settled at once).

By strong embedding, all involutions in N are actually in C : this is because N conjugates its involutions [BN94, Theorem 10.19] and $C = N^\circ$ contains one, say i , with connected centraliser $D = C_G^\circ(i) \leq C$. Then the set of involutions of C is $I_C = i^N$, with rank $\text{rk } C - \text{rk } D$. And the set $S_C = I_C \cdot I_C$ of strongly real elements of C has rank at most $2 \text{rk } C - 2 \text{rk } D$.

We first estimate $\text{rk } \Sigma$. If $1 \neq x = rs \in C$ is strongly real in G (i.e. with $r, s \in I$), then $r, s \in N$ so actually $r, s \in C$: hence $x \in S_C$. This shows:

$$\Sigma = \bigcup_G (S \cap C^g) = \bigcup_G S_{C^g} = \bigcup_G S_C^g,$$

whence $\text{rk } \Sigma \leq \text{rk } G - \text{rk } N + \text{rk } S_C \leq \text{rk } G + \text{rk } C - 2 \text{rk } D$.

Now consider the following restriction of the product map:

$$\mu : I_C \times (I \setminus N) \rightarrow S.$$

We contend that it has trivial fibres, and that the image avoids Σ . The former uses essentially the same argument as the latter: if $ir \in C^g$ in obvious notation, then $i, r \in N^g$ which forces $C = C_i = C^g = C_r$, a contradiction. Therefore:

$$\text{rk } I_C + \text{rk}(I \setminus N) = \text{rk } \mu(I_C \times (I \setminus N)) \leq \text{rk}(S \setminus \Sigma) < \text{rk } S.$$

(Notice that we used $\text{rk}(S \setminus \Sigma) < \text{rk } S$, not only $\text{rk } \Sigma = \text{rk } S$.)

Thanks to $\text{rk}(I \setminus N) = \text{rk } I$ and G -conjugacy of involutions ([BN94, Theorem 10.19] again) we derive the following contradiction:

$$\text{rk } S > \text{rk } C - \text{rk } D + \text{rk } G - \text{rk } D \geq \text{rk } \Sigma.$$

This shows that (i)–(vii) are equivalent.

(viii) \Rightarrow (ix). Whether N is strongly embedded or not, we know by [BN94, Theorem 10.19], resp. the implication (i) \Rightarrow (iii) (and torality principles), that all involutions are conjugate, say $I = i^G$. Since fibres of the map $g \mapsto i^g$ are translates of $C_G(i)$, there is a correspondence between the generic element of G and the generic involution (genericity over i). Moreover I^2 has degree 1.

Now if for the generic pair of involutions (r, s) the definable envelope $\langle rs \rangle_{\text{def}}$ is 2-divisible, then for $g \in G$ generic over i , so is $\langle ii^g \rangle_{\text{def}}$. By the generic version of the Borovik-Cartan polar decomposition (use the tools of [BBC07, §5]), $C_G(i)$ is connected: a contradiction.

(ix) \Rightarrow (viii). Suppose that $C_G(i)$ is connected. Let (i, j) be a generic pair of involutions. By (ix) there is $k \in I$ commuting with both, so $i \in C_G(k) = C_G^\circ(k) \leq C_k$, forcing $C_i = C_k = C_j$ likewise, violating genericity.

So (viii) and (ix) are equivalent. We finally establish the connection with (i)–(vii).

(ix) \Rightarrow (i). Let (i, j) be a generic pair of involutions. By assumption there is k commuting with both; say $C_k = C_i^g$. Now if N is strongly embedded, then $i \in C_i \cap N(C_k) \leq N(C_i) \cap N(C_i)^g$ implies that $g \in N(C_i)$, whence $C_i = C_k = C_j$ likewise, against genericity of (i, j) .

