HAL
open science

# Locally quadratic modules and minuscule representations

Adrien Deloro

## To cite this version:

Adrien Deloro. Locally quadratic modules and minuscule representations. Algebras and Representation Theory, 2017, 20 (4), pp.977-1005. 10.1007/s10468-017-9671-6 . hal-01989789

**HAL Id: hal-01989789**

**https://hal.science/hal-01989789v1**

Submitted on 22 Jan 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Locally quadratic modules and minuscule representations

Adrien Deloro

27th January 2017

*Being natural is simply a pose, and the most irritating pose I know.*

**Abstract**

We give a new, geometric proof of a theorem by Timmesfeld showing that for simple Chevalley groups, abstract modules where all roots act quadratically are direct sums of minuscule representations. Our proof is uniform, treats finite and infinite fields on an equal footing, and includes Lie rings.

The present article deals with Chevalley groups over arbitrary fields, the attached Lie rings, and some of their representations seen as abstract modules. By attached Lie ring we mean the $(+;[\cdot,\cdot])$-structure one gets when forgetting the linear structure from the Lie algebra; thanks to Chevalley bases, the Lie ring of a Chevalley group over an arbitrary field makes sense.

In order to handle modules over groups and Lie rings in a single statement one needs a bit of terminology and notation. If $V$ is any abelian group, then $\mathrm{End}(V)$ is naturally a Lie ring; now if $\mathfrak{g}$ is any Lie ring, a $\mathfrak{g}$-module structure on $V$ is simply a Lie ring homomorphism $\mathfrak{g} \to \mathrm{End}(V)$. As it is compatible with the abelian group structure, $V$ is actually a $(\mathfrak{g},\mathbb{Z})$-bimodule. Slightly abusing notation for a convenience dictated by analogy with the group case, we shall call $V$ a $\mathbb{Z}[\mathfrak{g}]$-module (although we define no ring denoted by $\mathbb{Z}[\mathfrak{g}]$, in contrast to the group case). So the phrase: "$V$ is a $\mathbb{Z}[\mathfrak{g}]$-module" merely means that $V$ is an abelian group acted on by the Lie ring $\mathfrak{g}$. Likewise, we call $\mathbb{K}[\mathfrak{g}]$-module any $\mathbb{K}$-vector space with a linear $\mathfrak{g}$-action; if $\mathfrak{g}$ happens to be a Lie $\mathbb{K}$-algebra this is the same thing as a representation of $\mathfrak{g}$ as a Lie $\mathbb{K}$-algebra.

**Notation.**

- *If $G$ is a group and $V$ is a $\mathbb{Z}[G]$-module, let $Z_V(G) = C_V(G) = \{v \in V : \forall g \in G, g \cdot v = v\}$ and $\lfloor G, V \rfloor = [G,V] = \langle g \cdot v - v : (g,v) \in G \times V\rangle$;*
- *if $\mathfrak{g}$ is a Lie ring and $V$ is a $\mathbb{Z}[\mathfrak{g}]$-module, let $Z_V(\mathfrak{g}) = \mathrm{Ann}_V(\mathfrak{g}) = \{v \in V : \forall z \in \mathfrak{g}, z \cdot v = 0\}$ and $\lfloor \mathfrak{g}, V \rfloor = \mathfrak{g} \cdot V = \langle z \cdot v : (z,v) \in \mathfrak{g} \times V\rangle$.*

**Main Theorem** (cf. [14]: see §1.4 below)**.** *Let $\mathbb{K}$ be a field of characteristic $\neq 2$ with more than three elements and $\mathbb{G}$ be one of the simple algebraic groups (of classical or exceptional type; untwisted). Let $G = \mathbb{G}_{\mathbb{K}}$ be the abstract group of $\mathbb{K}$-points of the functor $\mathbb{G}$ and $\mathfrak{g} = (\mathrm{Lie}\,\mathbb{G})_{\mathbb{K}}$ be the abstract Lie ring of $\mathbb{K}$-points of the functor $\mathrm{Lie}\,\mathbb{G}$. Let $\mathcal{G}$ be either $G$ or $\mathfrak{g}$ and $V$ be a $\mathbb{Z}[\mathcal{G}]$-module.*

*Suppose that all roots act quadratically. Then $V = Z_V(\mathcal{G}) \oplus \lfloor \mathcal{G}, V \rfloor$ and $\lfloor \mathcal{G}, V \rfloor$ can be equipped with a $\mathbb{K}$-vector space structure making it isomorphic to a direct sum of minuscule representations of $\mathcal{G}$ as a $\mathbb{K}[\mathcal{G}]$-module.*

**Remark.** *The Main Theorem equips a certain $\mathbb{Z}[\mathcal{G}]$-module with a compatible $\mathbb{K}$-linear structure such that the resulting $\mathbb{K}[\mathcal{G}]$-module structure is well-understood. But if $V$ is already given as a $\mathbb{K}[\mathcal{G}]$-module, then the compatible $\mathbb{K}$-linear structure we construct need not coincide with the given one, and one would then see* two *rival $\mathbb{K}$-linear structures on the same $\mathbb{Z}[\mathcal{G}]$-module.*

*For instance one could start with $V = {}^{\varphi}M$, the "twisted" version of a minuscule representation $M$ by some non-trivial base field automorphism $\varphi$. Then $V$ and $M$ are not isomorphic as*

*Institut de Mathématiques de Jussieu – Paris Rive Gauche; Université Pierre et Marie Curie ; 4, place Jussieu 75005 France

†adrien.deloro@imj-prg.fr; +33144279222

*$\mathbb{K}[\mathcal{G}]$-modules. But there is, on the underlying $\mathbb{Z}[\mathcal{G}]$-module $\check{V}$ obtained from $V$ by forgetting $\mathbb{K}$, a compatible $\mathbb{K}$-linear structure $\hat{\check{V}}$ constructed by the Main Theorem, and for which $\hat{\check{V}} \simeq M$ as $\mathbb{K}[\mathcal{G}]$-modules. (Of course $V$ and $\hat{\check{V}}$ are only $\mathbb{Z}[\mathcal{G}]$-isomorphic.)*

We view the result as a first instance of a more general "theorem-template" one should investigate systematically. We comment on that in §1.1 of the introduction hereafter. Then we move to explaining some notions: quadraticity in §1.2 and minuscule representations in §1.3. Our result and method are compared to those of Timmesfeld [14] in §1.4, then some further remarks are sketched in §1.5. The proof is in §2, which will begin with an overview of the argument.

We thank Patrick Polo for introducing us to the elegance of root data, and a first referee for very accurate comments, in particular on finite groups.

# 1 Introduction

Before we start mentioning Curtis-Phan-Tits Theorems and quadratic pairs, we must warn the reader: the present article is *not* in finite group theory. This will become clear when we discuss method in §1.4.

## 1.1 Local-to-Global results

There is in algebra a wide class of *local-to-global* identification results which determine the isomorphism type of an algebraic structure $\mathcal{A}$ from a collection of substructures $(\mathcal{A}_\alpha)_{\alpha\in\Psi}$ such that:

- the isomorphism type of the "atoms" $\mathcal{A}_\alpha$ is known;
- the "chemical bonds", i.e. the substructures generated by pairs $\langle \mathcal{A}_\alpha, \mathcal{A}_\beta\rangle$, have known isomorphism type.

The best example is given by the celebrated Curtis-Phan-Tits theorem(s) identifying a Chevalley group from a collection of subgroups of type $(\mathrm{P})\mathrm{SL}_2(\mathbb{K})$ provided these pairwise generate what they should according to the expected Dynkin diagram. See [13] for a powerful form, and [8] for a thorough account of all existing group-theoretic versions. (Curtis-Phan-Tits theorems themselves are not used anywhere in the present paper; they serve as an inspiration and an analogy.)

We wish to suggest that similar results should exist beyond the classical topic of group identification; as a matter of fact our Main Theorem resembles such a result for *representations*. It is not literally of the above type as the acting structure is already supposed to be Chevalley, viz. known in terms of atoms and bonds; what we identify is the acted module, under assumptions of a local nature. Namely, given an algebraic structure $\mathcal{G}$ which is either a simple (untwisted) Chevalley group $G$ with root $\mathrm{SL}_2$-subgroups $G_\alpha$ or the associated Lie ring $\mathfrak{g}$ with root $\mathfrak{sl}_2$-subrings $\mathfrak{g}_\alpha$, and an abstract $\mathbb{Z}[\mathcal{G}]$-module $V$, we identify *$V$ as a $\mathcal{G}$-module* under assumptions on *$V$ as a $\mathcal{G}_\alpha$-module* for the various roots $\alpha$. We treat only the simplest possible case where all root substructures act quadratically, i.e. where each $\mathcal{G}_\alpha$ essentially sees sums of trivial spaces and natural representations (the equivalence is explained in §1.2).

The resulting statement is — in the sole case where $\mathcal{G} = G$ is a group — already in [14]; we give a completely different proof which is not group-theoretic but more deeply structural, at the level of the root system, and thus remains valid for the associated Lie ring $\mathfrak{g}$.

Quite interestingly (but parenthetically as it is not our present concern), Curtis-Phan-Tits theorems do not seem to have been systematically investigated for Lie rings. It would be nice to have a result of the form: "if $\mathfrak{g}$ is a Lie ring with subrings $\mathfrak{g}_\alpha \simeq \mathfrak{sl}_2(\mathbb{K})$ generating what they should according to the Dynkin diagram $\Delta$, then $\mathfrak{g} \simeq (\mathrm{Lie}\,\mathbb{G})_\mathbb{K}$ for $\mathbb{G}$ the group with Dynkin diagram $\Delta$". We do not know of any such statement in the literature. The reason might be that abstract Lie rings deserved little attention, and that in the usual case of finite-dimensional Lie algebras the adjoint action is much handier a tool.

We mentioned the latter question only to demonstrate that the topic of local-to-global theorems goes well beyond abstract group identification results.

## 1.2 Quadratic Actions

Recalling our original inspiration will serve a double purpose: offering a convenient introduction to minuscule modules, and quoting the only prerequisites needed for the proof besides familiarity with Chevalley groups. It is the following theorem which was proved in the mid-eighties by S. Smith and F.G. Timmesfeld, independently. $U$ stands for a unipotent subgroup of $\mathrm{SL}_2(\mathbb{K})$, say the group of upper-triangular matrices with 1 on the diagonal; quadraticity of the $G$-module $V$ means that $[U, U, V] = 0$ (which does not depend on the unipotent subgroup by conjugacy).

**Timmesfeld's Quadratic Theorem** ([12, Exercise 3.8.1 of chapter I]; also [9])**.** *Let $\mathbb{K}$ be a field of characteristic $\neq 2$ with more than three elements, $G = \mathrm{SL}_2(\mathbb{K})$, and $V$ be a* quadratic *$G$-module. Then $V = C_V(G) \oplus [G, V]$, and there exists a $\mathbb{K}$-vector space structure on $[G, V]$ making it isomorphic to a direct sum of copies of* $\mathrm{Nat}\,\mathrm{SL}_2(\mathbb{K})$ *as a $\mathbb{K}[G]$-module.*

Since the field $\mathbb{K}$ is rather arbitrary, there are no character-theoretic nor Lie-theoretic methods available; $\mathrm{SL}_2(\mathbb{K})$ is seen as an abstract group with no extra structure, and the proof is therefore by computation. One fixes generators and works with the so-called Steinberg relations for $\mathrm{SL}_2(\mathbb{K})$.

The lack of Lie-theoretic information incidently suggests to ask the same question about $\mathfrak{sl}_2(\mathbb{K})$-modules. For the problem of linear reconstruction to make sense we view $\mathfrak{sl}_2(\mathbb{K})$ as a Lie ring, viz. an abelian group with a bracket (forgetting the underlying vector space structure); an $\mathfrak{sl}_2(\mathbb{K})$-module need not be a vector space over $\mathbb{K}$. We let $\mathfrak{u}_+$ and $\mathfrak{u}_-$ be the abelian subrings of upper-triangular, resp. lower-triangular, matrices with 0 on the diagonal. Quadraticity of the $\mathfrak{g}$-module $V$ now means that both $\mathfrak{u}_+^2 \cdot V = \mathfrak{u}_-^2 \cdot V = 0$ (see §1.5.1 for more on this two-sided assumption).

**Lie-ring analogue** ([4])**.** *Let $\mathbb{K}$ be a field of characteristic $\neq 2$, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{K})$, and $V$ be a quadratic $\mathfrak{g}$-module. Then $V = \mathrm{Ann}_V(\mathfrak{g}) \oplus \mathfrak{g} \cdot V$, and there exists a $\mathbb{K}$-vector space structure on $\mathfrak{g} \cdot V$ making it isomorphic to a direct sum of copies of* $\mathrm{Nat}\,\mathfrak{sl}_2(\mathbb{K})$ *as a $\mathbb{K}[\mathfrak{g}]$-module.*

We have two comments.

- Quadraticity is fundamental in our proof to get the analysis started (§2.2), but no more. The classical topic of quadratic pairs in the sense of Thompson [11] plays absolutely no role here.
- Results similar to the Quadratic Theorem and its Lie-ring analogue can be proved relaxing the quadraticity assumption to higher unipotence (resp., nilpotence) length of $U$ (resp., $\mathfrak{u}_\pm$) [5, 6]. This goes smoothly in the case of the Lie ring [5]. In the case of the group [6] the length cannot exceed 5, and computations are much more unpleasant than for the Quadratic Theorem.

  Existence of such results suggests that one ought to look for a generalisation of our Main Theorem to the case where root substructures act with length 3, using similar methods but now taking [5, 6] as the base case analysis. We shall return to this in §1.5.4.

## 1.3 Minuscule Representations

No deep understanding of what "minuscule" means is required to read the article. According to one of the possible equivalent definitions [2, Chap. VIII, §7.3, Definition 1], the minuscule representations of a semisimple Lie algebra are its irreducible representations such that the action of the Weyl group on the set of weights is transitive; the latter condition is equivalent to: every root element acts with $x^2 = 0$ [2, Chap. VIII, §7.3, Propositions 6 and 7].

In the simple case, the list of the minuscule weights can be determined from that of the fundamental weights, and the minuscule representations of the various simple Lie algebras are therefore known. They are as follows: all exterior powers of the natural representation for type $A_n$, the spin representation for type $B_n$, the natural representation for type $C_n$, the natural and the two half-spin representations for type $D_n$, two representations for type $E_6$, one for type $E_7$, none for types $E_8$, $F_4$, $G_2$ [2, Chap. VIII, end of §7.3]. (The contents of this list can be forgotten; what matters is that the list exists.)

It is tempting to see Timmesfeld's Quadratic Theorem and its Lie-ring analogue as identification results for the unique minuscule representation of the algebraic group $\mathbb{SL}_2$ among abstract $G$- or $\mathfrak{g}$-modules. And indeed, the statement is the natural extension of Timmesfeld's quadratic theorem to the other simple algebraic groups and their Lie algebras seen as Lie rings.

## 1.4 The method

We mentioned that Timmesfeld has already obtained a result similar to ours.

