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# On $\{a, b\}$ -edge-weightings of bipartite graphs with odd $a, b^*$

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## Abstract

For any  $S \subset \mathbb{Z}$  we say that a graph  $G$  has the  $S$ -property if there exists an  $S$ -edge-weighting  $w : E(G) \rightarrow S$  such that for any pair of adjacent vertices  $u, v$  we have  $\sum_{e \in E(v)} w(e) \neq \sum_{e \in E(u)} w(e)$ , where  $E(v)$  and  $E(u)$  are the sets of edges incident to  $v$  and  $u$ , respectively. This work focuses on  $\{a, a + 2\}$ -edge-weightings where  $a \in \mathbb{Z}$  is odd. We show that a 2-connected bipartite graph has the  $\{a, a + 2\}$ -property if and only if it is not a so-called odd multi-cactus. In the case of trees, we show that only one case is pathological. That is, we show that all trees have the  $\{a, a + 2\}$ -property for odd  $a \neq -1$ , while there is an easy characterization of trees without the  $\{-1, 1\}$ -property.

## 1 Introduction

Let  $G$  be an undirected graph. For an  $S$ -edge-weighting  $w : E(G) \rightarrow S$  of  $G$ , where  $S \subset \mathbb{Z}$ , each vertex  $v \in V(G)$  has *weighted degree* equal to the sum of the weights of its incident edges. We call  $w$  *neighbour sum-distinguishing* if no two adjacent vertices of  $G$  have the same weighted degree. For a set  $S$  of weights, we say that  $G$  has the  $S$ -property if it admits neighbour sum-distinguishing  $S$ -edge-weightings. The study of graphs having or not having the  $S$ -property for some sets  $S$  is highly related to the well-known **1-2-3 Conjecture** raised by Karonski, Luczak, and Thomason in 2004 [6]. That conjecture states that every connected graph different from  $K_2$ <sup>1</sup> has the  $\{1, 2, 3\}$ -property. A particular case of a list version of the 1-2-3 Conjecture (introduced by Bartnicki, Grytczuk, and Niwczyk [2]), even states that every graph should have the  $\{a, b, c\}$ -property for every distinct  $a, b, c \in \mathbb{N}$ . For more details on the progress towards the 1-2-3 Conjecture (and variants of it), please refer to [11] for a survey on this topic.

For any smaller set  $S \subset \mathbb{Z}$  of weights, *i.e.*, with  $|S| = 2$ , one can easily come up with examples showing that there do exist graphs not having the  $S$ -property (complete graphs are such examples).

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<sup>1</sup>This requirement is mandatory for any graph to be weightable; throughout this work, it is thus implicit, unless stated otherwise, that every considered graph does not have  $K_2$  as a connected component.

29 A natural question that has been investigated is about the existence of a good characterization  
30 of graphs that have the  $S$ -property for such smaller sets  $S$ . Here and further on, by a “good  
31 characterization” we mean a description in terms of a graph class whose members can be recognized  
32 in polynomial time. Dudek and Wajc [5] settled the question in the negative, as they proved that,  
33 unless  $P=NP$ , there is no good characterization of graphs with the  $\{1, 2\}$ -property, and similarly  
34 for the  $\{0, 1\}$ -property. Later on, noticing that, for any two distinct sets  $S, S' \subset \mathbb{Z}$  of weights  
35 with  $|S|, |S'| = 2$ , any neighbour sum-distinguishing  $S$ -edge-weighting of a regular graph yields a  
36 neighbour sum-distinguishing  $S'$ -edge-weighting, Ahadi, Dehghan, and Sadeghi [1] proved that there  
37 is no good characterization of graphs with the  $\{a, b\}$ -property for any two distinct  $a, b \in \mathbb{Z}$ .

38 From this point on, it thus made sense investigating, for any two distinct  $a, b \in \mathbb{Z}$ , sufficient  
39 conditions for graphs to have the  $\{a, b\}$ -property. A special focus has been dedicated to bipartite  
40 graphs, as 1) the aforementioned NP-completeness results were not known to hold in the bipartite  
41 context, and 2) bipartite graphs form one of the rare graph classes for which the 1-2-3 Conjecture is  
42 relatively well understood (see [6]). As a first step, several works [3, 4, 7, 8, 9] investigated whether  
43 there is a good characterization of bipartite graphs with the  $\{1, 2\}$ -property. Back then, it was  
44 believed that such a good characterization should exist, as, notably, all 3-connected bipartite graphs  
45 were proved to have the  $\{1, 2\}$ -property [9]. It was not until quite recently that Thomassen, Wu, and  
46 Zhang proved that, indeed, bipartite graphs without the  $\{1, 2\}$ -property are easy to describe [13].  
47 Namely, only so-called **odd multi-cacti** are bipartite and do not have the  $\{1, 2\}$ -property. These  
48 graphs are defined as follows (the comprehensive definition is from [10]; refer to Figure 2 later on  
49 for an illustration):

50 *“Take a collection of cycles of length 2 modulo 4, each of which has edges coloured alternately*  
51 *red and green. Then form a connected simple graph by pasting the cycles together, one by one, in a*  
52 *tree-like fashion along green edges; the resulting graph is an odd multi-cactus. The graph with one*  
53 *green edge and two vertices ( $K_2$ ) is also an odd multi-cactus. When replacing a green edge of an*  
54 *odd multi-cactus by a green edge of any multiplicity, we again obtain an odd multi-cactus.”*

55 One main ingredient behind Thomassen *et al.*'s result is the nice observation, already made back  
56 in [3], that, when  $a$  and  $b$  are integers with distinct parity, every bipartite graph  $G$  with bipartition  
57  $(X, Y)$  such that at least one of  $X$  and  $Y$  has even cardinality has the  $\{a, b\}$ -property. This is  
58 because, in such a case, one can easily construct  $\{a, b\}$ -edge-weightings of  $G$  where all vertices in  
59  $X$  have odd weighted degree while those in  $Y$  have even weighted degree. These observations also  
60 imply that, for  $a$  and  $b$  with distinct parity, bipartite graphs without the  $\{a, b\}$ -property have their  
61 two partite sets of odd cardinality, and they thus have even order.

62 Reusing some of Thomassen *et al.*'s ideas, Lyngsie later considered the  $\{0, 1\}$ -property for bi-  
63 partite graphs [10]. His main result is a good characterization of 2-edge-connected bipartite graphs  
64 without the  $\{0, 1\}$ -property, which turns out to be nothing but the class of odd multi-cacti. This  
65 result was established, in particular, through aforementioned tools and results for cases where  $a$  and  
66  $b$  have different parities. However, both Thomassen *et al.* and Lyngsie observed that there exist  
67 infinitely many separable (*i.e.*, with cut-vertices) bipartite graphs without the  $\{0, 1\}$ -property.

68 Although they are far from covering all the cases of  $a$  and  $b$ , the previous series of results show  
69 two things. First, that, when considering 2-connected bipartite graphs without the  $\{a, b\}$ -property,  
70 one should pay attention to odd multi-cacti. Second, that separable bipartite graphs without the

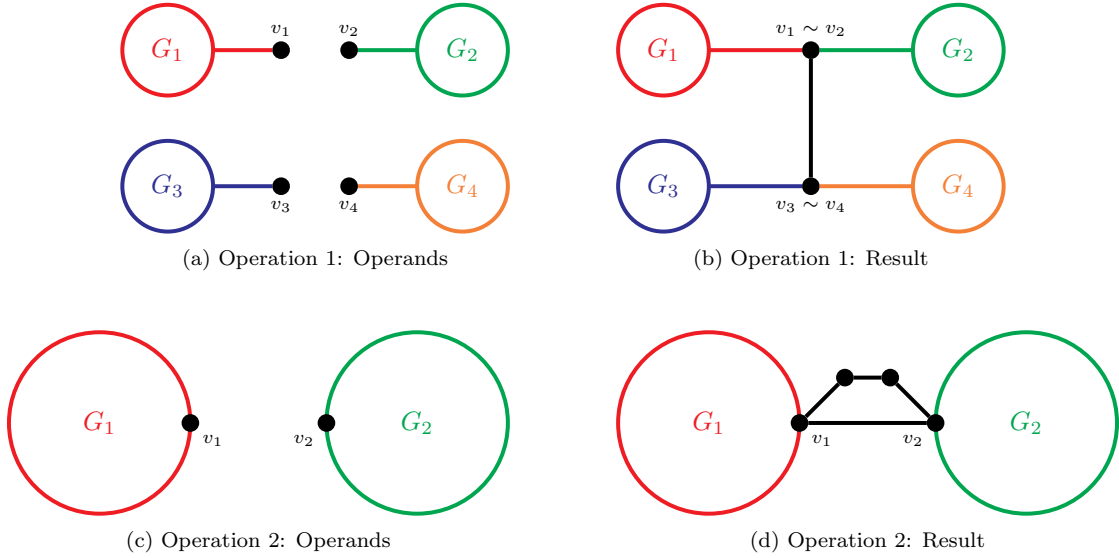


Figure 1: Constructing graphs without the  $\{-1, 1\}$ -property from graphs without that property.

71  $\{a, b\}$ -property and those without the  $\{a', b'\}$ -property may differ for different pairs  $a, b$  and  $a', b'$ .  
 72 This is already well illustrated by the class of trees: while they all have the  $\{1, 2\}$ -property [3],  
 73 infinitely many of them do not have the  $\{0, 1\}$ -property [10].

74 This paper is mainly devoted to studying  $\{a, b\}$ -properties where both  $a$  and  $b$  are odd. As a  
 75 first step, we focus on the cases where  $b = a + 2$ . We introduce mechanisms that are reminiscent of  
 76 the ones mentioned above (for  $a$  and  $b$  with distinct parity), which allow us to study the  $\{a, a + 2\}$ -  
 77 property for bipartite graphs and odd  $a \in \mathbb{Z}$ . One of the main results we get from these is that, for  
 78 any odd  $a$ , 2-connected bipartite graphs without the  $\{a, a + 2\}$ -property are precisely odd multi-cacti  
 79 again.