(i) \Rightarrow (ix). We make free use of all properties (i)–(vii). Let i be the involution of C . Consider the restriction of the multiplication map:

$$\mu : (N \setminus C) \times (I \setminus N) \rightarrow G.$$

The fibres are trivial since if $(k, r) \neq (\ell, s)$, in obvious notation, have the same image, then $1 \neq \ell k \in C$ is inverted by r : forcing $r \in N$, a contradiction. Since $\text{rk}(N \setminus C) = \text{rk } C$ and $\text{rk}(I \setminus N) = \text{rk } G - \text{rk } C$, we find that μ is generically onto G . Moreover a generic element $(k, r) \in (N \setminus C) \times (I \setminus N)$ is mapped to one of G , and in particular there is $g \in G$ such that $kr \in C^g$. Then k, r centralise i^g : the pair (k, r) has the required property, which carries through conjugation to any generic pair. \square

2.2 Fantasies

We now venture into the inconsistent.

Proposition 2 (Kaum hatte Hutter die Brücke überschritten...). *Let G be a connected, U_2^\perp , ranked group with involutions. Suppose that G has a definable, connected subgroup $C < G$ which is TI and almost self-normalising. Then the following are equivalent:*

- (x) every strongly real element is in $\bigcup_G C^g$;
- (xi) if i, j are any involutions, there is $k \in I$ commuting with both;
- (xii) $G = \bigcup_G C^g$;
- (xiii) G does not exist.

Remarks.

- Here again, $\text{SO}_3(\mathbb{R})$ has all properties—except finiteness of the Morley rank and perhaps inexistence.
- Using Proposition 1 it is a straightforward consequence of (xii) that $G = I \cdot I$; the latter is however not inconsistent as it holds in $\text{PGL}_2(\mathbb{K})$.

Proof. Each condition clearly implies at least one in Proposition 1, which we use freely.

Claim 1. We have equivalence of (x), (xi), and (xii).

Proof of Claim.

(x) \Rightarrow (xi). Let $i \neq j$ be involutions. Then we may assume that $ij \in C$: so i and j normalise C and centralise its unique involution.

(xi) \Rightarrow (xii). We cheat a bit as we first prove (x).

Let $x = ij$ be any strongly real element. By assumption, there is an involution k commuting with i and j , so $x \in C_G(k)$. As we know from Proposition 1, x is in $C_G^\circ(k)$ or an involution, so in any case, $x \in \bigcup_G C^g$.

We now prove that $I \cdot I \cdot I = I \cdot I$. Let i, j, k be any three involutions. By assumption, there is an involution r commuting with both i and j and an involution s commuting with both r and k . We may assume that $r \neq i, j$ as otherwise ij is an involution or 1, which implies that $ij \cdot k \in I \cdot I \cup I \subset I \cdot I$ (by Proposition 1). In particular, $ij \in C_G^\circ(r)$.

Now, if $s = r$, then $k \in C(r)$, and there are two options: $k = r$ or k inverts $C_G^\circ(r)$. In the former case, jk is an involution or 1, forcing $i \cdot jk \in I \cdot I$. In the latter case, $ijk \in I \cup \{1\}$ since $ij \in C_G^\circ(r)$. So we are left to consider when $s \neq r$.

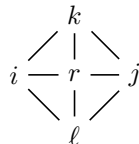
Of course, $ijk = ijs \cdot sk$. On one hand, as s inverts $ij \in C_G^\circ(r)$, ijs is an involution or 1; on the other hand, the same is true of sk .

Hence $\langle I \rangle = I \cdot I$ is a definable subgroup containing the generic element by Proposition 1: so $I \cdot I = G \subseteq \bigcup_G C^g$. \diamond

From now on we suppose (x)–(xii). First we strengthen (xi) as follows.

Claim 2. Let $i \neq j$ be two distinct involutions. Then there is a unique involution which commutes with and is distinct from both.

Proof of Claim. If i and j do not commute then an involution commuting with both (this exists by (xi)) must be distinct from both. If i and j do commute then ij is again distinct from both. We must now prove uniqueness. So let i, j, k, ℓ be four pairwise distinct involutions such that k and ℓ commute with i and j ; notice how perfectly symmetric the configuration is; we are after a contradiction. The relevant portion of the commuting-involution graph is as follows—the involution r , which is introduced below, may in fact be equal to one of the others.