**Fact** (Timmesfeld, [14])**.** *Let $G$ be a finite Lie-type group over $GF(q)$, $q = p^n$, $p \neq 2$, different from* $\mathrm{SL}_2(3)$*, with Dynkin diagram $\Delta = \Delta(I)$ and let $V$ a $\mathbb{Z}_pG$-module, on which the root groups of $G$ act quadratically, i.e. $[V, A_r, A_r] = 0$ for all roots $r$ of the root system of $G$. Then $V = C_V(G) \oplus [V, G]$ and $[V, G]$ is the direct sum of irreducible $\mathbb{Z}_pG$-modules $V_j$* [the list of which is as expected and explicitly given].

No statement is stronger than the other and we highlight a few differences. Despite his version being given for finite groups, Timmesfeld states that in the case of classical groups his proof applies to infinite fields of characteristic not 2 (see [14, p.36]). However his treatment of $E_6(\mathbb{K})$ and $E_7(\mathbb{K})$ does require finiteness of $\mathbb{K}$. He also deals with the field with three elements more thoroughly than we do (see §1.5.2 for a discussion of what our argument achieves in this case). And he handles twisted groups, notably $\mathrm{SU}_n$. We do not do the twist but the main reason is our lack of expertise of such groups; perhaps our method applies to these as well.

On the other hand, we have no restriction on the isomorphism type of $\mathbb{G}$ if $\mathbb{K}$ is infinite. We can also treat Lie rings by the same argument, and this is by no means a corollary to the group case since we fall short of any form of Lie correspondence in the present abstract setting.

Beyond statements, we wish to emphasize that our method seems robust and hopefully general as it takes place at a more geometric level. Our proof is indeed:

- group-independent, while [14] is a case division (we shall return to this in §1.5.3);
- completely field-independent as we said (characteristic 2 and $\mathbb{F}_3$ left aside);
- representation-independent. Our method linearises without caring for what the resulting representation will be (something determined afterwards), while the philosophy of [14] involves explicit module identification. In particular, complete reducibility is to us a late and virtually trivial by-product of the analysis;
- entirely self-contained modulo the Quadratic Theorem and its Lie-ring analogue (§1.2), while [14] requires non-trivial representation-theoretic information ([14, end of §1]; [14, Lemma 2.6] is one crux of the argument);
- effective since the linear structure is defined explicitly provided one has realised one root substructure of type $A_1$ and the global Weyl group;
- transparent — in our opinion. It is sometimes hard to tell which arguments on finite groups of Lie type come from finiteness miracles, and which are just consequences of general Chevalley theory in the spirit of Steinberg [10].

  But our proof obviously takes place at the general Chevalley level. It is about minuscule weights and transitivity of the Weyl group, which are the natural phenomena to investigate when one is talking about minuscule representations. The structure of the argument is explained at the beginning of §2.

Yet our proof has one thing in common with Timmesfeld's: it relies on the action of central involutions in root $\mathrm{SL}_2$-subgroups: this will be obvious in the proofs of Propositions 9 and 11. As a consequence it is essential for us to work in characteristic not 2. We do not know how to dispense with these involutions and it is not clear whether their role is that of mere accelerators or more essential. The latter question makes sense for art's sake but there is in any case no hope to extend the Main Theorem to characteristic 2 (the Lie ring being of course left aside), since complete reducibility for quadratic $\mathrm{SL}_2(\mathbb{K})$-modules fails in characteristic 2: as pointed out by a first referee, the action of a copy of $\mathrm{SL}_2(\mathbb{K})$ on the unipotent radical inside a parabolic subgroup of $\mathrm{Sp}_4(\mathbb{K})$ in characteristic 2 is not completely reducible. Hence the very cornerstone of the analysis, Proposition 6, fails in that case.

## 1.5 Remarks and Questions

Our last introductory subsection consists of remarks on the statement and its proof, together with a few questions on possible extensions. None is necessary in order to understand the proof in §2.

### 1.5.1 The Quadraticity Assumption

Our interpretation of "all roots act quadratically" is that all root $\mathbb{SL}_2$-substructures in one realisation of $\mathcal{G}$ act quadratically. This was explained in §1.2. For $\alpha$ a root let $\mathcal{U}_\alpha$ be the associated 1-parameter root substructure (i.e., root subgroup $U_\alpha$ or root Lie subring $\mathfrak{u}_\alpha$ isomorphic to $\mathbb{K}_+$; the reader with a doubt will find all notation in §2.2 of the proof).

1. As stated the assumption means: for $\alpha \in \Phi$ (the root system), $\lfloor \mathcal{U}_\alpha, \lfloor \mathcal{U}_\alpha, V \rfloor \rfloor = 0$.
2. One may try to restrict to positive roots: for $\alpha \in \Phi_+$ (positive roots), $\lfloor \mathcal{U}_\alpha, \lfloor \mathcal{U}_\alpha, V \rfloor \rfloor = 0$.
3. One may try to restrict to simple roots: for $\alpha \in \Phi_s$ (simple roots), $\lfloor \mathcal{U}_\alpha, \lfloor \mathcal{U}_\alpha, V \rfloor \rfloor = 0$.
4. One may try to restrict to root elements: for $\alpha \in \Phi$ (resp. $\Phi_+$, $\Phi_s$) and some $y_\alpha \in \mathcal{U}_\alpha$ not the identity, $\lfloor y_\alpha, \lfloor y_\alpha, V \rfloor \rfloor = 0$.

These slight variations can have unexpected effects.

- For instance, supposing that the positive Lie subring $\mathfrak{u}_+ \leq \mathfrak{sl}_2(\mathbb{K})$ acts quadratically does not fully guarantee that so does the negative Lie subring $\mathfrak{u}_-$. In characteristic neither 2 nor 3 these turn out to be equivalent [4, Variation 12] but in characteristic 3 one can construct $\mathfrak{sl}_2(\mathbb{K})$-modules with $\mathfrak{u}_+^2 \cdot V = 0 \neq \mathfrak{u}_-^2 \cdot V$ [4, §4.3]. (For a more general discussion of the non-equality of nilpotence orders of generators of $\mathfrak{u}_+$ and $\mathfrak{u}_-$ in $\mathrm{End}(V)$, see [5, §§3.2 and 3.3].) As a consequence, an assumption restricted to positive roots is too weak in the case of the Lie ring.
- The reader is now aware that in the case of the Lie ring, lifting the action of the Weyl group on roots to an action on the module is non-trivial. The caveat extends to Lie rings not of type $A_1$. It is therefore not clear whether all roots of the same length must have similar actions on $V$ since conjugacy under the Weyl group may fail to be compatible with the action on the module.

  So in the case of the Lie ring, even an assumption restricted to simple roots and their opposites could be too weak.
- Finally, it is the case that for an action of $\mathrm{SL}_2(\mathbb{K})$ in characteristic neither 2 nor 3, $(u-1)^2 = 0$ for some element $u \in U \setminus \{1\}$ implies $[U, U, V] = 0$ [4, Variation 7]; we do not know what happens in characteristic 3 (bear in mind that the field can be infinite; the "$\mathrm{SL}_2$-lemma" [12, V.1.12] requires some finite-dimensionality and there is no such assumption here).

  For an action of $\mathfrak{sl}_2(\mathbb{K})$ it suffices to be in characteristic not 2: $x^2 \cdot V = 0$ for some $x \in \mathfrak{u}_+ \setminus \{0\}$ does imply $\mathfrak{u}_+^2 \cdot V = 0$ [4, Variation 9] (but in characteristic 3 the latter does however not entail $\mathfrak{u}_-^2 \cdot V = 0$ as we just said). Hence an assumption restricted to root elements is too weak.

There are two conclusions. First, in the case of the group and characteristic not 3, it would be enough to suppose that one element in one root subgroup of each length is quadratic — which makes an assumption on at most two elements. Second, in the case of Lie rings, apparently minor changes in the hypothesis can give rise to pathologies. Here is our only positive claim.

**Corollary.** *Let $\mathbb{K}$ be a field of characteristic $\neq 2$ with more than three elements and $\mathbb{G}$ be one of the simple algebraic groups of type $A - D - E$* but not *$A_1$. Let $\mathfrak{g} = (\mathrm{Lie}\,\mathbb{G})_\mathbb{K}$ be the abstract Lie ring of $\mathbb{K}$-points of the functor* $\mathrm{Lie}\,\mathbb{G}$. *Let $V$ be a $\mathbb{Z}[\mathfrak{g}]$-module.*

*Suppose that one root element acts quadratically. Then $V$ is as in the Main Theorem.*

*Proof.* It suffices to prove that all root subrings act quadratically, and then apply the Main Theorem; it suffices to do it for $\mathbb{G}$ of type $A_2$, i.e. $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{K})$ with bracket denoted $[\![\cdot,\cdot]\!]$. Recall that one may not apply Weyl reflections, so a little computation is required.

However in type $A_2$ it suffices to show that if one root acts quadratically, then any root adjacent to it (i.e. with angle $\pm\frac{\pi}{3}$) acts quadratically too. Say that the root system is generated by $\alpha$ and $\beta$ with angle $\frac{2\pi}{3}$ and suppose that $\mathfrak{u}_\alpha$ acts quadratically; we shall prove that $\mathfrak{u}_{\alpha+\beta}$ does too.

We may fix generators $x_\gamma$ of the various root subrings $\mathfrak{u}_\gamma$ ($\gamma$ any root in the system) in such a way that $[\![x_\alpha, x_\beta]\!] = x_{\alpha+\beta}$. For any root $\gamma$ let $x'_\gamma = \frac{1}{2}x_\gamma \in \mathfrak{g}$, which makes sense since $\mathfrak{g}$ is a $\mathbb{K}$-vector space.

Our assumption is that $x_\alpha^2 = 0$ in $\mathrm{End}(V)$. In $\mathrm{End}(V)$ one sees:

$$x_\alpha x'_{\alpha+\beta} = x_\alpha [\![x_\alpha, x'_\beta]\!] = -x_\alpha x'_\beta x_\alpha = -[\![x_\alpha, x'_\beta]\!] x_\alpha = -x'_{\alpha+\beta} x_\alpha = -x_\alpha x'_{\alpha+\beta}$$

Hence $x_\alpha x_{\alpha+\beta} = 2x_\alpha x'_{\alpha+\beta} = 0$.

Then always in $\mathrm{End}(V)$:

$$x_{\alpha+\beta}^2 = [\![x_\alpha, x'_\beta]\!] x_{\alpha+\beta} = x_\alpha x'_\beta x_{\alpha+\beta} = x_\alpha x_{\alpha+\beta} x'_\beta = 0$$

So $x_{\alpha+\beta}^2 = 0$, which suffices as noted. □

The final word belongs to Timmesfeld, who has a nice Corollary [14, p.36] on *finite groups* of type $A-D-E$: to be under the assumptions of the Main Theorem it suffices that some $g \in G \setminus \{1\}$ (not assumed to be unipotent) acts quadratically. But the method clearly belongs to finite group theory and even involves non-trivial external material on quadratic pairs in characteristic 3.

### 1.5.2 The Field with Three Elements

It has been observed by a first referee that over $\mathbb{F}_3$ (where perfectness of $G_\alpha \simeq \mathrm{SL}_2(\mathbb{K})$ and the version of Timmesfeld's Quadratic Theorem we quoted in §1.2 fail; notice that the Lie-ring analogue remains however true), our method immediately yields the following variation.

**Corollary.** Let $\mathbb{K} = \mathbb{F}_3$ be the field with three elements *and* $\mathbb{G}$ *be one of the simple algebraic groups (of classical or exceptional type; untwisted). Let* $G = \mathbb{G}_{\mathbb{F}_3}$ *be the abstract group of* $\mathbb{F}_3$*-points of the functor* $\mathbb{G}$ *and* $\mathfrak{g} = (\mathrm{Lie}\,\mathbb{G})_{\mathbb{F}_3}$ *be the abstract Lie ring of* $\mathbb{F}_3$*-points of the functor* $\mathrm{Lie}\,\mathbb{G}$*. Let* $\mathcal{G}$ *be either* $G$ *or* $\mathfrak{g}$ *and* $V$ *be a* $\mathbb{Z}[\mathcal{G}]$*-module.*

*Suppose that all roots act quadratically.* If $\mathcal{G} = G$ is a group, suppose in addition that fundamental root (P)$\mathrm{SL}_2$-subgroups satisfy the conclusion of Timmesfeld's Quadratic Theorem. *Then the conclusion of the Main Theorem holds.*

The proof is exactly that of our Main Theorem: the hypothesis simply shortcuts the "local analysis", Proposition 6 of §2.2 in the problematic case. The rest of our argument does not require perfectness of $\mathrm{SL}_2(\mathbb{K})$ and can be kept without a change.

Interestingly, Timmesfeld in [14, §3] handles the case where $\mathbb{K} = \mathbb{F}_3$ without our extra assumption, but using specific computations: *over* $\mathbb{F}_3$ *and in Lie rank not* 1*, the additional requirement is not needed in the Corollary.* But this is more involved again; we have nothing better to say on the topic than Timmesfeld. In any case our focus is not on finite groups.

### 1.5.3 Towards More Groups

As we said our linearisation argument goes uniformly and does not require any form of module identification, so knowing the precise isomorphism type of $\mathbb{G}$ is never necessary. A careful reader will wonder how much information is really needed. The next few words are of a speculative nature.

- It is not clear whether one can avoid assuming simplicity of $\mathbb{G}$.

  Of course reductivity looks like a minimal requirement, but even at the semi-simple level things are not obvious. Say for instance $G = G_1 \times G_2$ acts on $V$ with quadratic roots. The method gives *two* linear structures on $V$, conveniently thought of as actions of isomorphic fields $\mathbb{K}_i$ (attached to $G_i$). The $\mathbb{K}_i$-structure from our proof is given by the action of the torus of $G_i$; now $G_{i+1}$ commutes $G_i$, so acts $\mathbb{K}_i$-linearly.

These linear structures need not coincide, so it is not clear at all whether a third, better linear structure on $V$ exists for which one would analyse further $V$ in terms of tensor products of minuscule representations of $G_1$ and $G_2$. It is a mystery to us; but for one thing, it is not in the scope of the method.

The situation can get even more confusing. Our arguments would apply equally to $G_1 = \mathbb{G}_1(\mathbb{K}_1)$ and $G_2 = \mathbb{G}_2(\mathbb{K}_2)$ over different fields of the same characteristic, a case in which nothing decent can be conjectured.

- The crux of the method is generation by root substructures. An infinite set of roots à la Kac-Moody could be a further direction to explore; we lack knowledge on the topic.

### 1.5.4 Towards More Modules

A more concrete question of interest is the following. Can one use the same or a similar method to identify other representations of the simple Chevalley groups? For instance, so-called quasi-minuscule representations are defined by the clause that the Weyl group is transitive in its action on *non-zero* weights. Can one characterise quasi-minuscule representations among abstract modules?

Being suddenly modest — can one, using the same method, characterise the adjoint action of a structure of type $A_2$? The following needs some notation from §2.2, namely Notation 5.