80 **Theorem 1.1.** *Let  $a, b \in \mathbb{Z}$  be odd integers with  $b = a + 2$ . A 2-connected bipartite graph  $G$  does*  
 81 *not have the  $\{a, b\}$ -property if and only if  $G$  is an odd multi-cactus.*

82 Similarly as for the  $\{0, 1\}$ -property, the structure of separable bipartite graphs without the  $\{a, b\}$ -  
 83 property for odd  $a$  and  $b$  does not appear obvious. As a first step, we give a special focus on the  
 84 case  $\{a, b\} = \{-1, 1\}$ . In that case, we can already point out two operations that, given bipartite  
 85 graphs without the  $\{-1, 1\}$ -property, clearly provide more separable bipartite graphs without the  
 86  $\{-1, 1\}$ -property (see Figure 1):

- 87 • Let  $G_1, G_2, G_3, G_4$  be four bipartite graphs without the  $\{-1, 1\}$ -property, and let  $v_1, v_2, v_3, v_4$   
 88 be any four degree-1 vertices of  $G_1, G_2, G_3, G_4$ , respectively. The operation (see Figure 1 (a)  
 89 and (b)) consists in considering the disjoint union  $G_1 \cup G_2 \cup G_3 \cup G_4$ , identifying the vertices  
 90  $v_1$  and  $v_2$ , identifying the vertices  $v_3$  and  $v_4$ , and adding an edge joining the two vertices  
 91 resulting from these identifications (*i.e.*,  $v_1 \sim v_2$  and  $v_3 \sim v_4$ ).
- 92 • Let  $G_1, G_2$  be two bipartite graphs without the  $\{-1, 1\}$ -property, and let  $v_1, v_2$  be any two  
 93 vertices of  $G_1, G_2$ , respectively. The operation (see Figure 1, (c) and (d)) consists in consid-

94 ering the disjoint union  $G_1 + G_2$ , adding the edge  $v_1, v_2$ , and further joining  $v_1, v_2$  by a path  
 95 with odd length at least 3.

96 In the case of trees, when  $a$  and  $b$  are any two non-zero integers that are both positive (or  
 97 negative), it is easy to see that  $K_2$  is the only tree without the  $\{a, b\}$ -property: consider a vertex  $v$   
 98 whose all neighbours  $u_1, \dots, u_{d-1}$  but one  $u_d$  (if any) are leaves, remove  $u_1, \dots, u_{d-1}$ , apply induction  
 99 to deduce a neighbour sum-distinguishing  $\{a, b\}$ -edge-weighting, and extend the weighting to the  
 100 edges  $vu_1, \dots, vu_{d-1}$  so that the conflict  $vu_d$  is avoided. Thus, when  $b = a + 2$  and  $a, b$  are odd,  
 101 only the case  $a = -1, b = 1$  is potentially non-trivial. In Section 3, we show that trees without the  
 102  $\{-1, 1\}$ -property can all be constructed through the first operation above (illustrated in Figure 1,  
 103 (a) and (b)) performed on  $K_2$ 's.

104 **Theorem 1.2.** *A tree does not have the  $\{-1, 1\}$ -property if and only if it can be constructed from  
 105 a disjoint union of  $K_2$ 's through repeated applications of the first operation above.*

106 In particular, the structure of trees without the  $\{-1, 1\}$ -property is very different and simpler  
 107 than that of trees without the  $\{0, 1\}$ -property (for more on the structure of these trees, see [10]).  
 108 Recall that all trees have the  $\{1, 2\}$ -property, as was shown, e.g., in [3].

109 **Terminology and notation.** Let  $G$  be a connected graph. For a given vertex  $v$  of  $G$  we denote  
 110 by  $E(v)$  the set of edges incident to  $v$ . A *bridge* in  $G$  is an edge whose removal results in two  
 111 components. Let  $w$  be an edge-weighting of  $G$ . Abusing the notation, the weighted degree of  $v$  in  
 112  $G$  by  $w$  will sometimes be denoted  $w(v)$  for convenience. We say that an edge  $uv$  of  $G$  is a *conflict*  
 113 by  $w$  if  $w(u) = w(v)$ . In other words,  $w$  is neighbour sum-distinguishing if no edge is a conflict.  
 114 In what follows, we will instead use the term *proper* in place of neighbour sum-distinguishing to  
 115 lighten the writing. By an  *$x$ -edge* of  $G$  (by  $w$ ), we mean an edge assigned weight  $x$  by  $w$ .

## 116 2 Proof of Theorem 1.1

117 In this section, we prove that for every odd integer  $a \in \mathbb{Z}$ , the class of 2-connected bipartite graphs  
 118 without the  $\{a, a + 2\}$ -property is exactly that of odd multi-cacti. Another way to define these  
 119 graphs is as follows. Start from  $K_2$ , the simple connected graph on two vertices, having its only  
 120 edge coloured green. Then, repeatedly apply an arbitrary number of the following operation (see  
 121 Figure 2 for an illustration). Consider any green edge  $uv$  of the current graph, and join  $u, v$  by a  
 122 new path  $P$  of length  $\ell \geq 1$  congruent to 1 modulo 4 whose edges are coloured red and green as  
 123 follows:

- 124 • if  $\ell = 1$ , *i.e.*,  $P$  has a unique edge, then this edge is green;
- 125 • if  $\ell \geq 5$ , then the edges of  $P$  are coloured red and green properly (*i.e.*, no two subsequent  
 126 edges have the same colour) so that the two end-edges are red.

127 Figure 2 notably shows that performing this operation multiple times for a same green edge is  
 128 allowed, and that adding paths of length 1 is similar to increasing the multiplicity of a green edge.  
 129 Note also that it is not possible to get two adjacent green edges with distinct ends at any point  
 130 of the process. Furthermore, every obtained graph is bipartite. An odd multi-cactus is any graph

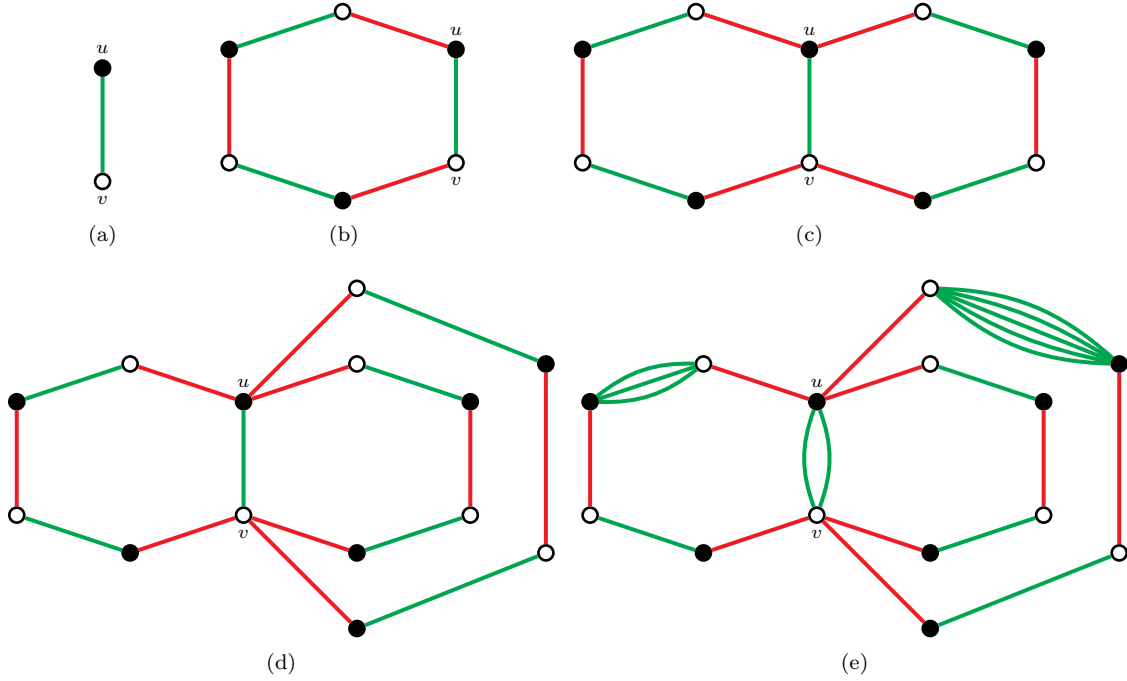


Figure 2: Constructing an odd multi-cactus through several steps, from  $K_2$  (a). Red-green paths with length at least 5 congruent to 1 modulo 4 are being attached onto the green edge  $uv$  through steps (b) to (d). In step (e), (green) paths of length 1 are added, which corresponds to increasing the multiplicity of some green edges.

131 that can be obtained during this process, no matter how many times the operation is applied. In  
132 particular,  $K_2$  itself is regarded as an odd multi-cactus.

133 In this section, we will implicitly use several properties of odd multi-cacti, such as:

134 **Observation 2.1.** *Let  $M$  be an odd multi-cactus with its edges being coloured red and green as*  
135 *described above. Then:*

- 136 •  $M$  is 2-connected;
- 137 • when replacing every (green) edge of  $M$  by an edge with multiplicity 1, a 2-degenerate graph  
138 (i.e., a graph in which every subgraph has a vertex of degree at most 2) is obtained;
- 139 • for every green edge  $uv$  of  $M$ , we have  $d_M(u) = d_M(v)$ .

140 Having the structure of odd multi-cacti in mind, it can be proved that the following holds true.

141 **Lemma 2.2.** *If  $G$  is not an odd multi-cactus and was obtained from an odd multi-cactus  $M$  by*  
142 *replacing a red edge with an edge of multiplicity at least 2 or by replacing a green edge by a path*  
143 *of length  $k \geq 5$  with  $k \equiv 1 \pmod{4}$ , then  $G$  has the  $\{a, b\}$ -property for any two distinct integers*  
144  *$a, b \in \mathbb{Z}$ .*

145 *Proof.* The proof is by induction on the order of  $M$ . So suppose that  $G$  is obtained from an odd  
146 multi-cactus  $M$  by replacing an edge  $e$  with either an edge of multiplicity at least 2 (if  $e$  is red in

147  $M$ ) or a path of length  $k \geq 5$  with  $k \equiv 1 \pmod{4}$  (if  $e$  is green in  $M$ ). It is easy to check that the  
148 statement is true if  $M$  is just a cycle with some multiple green edges. So we can focus on cases  
149 where  $M$  was obtained in the general way, *i.e.*, by pasting together cycles with length 2 modulo 4  
150 having possibly green edges of any multiplicity. Furthermore, it is easy to check that the statement  
151 is true if  $M$  was constructed by only pasting cycles together along one single green edge  $e'$  as in  
152 Figures 2 (c), (d) and (e) where there have only been pasted cycles together along the edge  $uv$ :  
153 in this case, since  $G$  is not an odd multi-cactus,  $G$  is either obtained from  $M$  by replacing a red  
154 edge with an edge of multiplicity at least 2, or  $e = e'$  must be a simple edge and  $G$  is obtained by  
155 replacing that edge  $e$  with a path of length  $k \geq 5$  with  $k \equiv 1 \pmod{4}$  and  $k \geq 5$ . In both cases it is  
156 easy to check that  $G$  has the  $\{a, b\}$ -property.