Let r be the unique involution in $C_G^\circ(k\ell) = C_r$. We see that i, j, k, ℓ all normalise C_r , so as at most one of them can equal r , we may suppose that $i, j, k \notin C_r$. Then by the structure of the Sylow 2-subgroup, the fact that $\{1, i, k, r\}$ forms a four-group forces $i = kr$; likewise $j = kr = i$, a contradiction. \diamond

And now let us introduce an incidence geometry.

Notation 3. Let Γ be the incidence structure having:

- for points the elements of G ,
- for lines the translates of the various conjugates of C ,
- for planes the translates of I ,

and set-theoretic incidence relation (which is transitive).

Notice that G acts on the geometry Γ both on the left and on the right—as opposed to if we had focused on the mere plane of involutions I , where only one copy of G acts (by conjugation).

Claim 4. The action of $\mathbb{G} = G \times G$ on Γ is flag-transitive.

Proof of Claim. Indeed, \mathbb{G} is clearly transitive on planes, and as the stabiliser of the plane I is the diagonal group $\mathbb{G}^\Delta \simeq G$ (acting on I as G does by conjugation), this stabiliser is transitive on the lines of I , which are precisely those of the form $(N \setminus C)^g$. Consequently, the joint stabiliser of I and $(N \setminus C)$ is N in its conjugation action, which is easily seen to be transitive on the points of $(N \setminus C)$ by 2-divisibility of C . \diamond

We also introduce a polarity, which will have the effect of considerably accelerating the proof.

Notation 5. Let $\check{N} = N \setminus C$ and define an operation \perp on Γ as follows:

- if g is a point, let $g^\perp = gI$;
- if $L = gC$ is a line, let $(gC)^\perp = g\check{N}$;
- if gI is a plane, let $(gI)^\perp = g$.

Claim 6. \perp is a well-defined polarity of Γ , compatible with the action of \mathbb{G} .

Proof of Claim. Clearly \perp is well-defined for a point. For a line, it amounts to having $C\check{N} = \check{N}$, an obvious claim. Finally, for a plane: if $gI = hI$ then $x = h^{-1}g$ is strongly real, inverted by all involutions, so every involution normalises C_x , which we take to be G if $x = 1$. In fact, as we now see that $I \cdot I \subseteq N_G(C_x)$, we get $C_x = G$ and $x = 1$.

Of course \perp is involutive on points and planes; on a line $L = gC$, fix k inverting C so that $\check{N} = kC$ and see that $L^{\perp\perp} = (g \cdot kC)^\perp = (gkC)^\perp = gk \cdot kC = gC = L$.

The polarity is clearly left G -covariant. It also is right-covariant: since I is a normal set, only the case of a line could fail to be clear. But by left-covariance, this reduces to proving conjugacy-covariance, which is obvious. It thus remains to check that the polarity preserves incidence. There are three cases to consider:

point-line: by flag-transitivity and \mathbb{G} -covariance, it suffices to prove it for $1 \in C$; now in usual notation, $C^\perp = kC \subseteq I = 1^\perp$;

point-plane: here we may work with $1 \in xI$, meaning that x is an involution; hence $(xI)^\perp = x \in I = 1^\perp$;

line-plane: translating again, suppose $C \subseteq xI$, forcing x to be an involution inverting C ; hence $(xI)^\perp = x \in \tilde{N} = C^\perp$. \diamond

Claim 7. Γ is a projective 3-space.

Proof of Claim. The notion has a variety of equivalent definitions. We adopt the following: a projective 3-space is a transitive point-line-plane incidence structure satisfying the following axioms [Har67].

- (S1) Every pair of distinct points lie on a unique line.
- (S2) Three non-collinear points lie in a unique plane.
- (S3) Every line and plane meet in at least one point.
- (S4) Two planes intersect in at least a line.
- (S5) There exist four non-coplanar points, no three of which are collinear.
- (S6) Every line has at least three points.