**Question.** *Let $\mathbb{K}$ be a field of characteristic $\neq 2, 3$; let $G = \mathrm{SL}_3(\mathbb{K})$ and $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{K})$; let $\mathcal{G} = G$ or $\mathfrak{g}$; let $V$ be a* simple *$\mathbb{Z}[\mathcal{G}]$-module.*

*Suppose that all roots act cubically (i.e., with length equal to 3). Suppose in addition that for fixed $\alpha \in \Phi$, the kernel $\ker \partial_{\alpha,\lambda}$ does not depend on $\lambda \in \mathbb{K} \setminus \{0\}$.*

*Can $V$ be equipped with a $\mathbb{K}$-vector space structure making it isomorphic to the adjoint representation of $\mathcal{G}$ as a $\mathbb{K}[\mathcal{G}]$-module?*

The results briefly mentioned at the end of §1.2 would serve as the basis. Removing module simplicity could be more painful.

# 2 The Proof

For the reader's convenience let us state our Main Theorem again.

**Main Theorem.** *Let $\mathbb{K}$ be a field of characteristic $\neq 2$ with more than three elements and $\mathbb{G}$ be one of the simple algebraic groups (of classical or exceptional type; untwisted). Let $G = \mathbb{G}_{\mathbb{K}}$ be the abstract group of $\mathbb{K}$-points of the functor $\mathbb{G}$ and $\mathfrak{g} = (\mathrm{Lie}\,\mathbb{G})_{\mathbb{K}}$ be the abstract Lie ring of $\mathbb{K}$-points of the functor $\mathrm{Lie}\,\mathbb{G}$. Let $\mathcal{G}$ be either $G$ or $\mathfrak{g}$ and $V$ be a $\mathbb{Z}[\mathcal{G}]$-module.*

*Suppose that all roots act quadratically. Then $V = Z_V(\mathcal{G}) \oplus \lfloor \mathcal{G}, V \rfloor$ and $\lfloor \mathcal{G}, V \rfloor$ can be equipped with a $\mathbb{K}$-vector space structure making it isomorphic to a direct sum of minuscule representations of $\mathcal{G}$ as a $\mathbb{K}[\mathcal{G}]$-module.*

Let us sketch the strategy.

Recall from [2, Chap. VIII, §7.3, Proposition 6] that an irreducible representation of a semi-simple Lie algebra is minuscule iff for any weight $\mu$ and root $\alpha$, one has (in classical notation) $\mu(h_\alpha) \in \{-1, 0, 1\}$. For our purpose the latter property seems more tractable at first than a definition in terms of the action of the Weyl group.

The proof will therefore focus on weights and weight spaces. Of course in the absence of a field action the definition requires some care; the relevant analogues of weights and weight spaces will be called *masses* and *spots* (Definition 7).

We shall first show that the module is the direct sum of its spots (Proposition 9); the decomposition of an element of the module can be computed effectively. We then study how Weyl group elements permute spots. One point should be noted: if $\mathcal{G} = G$, one can find elements lifting Weyl reflections in the normaliser of a maximal torus; if $\mathcal{G} = \mathfrak{g}$, we must go to the enveloping ring since there are no suitable elements inside the Lie ring. But there is a slight trick enabling us to encode these in any case.

The action of the Weyl group is then as expected (Proposition 11); the argument is the only step in the proof where we feel we actually do something. Then Proposition 13 quickly enables us to reduce to an isotypical summand, where the Weyl group acts transitively on masses.

Once this is done we may define a field action on one arbitrary spot and use transitivity of the Weyl group to carry it around (Notation 15; here again the linear structure is given explicitly). Linearity is easily proved in Proposition 16.

At this stage we shall have a $\mathbb{K}[\mathcal{G}]$-module with unknown weights and possibly infinite dimension. But the earlier analysis in terms of spots will pay: after showing that we are actually dealing with minuscule weights (Proposition 18), complete reducibility will be immediate.

## 2.1 Prelude

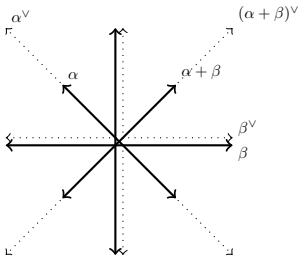
All necessary information on Chevalley groups can be found in [3] or [10]. We presume the reader familiar with root systems, the Weyl group, and how elements of the normaliser of a maximal torus permute the various root subgroups: this will be one of the key ingredients of the proof. But we also wish to use products of "root involutions" in algebraic groups and the reader will find some refreshments here.

Fix a realisation of a simple algebraic group $G = \mathbb{G}_{\mathbb{K}}$, and recall that central involutions of the various root $\mathrm{SL}_2$-subgroups can be computed from those attached to simple roots in the following way:

> since for any root $\alpha$, one has $i_\alpha = \alpha^\vee(-1)$ (where $\alpha^\vee$ is the cocharacter $\mathbb{K}^\times \to T$, the torus), it suffices to express coroots in the cobasis.

Consider for instance the case of $C_2$.

Let $\langle \cdot, \cdot \rangle$ be the abstract root datum pairing between roots and coroots, and $(\cdot, \cdot)$ be the standard Euclidean dot product on the root space (only the former will play a role in the remainder of the argument; the latter is of course more visual). Normalising in such a way that $\langle \delta, \epsilon^\vee \rangle$ is given by $(\delta, \epsilon^\vee)$, we may represent the dual system on the same picture where the dual of a short root $\gamma$ corresponds to $2\gamma$ (abusing language, a long root is "self-dual").




It is then clear *without computing* that $(\alpha + \beta)^\vee = \alpha^\vee + 2\beta^\vee$, and therefore $i_{\alpha+\beta} = i_\alpha i_\beta^2 = i_\alpha$.

The same picture allows of course to determine conjugates of root subgroups by elements of the Weyl group, still with no computations. (We hope the following notation to be standard; in any case it will be introduced in Notation 2 below.) Remember that $w_\gamma$ acts on root subgroups as $\sigma_\gamma$, the reflection in hyperplane $\gamma^\perp$, acts on roots: hence $w_\gamma U_\delta w_\gamma^{-1} = U_{\sigma_\gamma(\delta)}$ can be found graphically.

And of course there is something similar for the other rank 2 systems. This "visual computing" will be used freely in the argument, viz. in Propositions 9 and 11.

## 2.2 Local Analysis

The proof starts here. We may suppose the action to be non-trivial. Let us first realise $\mathcal{G}$, following the Chevalley-Steinberg ideology. We apologise for the necessarily heavy notation: we must introduce the relevant elements for the group and also for the Lie ring; once this is done we then introduce *common* designations and forget the specialised ones. This results in a rather awkward moment, but most of our notation can be disposed of quickly.

The idea is merely to realise the root (P)$\mathrm{SL}_2$-subgroups (resp., $\mathfrak{sl}_2$-subrings) and work with the elements:

$$u_{\alpha,\lambda} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad t_{\alpha,\lambda} = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}, \quad w_\alpha = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

in the case of the group, or:

$$x_{\alpha,\lambda} = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \quad y_{\alpha,\lambda} = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}, \quad h_{\alpha,\lambda} = \begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}$$

in the case of the Lie ring. Follows one ugly page just to say this. The well-versed reader may immediately skip to Notation 5.

As far as root data are concerned, we follow standard conventions.

**Notation 1** (naming the root datum: $L, \Phi, L^\vee, \Phi^\vee, \langle\cdot,\cdot\rangle, E, \sigma_\alpha, \Phi_+, \Phi_s$)**.**

- *Let $(L, \Phi, L^\vee, \Phi^\vee)$ be the root datum of $\mathbb{G}$ and $\langle\cdot,\cdot\rangle : L \times L^\vee \to \mathbb{Z}$ be the pairing; let $E = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}\Phi$.*
- *For $\alpha \in \Phi$, let $\sigma_\alpha$ be the linear map on $E$ mapping $e$ to $e - \langle e, \alpha^\vee\rangle\alpha$.*
- *Let $\Phi_+$ be a choice of positive roots and $\Phi_s$ be the (positive) simple roots.*

With this at hand we can realise $G$. General information on Chevalley groups can be found in [3] (in particular Chapters 5 and 6 there) or [10]; sometimes our notation differs as we use $u$ for unipotent elements and $t$ for toral (semi-simple) elements; for elements associated to the Weyl group we use $w$. A particularly thorough reference is [7, Exposé 23] but we shall avoid using geometric language.

**Notation 2** (realising $G$: $T, U_\alpha, G_\alpha, u_{\alpha,\lambda}, u_\alpha, w_\alpha, i_\alpha, t_{\alpha,\lambda}$)**.**

- *Fix an algebraic torus $T \leq G$; root subgroups will refer to this particular torus.*
- *For $\alpha \in \Phi$, let $U_\alpha$ be the root subgroup and $G_\alpha = G_{-\alpha} = \langle U_\alpha, U_{-\alpha}\rangle$ be the root $\mathrm{SL}_2$-subgroup.*
- *Realising enables us to fix isomorphisms $\varphi_\alpha : (\mathrm{P})\mathrm{SL}_2(\mathbb{K}) \simeq G_\alpha$ mapping upper-triangular matrices to $U_\alpha$ and diagonal matrices to $T \cap G_\alpha$ and such that $\varphi_\alpha^{-1} \circ \varphi_{-\alpha} = \varphi_{-\alpha}^{-1} \circ \varphi_\alpha$ is the inverse-transpose automorphism. Let:*

  $$u_{\alpha,\lambda} = \varphi_\alpha\left(\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\right), \quad t_{\alpha,\lambda} = \varphi_\alpha\left(\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}\right)$$

  *For simplicity write $u_\alpha = u_{\alpha,1}$.*
- *Let $w_\alpha = u_\alpha \cdot u_{-\alpha}^{-1} \cdot u_\alpha \in N_G(T)$ and $i_\alpha = w_\alpha^2$, an element with order at most 2.*

  *(It will be a consequence of Proposition 6 below that $i_\alpha$ has order* exactly *2.)*

In particular, it should be noted that $w_{-\alpha} = w_\alpha$ and $w_\alpha u_{\alpha,\lambda} w_\alpha^{-1} = u_{-\alpha,\lambda}$. Moreover, $t_{-\alpha,\lambda} = t_{\alpha,\lambda^{-1}}$. Importantly enough, $w_\alpha U_\beta w_\alpha^{-1} = U_{\sigma_\alpha(\beta)}$. Now to $\mathfrak{g}$.

**Notation 3** (realising $\mathfrak{g}$: $\mathfrak{t}, \mathfrak{u}_\alpha, \mathfrak{g}_\alpha, x_{\alpha,\lambda}, x_\alpha, h_{\alpha,\lambda}, h_\alpha$)**.**

- *Fix a decomposition $\mathfrak{g} = \mathfrak{t} \oplus \oplus_{\alpha\in\Phi}\mathfrak{u}_\alpha$ with $\mathfrak{t}$ a Cartan subring and $\mathfrak{u}_\alpha$ the root subrings. Let $\mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha} = \langle \mathfrak{u}_\alpha, \mathfrak{u}_{-\alpha}\rangle$ be the root $\mathfrak{sl}_2$-subring.*
- *Realising enables us to fix isomorphisms $\psi_\alpha : \mathfrak{sl}_2(\mathbb{K}) \simeq \mathfrak{g}_\alpha$ mapping upper-triangular matrices to $\mathfrak{g}_\alpha$ and diagonal matrices to $\mathfrak{t} \cap \mathfrak{g}_\alpha$ and such that $\psi_\alpha^{-1} \circ \psi_{-\alpha} = \psi_{-\alpha}^{-1} \circ \psi_\alpha$ is the oppose-transpose automorphism. Let:*

  $$x_{\alpha,\lambda} = \psi_\alpha\left(\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}\right), \quad h_{\alpha,\lambda} = \psi_\alpha\left(\begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}\right)$$

  *For simplicity write $x_\alpha = u_{\alpha,1}$ and $h_\alpha = h_{\alpha,1}$.*

**Remark 4.** *If one were to let* $y_{\alpha,\lambda} = \psi_\alpha\left(\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right)$*, one would have* $y_{\alpha,\lambda} = -x_{-\alpha,\lambda} = x_{-\alpha,-\lambda}$*. (The author finds computations less confusing to perform or check when working in the basis* $(h, x, y)$*.)*

Let us now provide uniform notation. The reason for choosing letter $\omega$ (which more classically stands for the fundamental weights, see [2]) is by analogy with $w$ for elements of the group lifting the Weyl group. Checking that $\omega_\alpha$ behaves as expected in the case of the Lie ring as well is not obvious and will be carried in Proposition 6.

**Notation 5** (realising $\mathcal{G}$ by assembling Notations 2 and 3: $\mathcal{T}, \mathcal{U}_\alpha, \mathcal{G}_\alpha, \omega_\alpha, \partial_{\alpha,\lambda}, \tau_{\alpha,\lambda}$)**.**

- *If* $\mathcal{G} = G$ *let* $\mathcal{T} = T$*, let* $\mathcal{U}_\alpha = U_\alpha$ *and* $\mathcal{G}_\alpha = G_\alpha$*. Also let* $\omega_\alpha = w_\alpha$*; for* $\lambda \in \mathbb{K}^\times$ *let* $\partial_{\alpha,\lambda} = u_{\alpha,\lambda} - 1$ *and* $\tau_{\alpha,\lambda} = t_{\alpha,\lambda}$*;*
- *if* $\mathcal{G} = \mathfrak{g}$ *let* $\mathcal{T} = \mathfrak{t}$*, let* $\mathcal{U}_\alpha = \mathfrak{u}_\alpha$ *and* $\mathcal{G}_\alpha = \mathfrak{g}_\alpha$*. Also let* $\omega_\alpha = 1 - h_\alpha^2 + x_\alpha + x_{-\alpha}$*; for* $\lambda \in \mathbb{K}_+$ *let* $\partial_{\alpha,\lambda} = x_{\alpha,\lambda}$ *and* $\tau_{\alpha,\lambda} = h_{\alpha,\lambda}$*.*

Let us apologise again for this notation storm; on second thought, the reader will find that we mostly wanted to have toral ("$\tau$") and root ("$\partial$") elements, and to encode Weyl elements ("$\omega$") in a consistent way.