157 Thus, we can assume that  $M$  was obtained by pasting together at least three cycles and that  
158 there are at least two disjoint edges to which other cycles have been pasted to. This implies that  
159 there are at least two disjoint cycles of length congruent to 2 modulo 4 where all vertices except  
160 two which are adjacent have exactly two neighbours. One of these cycles  $C = v_1v_2\dots v_nv_1$  does not  
161 contain  $e$ . By possibly relabelling the vertices we can assume that all vertices of  $C$  except  $v_1$  and  
162  $v_2$  only have two distinct neighbours. By induction the graph  $G'$  obtained from  $G$  by replacing the  
163 path  $v_2v_3\dots v_n$  with an edge  $e''$  has the  $\{a, b\}$ -property, but any proper  $\{a, b\}$ -edge-weighting of  $G'$   
164 can be converted to a proper  $\{a, b\}$ -edge-weighting of  $G$  by assigning the same weight to an edge in  
165  $G$  as in  $G'$  and assigning the weight assigned to  $e''$  in  $G'$  to the edges  $v_2v_3$  and  $v_{n-1}v_n$ , and finally  
166 assigning the weights of the remaining edges  $v_2v_3, \dots, v_{n-2}v_{n-1}$  in a way avoiding conflicts inside  
167  $C$ . □

168 We now introduce or recall results that will be needed during the course of our main proof below.  
169 The following first observation is obvious and implies that studying the  $\{a, b\}$ -property only makes  
170 sense when  $\gcd(a, b) = 1$ .

171 **Observation 2.3.** *Let  $w$  be a proper  $\{a, b\}$ -edge-weighting of a graph  $G$ . If we multiply all edge*  
172 *weights of  $w$  by a non-zero integer  $\alpha$ , then we get a proper  $\{\alpha a, \alpha b\}$ -edge-weighting of  $G$ .*

173 In what follows, given a graph  $G$  and a mapping  $f : V(G) \rightarrow \mathbb{Z}_k$ , by an  $f$ -factor modulo  $k$  we  
174 mean a spanning subgraph  $H$  of  $G$  such that, for every  $v \in V(G)$ , we have  $d_H(v) \equiv f(v) \pmod{k}$ .

175 **Lemma 2.4** (Thomassen [12]). *Let  $G$  be a connected graph. If  $f : V(G) \rightarrow \mathbb{Z}_2$  is a mapping*  
176 *satisfying  $\sum_{v \in V(G)} f(v) \equiv 0 \pmod{2}$ , then  $G$  contains an  $f$ -factor modulo 2.*

177 When dealing with bipartite graphs with a bipartition set of even cardinality, and when  $a$  and  $b$   
178 have distinct parity,  $f$ -factors modulo 2 can be employed as a convenient tool to deduce proper  $\{a, b\}$ -  
179 edge-weightings in quite an easy way (see [10, 13]). More precisely, let  $(X, Y)$  be the bipartition of a  
180 bipartite graph  $G$  where  $|X|$  is even. Lemma 2.4 (when applied onto the function  $f$  where  $f(x) = 1$   
181 for  $x \in X$  and  $f(y) = 0$  for  $y \in Y$ ) implies that  $G$  has a spanning subgraph  $H$  where all of the  
182 vertices in  $X$  have odd degree, while all of the vertices in  $Y$  have even degree. From this, it is easy  
183 to see that, assuming  $a$  is odd and  $b$  is even, assigning weight  $a$  to all of the edges in  $E(H)$  and  
184 weight  $b$  to all of the edges in  $E(G) \setminus E(H)$  yields a proper  $\{a, b\}$ -edge-weighting of  $G$ .

185 The upcoming new tools and concepts (in particular that of *mod-4 vertex-colourings*) are the  
186 key to generalize this approach to odd  $a, b \in \mathbb{Z}$  when  $|a - b| = 2$ .

187 **Definition 2.5.** A mod-4 vertex-colouring of a graph  $G$  is a vertex-colouring  $c : V(G) \rightarrow \{1, 2\}$  of  
 188  $G$  satisfying the following conditions for any  $uv \in E(G)$  where  $d(u)$  and  $d(v)$  have the same parity:

189 1.  $d(u) \equiv d(v) \pmod{4} \Rightarrow c(u) \neq c(v)$ .

190 2.  $d(u) \not\equiv d(v) \pmod{4} \Rightarrow c(u) = c(v)$ .

191 In the next result, we prove that every bipartite graph  $G$  admits a mod-4 vertex-colouring  $c$ .  
 192 It is important to point out that, in general,  $c$  might be far from fitting with the bipartition of  $G$ .  
 193 Actually,  $G$  might have many edges whose two ends have the same colour by  $c$ .

194 **Lemma 2.6.** Every bipartite graph has a mod-4 vertex-colouring.

195 *Proof.* It suffices to prove the lemma for connected bipartite graphs where all vertices have odd  
 196 degree or where all vertices have even degree (as otherwise we can consider, still in the whole graph,  
 197 the vertices with even degree first, and then those with odd degree). So let  $G$  be a connected  
 198 bipartite graph where all vertex degrees have the same parity. Let  $v$  be a vertex in  $G$  and let  
 199  $D_0, D_1, \dots, D_m$  denote the distance classes of  $G$  from  $v \in D_0$ . Since  $G$  is bipartite, each  $D_i$  is an  
 200 independent set. Now give  $v$  colour 1 and colour the distance classes in the given order starting  
 201 with  $D_1$ , then  $D_2$  and so on until we reach a vertex  $v' \in D_{i'}$  we cannot assign a colour without  
 202 violating conditions 1 or 2 in Definition 2.5. If this happens one or both of the following two cases  
 203 have occurred:

204 1. there are two neighbours  $v_1, v_2 \in D_{i'-1}$  of  $v'$  with  $d(v_1) \equiv d(v_2) \pmod{4}$  and  $c(v_1) \neq c(v_2)$ ;

205 2. there are two neighbours  $v_1, v_2 \in D_{i'-1}$  of  $v'$  with  $d(v_1) \not\equiv d(v_2) \pmod{4}$  and  $c(v_1) = c(v_2)$ .

206 Let us first assume that we are in the first case and let  $P_1, P_2$  be two internally disjoint shortest  
 207 paths towards  $v$  starting with  $v'v_1$  and  $v'v_2$  respectively and ending in a common vertex  $v'' \in D_{i''}$ .  
 208 That is,  $v''$  is the first vertex on both  $P_1$  and  $P_2$  that is encountered when going from  $v_1$  towards  
 209  $v$  along  $P_1$ ; possibly  $v'' = v$ . All the vertices of  $P_1$  and  $P_2$  except  $v'$  are coloured without violating  
 210 conditions 1 and 2 in Definition 2.5, and  $P_1$  and  $P_2$  have the same length. The parity of the number  
 211 of times the degree modulo 4 changes when walking from  $v''$  to  $v_1$  on  $P_1$  is the same as the parity  
 212 of the number of times the degree modulo 4 changes when walking from  $v''$  to  $v_2$  on  $P_2$ . Thus, the  
 213 parity of the number of times the degree modulo 4 does not change when walking from  $v''$  to  $v_1$   
 214 on  $P_1$  is the same as the parity of the number of times the degree modulo 4 does not change when  
 215 walking from  $v''$  to  $v_2$  on  $P_2$ . Since conditions 1 and 2 in Definition 2.5 are not violated, this implies  
 216 that the parity of the number of times the colour changes when walking from  $v''$  towards  $v'$  is the  
 217 same when walking along  $P_1$  as when walking along  $P_2$ . Thus,  $c(v_1) = c(v_2)$ , a contradiction. The  
 218 second case above can be dealt with in a similar way.  $\square$

219 Let  $a, b \in \mathbb{Z}$  be two odd integers with  $b = a + 2$ . Let  $G$  be a graph and  $X, Y$  be two disjoint  
 220 subsets of its vertices. By an  $(X, Y)$ - $a$ -parity  $\{a, b\}$ -edge-weighting of  $G$ , we mean an  $\{a, b\}$ -edge-  
 221 weighting where all vertices in  $X$  are incident to an odd number of  $a$ -edges and all vertices in  $Y$  are  
 222 incident to an even number of  $a$ -edges.  $(X, Y)$ - $b$ -parity  $\{a, b\}$ -edge-weightings are defined similarly,  
 223 but with respect to the incident  $b$ -edges. In the following result, we establish a crucial connection  
 224 between mod-4 vertex-colourings and  $(X, Y)$ -parity  $\{a, b\}$ -edge-weightings, leading to the existence  
 225 of proper  $\{a, b\}$ -edge-weightings.



226 **Lemma 2.7.** *Let  $G$  be a connected bipartite graph and let  $a, b \in \mathbb{Z}$  be odd integers with  $b = a + 2$ .  
 227 If  $G$  has a mod-4 vertex-colouring where at least one of the two colour classes has even size, then  $G$   
 228 has the  $\{a, b\}$ -property. Consequently, if  $G$  does not have the  $\{a, b\}$ -property, then, in every mod-4  
 229 vertex-colouring, the two colour classes have odd size.*

230 *Proof.* Let  $G$  be a connected bipartite graph, and  $c$  a mod-4 vertex-colouring of  $G$ . We denote by  
 231  $X$  and  $Y$  the sets of vertices with colour 1 and 2, respectively. Assume  $|X|$  is even. By Lemma 2.4  
 232 there is an  $\{a, b\}$ -edge-weighting  $w : E(G) \rightarrow \{a, b\}$  such that all vertices in  $X$  are incident to an odd  
 233 number of  $b$ -edges and all vertices in  $Y$  are incident to an even number of  $b$ -edges. This corresponds  
 234 to our notion of an  $(X, Y)$ - $b$ -parity  $\{a, b\}$ -edge-weighting. The possible weighted degrees of a vertex  
 235  $v$  of even degree and colour 1 induced by such an edge-weighting are  $\{a(d(v) - 1) + b, a(d(v) - 1) +$   
 236  $b + 4, a(d(v) - 1) + b + 8, \dots, a + b(d(v) - 1)\}$  and the possible weighted degrees of a vertex  $v'$  of even  
 237 degree and colour 2 induced by such an edge-weighting are  $\{ad(v'), ad(v') + 4, ad(v') + 8, \dots, bd(v')\}$ .  
 238 The possible weighted degrees of a vertex  $u$  of odd degree and colour 1 induced by such an edge-  
 239 weighting are  $\{a(d(u) - 1) + b, a(d(u) - 1) + b + 4, a(d(u) - 1) + b + 8, \dots, bd(u)\}$  and the possible  
 240 weighted degrees of a vertex  $u'$  of odd degree and colour 2 induced by such an edge-weighting are  
 241  $\{ad(u'), ad(u') + 4, ad(u') + 8, \dots, a + b(d(u') - 1)\}$ . Let  $xy \in E(G)$ . We will show that  $w(x) \neq w(y)$ .  
 242 To do this we distinguish two distinct cases (note that we can assume that  $x$  and  $y$  have the same  
 243 degree parity, as otherwise  $w(x)$  cannot be equal to  $w(y)$ ):

- 244 1.  $x$  and  $y$  have the same colour by  $c$ .
- 245 2.  $x$  and  $y$  have distinct colours by  $c$ .