First notice the following compatibility property: if $a \neq b$ are two distinct points of a plane Π and L is some line through a and b , then $L \subseteq \Pi$. (This is a consequence of Hartshorne's axioms, but also something we need on our way towards them; notice that we make no claims on uniqueness of L yet.) By flag transitivity, it suffices to deal with $b = 1$ and $a \in C \setminus \{1\}$. Then a plane containing $1 \neq a$ must be of the form kI where k is an involution inverting a ; notice that even if $a = i$ is the involution in C , then $k = i$ is forbidden. So k inverts C , and $L = C \subseteq kI = \Pi$, as claimed.

We turn to proving the axioms. The last two are trivial; Axiom (S1) is exactly $G = \bigsqcup_{G/N} C^g$, since for $x \neq y \in G$ only $x C_{x^{-1}y}$ will contain both. Axiom (S4) is a weakening of (S1) $^\perp$. Axiom (S2) follows from another property.

(T) If a point does not lie on a line, then there is a unique plane containing both.

(Existence in (S2) follows from (T) modulo (S1); uniqueness also requires the ‘‘compatibility’’ above). Finally, Axiom (S3) is a weakening of the dual of (T), which we now prove in our setting.

By flag transitivity, we may suppose that $g \in G$ does not lie on $L = C$. Let $C_g = C_G^\circ(g) \neq C$ and $i_g \neq i$ be the involution in C_g . As we know from Claim 2, there is an involution j commuting to both i and i_g and distinct from both. Then j normalises C and C_g but lies in neither, so it inverts both. In particular, $g \in jI$ and $C \subseteq jI$, so the plane jI meets the requirements. We also contend that this plane is unique. If xI contains C (hence 1), then x is an involution, and if xI also contains g , then it contains i_g since this lies on the line through 1 and g . Thus, x centralises, i and i_g , so uniqueness of the plane amounts to uniqueness of j above, also given by Claim 2. \diamond

Remark. The Veblen-Young axioms for a projective space provide an alternative to Hartshorne's (disappointingly asymmetric) axioms above. To use the former, one needs to show that if a, b, c, d are distinct points for which the line through a and b meets the line through c and d , then the line through a and c meets the line through b and d .

Using the above ‘‘compatibility’’ property together with (T), this is rather trivial. Indeed, we may assume a, b, c are noncollinear, so they lie in a plane, which we may take to be I . By assumption, the line through c and d has two points in I , so the entire line, hence d , is in I . Thus, what we are really showing is that two lines in I must meet, which is just expressing that two distinct involutions commute with a third.

But of course this alternative approach still uses our accurate knowledge of involutions.

Claim 8. Contradiction.

Proof of Claim. G acts regularly by left-translation on the points of Γ . But Γ is a 3-dimensional projective space, hence coordinatisable as $G \simeq \mathbb{P}^3(\mathbb{K})$; now G acts regularly on this, implying $G \leq \text{Aut}(\mathbb{P}^3(\mathbb{K})) \simeq \text{PGL}_4(\mathbb{K}) \rtimes \text{Aut}(\mathbb{K})$. Of course \mathbb{K} is algebraically closed since Desargues field interpretation is definable (something obvious with Hilbert’s proof [Hil99], but harder to see with Artin’s method [Art57]—unless one understands Hrushovski’s group configuration theorem); since there are no definable groups of field automorphisms in our ranked context [BN94, Theorem 8.3], one finally finds $G \leq \text{PGL}_4(\mathbb{K})$.

Consider the Zariski-closure \overline{C} of C inside $\text{PGL}_4(\mathbb{K})$; being closed, connected, and soluble (the latter can be seen as an instance of [BN94, Corollary 5.38]), it has a fixed point in its action on the complete variety $\mathbb{P}^3(\mathbb{K})$ [Hum75, Theorem 21.2]. Hence so does $C \leq \overline{C}$: violating regularity. \diamond

Theorem A and Corollaries A.1 and A.2 (the latter, modulo local analysis) as well are now proved. This we hope will deter our colleagues from ill-advisedly invoking Bachmann’s Theorem in the future. \square

We wish to make some more comments on the final contradiction.