**Proposition 6** (local analysis)**.** *For* $\alpha \in \Phi$*, one has* $V = Z_V(\mathcal{G}_\alpha) \oplus [\mathcal{G}_\alpha, V]$ *and* $\lfloor \mathcal{G}_\alpha, V \rfloor = \lfloor \mathcal{U}_\alpha, V \rfloor \oplus \lfloor \mathcal{U}_{-\alpha}, V \rfloor$ *can be equipped with a* $\mathbb{K}$*-vector space structure making it isomorphic to a direct sum of natural representations of* $\mathcal{G}_\alpha$ *as a* $\mathbb{K}[\mathcal{G}_\alpha]$*-module.*

*Consequently:*

- *in the case of the group,* $C_V(G_\alpha) = C_V(i_\alpha)$ *and* $[G_\alpha, V] = [i_\alpha, V]$*;*
- $\omega_\alpha = \omega_{-\alpha}$ *is a bijection fixing* $Z_V(\mathcal{G}_\alpha)$ *pointwise and mapping* $\lfloor \mathcal{U}_\alpha, V \rfloor$ *to* $\lfloor \mathcal{U}_{-\alpha}, V \rfloor$ *and conversely;* $\omega_\alpha^2$ *acts as* $-1$ *on* $\lfloor \mathcal{G}_\alpha, V \rfloor$*;*
- *for* $v \in \lfloor \mathcal{U}_\alpha, V \rfloor = Z_{\lfloor \mathcal{G}_\alpha, V \rfloor}(\mathcal{U}_\alpha)$*, one has* $\partial_{\alpha,\lambda}\omega_\alpha v = -\tau_{\alpha,\lambda} v$*;*
- *for* $\alpha \in \Phi$*, one has* $\omega_\alpha \partial_{\alpha,\lambda} \omega_\alpha^{-1} = \partial_{-\alpha,\lambda}$*; moreover* $\omega_\alpha$ *normalises the image of* $\mathcal{T}$ *in* $\mathrm{End}(V)$*.*

Before the proof, bear in mind that for the Corollary in §1.5.2 above, i.e. in the case $\mathcal{G} = \mathbb{G}_{\mathbb{F}_3}$ of the group over the field with three elements, we assume the main conclusion of Proposition 6, and the few remaining details follow as below. There will be no more mention of the order of the field in our argument as perfectness plays no further role.

*Proof.* By assumption $\mathcal{G}_\alpha$ acts quadratically. So most claims follow from Timmesfeld's Quadratic Theorem and its Lie-ring analogue from §1.2, and inspection in the natural $\mathbb{SL}_2$-module (possibly with a few computations).

We urge the reader not to underestimate the fact that $\partial_{\alpha,\lambda}\omega_\alpha v = -\tau_{\alpha,\lambda} v$ for $v \in \lfloor \mathcal{U}_\alpha, V \rfloor$: here we can see it by inspection again, but this rather deep equation is the crux of the Quadratic Theorem. (The curious reader interested in extending this remarkable formula to other rational representations of $\mathrm{SL}_2(\mathbb{K})$ may be directed to the proof of [6, Theorem 2].)

We now prove the final statement, and begin with $\omega_\alpha \partial_{\alpha,\lambda} \omega_\alpha^{-1} = \partial_{-\alpha,\lambda}$. This is clear in the case of the group (the formula holds in $G_\alpha$, replacing $\partial$ by $u$); for the Lie ring, proceed piecewise. On $\mathrm{Ann}_V(\mathfrak{g}_\alpha)$ this is clear as both hands are zero. So let us work on $\mathfrak{g}_\alpha \cdot V$, where $h_\alpha^2 = 1$, so that $\omega_\alpha$ simplifies into $x_\alpha + x_{-\alpha}$. Let $[\![f, g]\!] = fg - gf$ in $\mathrm{End}(V)$ (we avoid $[\cdot, \cdot]$ which we reserve for group commutators). Then using quadraticity of $\mathfrak{g}_\alpha$ and with a possible look at Remark 4 one can check that on $\mathfrak{g}_\alpha \cdot V$:

$$\begin{aligned}
\omega_\alpha \partial_{\alpha,\lambda} \omega_\alpha^{-1} &= -(x_\alpha + x_{-\alpha}) x_{\alpha,\lambda} (x_\alpha + x_{-\alpha}) = -x_{-\alpha} x_{\alpha,\lambda} x_{-\alpha} \\
&= -[\![x_{-\alpha}, x_{\alpha,\lambda}]\!] x_{-\alpha} = -h_{\alpha,\lambda} x_{-\alpha} = -[\![x_{-\alpha,\lambda}, x_\alpha]\!] x_{-\alpha} \\
&= -x_{-\alpha,\lambda} [\![x_\alpha, x_{-\alpha}]\!] = x_{-\alpha,\lambda} h_\alpha = x_{-\alpha,\lambda} = \partial_{-\alpha,\lambda}
\end{aligned}$$

We now show that $\omega_\alpha$ normalises the image in $\mathrm{End}(V)$ of $\mathcal{T}$: here again the case of the group is obvious so we turn to the Lie ring. Let $\alpha, \beta$ be roots. Then:

- first, $[\![\omega_\alpha, \tau_{\beta,\lambda}]\!] = [\![x_\alpha + x_{-\alpha}, h_{\beta,\lambda}]\!] = x_{-\alpha,\langle\alpha,\beta^\vee\rangle\lambda} - x_{\alpha,\langle\alpha,\beta^\vee\rangle\lambda}$;
- by piecewise inspection one has $x_{-\alpha,\lambda}x_\alpha - x_{\alpha,\lambda}x_{-\alpha} = h_{\alpha,\lambda}$ and $x_{\alpha,\lambda}h_\alpha = -x_{\alpha,\lambda}$ in $\mathrm{End}(V)$; also notice that $\omega_\alpha^{-1} = -x_\alpha - x_{-\alpha} + 1 - h_\alpha^2$;
- therefore $(x_{-\alpha,\lambda} - x_{\alpha,\lambda})\omega_\alpha^{-1} = (x_{-\alpha,\lambda} - x_{\alpha,\lambda})(-x_\alpha - x_{-\alpha}) = -x_{-\alpha,\lambda}x_\alpha + x_{\alpha,\lambda}x_{-\alpha} = -h_{\alpha,\lambda}$;
- as a consequence $\omega_\alpha\tau_{\beta,\lambda}\omega_\alpha^{-1} = ([\![\omega_\alpha, \tau_{\beta,\lambda}]\!] + \tau_{\beta,\lambda}\omega_\alpha)\,\omega_\alpha^{-1} = (x_{-\alpha,\langle\alpha,\beta^\vee\rangle\lambda} - x_{\alpha,\langle\alpha,\beta^\vee\rangle\lambda})\omega_\alpha^{-1} + h_{\beta,\lambda} = -h_{\alpha,\langle\alpha,\beta^\vee\rangle\lambda} + h_{\beta,\lambda}$.

Hence $\omega_\alpha$ normalises the image of $\mathfrak{t}$. □

Notice that if $\mathcal{G} = G$, then since $\mathbb{G}$ is simple as an algebraic group and the action is non-trivial, for any $\alpha \in \Phi$ the root $\mathbb{SL}_2$-subgroup $G_\alpha$ must act non-trivially on $V$. As a consequence $G_\alpha \simeq \mathrm{SL}_2(\mathbb{K})$ and $i_\alpha$ is a genuine involution.

## 2.3 Spots and Masses

Capturing weight spaces requires a little care in the absence of a linear structure.

In the case of the Lie ring $\mathcal{G} = \mathfrak{g}$ there is a straightforward approach. Diagonalise all operators $h_\alpha$, for $\alpha \in \Phi_s$, simultaneously (the reader with a doubt will find what we precisely mean at the very beginning of the proof of Proposition 9). The various eigenspaces will be the weight spaces for the action of the Cartan subring $\mathfrak{t}$. But there is no such argument in the case of the group $\mathcal{G} = G$. Yet returning to the Lie ring one sees by inspection that $\ker(h_\alpha - 1) = \mathfrak{u}_\alpha \cdot V = \lfloor \mathcal{U}_\alpha, V \rfloor$ whereas $\ker(h_\alpha) = \mathrm{Ann}_V(\mathfrak{g}_\alpha) = Z_V(\mathcal{G}_\alpha)$. This suggests a general method.

Recall from Notation 1 that $\Phi$ (resp. $\Phi^\vee$) denotes the root (resp. dual root) system and $E$ the underlying Euclidean space. We had also let $\langle\cdot,\cdot\rangle : L \times L^\vee \to \mathbb{Z}$ denote the pairing.

**Definition 7.**

- For $\mu \in E$ and $\alpha$ a root define $V_{(\mu,\alpha^\vee)}$ as follows:
  - if $\langle\mu,\alpha^\vee\rangle = -1$ let $V_{(\mu,\alpha^\vee)} = \lfloor \mathcal{U}_{-\alpha}, V \rfloor$;
  - if $\langle\mu,\alpha^\vee\rangle = 0$ let $V_{(\mu,\alpha^\vee)} = Z_V(\mathcal{G}_\alpha)$;
  - if $\langle\mu,\alpha^\vee\rangle = 1$ let $V_{(\mu,\alpha^\vee)} = \lfloor \mathcal{U}_\alpha, V \rfloor$;
  - if $\langle\mu,\alpha^\vee\rangle \notin \{-1,0,1\}$ let $V_{(\mu,\alpha^\vee)} = \{0\}$.
- For $\mu \in E$ let $S_\mu = \bigcap_{\alpha\in\Phi_s} V_{(\mu,\alpha^\vee)}$ be the *spot with mass* $\mu$.
- Let $M = \{\mu \in E : S_\mu \neq \{0\}\}$ be the set of *masses*. (Being a mass certainly implies: $\forall\alpha \in \Phi_s, \langle\mu,\alpha^\vee\rangle \in \{-1,0,1\}$, but the condition is not sufficient.)

Notice how in the presence of a field, a mass $\mu$ will become a weight. (The idea in taking the intersection over the set of *simple* roots $\Phi_s$ is that the behaviour of simple roots should determine that of all roots; we shall neither need nor prove this.)

**Remark 8.**

1. *$S_\mu$ is a $\mathcal{T}$-submodule of $V$.*
2. *Suppose $\mathcal{G} = \mathfrak{g}$ and $\langle\mu,\alpha^\vee\rangle \in \{-1,0,1\}$. Then $V_{(\mu,\alpha^\vee)} = \ker(h_\alpha - \langle\mu,\alpha^\vee\rangle)$.*

We now show that $V$ is the direct sum of its spots.

**Proposition 9.** $V = \oplus_{\mu\in M} S_\mu$.

**Remark 10.** *As a matter of fact during the proof we shall see:*

- *that the components $v_\mu \in S_\mu$ in a decomposition $v = \sum_{\mu\in M} v_\mu$ all lie in $\langle\mathcal{G}\cdot v\rangle$;*
- *that if $\mathcal{G} = G$ and $\mathbb{G} = G_2$, then $V = S_0$ (with $0$ the null mass).*

*Proof of Proposition 9.* In the case of the Lie ring $\mathfrak{g}$ this is obvious. Using the Lie-ring analogue to the Quadratic Theorem, for any root $\alpha \in \Phi_s$, $V$ decomposes as $\ker(h_\alpha + 1) \oplus \ker(h_\alpha) \oplus \ker(h_\alpha - 1)$, which — despite lack of a $\mathbb{K}$-linear structure so far — we suggestively call "diagonalising $h_\alpha$". Since the various $h_\alpha$ (always $\alpha \in \Phi_s$) commute, diagonalisation is simultaneous. The various $\{-1, 0, 1\}^{\Phi_s}$-eigenspaces are the spots by Remark 8.

We then focus on the case of the group $G$; nothing so quick is available, since no toral element in $G_\alpha$ suffices to determine the value of $\langle \mu, \alpha^\vee \rangle$: looking at the involution can distinguish 0 from $\pm 1$, but no further. We need a closer look.

Bear in mind from Proposition 6 that for any $\alpha \in \Phi$, one has $C_V(i_\alpha) = C_V(G_\alpha)$ and $[i_\alpha, V] = [G_\alpha, V]$; also $[U_\alpha, V] = C_{[G_\alpha, V]}(U_\alpha)$. Finally if $v \in [U_\alpha, V]$, then $\partial_\alpha w_\alpha v = -v$.

*Claim* 1. The sum is direct.

*Proof of Claim.* Let $\sum_{\mu \in M_1} v_\mu = 0$ be an identity minimal with respect to: for all $\mu \in M_1$, $v_\mu \in S_\mu \setminus \{0\}$. Let $\nu \in M_1$ (if any) and $\alpha \in \Phi_s$ be fixed.

- If $\langle \nu, \alpha^\vee \rangle = 0$ then $i_\alpha v_\nu = v_\nu$, so that $\sum_{\mu \in M_1} (i_\alpha v_\mu - v_\mu) = 0$ is a shorter relation. It follows that $i_\alpha v_\mu = v_\mu$ for all $\mu \in M_1$, meaning $\langle \mu, \alpha^\vee \rangle = 0$.
- If $\langle \nu, \alpha^\vee \rangle = 1$ then $\partial_\alpha w_\alpha v_\nu = -v_\nu$; notice that whenever $\langle \mu, \alpha^\vee \rangle \neq 1$, one has $\partial_\alpha w_\alpha v_\mu = 0$. So minimality again forces $\langle \mu, \alpha^\vee \rangle = 1$ for all $\mu \in M_1$.
- There is a similar argument if $\langle \nu, \alpha^\vee \rangle = -1$.

This shows that all $\mu \in M_1$ coincide on all $\alpha \in \Phi_s$, a spanning set of $E$: $M_1$ is at most a singleton, hence empty, as desired. ◊

Now let $R(V) = \oplus_{\mu \in M} S_\mu$ and fix $v \in V$; we aim at showing $v \in R(V)$. This we do by induction on the rank of $\mathbb{G}$, or equivalently, on the Dynkin diagram.

Let $\alpha \in \Phi_s$ be extremal in the Dynkin diagram and $\beta$ be its neighbour; *we may suppose $\alpha$ not to be longer than $\beta$*. By induction, we know the result for the subgroup with Dynkin diagram $\Phi_s \setminus \{\alpha\}$. Hence we may assume that for any $\gamma \in \Phi_s \setminus \{\alpha\}$, the element $v$ is already *decomposed* under the action of $G_\gamma$, viz.:

$$v \in C_V(i_\gamma) \cup [U_\gamma, V] \cup [U_{-\gamma}, V]$$

*Claim* 2. We may suppose $v \in [i_\alpha, V]$.

*Proof of Claim.* Write $v = v_0 + v_\pm$ with respect to the action of $i_\alpha$, meaning $v_0 \in C_V(i_\alpha) = C_V(G_\alpha)$ and $v_\pm \in [i_\alpha, V] = [G_\alpha, V]$; as a matter of fact $v_\pm = \frac{1}{2}[i_\alpha, v]$ (which makes sense since $[i_\alpha, V]$ is a vector space over $\mathbb{K}$). Since $i_\alpha$ centralises $G_\gamma$ for $\gamma \in \Phi_s \setminus \{\alpha, \beta\}$, $v_0$ remains decomposed under the action of such root $\mathrm{SL}_2$-subgroups; by construction, it is decomposed under that of $G_\alpha$. Now $i_\alpha$ normalises $U_\beta$ and $U_{-\beta}$ (hence also $G_\beta$). As a consequence:

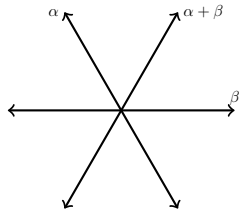
- if $v \in C_V(G_\beta)$ then $i_\alpha v$, $v_\pm$, and $v_0$ lie in $C_V(G_\beta)$;
- if $v \in [U_\beta, V]$ then $i_\alpha v$, $v_\pm$, and $v_0$ lie in $[U_\beta, V]$;
- there is a similar argument if $v \in [U_{-\beta}, V]$.