246 First assume that  $x$  and  $y$  have the same colour. Since  $c$  is a mod-4 vertex-colouring we have  
 247 that  $d(x) \not\equiv d(y) \pmod{4}$ . Note that by the above it suffices to show that  $ad(x) \not\equiv ad(y) \pmod{4}$  and  
 248 this is trivially true since  $\gcd(a, 4) = 1$ . Now assume that  $x$  and  $y$  have distinct colours. Since  $c$  is  
 249 a mod-4 vertex-colouring we have that  $d(x) \equiv d(y) \pmod{4}$ . Note that by the above it suffices to  
 250 show that  $ad(x) \equiv ad(y) \pmod{4}$ , and as mentioned above this follows since  $\gcd(a, 4) = 1$ .  $\square$

251 From the previous proof, we can also extract the following:

252 **Observation 2.8.** *Let  $a, b \in \mathbb{Z}$  be odd integers with  $b = a + 2$  and let  $uv$  be an edge in a graph  $G$   
 253 whose edges are weighted with  $a$  and  $b$ . If either*

- 254 1.  $d(u)$  and  $d(v)$  have distinct parity, or
- 255 2.  $d(u) \equiv d(v) \pmod{4}$  and  $v$  is incident to an odd number of  $a$ -edges while  $u$  is incident to an  
 256 even number of  $a$ -edges, or
- 257 3.  $d(u) \not\equiv d(v) \pmod{4}$  and both  $v$  and  $u$  are incident to an odd or even number of  $a$ -edges,

258 then  $u$  and  $v$  have distinct weighted degrees. This is also true if one considers the parity of the  
 259 numbers of incident  $b$ -edges instead of the parity of the numbers of incident  $a$ -edges.

260 Let  $G$  be a graph and  $w$  an  $\{a, b\}$ -edge-weighting of  $G$ . By *swapping* (the weight of) an edge, we  
 261 mean changing its weight to  $a$  if it is a  $b$ -edge, or changing its weight to  $b$  otherwise. By swapping  
 262 a path or a cycle, we mean swapping all of its edges. For a vertex  $v$  in a cycle  $C$  of  $G$ , it can be

263 observed that the parity of the number of  $a$ -edges (and similarly  $b$ -edges) incident to  $v$  is not altered  
 264 upon swapping  $C$ . In the proof of our main result below, this fact will be used a lot to get rid of  
 265 conflicts in the following way.

266 Let  $X, Y$  be the two colour classes of a mod-4 vertex-colouring of  $G$  and assume that, for some  
 267 vertex  $v \in X$ ,  $w$  is an  $(X \setminus \{v\}, Y \cup \{v\})$ - $a$ -parity  $\{a, b\}$ -edge-weighting, *i.e.*, all vertices in  $X \setminus \{v\}$   
 268 are incident to an odd number of  $a$ -edges and all vertices in  $Y \cup \{v\}$  are incident to an even number of  
 269  $a$ -edges. According to Observation 2.8, all conflicts (if any) involve  $v$ . So let  $uv$  be a conflict. To get  
 270 rid of this conflict while controlling the possible creation of new conflicts, we will swap particular  
 271 cycles of  $G$ . Let  $C$  be a cycle of  $G$  going through  $u$  using two edges  $e, e'$  incident to  $u$ . If  $e, e'$  are  
 272 assigned the same weight by  $w$ , then  $C$  is called  $u$ -changing.  $C$  will be called  $v$ -avoiding if it does  
 273 not go through  $v$ .

274 **Observation 2.9.** *Let  $G$  be a graph,  $X, Y$  be the two colour classes of a mod-4 vertex-colouring of*  
 275  *$G$ , and let  $w$  be an  $(X \setminus \{v\}, Y \cup \{v\})$ - $a$ -parity  $\{a, b\}$ -edge-weighting for some vertex  $v$ , where  $a, b \in \mathbb{Z}$*   
 276 *are odd integers with  $b = a + 2$ . If  $uv$  is a conflict, then, by swapping a  $u$ -changing  $v$ -avoiding cycle*  
 277  *$C$  of  $G$ , we get rid of this conflict. Furthermore, any remaining/arising conflicts involve  $v$ .*

278 *Proof.* According to Observation 2.8, all original conflicts by  $w$  must involve  $v$ . When swapping  $C$ ,  
 279 the weighted degree of  $u$  is altered since  $C$  is  $u$ -changing, while the weighted degree of  $v$  is unaltered  
 280 since  $C$  is  $v$ -avoiding. So we get rid of the conflict  $uv$ . Furthermore, it can be noticed that, upon  
 281 swapping any cycle of  $G$ , the parities of the number of  $a$ -edges (and similarly  $b$ -edges) incident to the  
 282 vertices are unaltered. Therefore, we get another  $(X \setminus \{v\}, Y \cup \{v\})$ - $a$ -parity  $\{a, b\}$ -edge-weighting,  
 283 and Observation 2.8 indicates that, after the swapping of  $C$ , all conflicts (if any) in the resulting  
 284  $\{a, b\}$ -edge-weighting must involve  $v$ . □

285 We finish with a few general lemmas to be used in particular cases of our upcoming main proof.

286 **Lemma 2.10.** *Let  $G$  be a 2-connected bipartite graph,  $X, Y$  be the two colour classes of a mod-4*  
 287 *vertex-colouring of  $G$ , and let  $a, b \in \mathbb{Z}$  be odd integers with  $b = a + 2$ . If both  $X$  and  $Y$  have odd*  
 288 *size and  $v \in X$  is such that  $G - v - N(v)$  is connected, then there is an  $(X \setminus \{v\}, Y \cup \{v\})$ - $a$ -parity*  
 289  *$\{a, b\}$ -edge-weighting of  $G$  where all edges incident to  $v$  have weight  $b$  and every vertex  $u \in N(v)$  is*  
 290 *incident to at most  $1 + M(uv)$   $b$ -edges, where  $M(uv)$  denotes the multiplicity of the edge  $uv$ .*

291 *Proof.* Suppose  $G' = G - v - N(v)$  is connected. Let  $G''$  be obtained from  $G - v$  by, for every vertex  
 292  $u \in N(v)$ , removing all edges but one incident to  $u$  in  $G - v$ . For each  $u \in N(v)$ , let  $e_u$  be the  
 293 unique edge incident to  $u$  in  $G''$  and let  $n(u)$  denote the unique neighbour of  $u$  in  $G''$ . Note that  
 294 since  $G'$  is connected, then so is  $G''$ . Let  $S$  denote the set of edges in  $G$  not incident to  $v$  and not in  
 295  $G''$ . That is,  $S$  is the set of edges removed from  $G - v$  to obtain  $G''$ . Let  $G[S]$  denote the subgraph  
 296 of  $G$  induced by the edges in  $S$  and let  $Z$  denote the vertices of odd degree in  $G[S]$ . Clearly  $|Z|$  is  
 297 even, so, since  $X \setminus \{v\}$  has even size, the set  $X' = (X \setminus (Z \cup \{v\})) \cup Z \cap Y$  also has even size. Thus,  
 298 Lemma 2.4 implies that there is an  $(X', V(G'') \setminus X')$ - $a$ -parity  $\{a, b\}$ -edge-weighting of  $G''$ . We now  
 299 extend this weighting to  $G$  by assigning weight  $a$  to all edges in  $S$  and weight  $b$  to all edges in  $E(v)$ ;  
 300 this results in a desired edge-weighting of  $G$ . □

301 **Lemma 2.11.** *Let  $G$  be a 2-connected bipartite graph. If there is a vertex  $v \in V(G)$  of degree at*  
 302 *least 4 and with  $|N(v)| \geq 3$  such that  $G - v - N(v)$  is connected, then  $G$  has the  $\{-1, 1\}$ -property.*

303 *Proof.* By Lemma 2.10, there is an  $(X \setminus \{v\}, Y \cup \{v\})$ - $(-1)$ -parity  $\{-1, 1\}$ -edge-weighting of  $G$ , where  
304 all edges incident to  $v$  have weight 1 and any vertex  $u \in N(v)$  is incident to at most  $1 + M(uv)$   
305 1-edges, where  $M(uv)$  denotes the multiplicity of the edge  $uv$ . Observation 2.8 implies that the only  
306 potential conflicts are between  $v$  and its neighbours. But the weighted degree of  $v$  is  $d(v)$  and since  
307  $|N(v)| \geq 3$ , we have for any  $u \in N(v)$  that the multiplicity of  $uv$  is less than  $d(v) - 1$ . Thus, the  
308 weighted degree of any  $u \in N(v)$  is less than  $d(v)$  and therefore, there can be no conflicts.  $\square$

309 **Lemma 2.12** (Thomassen, Wu, Zhang [13]). *Let  $q$  be a natural number such that  $q \geq 4$ . Let  $G$  be a*  
310 *connected graph and let  $A$  be an independent set of at most  $q$  vertices such that each vertex in  $A$  has*  
311 *degree at least  $q - 1$ , or, each vertex in  $A$ , except possibly one, has degree at least  $q$ . Assume that no*  
312 *vertex in  $A$  is adjacent to a bridge in  $G$ . Then, for each vertex  $a$  of  $A$ , there is an edge  $e_a$  incident*  
313 *with  $a$  such that the deletion of all  $e_a$ ,  $a \in A$ , results in a connected graph unless  $|A| = q = 4$ , all*  
314 *vertices of  $A$  have degree 3, and  $G - A$  has six components each of which is joined to two distinct*  
315 *vertices of  $A$ .*

316 Let  $a$  and  $b$  be two odd integers with  $b = a + 2$ . In some cases Lemmas 2.4 and 2.12 work well  
317 together when trying to construct a proper  $\{a, b\}$ -edge-weighting of a connected bipartite graph  
318  $G$ : suppose  $c : V(G) \rightarrow \{1, 2\}$  is a mod-4 vertex-colouring of  $G$  and let  $X$  and  $Y$  denote the sets  
319 of vertices in  $G$  of colour 1 and 2 respectively and assume that both  $X$  and  $Y$  have odd size.  
320 Furthermore, suppose that the degree of a vertex  $v \in X$  is at least 4 and no vertex in  $N(v)$  has  
321 degree strictly larger than 4. Let  $A$  be the vertices in  $N(v)$  with the same degree as  $v$  and suppose  
322 that no vertex in  $A$  is incident to a bridge in  $G - v$ , the graph  $G - v$  is connected, and we are  
323 not in the exceptional case of Lemma 2.10, that is, for each  $u \in A$  there is an edge  $e_u$  such that  
324  $G - \cup_{u \in A} \{e_u\}$  is connected. Define  $S = \cup_{u \in A} \{e_u\}$  and let  $Z$  denote the set of vertices in  $G$  which  
325 have odd degree in the subgraph of  $G$  induced by  $S$  (note that  $A \subset Z$ ). Since  $X \setminus \{v\}$  and  $Z$   
326 have even size, the set  $X' = (X \setminus (Z \cup \{v\})) \cup Z \cap Y$  also has even size. Thus, Lemma 2.4 implies  
327 that there is an  $(X', V(G - v) \setminus X')$ - $a$ -parity  $\{a, b\}$ -edge-weighting of  $G - v$ . We can extend this  
328 edge-weighting to  $G$  by assigning weight  $a$  to all edges in  $S$  and weight  $b$  to all edges in  $E(v)$  to  
329 obtain an  $(X \setminus \{v\}, Y \cup \{v\})$ - $a$ -parity  $\{a, b\}$ -edge-weighting of  $G$  where all edges incident to  $v$  have  
330 weight  $b$  and every vertex  $u \in N(v)$  is incident to at least one  $a$ -edge. Thus, the weighted degree  
331 of  $v$  is greater than that of its neighbours and Observation 2.8 implies that the edge-weighting is  
332 proper.