Remarks.

- One does not need the full strength of Borel’s theorem: since C is abelian and the action of $\text{PGL}_4(\mathbb{K})$ on $\mathbb{P}^3(\mathbb{K})$ is a factor of the natural representation of $\text{GL}_4(\mathbb{K})$, we may trigonalize simultaneously and find a fixed point for the preimage of C in $\text{GL}_4(\mathbb{K})$. So all we use geometrically seems to be the 2-nilpotent version of the Lie-Kolchin theorem.
- It is well-known that Hilbert interpretation does not rely on Chevalley-Zilber indecomposable generation (parenthetically said, and contrary to widespread belief, Schur-Zilber field interpretation does not either: see [DW18]), so definable coordinatisation would carry to other model-theoretic settings, say finite-dimensional theories.
- On the other hand, to find a fixed point one *does* need algebraic closedness of \mathbb{K} , which is typical of \aleph_1 -categorical behaviour. And indeed, real closed fields do not satisfy Borel’s theorem; and indeed, $\text{SO}_3(\mathbb{R})$ does exist.

In the \mathfrak{o} -minimal case, one may argue in favor of using Bachmann’s theorem in order to explicitly identify a real form of PGL_2 ; but not in finite Morley rank in order to prove a contradiction.

3 The B-Sides

Here is the other theorem we announced in the introduction. We remind the reader that *hereditarily conjugate* means that every definable connected subgroup of G enjoys Borel conjugacy, viz. conjugacy of its maximal definable, connected, soluble subgroups.

Theorem B. *Let G be a connected, U_2^\perp , ranked group in which all Borel subgroups are nilpotent and hereditarily conjugate. Then the (possibly trivial) 2-torsion of G is central.*

We shall first prove Theorem B, then its Corollaries B.1 and B.2, which will be recalled hereafter.

3.1 Proof of Theorem B

Proof. Notice that any definable, connected subgroup of G , or any quotient by a definable, normal, soluble subgroup of G still satisfies the assumptions.

Claim 1. We may assume that G is centreless.

Proof of Claim. Suppose that $Z(G)$ is finite, so that $G/Z(G)$ is centreless [BN94, Lemma 6.1]. Suppose in addition that the result holds of $G/Z(G)$: being centreless it has no involutions, meaning that 2-torsion of G lies in $Z(G)$, as desired.

So now suppose that $Z(G)$ is infinite: then working inductively, the 2-torsion of $G/Z(G)$ is central, meaning that the 2-torsion of G is in $Z_2(G)$. Hence the maximal 2-torus of G is normal in

G , hence central [BN94, Theorem 6.16]; this means that $G/Z(G)$ has no non-trivial 2-torus, which by [BBC07, Theorem 1] implies that it actually has no involutions. Therefore all the 2-torsion of G was in $Z(G)$. \diamond

Suppose that G contains involutions; let $T > 1$ be a maximal 2-torus of G , and $C = C(T)$ be its centraliser, which we know by [AB08, Theorem 1] to be connected. Also let $N = N(T) = N(C(T))$ be its normaliser, a finite extension of C .

Claim 2. C is a TI-subgroup of G .

Proof of Claim. Let C_1, C_2 be conjugates of C meeting in $x \neq 1$. Then $H = C_G^\circ(x)$ is a definable, connected, proper subgroup of G ; by induction, its 2-torsion is central. Since it contains both T_1 and T_2 , this forces $T_1 = T_2$ and $C_1 = C_2$. \diamond

Claim 3. If $x \in G$ is a strongly real element then $x^2 \in \bigcup_G C^g$.