As a conclusion, $v_0$ is decomposed under the action of $G_\beta$ as well: hence $v_0 \in R(V)$. We may therefore assume $v = v_\pm \in [i_\alpha, V]$. ◊

It follows from inspection in $\mathrm{Nat}\,\mathrm{SL}_2(\mathbb{K})$ that $v = v_+ + v_-$ with $v_+ = -\partial_\alpha w_\alpha v \in [U_\alpha, V]$ and $v_- = -w_\alpha \partial_\alpha v \in [U_{-\alpha}, V]$. We aim at showing $v_+, v_- \in R(V)$. By construction the latter elements are already decomposed under the action of $G_\alpha$ and $G_\gamma$ for $\gamma \in \Phi_s \setminus \{\alpha, \beta\}$, but it remains to see what happened under the action of $G_\beta$. This we do dividing three cases (remember that we assumed $\alpha$ not to be longer than $\beta$). We use classical notation for Dynkin diagrams: $-$, $=$, and $\equiv$; an arrow goes from a long root to a short root.

*Claim* 3. If $\alpha - \beta$, then $v \in R(V)$.

*Proof of Claim.*

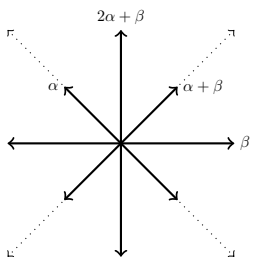



Here $i_{\alpha+\beta} = i_\alpha i_\beta$ (the reader may wish to return to §2.1); also notice that $i_\beta w_\alpha = w_\alpha i_\alpha i_\beta$. Since $\langle \alpha, \beta^\vee \rangle = -1$, the involution $i_\beta$ inverts $U_\alpha$; observe that $i_\beta \partial_\alpha w_\alpha v = -\partial_\alpha i_\beta w_\alpha v = -\partial_\alpha w_\alpha i_\alpha i_\beta v$.

- If $v \in C_V(i_\beta)$, then $\partial_\alpha w_\alpha v \in C_V(i_\beta) = C_V(G_\beta)$; hence $v_+, v_- \in R(V)$: we are done.
- If $v \in [U_\beta, V]$, then both $i_\beta$ and $i_\alpha$ invert $v$: hence $i_{\alpha+\beta} = i_\alpha i_\beta$ centralises it, so that $v = w_{\alpha+\beta} v \in [w_{\alpha+\beta} U_\beta w_{\alpha+\beta}^{-1}, V] = [U_{-\alpha}, V]$. Hence $v = v_-$ is already decomposed under the action of both $G_\alpha$ and $G_\beta$: therefore $v \in R(V)$.
- Likewise, if $v \in [U_{-\beta}, V]$, then $v = v_+ \in [U_\alpha, V]$ and $v \in R(V)$ again. $\Diamond$

*Claim* 4. If $\alpha \Leftarrow \beta$, then $v \in R(V)$.

*Proof of Claim.*



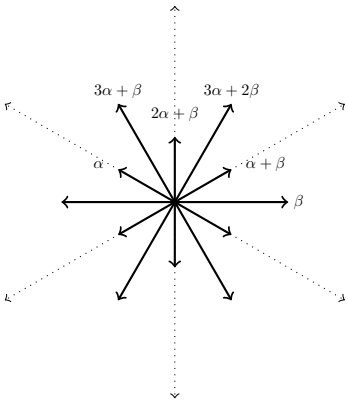

Now $i_{2\alpha+\beta} = i_\alpha i_\beta$ and $i_\beta w_\alpha = w_\alpha i_{2\alpha+\beta}$. Since $\langle \alpha, \beta^\vee \rangle = -1$, the involution $i_\beta$ inverts $U_\alpha$, and one still has $i_\beta \partial_\alpha w_\alpha v = -\partial_\alpha w_\alpha i_\alpha i_\beta v$.

- If $v \in C_V(i_\beta)$, then $i_\beta \partial_\alpha w_\alpha v = \partial_\alpha w_\alpha v$, so $v_+$ lies in $C_V(i_\beta)$; since $v$ as well, so does $v_-$. As a consequence $v_+, v_- \in R(V)$.
- If $v \in [U_\beta, V]$, then $w_\alpha v \in [w_\alpha U_\beta w_\alpha^{-1}, V] = [U_{2\alpha+\beta}, V]$. Now $[U_\alpha, U_{2\alpha+\beta}] = 1$ in the group, so by the three subgroups lemma $v_+ = -\partial_\alpha w_\alpha v \in [U_{2\alpha+\beta}, V] \leq [i_{2\alpha+\beta}, V]$.

  However $i_\alpha i_\beta \partial_\alpha w_\alpha v = \partial_\alpha w_\alpha v$ so $v_+ \in C_V(i_\alpha i_\beta) = C_V(i_{2\alpha+\beta})$. This shows $v_+ = 0$, and therefore $v = v_- \in R(V)$.
- There is a similar argument showing $v = v_+ \in R(V)$ if $v \in [U_{-\beta}, V]$. $\Diamond$

*Claim* 5. If $\alpha \Leftarrow \beta$, then $v = 0 \in R(V)$.

*Proof of Claim.*




Finally $i_{\alpha+\beta} = i_\alpha i_\beta = i_{3\alpha+\beta}$; also $i_{2\alpha+\beta} = i_\beta$ and $i_{3\alpha+2\beta} = i_\alpha$.

- If $v \in C_V(i_\beta)$ then it can be checked that $i_\beta \partial_\alpha w_\alpha v = -\partial_\alpha w_\alpha i_\alpha i_\beta v = \partial_\alpha w_\alpha v$, implying that $v_+ = w_\beta v_+ \in [w_\beta U_\alpha w_\beta^{-1}, V] = [U_{\alpha+\beta}, V]$ and $v_+ = w_{2\alpha+\beta} v_+ \in [w_{2\alpha+\beta} U_\alpha w_{2\alpha+\beta}^{-1}, V] = [U_{-\alpha-\beta}, V]$, so $v_+ = 0$. One can show $v_- = 0$ as well; hence $v = 0 \in R(V)$.
- If $v \in [U_\beta, V]$, then both $i_\alpha$ and $i_\beta$ invert $v$; as a consequence one has $v = w_{\alpha+\beta} v \in [w_{\alpha+\beta} U_\beta w_{\alpha+\beta}^{-1}, V] = [U_{-3\alpha-2\beta}, V]$ and $v = w_{3\alpha+\beta} v \in [w_{3\alpha+\beta} U_\beta w_{3\alpha+\beta}^{-1}, V] = [U_{3\alpha+2\beta}, V]$, so $v = 0 \in R(V)$.
- There is a similar argument if $v \in [U_{-\beta}, V]$.

Notice that in case $\mathbb{G} = G_2$ we proved $v = v_0$ in the above notation. This means that $V$ is centralised by $i_\alpha$ and therefore by $G_\alpha$, so by simplicity of $\mathbb{G}$ the action of $G$ on $V$ is actually trivial. ◊

This completes the proof of Proposition 9. □

## 2.4 Weyl Group Action

By Proposition 9, $V$ is the direct sum of its various mass spots, which mimicks the decomposition into weight spaces.

We now wish to see how the Weyl group permutes spots: it is as expected, with the major warning that it is not entirely clear what this means in the case of the Lie ring (see §1.5.1 for why this is not meaningful a priori for the Lie ring, and remember our contortions in Notation 5). Our approach is elementary again.

In Notation 1, for any $\alpha \in \Phi$ we introduced the reflection on the root space $\sigma_\alpha(e) = e - \langle e, \alpha^\vee \rangle \alpha$. Also remember from Notation 5 that we have let $\omega_\alpha = w_\alpha$ if $\mathcal{G} = G$ and $\omega_\alpha = 1 - h_\alpha^2 + x_\alpha - x_{-\alpha}$ if $\mathcal{G} = \mathfrak{g}$; the action of $\omega_\alpha$ is as expected by Proposition 6.

Before the statement, observe that will shall be working with simple roots throughout. The author did not think about extending to other roots; in any case this will not be necessary.

**Proposition 11.** *For all $(\alpha, \mu) \in \Phi_s \times M$, one has $\omega_\alpha S_\mu = S_{\sigma_\alpha(\mu)}$.*

*Proof.* The case of the Lie ring is straightforward and will be dealt with quickly.

*Claim* 1. We may suppose $\mathcal{G} = G$.

*Proof of Claim.* Suppose $\mathcal{G} = \mathfrak{g}$; let $\mu \in M$ be a mass; let $\alpha, \beta$ be any two (possibly equal) simple roots. First notice that in the Lie ring $(\mathrm{End}(V), +, [\![\cdot,\cdot]\!])$, one has $[\![h_\beta, \omega_\alpha]\!] = [\![h_\beta, x_\alpha + x_{-\alpha}]\!] = \langle \alpha, \beta^\vee \rangle (x_\alpha - x_{-\alpha})$. On the other hand, as one checks by piecewise inspection with the help of Proposition 6, for $v \in V_{(\mu,\alpha^\vee)}$ holds: $(x_{-\alpha} - x_\alpha) v = -\langle \mu, \alpha^\vee \rangle \omega_\alpha v$. So for $v \in S_\mu \leq V_{(\mu,\alpha^\vee)} \cap V_{(\mu,\beta^\vee)}$, one has:

$$\begin{aligned}(h_\beta - \langle \sigma_\alpha(\mu), \beta^\vee \rangle)\omega_\alpha v &= (\omega_\alpha h_\beta + \langle \alpha, \beta^\vee \rangle (x_\alpha - x_{-\alpha}) - (\langle \mu, \beta^\vee \rangle - \langle \mu, \alpha^\vee \rangle \langle \alpha, \beta^\vee \rangle)\omega_\alpha)\, v \\ &= (\langle \mu, \beta^\vee \rangle - \langle \alpha, \beta^\vee \rangle \langle \mu, \alpha^\vee \rangle - \langle \mu, \beta^\vee \rangle + \langle \mu, \alpha^\vee \rangle \langle \alpha, \beta^\vee \rangle)\, \omega_\alpha v \\ &= 0\end{aligned}$$

showing that $\omega_\alpha S_\mu \leq \ker(h_\beta - \langle \sigma_\alpha(\mu), \beta^\vee \rangle)$.

We claim that $\ker(h_\beta - \langle \sigma_\alpha(\mu), \beta^\vee \rangle) = V_{(\sigma_\alpha(\mu),\beta^\vee)}$; by construction (see Remark 8) it suffices to see why $\langle \sigma_\alpha(\mu), \beta^\vee \rangle \in \{-1, 0, 1\}$. But let $\gamma \in E$ satisfy $\gamma^\vee = \sigma_\alpha^\vee(\beta^\vee) = \beta^\vee - \langle \alpha, \beta^\vee \rangle \alpha^\vee$; we know that $\gamma \in \Phi$ (not necessarily simple though). Now,

$$h_\gamma = \gamma^\vee(1) = \beta^\vee(1) - \langle \alpha, \beta^\vee \rangle \alpha^\vee(1) = h_\beta - \langle \alpha, \beta^\vee \rangle h_\alpha$$

acts on $S_\mu$ as the integer

$$\langle \mu, \beta^\vee \rangle - \langle \alpha, \beta^\vee \rangle \langle \mu, \alpha^\vee \rangle = \langle \sigma_\alpha(\mu), \beta^\vee \rangle$$

Since $\mathfrak{g}_\gamma$ is quadratic — bear in mind the assumption was on *all* roots — this integer remains in $\{-1, 0, 1\}$, as desired.

Therefore $\omega_\alpha S_\mu \leq \ker(h_\beta - \langle \sigma_\alpha(\mu), \beta^\vee \rangle) = V_{(\sigma_\alpha(\mu),\beta^\vee)}$. Since this holds for any $\beta \in \Phi_s$, one has $\omega_\alpha S_\mu \leq S_{\sigma_\alpha(\mu)}$. Since this holds for any mass $\mu \in M$, one also finds $\omega_\alpha S_{\sigma_\alpha(\mu)} \leq S_\mu$, proving equality. ◊

We move to the case of the group, for which there is no such argument: exactly as in Proposition 9, no toral element in $G_\alpha$ suffices to determine the value of $\langle \mu, \alpha^\vee \rangle$.

*Claim* 2. We may assume $\langle \mu, \alpha^\vee \rangle = 1$; it is enough to prove that for any $\beta \in \Phi_s$,

$$\omega_\alpha S_\mu \leq V_{(\mu-\alpha,\beta^\vee)} \quad (*)$$

*Proof of Claim.* First suppose $\langle \mu, \alpha^\vee \rangle = 0$. Then $\sigma_\alpha(\mu) = \mu$ and $\omega_\alpha$ acts as Id on $S_\mu$: there is nothing to prove. We then turn to $\langle \mu, \alpha^\vee \rangle = \pm 1$. Observe how it suffices to check $\omega_\alpha S_\mu \leq S_{\sigma_\alpha(\mu)}$: then one will find $S_\mu = \omega_\alpha^2 S_\mu \leq \omega_\alpha S_{\sigma_\alpha(\mu)} \leq S_\mu$, proving equality.

So it suffices to see that $\omega_\alpha S_\mu \leq S_{\sigma_\alpha(\mu)}$; without loss of generality, we may assume $\langle \mu, \alpha^\vee \rangle = 1$, so that $\sigma_\alpha(\mu) = \mu - \alpha$. We then wish to show $\omega_\alpha S_\mu \leq S_{\mu-\alpha}$. This we shall do by taking another simple root $\beta \in \Phi_s$ and showing that the action of $G_\beta$ on $\omega_\alpha S_\mu$ is as expected, viz. condition $(*)$ above. $\diamondsuit$

We start a case divison based on the nature of the bound between $\alpha$ and $\beta$ in the Dynkin diagram. Now there are five cases.

*Claim* 3. If $\beta$ is not bound to $\alpha$ then $(*)$ holds.

*Proof of Claim.* If $\beta$ equals $\alpha$ then with the assumption that $\langle \mu, \alpha^\vee \rangle = 1$, one finds $S_\mu \leq V_{(\mu,\alpha^\vee)} = \lfloor \mathcal{U}_\alpha, V \rfloor$, and:

$$\omega_\alpha S_\mu \leq \omega_\alpha \lfloor \mathcal{U}_\alpha, V \rfloor = \lfloor \mathcal{U}_{-\alpha}, V \rfloor = V_{(\sigma_\alpha(\mu),\alpha^\vee)} \qquad \diamondsuit$$

If $\beta$ is neither bound nor equal to $\alpha$, then $(*)$ is obvious since the images of $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ in $\mathrm{End}(V)$ commute, and $\langle \sigma_\alpha(\mu), \beta^\vee \rangle = \langle \mu, \beta^\vee \rangle$.

*Claim* 4. If $\alpha - \beta$, then $(*)$ holds.

*Proof of Claim.* There is a picture on page 13; in particular bear in mind that $i_{\alpha+\beta} = i_\alpha i_\beta$. Also notice that $\langle \mu - \alpha, \beta^\vee \rangle = \langle \mu, \beta^\vee \rangle + 1$.