333 We are now ready to prove the main result of this section. Let us emphasize that the main steps  
334 in the proof follow the lines of those in the proofs of the main results in [10] and [13]. In particular,  
335 Claims 2, 3, 4, 5, as stated below, can also be found in [10]. The proofs are rather different, though,  
336 as most arguments used to deal with the  $\{0, 1\}$ -property do not apply immediately for the  $\{a, b\}$ -  
337 property when  $a, b \in \mathbb{Z}$  are odd and  $b = a + 2$ . Instead, some of the tools and results we have  
338 introduced earlier are used. Also, the end of the proof in our case is more straightforward than  
339 those for the  $\{1, 2\}$ - and  $\{0, 1\}$ -properties.

340 *Proof of Theorem 1.1.* Suppose the theorem is false and, for some odd  $a \in \mathbb{Z}$  and  $b = a + 2$ , let  $G$   
341 be a counterexample which has smallest order possible. By possibly multiplying the weights by  $-1$   
342 we can assume  $b > 0$ . Let  $c$  be a mod-4 vertex-colouring of  $G$  (such a colouring exists by Lemma

2.6), and let  $X$  denote the set of vertices with colour 1 and  $Y$  denote the set of vertices with colour 2. By Lemma 2.7, we can assume that both  $X$  and  $Y$  have odd size.

**Claim 1.**  $G$  has no multiple edge  $uv$  where both  $u$  and  $v$  have only two distinct neighbours.

*Proof of the claim.* Suppose  $uv$  is a multiple edge and both  $u$  and  $v$  have only two distinct neighbours. By the minimality of  $G$ , Lemma 2.2, and the fact that  $G$  is not an odd multi-cactus, the graph obtained from  $G$  by replacing  $uv$  with one non-multiple edge has a proper  $\{a, b\}$ -edge-weighting  $w$ . But since the multiplicity of  $uv$  in  $G$  is at least 2 and since  $u$  and  $v$  can each be in only one conflict distinct from  $uv$ , we can obtain a proper  $\{a, b\}$ -edge-weighting of  $G$  from  $w$  by weighting the edges joining  $u$  and  $v$  to avoid the conflicts involving  $u$  and  $v$  (such an edge-weighting exists because there are at least three possible sums for the weight of edges joining  $u$  and  $v$ ).  $\diamond$

Since  $G$  is 2-connected, it has minimum degree at least 2. In what follows, by a *suspended path* of  $G$ , we mean a path  $v_1x_1\dots x_kv_2$  where all internal vertices  $x_1, \dots, x_k$  have degree 2 and  $v_1$  and  $v_2$  have degree at least 3.

**Claim 2.**  $G$  has no suspended path of length 2.

*Proof of the claim.* Suppose the claim is false and let  $v_1xv_2$  be a suspended path in  $G$ , where  $d(x) = 2$  and  $d(v_1), d(v_2) \geq 3$ . We can assume  $x \in X$ . Define  $G' = G - x$ . Recall that  $G'$  is connected since  $G$  is 2-connected. Lemma 2.4 implies that there is an  $(X \setminus \{x\}, Y \cup \{x\})$ - $a$ -parity  $\{a, b\}$ -edge-weighting  $w$  of  $G$ , where  $w(v_1x) = w(v_2x) = a$ . More precisely,  $w$  can be obtained as follows (recall that  $|V(G') \cap X|$  is even):

- If  $v_1, v_2 \in X$ , then, according to Lemma 2.4, there is an  $(X \setminus \{v_1, v_2, x\}, Y \cup \{v_1, v_2\})$ - $a$ -parity  $\{a, b\}$ -edge-weighting of  $G'$ . Then, assigning weight  $a$  to  $v_1x$  and  $v_2x$  gives the desired weighting of  $G$ .
- If  $v_1, v_2 \in Y$ , then the same conclusion can be reached when applying Lemma 2.4 so that we start from an  $(X - \{x\} \cup \{v_1, v_2\}, Y \setminus \{v_1, v_2\})$ - $a$ -parity  $\{a, b\}$ -edge-weighting of  $G'$ .
- If  $v_1 \in X$  and  $v_2 \in Y$  (resp.  $v_2 \in X$  and  $v_1 \in Y$ ), then, again, we can get the same conclusion after applying Lemma 2.4 from an  $(X \setminus \{v_1, x\} \cup \{v_2\}, Y \cup \{v_1\} \setminus \{v_2\})$ - $a$ -parity  $\{a, b\}$ -edge-weighting (resp.  $(X \setminus \{v_2\} \cup \{v_1\}, Y \cup \{v_2\} \setminus \{v_1\})$ - $a$ -parity  $\{a, b\}$ -edge-weighting) of  $G'$ .

Observation 2.8 implies that the only conflicts that can arise are  $xv_1$  and  $xv_2$ . So we can assume that  $x$  and  $v_1$  are in conflict, *i.e.*, both  $x$  and  $v_1$  have weighted degree  $2a$ . This implies that  $v_1$  has even degree at least 4 (by the definition of a suspended path, and only vertices with degree of the same parity can be in conflict) and  $a < 0 < b$  and hence  $a = -1$  and  $b = 1$ . Let  $u_1, u_2$  be two neighbours of  $v_1$  in  $G'$  such that  $u_1v_1$  and  $u_2v_1$  have weight  $-1$  and let  $C$  be a cycle in  $G'$  using the edges  $u_1v_1$  and  $v_1u_2$ . Such a cycle exists as, because  $G$  is 2-connected, there is a path from  $u_1$  to  $u_2$  in  $G - v_1$ . Because  $C$  is  $v_1$ -changing and  $x$ -avoiding, if we swap all weights on  $C$  we do not create new conflicts in  $G'$  and we lose the conflict  $xv_1$  (recall Observation 2.9). In particular,  $x$  remains of weighted degree  $-2$  while  $v_1$  becomes of weighted degree 2.

Thus, we can now assume that  $xv_2$  is a conflict. This implies that  $v_2$  also has even degree at least 4. We can now get rid of this conflict in the same way as we got rid of the conflict  $(v_1, x)$ , unless all

382  $v_2$ -changing cycles in  $G'$  (that thus use two edges in  $E(v_2)$  having the same weight) all use two edges  
 383 in  $E(v_1)$  both having weight 1 (in this case we only move the conflict from  $xv_2$  to  $xv_1$ ). Since  $xv_2$  is  
 384 a conflict,  $v_2$  must be incident to at least two  $-1$ -edges in  $G'$  and at least one  $1$ -edge. Furthermore,  
 385 as mentioned above, we can assume that any  $v_2$ -changing cycle in  $G'$  contains two edges incident  
 386 to  $v_1$  having weight 1.

387 Assume that  $v_1$  is incident to a  $-1$ -edge  $e$  in  $G'$ . Since  $G$  is 2-connected, there is, in  $G - v_1$ , a  
 388 path from  $v_2$  to the end of  $e$  different from  $v_1$ . From the existence of that path, we get that there is  
 389 a path  $P$  in  $G'$  from  $v_1$  to  $v_2$  using  $e$ . If the weight on the last edge  $e'$  of  $P$  (the one incident to  $v_2$ )  
 390 is 1, then swapping the weights on the cycle  $P \cup v_1x \cup xv_2$  yields a proper edge-weighting; so we can  
 391 assume  $e'$  has weight  $-1$ . Since  $xv_2$  is a conflict and  $v_2$  has even degree at least 4, vertex  $v_2$  must  
 392 be incident to a  $-1$ -edge  $e'' \neq e'$  in  $G'$ . Now, because  $G$  is 2-connected, the graph  $G - v_2$  has a path  
 393  $P'$  joining the end of  $e'$  different from  $v_2$  and the end of  $e''$  different from  $v_2$ . Note that if  $P'$  does  
 394 not contain  $v_1$ , then we would get a cycle whose weights can be swapped to immediately deduce a  
 395 proper edge-weighting of  $G$ . The same conclusion holds if  $P'$  and  $P$  intersect for the first time on  
 396 a vertex different from  $v_1$ . So  $v_1$  is the first intersection point between  $P$  and  $P'$ , in which case we  
 397 deduce a cycle of  $G$  containing  $v_2$  as well as all of  $e, e', e''$  (first go from  $v_2$  to  $v_1$  along  $P'$ , before  
 398 going back to  $v_2$  along  $P$ ); when swapping the weights along that cycle, we get rid of all conflicts  
 399 between  $x$  and  $v_1, v_2$ .

400 We are left with the case where  $v_1$  is not incident to a  $-1$ -edge  $e$  in  $G'$ . Thus, we deduce that  
 401  $v_1$  has degree 4 and, by symmetry,  $v_2$  also has degree 4. Note that this implies that  $v_1, v_2 \in X$   
 402 and furthermore, that all four edges incident to  $v_1$  except  $v_1x$  have weight 1. Also, according to all  
 403 hypotheses so far,  $v_1$  has weighted degree 2,  $v_2$  has weighted degree  $-2$ , and so two edges incident  
 404 to  $v_2$  in  $G'$  have weight  $-1$  while the last edge incident to  $v_2$  in  $G'$  has weight 1. We now consider  
 405 the graph  $G'' = G' - v_1 - v_2$ . If  $G''$  is connected, then we can find in  $G'$  a cycle including the two  
 406  $-1$ -edges incident to  $v_2$  (thus  $v_2$ -changing), and not passing through  $v_1$ ; then, as earlier, we can  
 407 swap the weights along this cycle to get rid of the conflict  $v_2x$ . So we may assume the two  $-1$ -edges  
 408 incident to  $v_2$  in  $G'$  are incident to two distinct components of  $G''$ . This leaves us with the following  
 409 three cases to consider:

410 **Case 1:**  $G''$  has two components  $C_1, C_2$ , such that  $v_1$  has two neighbours in  $C_1$  and one neighbour  
 411 in  $C_2$ , and  $v_2$  has two neighbours in  $C_2$  and one neighbour in  $C_1$ .

412 Let  $e_{1,a}$  and  $e_{1,b}$  denote the two edges incident to  $v_1$  going to  $C_1$ . Recall that  $e_{1,a}, e_{1,b}$  have  
 413 weight 1. Since  $C_1$  is connected, there is a path from the end of  $e_{1,a}$  different from  $v_1$  to the end  
 414 of  $e_{1,b}$  different from  $v_1$ . We swap all weights along the cycle formed by this path and  $e_{1,a}, e_{1,b}$  to  
 415 get another edge-weighting of  $G$  where  $v_1$  has weighted degree  $-2$ ; so, now, both  $xv_1$  and  $xv_2$  are  
 416 conflicts.