Proof of Claim. Say $x = ij$ for distinct (possibly commuting) involutions i, j . Let $H = C_G^\circ(x)$, a definable, connected, proper subgroup. If i inverts H , then let $A = H$; if not, invoke Reineke's Theorem [BN94, Corollary 6.5] and let A be any infinite, definable, connected, subgroup of $C_H^\circ(i) > 1$. In either case A is an infinite, abelian, $\langle x, i \rangle$ -invariant group. We contend that A is contained in a unique conjugate of C .

If we fix one Borel subgroup B containing T , then by nilpotence $B \leq C$; by conjugacy, all Borel subgroups are contained in conjugates of C (equality is not accessible a priori, since there are no assumptions on the inner structure of C). But A certainly extends to a Borel subgroup, hence also to a conjugate of C .

Hence we may suppose that $A \leq C$; as the latter is TI, it also is $\langle x, i \rangle$ -invariant. Now there are two cases.

- Suppose that i inverts C . Let $t \in T \leq C$ and compute:

$$t^x = t^{-xi} = t^{-ix^{-1}} = t^{x^{-1}},$$

so that x^2 centralises t , and $x^2 \in C(T) = C$.

- Suppose not. As there is a 2-torus T_i containing i , with centraliser C_i , we see that $C \cap C_i \neq 1$, so by disjunction $C = C_i$. This means that $i \in T$. Bear in mind that x normalises T , so that:

$$x^{-1} = x^i = ixi = xi^x i \in xT.$$

This time $x^2 \in T \leq C$.

In either case, x^2 lies in C . \diamond

Claim 4. If $x \in G$ is a strongly real element, then $x \in \bigcup_G C^g$.

Proof of Claim. If x is an involution we are done. We may suppose that $1 \neq x^2 \in C$, and in particular $x \in N$. If N/C has odd order then we are done. Now suppose that N/C has even order, so Proposition 1 applies. Write the smallest definable subgroup containing x as $\langle x \rangle_{\text{def}} = \langle z \rangle_{\text{def}} \oplus \langle \alpha \rangle$ with $x = z\alpha$ so that $\langle z \rangle_{\text{def}}$ is a 2-divisible group and α a 2-element. Observe how z normalises C . Since N/C has order 2, we find $z \in C$. If $\alpha \in C$ we are done. Otherwise $\alpha \in N \setminus C$ forces α to be an involution inverting C , hence also z . So $x = z\alpha$ is an involution, which proves $C_x = C_\alpha$. \diamond

Theorem A now provides the desired contradiction. This completes the proof of Theorem B. \square

3.2 Corollaries of Theorem B

Corollary B.1. *Let \mathbb{K} be a ranked field of characteristic 0 and $G \leq \mathrm{GL}_n(\mathbb{K})$ be a simple, definable, non Zariski-closed subgroup. Then G has no involutions.*

Proof. Clearly G is U_2^\perp . By [Poi01, Théorème 3], Borel subgroups are abelian (hence nilpotent). Moreover, Mustafin [Mus04, Proposition 2.11] proved conjugacy of the Borel subgroups in all definably linear groups in the ranked setting, so the property is hereditary. Hence we are under the assumptions of Theorem B; conclude by simplicity. \square

Corollary B.2. *Let G be a connected ranked group in which all Borel subgroups are good tori. Then the (possibly trivial) 2-torsion of G is central.*

Proof. Here again G is necessarily U_2^\perp ; moreover, Borel subgroups are maximal good tori, hence abelian and conjugate. And since any definable, connected subgroup of a good torus is still one, the property is hereditary. \square

Remarks.

- It is unclear to us what would happen to Corollary B.2 with decent tori instead of good tori since hereditary properties are then lost. The situation is even worse with Cartan subgroups (i.e. centralisers of maximal decent tori).
- Despite vague attempts, we could not prove the following.

Conjecture (Borovik-Burdges [BB08, Conjecture 1]). *There is no simple ranked group in which all strongly real elements lie in $\bigcup_G C^g$ for C a Cartan subgroup.*

This does not seem directly related to the A_1 -conjecture, which we hope to return to shortly.

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