- Suppose $\langle \mu, \beta^\vee \rangle = -1$; notice that $\langle \mu - \alpha, \beta^\vee \rangle = 0$. Since both $i_\alpha$ and $i_\beta$ invert $S_\mu$, one has $S_\mu \leq C_V(i_{\alpha+\beta}) = C_V(G_{\alpha+\beta})$, and $w_\alpha S_\mu \leq C_V(w_\alpha G_{\alpha+\beta} w_\alpha^{-1}) = C_V(G_\beta) = V_{(\mu-\alpha,\beta^\vee)}$.
- Now suppose $\langle \mu, \beta^\vee \rangle = 0$; hence $\langle \mu - \alpha, \beta^\vee \rangle = 1$. Then $i_\beta$ centralises $S_\mu$, so $S_\mu = w_\beta S_\mu \leq [w_\beta U_\alpha w_\beta^{-1}, V] = [U_{\alpha+\beta}, V]$. Hence $w_\alpha S_\mu \leq [w_\alpha U_{\alpha+\beta} w_\alpha^{-1}, V] = [U_\beta, V] = V_{(\mu-\alpha,\beta^\vee)}$.
- Finally suppose $\langle \mu, \beta^\vee \rangle = 1$; notice that now $\langle \mu - \alpha, \beta^\vee \rangle = 2$ and there is a contradiction in the air. Here again, both $i_\alpha$ and $i_\beta$ invert $S_\mu$ so $i_{\alpha+\beta}$ centralises it. Therefore $S_\mu = w_{\alpha+\beta} S_\mu \leq [w_{\alpha+\beta} U_\alpha w_{\alpha+\beta}^{-1}, V] = [U_{-\beta}, V]$, and $S_\mu \leq [U_\beta, V] \cap [U_{-\beta}, V] = 0$. This is a contradiction to $\mu \in M$, that is, $S_\mu \neq 0$ (see Definition 7). $\diamondsuit$

*Claim* 5. If $\alpha \Leftarrow \beta$ then $(*)$ holds.

*Proof of Claim.* There is a picture on page 13; one has $i_{\alpha+\beta} = i_\alpha$ and $i_{2\alpha+\beta} = i_\alpha i_\beta$. Notice that $\langle \mu - \alpha, \beta^\vee \rangle = \langle \mu, \beta^\vee \rangle + 1$.

- Suppose $\langle \mu, \beta^\vee \rangle = -1$, so that $\langle \mu - \alpha, \beta^\vee \rangle = 0$. Both $i_\alpha$ and $i_\beta$ invert $S_\mu$, so $S_\mu \leq C_V(i_\alpha i_\beta) = C_V(i_{2\alpha+\beta})$ and $w_\alpha S_\mu \leq C_V(w_\alpha G_{2\alpha+\beta} w_\alpha^{-1}) = C_V(G_\beta)$, as desired.
- Now suppose $\langle \mu, \beta^\vee \rangle = 0$, so that $\langle \mu - \alpha, \beta^\vee \rangle = 1$. Then $U_\alpha$, $U_\beta$, and therefore $U_{2\alpha+\beta}$ as well, centralise $S_\mu$. On the other hand $i_{2\alpha+\beta} = i_\alpha i_\beta$ inverts it, so $S_\mu \leq [i_{2\alpha+\beta}, C_V(U_{2\alpha+\beta})] = [U_{2\alpha+\beta}, V]$ and $w_\alpha S_\mu \leq [w_\alpha U_{2\alpha+\beta} w_\alpha^{-1}, V] = [U_\beta, V] = V_{(\mu-\alpha,\beta^\vee)}$.
- Finally suppose $\langle \mu, \beta^\vee \rangle = 1$, so that $\langle \mu - \alpha, \beta^\vee \rangle = 2$. Here again both $i_\alpha$ and $i_\beta$ invert $S_\mu$: so $i_{\alpha+2\beta}$ centralises it, and therefore $S_\mu = w_{\alpha+2\beta} S_\mu \leq [w_{\alpha+2\beta} U_\alpha w_{\alpha+2\beta}^{-1}, V] = [U_{-\alpha-\beta}, V]$. But $U_\alpha$, $U_\beta$, and therefore $U_{\alpha+\beta}$ as well, centralise $S_\mu$, showing $S_\mu = 0$: against $\mu \in M$. $\diamondsuit$

*Claim* 6. If $\alpha \Rightarrow \beta$ then $(*)$ holds.

*Proof of Claim.* Be careful that $\beta$ is now the short root; hence $i_{\alpha+\beta} = i_\beta$ and $i_{\alpha+2\beta} = i_\alpha i_\beta$. Notice that $\langle \mu - \alpha, \beta^\vee \rangle = \langle \mu, \beta^\vee \rangle + 2$.

- Suppose $\langle \mu, \beta^\vee \rangle = -1$, so that $\langle \mu - \alpha, \beta^\vee \rangle = 1$. Then both $i_\alpha$ and $i_\beta$ invert $S_\mu$: so $i_{\alpha+2\beta}$ centralises it, and therefore $S_\mu = w_{\alpha+2\beta} S_\mu \leq [w_{\alpha+2\beta} U_{-\beta} w_{\alpha+2\beta}^{-1}, V] = [U_{\alpha+\beta}, V]$. Hence $w_\alpha S_\mu \leq [w_\alpha U_{\alpha+\beta} w_\alpha^{-1}, V] = [U_\beta, V]$.
- Now suppose $\langle \mu, \beta^\vee \rangle = 0$, so that $\langle \mu - \alpha, \beta^\vee \rangle = 2$. Then $i_{\alpha+\beta} = i_\beta$ centralises $S_\mu$: as a consequence $S_\mu = w_{\alpha+\beta} S_\mu \leq [w_{\alpha+\beta} U_\alpha w_{\alpha+\beta}^{-1}, V] = [U_{-\alpha-2\beta}, V]$. But $U_{\alpha+2\beta} \leq \langle U_\alpha, U_\beta \rangle$ centralises $S_\mu$, showing $S_\mu = 0$. This is a contradiction to $\mu \in M$.
- Finally, the case $\langle \mu, \beta^\vee \rangle = 1$ was dealt with in the previous configuration. $\Diamond$

We have already proved inconsistency of a configuration of type $G_2$ (Remark 10).
This completes the proof of Proposition 11. $\square$

As a consequence (and this was not obvious a priori), the Weyl group does act on the set of masses $M \subseteq E$; in particular for $\mu \in M$ and $\alpha, \beta \in \Phi_s$, one has $\langle \sigma_\alpha(\mu), \beta^\vee \rangle \in \{-1, 0, 1\}$. Therefore if $\mu \in M$ and $\alpha, \beta \in \Phi_s$ are adjacent in the Dynkin diagram, one cannot have $\langle \mu, \alpha^\vee \rangle = \langle \mu, \beta^\vee \rangle = 1$. (Notice that the proof of Proposition 11 did remove such configurations.)

## 2.5 Intermezzo — Isotypical Summands

We know from Propositions 9 and 11 that $V$ is the direct sum of its spots, and that elements of $\mathrm{End}(V)$ which stand for the standard generators of the Weyl group act as expected. This enables us to reduce to a single orbit in Proposition 13.

**Notation 12.**

- *Let $\mu \in M$ and $\mathrm{cl}(\mu)$ be the orbit of $\mu$ under the action of the Weyl group of $\mathbb{G}$;*
- *let $V_{\mathrm{cl}(\mu)} = \oplus_{\nu \in \mathrm{cl}(\mu)} S_\nu$.*

**Proposition 13.** *$V_{\mathrm{cl}(\mu)}$ is $\mathcal{G}$-invariant.*

*Proof.* It suffices to prove invariance under all maps $\partial_{\pm\alpha,\lambda}$ for $(\alpha, \lambda) \in \Phi_s \times \mathbb{K}$. So let $\nu \in \mathrm{cl}(\mu)$ and $v \in S_\nu$.

- If $\langle \nu, \alpha^\vee \rangle = 0$ then $S_\nu \leq Z_V(\mathcal{G}_\alpha)$ is annihilated by $\partial_{\pm\alpha,\lambda}$.
- Now suppose $\langle \nu, \alpha^\vee \rangle = 1$. Then $S_\nu \leq \lfloor \mathcal{U}_\alpha, V \rfloor$ is annihilated by $\partial_{\alpha,\lambda}$. Recall from Proposition 6 that in $\mathrm{End}(V)$ holds $\partial_{-\alpha,\lambda} = \omega_\alpha \partial_{\alpha,\lambda} \omega_\alpha^{-1}$. As a consequence,

$$\partial_{-\alpha,\lambda} v = \omega_\alpha \partial_{\alpha,\lambda} \omega_\alpha^{-1} v = -\omega_\alpha \partial_{\alpha,\lambda} \omega_\alpha v = \omega_\alpha \tau_{\alpha,\lambda} v \quad \in \quad \omega_\alpha S_\nu = S_{\sigma_\alpha(\nu)} \leq V_{\mathrm{cl}(\mu)}$$

- There is a similar argument if $\langle \nu, \alpha^\vee \rangle = -1$. $\square$

## 2.6 Linear Structure

By Proposition 13 we may suppose $V = V_{\mathrm{cl}(\mu_0)}$ for some $\mu_0 \in M$; if $\mu_0 = 0$ then $\mathrm{cl}(\mu_0) = \{0\}$ and $V = Z_V(\mathcal{G})$: we are done. So we may suppose $\mu_0 \neq 0$ and will construct a linear action. Notice that our construction is as uniform as one can hope since it depends neither on the field nor on the root system. Determination of the isomorphism type of $V$ as a representation will be handled in §2.7.

**Notation 14.**

- *Let $\alpha_0 \in \Phi_s$ with $\langle \mu_0, \alpha_0^\vee \rangle = 1$ (up to taking $\sigma_{\alpha_0}(\mu_0)$ instead of $\mu_0$ there is one such).*
- *For $\gamma = (\alpha_1, \ldots, \alpha_d) \in \Phi_s^d$, let $\sigma_\gamma = \sigma_{\beta_d} \circ \cdots \circ \sigma_{\alpha_1}$ and $\omega_\gamma = \omega_{\alpha_d} \ldots \omega_{\alpha_1} \in \mathrm{End}(V)$.*

  *(Be careful that despite the notation, the element $\sigma_\gamma$ of the Weyl group need not be a reflection.)*

We now define a field action piecewise on the various spots. Notice that whenever $\sigma_\gamma(\mu) = \nu$, then by Proposition 11, $\omega_\gamma$ restricts to a group isomorphism $S_\mu \to S_\nu$.

**Notation 15.** *Let $\lambda \in \mathbb{K}$ and $v \in S_\mu$ for some $\mu \in \mathrm{cl}(\mu_0)$. Take $\gamma \in \Phi_s^d$ with $\sigma_\gamma(\mu_0) = \mu$ and define:*

$$\lambda \cdot v = \omega_\gamma \tau_{\alpha_0,\lambda} \omega_\gamma^{-1} v$$

**Proposition 16.** *This turns $V = V_{\mathrm{cl}(\mu_0)}$ into a $\mathbb{K}[\mathcal{G}]$-module.*

*Proof.* Here again we make a series of claims.

*Claim* 1. Notation 15 is well-defined.

*Proof of Claim.* By the definition of $\mathrm{cl}(\mu_0)$ and since the reflections $\sigma_\alpha$ ($\alpha \in \Phi_s$) generate the Weyl group, there is at least one sequence $\gamma \in \Phi_s^d$ with $\sigma_\gamma(\mu_0) = \nu$. The problem is that the actual operator $\omega_\gamma$ may depend on $\gamma$: the basic example is $\sigma_{\alpha_0}^2(\mu_0) = \mu_0$, whereas $\omega_{\alpha_0}^2$ acts on $S_{\mu_0}$ as $-1$.

It suffices to show the following: if $\gamma, \gamma'$ are sequences such that $\sigma_\gamma(\mu_0) = \sigma_{\gamma'}(\mu_0)$, then there is $\varepsilon \in \{\pm 1\}$ with $(\omega_\gamma)_{|S_{\mu_0}} = \varepsilon(\omega_{\gamma'})_{|S_{\mu_0}}$. Notice by inspection that $(\omega_\alpha^{-1})_{S_\mu}$ equals $\pm(\omega_\alpha)_{S_\mu}$ (the sign is even given by $(-1)^{\langle \mu, \alpha^\vee \rangle}$ as one can see), so we may replace any $\omega_\alpha$ by its inverse in a product of type $\omega_\gamma$.

So applying $\omega_{\gamma'}^{-1}$ it therefore suffices to prove: if $\sigma_\gamma(\mu_0) = \mu_0$ then $\omega_\gamma$ acts as $\pm 1$ on $S_{\mu_0}$. (We may have missed something as this looks decently obvious but we failed to convey this feeling and have no better reason to offer the reader than the following argument.)

Write $\gamma = (\alpha_1, \ldots, \alpha_d)$; for $i \in \{1, \ldots, d\}$ let $\mu_i = \sigma_{\alpha_i}(\mu_{i-1})$. We suppose $\mu_d = \mu_0$ and shall prove that there is $\varepsilon \in \{\pm 1\}$ such that for any $v \in S_{\mu_0}$, one has $\omega_\gamma v = \varepsilon v$ (be careful that $\varepsilon$ will depend on both $\gamma$ and $\mu_0$). The proof will be by induction on $d$. For convenience let $k_i = \langle \mu_{i-1}, \alpha_i^\vee \rangle \in \{-1, 0, 1\}$; by definition, $\mu_i = \mu_{i-1} - k_i \alpha_i$.

First suppose that there is $i \in \{1, \ldots, d\}$ with $k_i = 0$. Let $\gamma' = (\alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_d)$ (i.e., remove $\alpha_i$ from the sequence). By assumption, $\mu_i = \mu_{i-1}$; hence $\sigma_{\gamma'}(\mu_0) = \sigma_\gamma(\mu_0) = \mu_0$. Also recall that $k_i = \langle \mu_{i-1}, \alpha_i^\vee \rangle = 0$ implies that $S_{\mu_{i-1}} \leq Z_V(\mathcal{G}_{\alpha_i})$: hence $\omega_{\alpha_i}$ fixes $S_{\mu_{i-1}}$ pointwise. So $\omega_\gamma v = \omega_{\gamma'} v$ and we may apply induction to conclude.

Now suppose there is $i \in \{1, \ldots, d-1\}$ with $k_{i+1} = -k_i$. The left-hand side is:

$$k_{i+1} = \langle \mu_i, \alpha_{i+1}^\vee \rangle = \langle \sigma_{\alpha_i}(\mu_{i-1}), \alpha_{i+1}^\vee \rangle = \langle \mu_{i-1}, \alpha_{i+1}^\vee \rangle - k_i \langle \alpha_i, \alpha_{i+1}^\vee \rangle$$

Hence $\langle \mu_{i-1}, \alpha_{i+1}^\vee \rangle = k_i(\langle \alpha_i, \alpha_{i+1}^\vee \rangle - 1) \in \{-1, 0, 1\}$.