417 Now let  $e_{1,c}$  denote the edge incident to  $v_1$  going to  $C_2$ , and  $e_{2,a}$  denote the  $1$ -edge incident to  
 418  $v_2$  going to  $C_2$ . Both these edges are weighted 1. Since  $C_2$  is connected, there is a path  $P$  from  
 419 the end of  $e_{1,c}$  different from  $v_1$  to the end of  $e_{2,a}$  different from  $v_2$ . Now consider the cycle of  $G$   
 420 starting in  $x$ , going through  $xv_2$  and  $e_{2,a}$ , then going along  $P$ , and finally going through  $e_{1,c}$  and  $v_1x$ .  
 421 When swapping all weights along this cycle, note that  $v_1, v_2$  remain of weighted degree  $-2$ , while  
 422  $x$  becomes of weighted degree 2. So the  $\{-1, 1\}$ -edge-weighting of  $G$  becomes proper according to  
 423 Observation 2.9.

424 **Case 2:**  $G''$  has two components  $C_1, C_2$ , such that both  $v_1$  and  $v_2$  have two neighbours in  $C_1$  and  
 425 one neighbour in  $C_2$ .

426 Let  $v_{1,a}, v_{1,b}$  denote the two neighbours of  $v_1$  in  $C_1$ , and let  $v_{1,c}$  denote the neighbour of  $v_1$  in  
 427  $C_2$ . Note that for one of  $v_{1,a}, v_{1,b}$ , say  $v_{1,a}$ , the graph  $G''' = G - v_1 - v_{1,a} - v_{1,c}$  is connected, for  
 428 if  $G'$  is disconnected, then it must be the case that  $v_{1,a}$  is a cut-vertex in  $G - v_1$  and then it is  
 429 easy to see that  $G - v_1 - v_{1,b} - v_{1,c}$  is connected, and we can just rename  $v_{1,a}$  and  $v_{1,b}$  accordingly.  
 430 Note that, by Lemma 2.11, we can assume that  $G''' - v_{1,b}$  is disconnected. Let  $L_1, \dots, L_n$  denote the  
 431 components of  $G''' - v_{1,b}$ , where  $v_2 \in V(L_1)$ . Since  $v_{1,b}$  is not a cut-vertex in  $G$ , it follows that  $v_{1,a}$   
 432 has a neighbour in each of the components  $L_i$  for  $i \geq 2$ .

433 Let us now consider the graph obtained from  $G$  by removing the vertex  $v_1$ , and, for each of  
 434  $v_{1,a}, v_{1,c}$ , removing all remaining incident edges but one. Note that, in that graph,  $x$  also has de-  
 435 gree 1. Lemma 2.4 implies that this graph has an  $(X \setminus \{v_1\}, Y)$ - $(-1)$ -parity  $\{-1, 1\}$ -edge-weighting.  
 436 By then assigning weight  $-1$  to all removed edges incident to  $v_1$  and weight 1 to all remaining edges  
 437 (incident to one of  $v_{1,a}, v_{1,c}$ ), we get that  $G$  has an  $(X \setminus \{v_1\}, Y)$ - $(-1)$ -parity  $\{-1, 1\}$ -edge-weighting  
 438 where all four edges incident to  $v_1$  are weighted  $-1$ , and each of  $v_{1,a}, v_{1,c}$  is incident to at most two  
 439  $-1$ -edges. Now the only possible conflict is  $v_1 v_{1,b}$ , so we can assume this is indeed a conflict, and  
 440 hence, both  $v_1$  and  $v_{1,b}$  have weighted degree  $-4$ . We can also assume that we cannot swap the  
 441 weights in a cycle in  $G'''$  containing two edges incident to  $v_{1,b}$  having the same weight, and hence,  
 442  $v_{1,b}$  has at most two neighbours in each of  $L_i$  for  $i = 1, \dots, n$ . Since the weighted degree of  $v_{1,b}$  is  
 443  $-4$ , there must be some components in  $G''' - v_{1,b}$  which are incident to strictly more  $-1$ -edges in  
 444  $E(v_{1,b})$  than 1-edges in  $E(v_{1,b})$ . Again, since we can assume that there is no cycle in  $G'''$  containing  
 445 two edges incident to  $v_{1,b}$  having the same weight, we can also deduce that no two edges incident to  
 446  $v_{1,b}$  having the same weight go to the same component in  $G''' - v_{1,b}$ . Thus, there are at least three  
 447 components  $L'_1, L'_2, L'_3$  in  $G''' - v_{1,b}$ , each of which is incident to only one edge in  $E(v_{1,b})$  and each  
 448 of these edges has weight  $-1$ . We can assume that  $L'_1$  and  $L'_2$  are distinct from  $L_1$ , and, since  $v_{1,b}$   
 449 is not a cut-vertex in  $G$ , the vertex  $v_{1,a}$  has a neighbour  $u'_i$  in each  $L'_i$  for  $i = 1, 2$ . There is now a  
 450 cycle  $C$  in  $G''' + v_{1,a}$  containing two edges incident to  $v_{1,b}$  having weight  $-1$  and containing the two  
 451 edges  $v_{1,a}u'_1$  and  $v_{1,a}u'_2$ . If we swap the weights on  $C$ , then the only possible conflict is  $v_1 v_{1,a}$  in the  
 452 case where  $v_{1,a}$  is a vertex of degree 4 and  $v_1 v_{1,a}$  has weight  $-1$ , both  $v_{1,a}u'_1$  and  $v_{1,a}u'_2$  have weight  
 453 1, and  $v_{1,a}$  is incident to some fourth  $-1$ -edge  $v_{1,a}u'$ . We can assume that the component  $L'$  to  
 454 which  $u'$  belongs in  $G''' - v_{1,b}$  is not incident to a  $-1$ -edge of  $v_{1,b}$ , since otherwise, we could modify  
 455  $C$  to contain the edge  $v_{1,a}u'$ . Note that this also implies that  $L'_3 = L_1$ . Since  $v_{1,b}$  had weighted  
 456 degree  $-4$  this implies that there is another component  $L'_4$  distinct from all of  $L'_1, L'_2, L'_3$ , which is  
 457 incident to an edge in  $E(v_{1,b})$  having weight  $-1$ . The vertex  $v_{1,a}$  must have a neighbour  $u''$  in this  
 458 component  $L'_4$  and, since we can assume that we cannot modify  $C$  to contain  $v_{1,a}u'$ , we must have  
 459  $u'' \neq u'$ . This contradicts  $v_{1,a}$  having degree 4.

460 **Case 3:**  $G''$  has three components  $C_1, C_2, C_3$ , such that  $v_1$  and  $v_2$  have one neighbour in each of  
 461 these three components.

462 In that case,  $G$  has the  $\{-1, 1\}$ -property according to Lemma 2.11 as  $v_1$  has even degree 4, and  
 463 it can be checked that  $G - v_1 - N_G(v_1)$  remains connected due to the 2-connectedness of  $G$ . In  
 464 particular, for  $i = 1, 2, 3$ , note that the edge incident to  $v_1$  going to  $C_i$  and the edge incident to  $v_2$   
 465 going to  $C_i$  cannot share an end. A contradiction.  $\diamond$

466 **Claim 3.**  $G$  has no suspended path of length 4.

467 *Proof of the claim.* Suppose the claim is false and let  $v_1x_1x_2x_3v_2$  be a suspended path in  $G$ ,  
 468 where  $d(x_1) = d(x_2) = d(x_3) = 2$  and  $d(v_1), d(v_2) \geq 3$ . We can assume  $x_2 \in X$ , which implies  
 469 that  $x_1, x_3 \in Y$ . Define  $G' = G - x_1 - x_2 - x_3$ . Using Lemma 2.4 similarly as in the proof of  
 470 Claim 2, we can come up with an  $(X \setminus \{x_1\}, Y \setminus \{x_2, x_3\})$ - $a$ -parity  $\{a, b\}$ -edge-weighting  $w$  of  $G$   
 471 where  $w(v_1x_1) = w(x_3v_2) = a$  and  $w(x_1x_2) = w(x_2x_3) = b$ . One can check that, by slightly  
 472 modifying the exact same arguments used in the proof of Claim 2, we can eventually remove all  
 473 conflicts from  $w$ , or deduce another proper  $\{a, b\}$ -edge-weighting of  $G$ .  $\diamond$

474 **Claim 4.**  $G$  has no suspended path of length at least 5.

475 *Proof of the claim.* Suppose the claim is false and let  $v_1x_1x_2x_3x_4v_2$  be a path in  $G$ , where  
 476  $x_1, x_2, x_3, x_4$  all have degree 2, and  $v_1, v_2$  here might be of degree 2. Let  $G'$  be obtained from  
 477  $G$  by replacing  $v_1x_1x_2x_3x_4v_2$  by an edge  $e = v_1v_2$  even if that edge is already there. If  $G'$  has the  
 478  $\{a, b\}$ -property, then so does  $G$ . Indeed, assume there is a proper  $\{a, b\}$ -edge-weighting of  $G$  where  
 479 the weight of  $e$  is, say  $a$ , and consider that weighting back in  $G$ . We start the extension to the five  
 480 edges by assigning weight  $a$  to  $v_1x_1$  and  $x_4v_2$ , so that  $v_1$  and  $v_2$  keep the same weighted degree as  
 481 in  $G'$ . Since  $v_1v_2$  is an edge in  $G'$ , note that  $v_1$  and  $v_2$  have different weighted degrees. From this,  
 482 we deduce that either  $v_1$  has weighted degree different from  $2a$  and  $v_2$  has weighted degree different  
 483 from  $a + b$ , or conversely. Assume the first situation holds. Then, we can achieve the weighting of  
 484  $G$  by assigning weight  $a$  to  $x_1x_2$  and weight  $b$  to  $x_2x_3$  and  $x_3x_4$ .

485 So we can assume that  $G'$  does not have the  $\{a, b\}$ -property and is thus an odd multi-cactus by  
 486 the minimality of  $G$ . The edge  $e$  cannot be red in  $G'$ , since then  $G$  would also be an odd multi-  
 487 cactus. Thus  $e$  is green and Lemma 2.2 implies that  $G$  has the  $\{a, b\}$ -property.  $\diamond$

488  
 489 By Claims 1, 2, 3, 4, all degree-2 vertices in  $G$  (if any) lie on suspended paths of length 3. In  $G$   
 490 we replace all suspended paths of length 3 by edges (even if the two ends were already adjacent) to  
 491 form a bipartite multigraph  $G^*$ . Edges arising from suspended paths of length 3, we call *blue edges*.  
 492 Every other edge of  $G^*$ , *i.e.*, which was already present in  $G$ , we call a *white edge*.

493 Note that  $G^*$  is bipartite, 2-connected, has minimum degree at least 3, and it may have more  
 494 multiple edges than  $G$  has. Also, note that for every vertex  $v$  in  $G^*$ , we have  $d_{G^*}(v) = d_G(v)$ .  
 495 In general, it is not easy to deduce a proper  $\{a, b\}$ -edge-weighting of  $G$  from one of  $G^*$  (typically  
 496 because of blue edges); however, information on the structure of  $G$  can be deduced from that of  $G^*$ .  
 497 In particular, we will study the existence of paths or cycles in  $G^*$  to deduce that of corresponding  
 498 paths or cycles in  $G$  (where any traversed blue edge in  $G^*$  is replaced by the corresponding path of  
 499 length 3 in  $G$ ).

500 If the deletion of some pair of adjacent vertices  $u, v$  disconnects  $G^*$ , then let  $z_0y_0 \in E(G^*)$  be  
 501 such that  $G - z_0 - y_0$  is disconnected and such that some component  $H$  of  $G^* - z_0 - y_0$  has smallest  
 502 possible order. The union of that component  $H$  and  $z_0, y_0$  together with all edges connecting them  
 503 is denoted  $B$ . In case  $G$  has no pair of adjacent vertices whose removal disconnects the graph, we  
 504 define  $H = B = G^*$ , and  $y_0, z_0$  do not exist.

505 **Claim 5.** For every vertex  $v$  of  $H$ , we have  $d_{G^*}(v) = 3$ .

506 *Proof of the claim.* Suppose the claim is false and let  $w_0$  be a vertex in  $H$  of maximum degree  
507  $d = d(w_0)$  at least 4. Without loss of generality, we can suppose that  $w_0 \in X$ . Assume first that  
508  $w_0$  is adjacent to none of  $z_0, y_0$  (this is the case if these two vertices do not exist). By the remark  
509 following Lemma 2.12, we can assume that we are in the exceptional case when considering  $G - w_0$   
510 and defining  $A$  to be the set of vertices in  $N(w_0)$  which have the same degree as  $w_0$ . Thus,  $w_0$  and  
511 all vertices in  $N(w_0)$  have degree 4 and  $G - w_0 - N(w_0)$  has exactly six components. We can now  
512 choose another vertex of degree  $d$  as  $w_0$  by choosing  $w_0$  such that the order of the component of  
513  $G - w_0 - N(w_0)$  containing  $z_0$  is maximum and avoid the exceptional case in Lemma 2.12. Again,  
514 the remark following Lemma 2.12 shows how to find a proper  $\{a, b\}$ -edge-weighting of  $G$ .

515 So we can assume  $w_0$  is adjacent to, say,  $z_0$  and that all the neighbours of  $w_0$  in  $H$  which have  
516 the same degree as  $w_0$  are adjacent to  $y_0$  (due to the bipartiteness of  $G^*$ ). Since  $y_0, z_0$  thus exist,  
517 we have  $G \neq H$ . We can also assume that all vertices in  $H$  having maximum degree are adjacent  
518 to  $z_0$  or  $y_0$  (as otherwise, the previous situation would apply). Note that this implies that we can  
519 never be in the exceptional case in Lemma 2.12 when we delete a vertex  $v$  in  $H$  of maximum degree  
520 and define  $A$  to be the neighbours of  $v$  with the same degree as  $v$ .

521 As pointed out earlier, there is an  $(X \setminus \{w_0\}, Y)$ - $a$ -parity  $\{a, b\}$ -edge-weighting of  $G$  where all  
522 edges incident to  $w_0$  are weighted  $b$  and every neighbour of  $w_0$  with degree  $d$  is incident to at least  
523 one  $a$ -edge. This edge-weighting is proper unless  $z_0 w_0$  is a conflict, which occurs only if the degree  
524 of  $z_0$  is strictly greater than that of  $w_0$ . Note that we can assume that  $z_0$  is incident to exactly  
525 one edge going to each component other than  $H$  in  $G - y_0 - z_0$ , since otherwise, we could deduce  
526 an  $\{a, b\}$ -edge-weighting of  $G$  as above with the extra condition that two edges  $e, e'$  incident to  $z_0$   
527 going to a component  $C$  of  $G - y_0 - z_0$  other than  $H$  are weighted  $b$ . Then, if  $z_0 w_0$  is a conflict, we  
528 could get rid of it by swapping the weights along a cycle going through  $z_0$  and  $C$  via  $e, e'$  (so that  
529 it is  $z_0$ -changing) and not going through  $H$  (so that it is  $w_0$ -avoiding). So  $z_0$  is incident to exactly  
530 one edge going to each component other than  $H$  in  $G - y_0 - z_0$ . We can also assume that there is at  
531 most one component  $C$  other than  $H$  in  $G - y_0 - z_0$ , since otherwise, we could reach the exact same  
532 conclusion by deducing an  $\{a, b\}$ -edge-weighting of  $G$  as before with the extra condition that two  
533 edges  $e, e'$  incident to  $z_0$  going to two different components  $C, C'$  distinct from  $H$  are weighted  $b$ .  
534 In case  $z_0 w_0$  is a conflict, we could again get rid of it by swapping the weights along a cycle going  
535 through  $z_0$ , in  $C$  via  $e$ , back to  $y_0$ , in  $C'$ , and back to  $z_0$  via  $e'$ . This would be correct since such a  
536 cycle would not go through  $H$ , and thus, would be  $w_0$ -avoiding. Similarly, we can assume that the  
537 multiplicity of  $z_0 y_0$  is 1.

538 Let us denote by  $z_1$  the unique neighbour of  $z_0$  in  $C$ . By swapping the weights along a cycle  
539 through  $C$  containing  $z_0, y_0$  and the edge  $z_0 z_1$ , and not going through  $H$ , we can further assume that  
540 the edge  $z_0 z_1$  is weighted  $a$ . For a similar reason, we can assume that the edge  $y_0 z_0$  is weighted  $b$ .  
541 Recall that  $w_0$  and all the neighbours of  $w_0$  with degree  $d$  are all incident to an even number of  
542  $a$ -edges; thus, each of the neighbours of  $w_0$  with degree  $d$  is incident to at least two  $a$ -edges.

543 To get rid of the conflict  $z_0 w_0$ , we would like to swap the weights along a  $z_0$ -changing cycle in  
544  $G - w_0$  (thus,  $w_0$ -avoiding). According to Observation 2.9, recall that this would not alter the parity  
545 of the number of incident  $a$ 's of any vertex in  $V(G) \setminus \{w_0\}$ . Furthermore, this would get rid of the  
546 conflict  $z_0 w_0$ . However, this swapping process can create a conflict between  $w_0$  and a neighbour  $v$   
547 of  $w_0$  with degree  $d$ ; but such a conflict can only arise when the cycle goes through the only two  
548  $a$ -edges incident to  $v$ . For a neighbour  $v$  of  $w_0$  with degree  $d$  that is incident to only two  $a$ -edges,



549 we call this pair of edges a *forbidden pair*. Our goal in what follows is to show that  $G - w_0$  has a  
 550  $z_0$ -changing cycle not containing any forbidden pair of edges.

551 Let us denote by  $v_1, \dots, v_m$  the neighbours of  $w_0$  with degree  $d$ . As mentioned earlier, recall that  
 552 the  $v_i$ 's are all adjacent to  $y_0$ . Since  $z_0 w_0$  is a conflict, recall that  $d(z_0)$  and  $d(w_0)$  have the same  
 553 parity. Furthermore, since  $d(z_0) > d(w_0) > 3$ , it follows that  $d(z_0) \geq 6$ , and, because the only  
 554 neighbours of  $z_0$  outside  $H$  are  $y_0$  and  $z_1$ , there is a vertex  $z_2 \neq w_0$  in  $N(z_0) \cap V(H)$ . Note that,  
 555 to find the desired cycle through  $z_0$  in  $G - w_0$ , it suffices to find a path  $P$  from  $y_0$  to a vertex  $z'$  in  
 556  $N(z_0) \cap V(H)$  in the connected graph  $G - z_0 - w_0$  (which is connected by the minimality of  $H$ ) not  
 557 using any forbidden pair of edges. Indeed, if the weight on  $z_0 z'$  is  $b$ , then we can define our cycle  
 558 to be  $P \cup \{z_0 y_0, z' z_0\}$ , while, if the weight on  $z_0 z'$  is  $a$ , then we can define our cycle to be  $P \cup P_c$ ,  
 559 where  $P_c$  is a path from  $z_0$  to  $y_0$  in  $G - H - z_0 y_0$  (thus, through  $C$ ). Since the graph  $G - z_0 - w_0$  is  
 560 connected, there is a path  $P_1$  from  $z_2$  to  $y_0$ . We can assume that  $P_1$  uses forbidden pairs of edges.  
 561 Without loss of generality, let  $p v_1$  and  $v_1 q$  be the first forbidden pair of edges  $P_1$  used when going  
 562 from  $z_2$  to  $y_0$ . Since  $v_1$  is adjacent to  $y_0$ , it follows that  $q = y_0$ , since otherwise, we have found a  
 563 path from  $y_0$  to  $z_2$  not using any forbidden pair of edges. Thus, we can assume that all paths from  
 564  $y_0$  to a vertex in  $N(z_0) \cap V(H)$  use exactly one forbidden pair of edges. Now we look at all such  
 565 paths using only one pair of forbidden edges  $y_0 v_i$  and  $v_i p$  (for  $i \in \{1, \dots, m\}$ ) and consider one such  
 566 path  $P$  that goes through the most neighbours of  $w_0$ . Let  $y_0 v_i$  and  $v_i p$  be the pair of forbidden  
 567 edges that  $P$  contains.

568 First suppose that  $v_i$  has a neighbour  $v'_i$  distinct from  $y_0, p, w_0$ . The edge  $v_i v'_i$  must have weight  
 569  $b$ . Since  $G - w_0 - v_i$  is connected, it has a path  $P'$  from  $v'_i$  to a vertex in  $N(z_0) \cap V(H)$ . The path  
 570  $P'$  must use a forbidden pair of edges, as otherwise, the graph induced by  $E(P) \cup E(P')$  would  
 571 contain a desired path from  $y_0$  to a vertex in  $N(z_0) \cap V(H)$  avoiding forbidden pairs of edges. Let  
 572 the first pair of forbidden edges  $P'$  used when starting from  $v'_i$  be  $q v$  and  $vr$ . The subpath  $P'_1$  of  $P'$   
 573 from  $v'_i$  to  $v$  must be disjoint from  $P$ , since otherwise, the graph induced by  $E(P) \cup E(P'_1)$  contains  
 574 a desired path from  $y_0$  to  $N(z_0) \cap V(H)$  avoiding forbidden pairs of edges. Furthermore, we must  
 575 have that  $r = y_0$ , since otherwise, the path  $P''$  defined to be  $y_0 v$  together with the subpath of  $P'$   
 576 from  $v$  to  $v'_i$  followed by  $v'_i v_i$  and the subpath of  $P$  from  $v_i$  to  $N(z_0) \cap V(H)$  is a desired path from  
 577  $y_0$  to  $N(z_0) \cap V(H)$  avoiding forbidden pairs of edges. Now the path  $P''$  contradicts the maximality  
 578 of  $P$ .