- If $\langle \alpha_i, \alpha_{i+1}^\vee \rangle = 2$ then $\alpha_{i+1} = \alpha_i$. Let $\gamma' = (\alpha_1, \ldots, \widehat{\alpha_i}, \widehat{\alpha_{i+1}}, \ldots, \alpha_d)$; clearly $\sigma_{\gamma'}(\mu_0) = \mu_0$ and $\omega_\gamma v = -\omega_{\gamma'} v$; apply induction.
- Otherwise $\langle \alpha_i, \alpha_{i+1}^\vee \rangle \leq 0$ and the above equality forces $\langle \alpha_i, \alpha_{i+1}^\vee \rangle = 0$: the roots are not adjacent, implying that $\sigma_{\alpha_i}$ and $\sigma_{\alpha_{i+1}}$ on the one hand, $\omega_{\alpha_i}$ and $\omega_{\alpha_{i+1}}$ on the other hand, commute. So swapping $\alpha_i$ and $\alpha_{i+1}$ in $\gamma$, the new sequence $\gamma' = (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_d)$ enjoys both $\sigma_{\gamma'}(\mu_0) = \sigma_\gamma(\mu_0)$ and $\omega_{\gamma'} v = \omega_\gamma v$. However, one has $k_i' = k_{i+1}$ and $k_{i+1}' = k_i$ with obvious notation. (The careful reader will note that $\mu_i$ changes, but $\mu_i$ is a mere gadget in our argument.)

Inductively applying the previous, we may suppose that there is $\ell \leq d$ with $k_1 = \cdots = k_\ell = -k_{\ell+1} = \cdots = -k_d$. Now $\mu_0 = \sigma_\gamma(\mu_0) = \mu_0 - k_1(\alpha_1 + \cdots + \alpha_\ell - \alpha_{\ell+1} \cdots - \alpha_d)$. Since simple roots are linearly independent in the vector space they span, there is $i \leq \ell$ maximal with $\alpha_i = \alpha_{\ell+1}$. But like above, we see that $\alpha_{\ell+1}$ is never adjacent to $\alpha_j$ for $j \in \{i+1, \ldots, \ell\}$. In particular the sequence $\gamma' = (\alpha_1, \ldots, \alpha_i, \alpha_{\ell+1}, \alpha_{i+1}, \ldots, \alpha_\ell, \alpha_{\ell+2}, \ldots, \alpha_d)$ obtained from $\gamma$ by moving $\alpha_{\ell+1}$ right after $\alpha_i$ enjoys both $\sigma_{\gamma'}(\mu_0) = \mu_0$ and $\omega_{\gamma'} v = \omega_\gamma v$. Now $\gamma'$ bears a redundancy, viz. $k_{i+1}' = 0$; conclude as above, by induction. ◊

*Claim* 2. Notation 15 defines a field action.

*Proof of Claim.* We argue piecewise; it clearly suffices to prove the claim in the action on $S_{\mu_0}$. Additivity in $v$ is obvious, so we now fix $v \in S_{\mu_0}$. Since $\alpha_0$ is the only root involved in the argument, we shall conveniently let $\alpha = \alpha_0$.

If $\mathcal{G} = \mathfrak{g}$, then additivity in $\lambda$ is obvious since $\tau_{\alpha,\lambda} = h_{\alpha,\lambda}$; we turn to multiplicativity. Observe how, since $v \in S_{\mu_0} \leq V_{(\mu_0,\alpha^\vee)} = \mathfrak{u}_\alpha \cdot V$, and looking if necessary at Remark 4:

$$\lambda \cdot v = h_{\alpha,\lambda} v = x_{-\alpha} x_{\alpha,\lambda} v - x_{\alpha,\lambda} x_{-\alpha} v = -x_{\alpha,\lambda} x_{-\alpha} v = -x_\alpha x_{-\alpha,\lambda} v$$

so that, using quadraticity of $\mathfrak{u}_\alpha$:

$$\begin{aligned}\lambda(\lambda' v) &= x_\alpha x_{-\alpha,\lambda} x_{\alpha,\lambda'} x_{-\alpha} v \\ &= x_\alpha h_{\alpha,\lambda\lambda'} x_{-\alpha} v \\ &= -2x_{\alpha,\lambda\lambda'} x_{-\alpha} v + h_{\alpha,\lambda\lambda'} x_\alpha x_{-\alpha} v \\ &= 2h_{\alpha,\lambda\lambda'} v - h_{\alpha,\lambda\lambda'} v \\ &= (\lambda\lambda')v\end{aligned}$$

as desired.

If $\mathcal{G} = G$, then multiplicativity in $\lambda$ is now obvious since $\tau_{\alpha,\lambda} = t_{\alpha,\lambda}$; we turn to additivity. But remember from Proposition 6 that $\partial_{\alpha,\lambda} w_\alpha v = -t_{\alpha,\lambda} v$, so that, using quadraticity of $U_\alpha$:

$$\begin{aligned}(\lambda + \lambda')v &= -\partial_{\alpha,\lambda+\lambda'} w_\alpha v \\ &= -(u_{\alpha,\lambda+\lambda'} - 1) w_\alpha v \\ &= -(u_{\alpha,\lambda} u_{\alpha,\lambda'} - 1) w_\alpha v \\ &= -(\partial_{\alpha,\lambda} + \partial_{\alpha,\lambda'} + \partial_{\alpha,\lambda}\partial_{\alpha,\lambda'}) w_\alpha v \\ &= -\partial_{\alpha,\lambda} w_\alpha v - \partial_{\alpha,\lambda'} w_\alpha v \\ &= \lambda v + \lambda' v\end{aligned}$$

as desired. ◊

*Claim* 3. The action of $\mathcal{G}$ on the $\mathbb{K}$-vector space $V$ is linear.

*Proof of Claim.* Remark that all operators $\omega_\beta$ for $\beta \in \Phi_s$ are linear by construction (and well-definedness of the action).

It could be tempting to prove linearity of one root $\mathbb{SL}_2$-substructure, say $\mathcal{G}_{\alpha_0}$, and of the Weyl group. The problem is that properly speaking, the Weyl group (the group of automorphisms *of the root system* generated by $\{\sigma_\beta : \beta \in \Phi_s\}$) does not act on $V$. Of course we just observed that $\omega_\beta$ does act linearly; the problem remains to see why the image of $\mathcal{G}$ in $\mathrm{End}(V)$ is generated by $\mathcal{G}_{\alpha_0}$ and the operators $\{\omega_\beta : \beta \in \Phi_s\}$. This is obvious in the case of the group but not entirely so in the case of the Lie ring. And though the latter question may be of independent interest, we take a side approach.

We shall first prove that all operators $\tau_{\beta,\lambda}$ for $(\beta, \lambda) \in \Phi_s \times \mathbb{K}^\times$ are linear. Notice that since $h_{-\beta,\lambda} = -h_{\beta,\lambda}$ and $t_{-\beta,\lambda} = t_{\beta,\lambda}^{-1}$ (see our realisation in §2.2 if necessary), this will actually imply linearity of $\tau_{\pm\beta,\lambda}$.

In the case of the group $\mathcal{G} = G$, assuming $\nu = \sigma_\gamma(\mu_0)$ and letting $v \in S_\nu$:

$$\begin{aligned}\tau_{\beta,\lambda}(\lambda' \cdot v) &= \tau_{\beta,\lambda}\omega_\gamma \tau_{\alpha_0,\lambda'} \omega_\gamma^{-1} v \\ &= \omega_\gamma \tau_{\alpha_0,\lambda'} \omega_\gamma^{-1} \tau_{\beta,\lambda} v \\ &= \lambda' \cdot (\tau_{\beta,\lambda} v)\end{aligned}$$

since $\omega_\gamma \tau_{\alpha_0,\lambda'} \omega_\gamma^{-1} \in \mathcal{T} \leq C_G(\tau_{\beta,\lambda})$.

In the case of the Lie ring $\mathcal{G} = \mathfrak{g}$ remember from Proposition 6 that in $\mathrm{End}(V)$ the operators $\omega_\alpha$ (and therefore operators $\omega_\gamma$ as well) normalise the image of the Cartan subring $\mathfrak{t}$ which is abelian. So we can carry exactly the same argument.

Hence $\mathcal{T}$ acts linearly in any case.

We can now deduce that all elements $\partial_{\pm\beta,\lambda}$ for $(\beta, \lambda) \in \Phi_s \times \mathbb{K}_+$ are linear. This will suffice for the linearity of $\mathcal{G}$. Fix $\nu \in \mathrm{cl}(\mu_0)$ and $v \in S_\nu$; also take $\lambda' \in \mathbb{K}$. We show that $\partial_{\pm\beta,\lambda}(\lambda' \cdot v) = \lambda' \cdot \partial_{\pm\beta,\lambda} v$. If $\langle \nu, \beta^\vee \rangle = 0$ there is nothing to prove as $\partial_{\pm\beta,\lambda}$ acts trivially on $S_\nu$. By symmetry we may assume $\langle \nu, \beta^\vee \rangle = -1$. Then $\partial_{-\beta,\lambda}$ acts as the zero map on $S_\nu$ and therefore is linear. Now $\omega_\beta S_\nu = S_{\sigma_\beta(\nu)} \leq \lfloor \mathcal{U}_\beta, V \rfloor$ so for any $v \in S_\nu$ one has by Proposition 6:

$$\partial_{\beta,\lambda} v = -\partial_{\beta,\lambda} \omega_\beta^2 v = \tau_{\beta,\lambda} \omega_\beta v$$

In particular, thanks to linearity of $\mathcal{T}$ and of $\omega_\beta$:

$$\partial_{\beta,\lambda}(\lambda' v) = \tau_{\beta,\lambda} \omega_\beta (\lambda' v) = \lambda' \cdot \tau_{\beta,\lambda} \omega_\beta v = \lambda' \cdot \partial_{\beta,\lambda} v$$

which proves linearity of $\partial_{\beta,\lambda}$. ◊

This completes the proof of Proposition 16. □

**Remark 17.** *The linear structure may seem to depend on both $\mu_0$ and $\alpha_0$. It actually depends on neither. This can be seen as a consequence of the remainder of the argument (but will not be used).*

## 2.7 Complete reducibility

So far we could assume that $V = V_{\mathrm{cl}(\mu_0)}$ with $\mu_0 \neq 0$, and turned it into a $\mathbb{K}[\mathcal{G}]$-module. Complete reducibility of $V$, and therefore its isomorphism type, is however not perfectly clear. The reader may even object that if $\mathbb{K} = \mathbb{F}_p$ (the field with $p$ elements), constructing a compatible $\mathbb{K}$-vector space structure on $V$ was not very impressive.

But actually we did much more than retrieving a linear structure: we explicitly performed the weight space decomposition of $V$. We contend that in order to conclude to complete reducibility and identification, it actually suffices to determine the weights involved — which we do now by elementary means without invoking [2, Chap. VIII, §7.3], as we promised that the present work would be self-contained.

**Proposition 18.** *$\mu_0$ is a minuscule weight.*

The list of minuscule weights is given in [2, Chap. VIII, end of §7.3]; we mentioned it in §1.3.

**Remark 19.** *As a matter of fact very little of Proposition 18 is actually needed to prove complete reducibility: Claims 1 and 2 suffice, namely the fact that up to the Weyl group action, we may assume $\mu_0$ to be non-negative on $\Phi_s$ and positive at exactly one $\alpha_0 \in \Phi_s$.*

*The rest of the proof will actually retrieve the list of minuscule weights.*

*Proof of Proposition 18.*

*Claim* 1. We may suppose that for all $\beta \in \Phi_s$, $\langle \mu_0, \beta^\vee \rangle \geq 0$.

*Proof of Claim.* This is because the topological closure of the positive chamber is a fundamental domain for the action of $W$ on $E$ [1, Chap. V, §3.3, Théorème 2]. ◊

Remember that a consequence of Proposition 11 we made explicit after its proof is that if $\mu \in M$ and $\alpha, \beta \in \Phi_s$ are adjacent in the Dynkin diagram, one cannot have $\langle \mu, \alpha^\vee \rangle = \langle \mu, \beta^\vee \rangle = 1$. This will be used repeatedly in the argument.

*Claim* 2. There is exactly one $\alpha \in \Phi_s$ with $\langle \mu_0, \alpha^\vee \rangle = 1$.

*Proof of Claim.* Suppose that there are a segment $\Sigma$ of the Dynkin diagram and a mass $\mu$ with both $\forall \gamma \in \Sigma$, $\langle \mu, \gamma^\vee \rangle \geq 0$ and two distinct $\alpha, \beta \in \Sigma$ with $\langle \mu, \alpha^\vee \rangle = \langle \mu, \beta^\vee \rangle = 1$. We may suppose the distance between $\alpha$ and $\beta$ to be minimal.

Notice that by Proposition 11, $\alpha$ and $\beta$ are not adjacent. Let $\gamma$ be the neighbour of $\alpha$ in $[\alpha\beta]$; by assumption, $\langle \mu, \gamma^\vee \rangle \geq 0$; by minimality, $\langle \mu, \gamma^\vee \rangle = 0$.

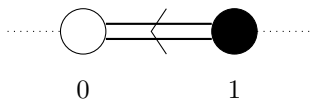
Let $\nu = \sigma_\alpha(\mu)$; clearly $\nu$ takes non-negative values on $[\gamma\beta]$ and $\langle \nu, \gamma^\vee \rangle = \langle \nu, \beta^\vee \rangle = 1$, against minimality of $[\alpha\beta]$. ◊

Let $\alpha_0$ be the unique simple root with $\langle \mu_0, \alpha_0^\vee \rangle = 1$. We shall draw Dynkin diagrams and label each simple root $\alpha$ with the value $\langle \mu, \alpha^\vee \rangle$.

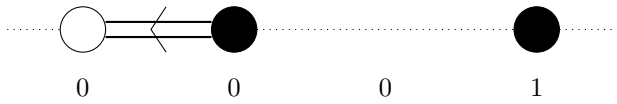
In case $\mathbb{G} = A_n$, we are done (see the list of minuscule weights for $A_n$); let us now handle types $B_n$ and $C_n$.

*Claim* 3. If the Dynkin diagram contains a double bond, then $\alpha_0$ is the extremal short root.

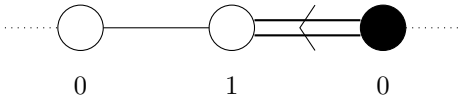
*Proof of Claim.* Notice that by Proposition 11, the following is inconsistent for any mass $\mu$:



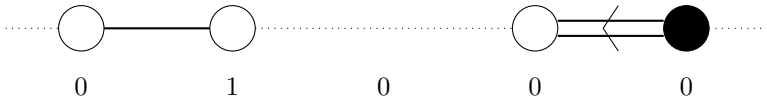
Therefore, inductively reflecting along the coroot with value 1, the following is inconsistent as well:




On the other hand, reflecting in the middle then in the left root, the following is inconsistent too:




Inductively reflecting in the next-to-left then in the left coroot, so is the following:

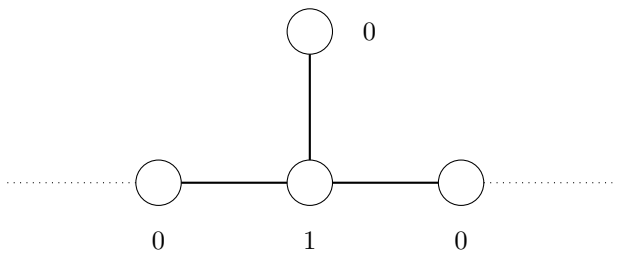



◊

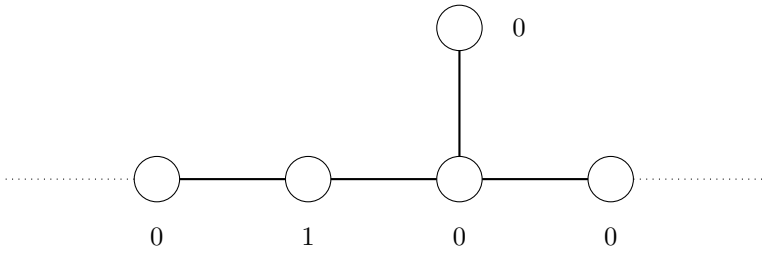
In particular this covers the cases of $B_n$ and $C_n$. We move to types $D_n$ and $E_n$.

*Claim* 4. If $\mathbb{G} = D_n$ or $E_n$ then $\alpha_0$ is extremal.

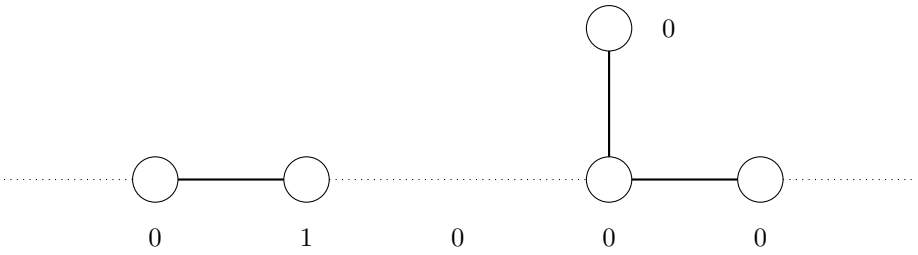
*Proof of Claim.* The following is easily seen inconsistent:




Therefore so is the following:




By induction so is the following:



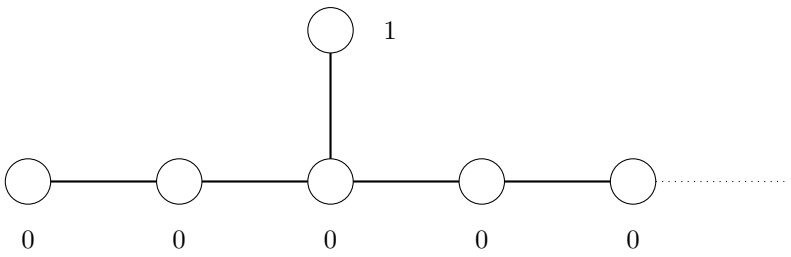

◊

This covers case $D_n$. We are not done with case $E_n$.

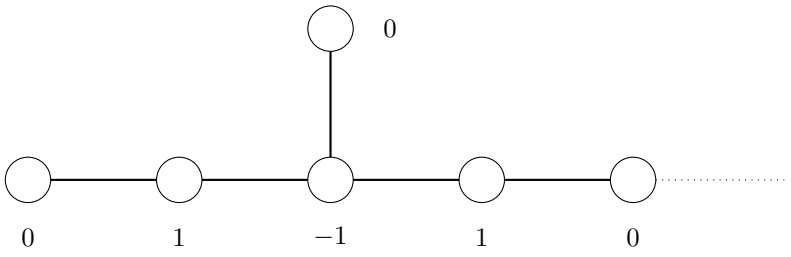
*Claim* 5. If $\mathbb{G} = E_n$ then $n = 6$ or $7$ and $\alpha_0$ is one of the roots (resp. the root) farthest from the arity 3 root.

*Proof of Claim.* We know from Claim 4 that $\alpha_0$ is extremal but there remains a number of configurations to kill.

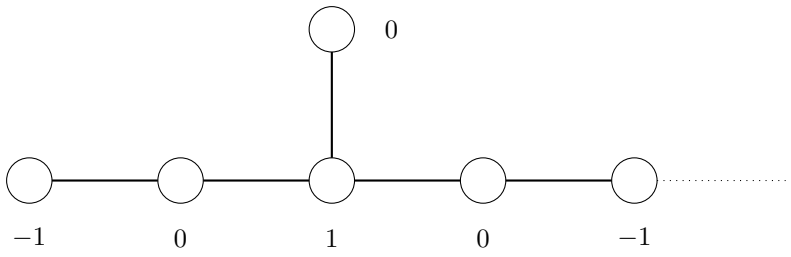
First, we shall check the following is inconsistent:




We see this by bringing the diagram into the following state:

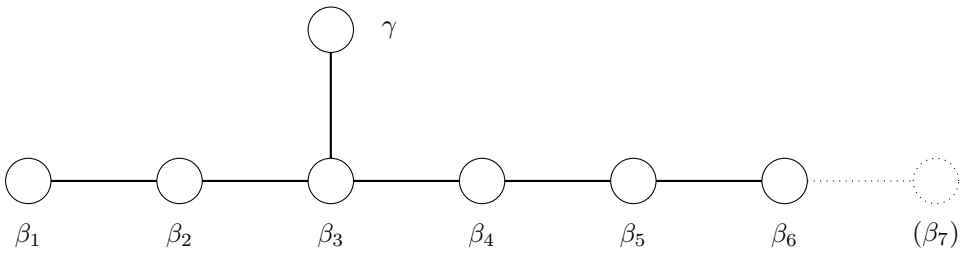



Then into:




an inconsistent configuration as we know from the proof of Claim 4.

The counting reader will find three more configurations to kill: one for $E_7$, two for $E_8$. We can remove two simultaneously. Perhaps we ought to make our notation more compact. Consider the diagram:




We tabulate consecutive masses until we reach inconsistency (each row describes a mass obtained

from the previous by reflecting in a coroot with value 1; an empty cell is an unchanged value):

| $\gamma$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $(\beta_7)$ |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | |
| | $-1$ | 1 | | | | | |
| | 0 | $-1$ | 1 | | | | |
| 1 | | 0 | $-1$ | 1 | | | |
| | | | 0 | $-1$ | 1 | | |
| | | | | 0 | $-1$ | 1 | |
| $-1$ | | | 1 | | | | |
| 0 | | 1 | $-1$ | 1 | | | |
| | | | 0 | $-1$ | 0 | | |
| | 1 | $-1$ | 1 | | | | |
| 1 | | 0 | $-1$ | 0 | | | |
| $-1$ | | | 0 | | | | |

In the final state, the value at $\beta_i^\vee$ for $i \in \{1, \ldots, 6\}$ is non-negative, and positive at both $\beta_1^\vee$ and $\beta_6^\vee$: this contradicts Claim 2.

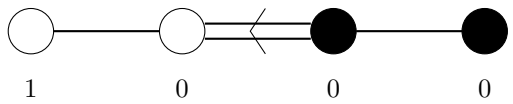
So there remains only one $E_8$ configuration, which we handle as follows.

| $\gamma$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\beta_7$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| | | | | | | 1 | $-1$ |
| | | | | | 1 | $-1$ | 0 |
| | | | | 1 | $-1$ | 0 | |
| | | | 1 | $-1$ | 0 | | |
| 1 | | 1 | $-1$ | 0 | | | |
| $-1$ | | | 0 | | | | |
| | 1 | $-1$ | 1 | | | | |
| | $-1$ | 0 | | | | | |
| 0 | | 1 | $-1$ | 1 | | | |
| | 0 | $-1$ | 0 | | | | |
| | | | 1 | $-1$ | 1 | | |
| 1 | | 0 | $-1$ | 0 | | | |
| $-1$ | | | 0 | | | | |
| | | | | 1 | $-1$ | 1 | |
| | | | 1 | $-1$ | 0 | | |
| 0 | | 1 | $-1$ | 0 | | | |
| | 1 | $-1$ | 0 | | | | |
| | $-1$ | 0 | | | | | |
| | | | | | 1 | $-1$ | 1 |
| | | | | 1 | $-1$ | 0 | |
| | | | 1 | $-1$ | 0 | | |
| 1 | | 1 | $-1$ | 0 | | | |
| $-1$ | | | 0 | | | | |

and $[\beta_2\beta_7]$ proves inconsistency of the final state. ◊

*Claim* 6. For $\mathbb{G} = F_4$ the configuration is inconsistent.

*Proof of Claim.* By Claim 3 only the following need be considered:




We leave it to the reader to push the configuration to inconsistency. ◊

*Claim* 7. For $\mathbb{G} = G_2$ the configuration is inconsistent.

*Proof of Claim.* By Proposition 11, $\alpha_0$ cannot be the long simple root (call it $\beta$) and is therefore the short root; reflecting in $\alpha_0^\vee$ we find $\langle \sigma_{\alpha_0}(\mu_0), \beta^\vee \rangle = -\langle \alpha_0, \beta^\vee \rangle = 1$, and then $\langle \sigma_\beta \sigma_{\alpha_0}(\mu_0), \alpha_0^\vee \rangle = -1 + 3 \notin \{-1, 0, 1\}$: a contradiction. $\Diamond$

This shows that $\mu_0$ is one of the minuscule weights. $\square$

**Remark 20.** *It is a consequence of the first two Claims 1 and 2 of Proposition 18 that $\mu_0 \in \mathrm{cl}(\mu_0)$ and suitable $\alpha_0$ are uniquely determined.*

*In particular, once the realisation of $\mathcal{G}$ is fixed in Notation 5, there is no choice involved in our construction of the $\mathbb{K}$-linear structure.*

Let us now finish the proof of the Main Theorem. We claim that $\mathcal{T}$ acts by scalars on $V$. We first see this on $S_{\mu_0}$, by definition of the action (Notation 15) and the first two Claims 1 and 2 of Proposition 18. This is true of $\mathcal{T} \cap \mathcal{G}_{\alpha_0}$ by construction, and the other intersections $\mathcal{T} \cap \mathcal{G}_\beta$ for $\beta \in \Phi_s \setminus \{\alpha_0\}$ act trivially; together they do generate $\mathcal{T}$. Now since operators $\omega_\gamma$ (defined in Notations 5 and 14) normalise the image of $\mathcal{T}$ in $\mathrm{End}(V)$ by Proposition 6, we find that $\mathcal{T}$ acts by scalars on all of $V = V_{\mathrm{cl}(\mu_0)}$.

For $v \in S_{\mu_0} \setminus \{0\}$ let $M_v = \oplus \mathbb{K} \omega_\gamma v$, where the sum is taken over a minimal set of sequences $\gamma$ representing the orbit $\mathrm{cl}(\mu_0)$ (otherwise it would fail to be a direct sum). By Proposition 6 again and the equation there, we now understand the action of $\partial_{\beta,\lambda}$ for $(\beta, \lambda) \in \Phi_s \times \mathbb{K}^\times$ on all rooms. In particular $\langle \mathcal{G} \cdot v \rangle = M_v$.

The latter is irreducible, even as a $\mathbb{Z}[\mathcal{G}]$-module. Indeed, let $a \in M_v \setminus \{0\}$; we want to show $v \in \langle \mathcal{G} \cdot a \rangle$. Remember that $V = V_{\mathrm{cl}(\mu_0)} = \oplus_{\mu \in \mathrm{cl}(\mu_0)} S_\mu$. Write $a = \sum_\mu a_\mu$ for components $a_\mu \in S_\mu$. In the proof of Proposition 9, such components were obtained by using operators in $\mathcal{G}$ (see Remark 10); hence each $a_\mu \in M_v$. Now because $\Phi_s$ is a spanning set of $E$, there is a product, say $f$, of operators $\partial_\beta$ for $\beta \in \Phi$, such that $f(a) = f(a_{\mu_1}) \neq 0$ for *one* $\mu_1 \in \mathrm{cl}(\mu_0)$. So we may as well suppose $a = a_{\mu_1} \in M_v \cap S_{\mu_1}$. Since the Weyl group is transitive on $\mathrm{cl}(\mu_0)$ by definition (and since it can be encoded in $\mathbb{Z}[\mathcal{G}]$ even in the case of the Lie ring), we may suppose $\mu_1 = \mu_0$, i.e. $a \in M_v \cap S_{\mu_0}$, which is a vector line. Transitivity of $\mathcal{T}$ on the latter (minus 0) now proves $v \in \langle \mathcal{G} \cdot a \rangle$.

We then let $M$ denote a maximal direct sum of such modules $M_v$, for various $v \in S_{\mu_0} \setminus \{0\}$. The isomorphism type of $M$ is known: it is a direct sum of representations with highest weight $\mu_0$ (which is minuscule in the classical sense, as proved by Proposition 18). Finally $V = V_{\mathrm{cl}(\mu_0)} = \langle \mathcal{G} \cdot S_{\mu_0} \rangle = M$.

---

Future variations will see our return to model theory: we shall untensor a cubic $\mathrm{SL}_2(\mathbb{K})$-module in the finite Morley rank category.

# References


[1] Nicolas Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines.* Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968.

[2] Nicolas Bourbaki. *Éléments de mathématique. Fasc. XXXVIII: Groupes et algèbres de Lie. Chapitre VII: Sous-algèbres de Cartan, éléments réguliers. Chapitre VIII: Algèbres de Lie semi-simples déployées.* Actualités Scientifiques et Industrielles, No. 1364. Hermann, Paris, 1975.

[3] Roger W. Carter. *Simple groups of Lie type.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1989. Reprint of the 1972 original, A Wiley-Interscience Publication.

[4] Adrien Deloro. Quadratic actions. *Confluentes Math.*, 5(2):23–41, 2013.

[5] Adrien Deloro. Symmetric powers of Nat $\mathfrak{sl}_2(\mathbb{K})$. *Communications in Algebra*, 44(5):1981–2008, 2016.

[6] Adrien Deloro. Symmetric powers of Nat $\mathrm{SL}_2(\mathbb{K})$. *Journal of Group Theory*, 19(4):661–692, 2016.

[7] Philippe Gille and Patrick Polo, editors. *Schémas en groupes (SGA 3). Tome III. Structure des schémas en groupes réductifs.* Documents Mathématiques (Paris), 8. Société Mathématique de France, Paris, 2011. Revised and annotated edition of the 1970 French original.

[8] Ralf Gramlich. Developments in finite Phan theory. *Innov. Incidence Geom.*, 9:123–175, 2009.

[9] Stephen D. Smith. Quadratic action and the natural module for $\mathrm{SL}_2(k)$. *J. Algebra*, 127(1):155–162, 1989.

[10] Robert Steinberg. *Lectures on Chevalley groups.* Yale University, New Haven, Conn., 1968. Notes prepared by John Faulkner and Robert Wilson.

[11] John G. Thompson. Quadratic pairs. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 375–376. Gauthier-Villars, Paris, 1971.

[12] Franz Georg Timmesfeld. *Abstract root subgroups and simple groups of Lie type*, volume 95 of *Monographs in Mathematics.* Birkhäuser Verlag, Basel, 2001.

[13] Franz Georg Timmesfeld. The Curtis-Tits-presentation. *Adv. Math.*, 189(1):38–67, 2004.

[14] Franz Georg Timmesfeld. Complete reducibility of quadratic modules for finite Lie-type groups. *J. Algebra*, 355:35–60, 2012.