579 So we can assume that  $N(v_i) = \{y_0, p, w_0\}$ , which means, because  $v_i$  has degree  $d > 3$ , that  
 580 some of the edges  $v_i y_0, v_i p, v_i w_0$  have multiplicity more than 1. The multiplicity of both  $y_0 v_i$  and  
 581  $v_i p$  must be 1, since otherwise, we would get a desired path from  $z_2$  to  $y_0$  avoiding the forbidden  
 582 pair of edges  $y_0 v_i, v_i p$ . So the multiplicity of  $v_i w_0$  is at least 2. If the only neighbours of  $w_0$  are  $v_i$   
 583 and  $z_0$ , then we can swap the weights on two edges between  $w_0$  and  $v_i$  to avoid the conflict  $z_0 w_0$   
 584 and obtain a proper  $\{a, b\}$ -edge-weighting of  $G$ ; so we can assume that  $w_0$  is incident to a vertex  
 585  $v'$  in  $H$  distinct from  $v_i$ . Since  $d(v_i) = d(w_0)$ , the edges  $w_0 z_0$  and  $w_0 v'$  are not multiple and since  
 586  $z_0$  has degree at least 6, there are two edges  $z_0 z'_2, z_0 z''_2$  incident with  $z_0$  having the same weight  
 587 and where  $z'_2, z''_2 \in H$ . Possibly  $z'_2 = z''_2$  or  $z'_2 = z_2$ . The graph  $B - w_0 - z_0$  is connected, so it  
 588 contains a path  $P'_1$  from  $z'_2$  to  $y_0$  and a path  $P'_2$  from from  $z''_2$  to  $y_0$ . These paths  $P'_1$  and  $P'_2$  must  
 589 be internally disjoint, since otherwise, there would be a  $z_0$ -changing cycle in  $B$  not containing any  
 590 pair of forbidden edges. We can assume that both  $P'_1$  and  $P'_2$  contain a pair of forbidden edges,  
 591 since otherwise, there is a desired path from  $y_0$  to  $(N(z_0) - w_0) \cap V(H)$ . Hence, we can assume that

592  $P'_1$  contains  $y_0v_i$  and  $v_i p$  and  $P'_2$  contains a pair of forbidden edges  $qv'$ ,  $v'r$  incident to  $v'$ . Since  
593 this implies that  $y_0$  and  $v'$  are adjacent, we can assume that  $y_0 = q$ . The vertex  $v'$  must have a  
594 neighbour  $s$  distinct from  $y_0, w_0, r$  and since the graph  $B - w_0 - v'$  is connected, there is a path  $P''$   
595 in  $B - w_0 - v'$  from  $s$  to  $y_0$ . If  $P''$  contains the forbidden pair of edges  $y_0v_i$  and  $v_i p$ , then the graph  
596  $P'_1 \cup P'_2 \cup P''$  contains a  $z_0$ -changing cycle in  $B$  containing no forbidden pair of edges. Thus, we can  
597 assume that  $P''$  contains no pair of forbidden edges. Now  $P'_2 \cup P''$  contains a desired path from  $w_0$   
598 to  $(N(z_0) - w_0) \cap V(H)$ .  $\diamond$

599 **Claim 6.** *There is no vertex  $v \in V(H)$  such that  $G - v - N(v)$  is connected.*

600 *Proof of the claim.* Suppose  $v \in V(H)$  and that  $G' = G - v - N(v)$  is connected. We can assume  
601  $v \in X$ . By Lemma 2.10, there is an  $(X \setminus \{v\}, Y \cup \{v\})$ - $a$ -parity  $\{a, b\}$ -edge-weighting of  $G$ , where  
602 all edges incident to  $v$  have weight  $b$  and any vertex  $u \in N(v)$  is incident to at most  $1 + M(uv)$   
603  $b$ -edges, where  $M(uv)$  denotes the multiplicity of the edge  $uv$ . Note that Claim 1 implies that if  
604  $vv'$  is a multiple edge, then  $v' \in \{z_0, y_0\}$ . Thus, the only potential conflict is between  $v$  and one  
605 of  $z_0, y_0$ , say  $y_0$ . This implies that  $y_0$  must have odd degree at least 5. But since  $G'$  is connected  
606 there must then be a  $y_0$ -changing cycle in  $G - (N(v) \setminus \{y_0\})$  and swapping the weights on this cycle  
607 yields a proper  $\{a, b\}$ -edge-weighting of  $G$ .  $\diamond$

608 **Claim 7.** *There are no multiple edges between two vertices in  $H$ .*

609 *Proof of the claim.* Suppose  $uv$  is a multiple edge in  $H$ . We can assume  $v \in X$ . Since  $u$  and  $v$   
610 have degree 3 in  $G^*$  (by Claim 5), the multiplicity of  $uv$  is exactly 2. Let  $e$  and  $e'$  be the two edges  
611 between  $u$  and  $v$ . By Claim 1,  $e, e'$  are not both white. Thus, at least one of  $e, e'$ , say  $e$ , is a blue  
612 edge in  $H$ . Let  $v'$  denote, in  $G^*$ , the unique neighbour of  $v$  different from  $u$ .

613 Let  $G'$  be obtained from  $G - v$  by removing all edges but one edge  $e''$  incident to  $v'$ . Clearly  $G'$   
614 is connected since  $G - v - v'$  is connected by the minimality of  $H$ . Let  $S = E(v') \setminus \{vv', e''\}$ . Let  
615  $X'$  denote the set of vertices in  $G - v$  which are incident to an odd number of edges in  $S$ . Note  
616 that  $S$  has even size. Thus,  $X' = (X \setminus S) \cup (S \cap Y)$  has even size and Lemma 2.4 implies that  
617 there is an  $(X', V(G') \setminus X')$ - $a$ -parity  $\{a, b\}$ -edge-weighting of  $G'$ . We now extend this weighting to  
618  $G$  by assigning weight  $a$  to all edges in  $S$  and weight  $b$  to all edges incident to  $v$ . This gives an  
619  $(X \setminus \{v\}, Y \cup \{v\})$ - $a$ -parity  $\{a, b\}$ -edge-weighting of  $G$  where all vertices in  $N(v)$  are incident to at  
620 most two  $b$ -edges (the edge in the suspended path of length 3 joining  $u$  and  $v$  incident to  $u$  must  
621 have weight  $a$ ). Observation 2.8 implies that the only conflict can be between  $v$  and its neighbours,  
622 so the only possible conflict is  $vv'$  in the case where  $v' \in \{z_0, y_0\}$ , say  $v' = y_0$ , and  $y_0$  has degree at  
623 least 5. But since  $G - v$  is connected there must then be a  $y_0$ -changing cycle avoiding  $v$  and  $u$  and  
624 swapping the weights on this cycle yields a proper  $\{a, b\}$ -edge-weighting of  $G$ .  $\diamond$

625 **Claim 8.** *Every vertex of  $H$  is incident to at most one blue edge.*

626 *Proof of the claim.* Recall that every vertex  $v$  of  $H$  has degree 3, by Claim 5. If  $v$  is incident to  
627 three blue edges, then  $G - v - N(v)$  is connected, which contradicts Claim 6. So now assume  $v$  is  
628 incident to two blue edges. Let  $uv$  denote the white (third) edge incident to  $v$ . Still by Claim 6, the  
629 graph  $G - v - N(v)$  cannot be connected, which means that  $u, v$  is a pair contradicting the choice  
630 of  $y_0, z_0$ .  $\diamond$

631 We now have all the tools in hand for finishing the proof. Two cases must be considered:

632 **Case 1:** *There is a vertex  $v \in V(H)$  not adjacent to any of  $z_0, y_0$ .*

633 Recall that, according to Claim 6, whenever removing from  $G$  a vertex of  $H$  and its neighbour-  
634 hood, we get a disconnected graph. Let  $v$  be a vertex not adjacent to  $z_0, y_0$  such that the component  
635  $K$  of  $G' = G - v - N(v)$  containing  $z_0$  and  $y_0$  has maximum order. We can assume that  $v \in X$ .  
636 Note that there must be a vertex  $v' \in V(H)$  distinct from  $v$  with  $N(v') = N(v)$  such that the  
637 components of  $G'$  are exactly  $K$  and the isolated vertex  $v'$ . This is because otherwise there would  
638 be a vertex  $v'' \neq v$  in  $H$  such that  $G - v'' - N(v'')$  has a bigger  $K$ , a contradiction to our choice of  
639  $v$ .

640 Let  $e_1 = vv_1, e_2 = vv_2, e_3 = vv_3$  denote the three edges incident to  $v$ . Since  $v, v', v_1, v_2, v_3$  all  
641 belong to  $H$ , by Claim 7, all these vertices are distinct. Furthermore, since  $N(v) = N(v')$ , we can  
642 also assume that none of  $vv_1, vv_2, vv_3$  are blue. Hence,  $v, v' \in X$  and  $v_1, v_2, v_3 \in Y$ . The graph  
643  $G - v$  is connected and so is the graph  $G'$  obtained from it by removing the edges  $v'v_1, v'v_2$ . Lemma  
644 2.4 implies that there is an  $\{a, b\}$ -edge-weighting of  $G'$  where the vertices in  $X \setminus \{v\} \cup \{v_2, v_3\}$  are  
645 incident to an odd number of  $a$ -edges and the vertices in  $Y \setminus \{v_2, v_3\}$  are incident to an even number  
646 of  $a$ -edges. In particular, in  $G'$  the only  $a$ -edge incident to  $v'$  is  $v'v_1$ . We extend this weighting  
647 to  $G$  by assigning weight  $a$  to  $v'v_2, v'v_3$ , and weight  $b$  to all three edges incident to  $v$ . That way,  
648 we get an  $(X \setminus \{v\}, Y \cup \{v\})$ - $a$ -parity  $\{a, b\}$ -edge-weighting of  $G$  where all three edges incident to  
649  $v$  have weight  $b$  and all three edges incident to  $v'$  have weight  $a$ . Observation 2.8 implies that the  
650 only potential conflicts are between  $v$  and its neighbours, but since all vertices in  $N(v)$  are incident  
651 to at least one  $a$ -edge (the one incident to  $v'$ ) this edge-weighting is proper.

652 **Case 2:** *All vertices in  $H$  are adjacent to  $z_0$  or  $y_0$ .*

653 By Claim 6, we can assume that, for every vertex  $v \in V(H)$ , the graph  $G' = G - v - N(v)$  is  
654 disconnected. First, suppose  $z_0$  is joined in  $G^*$  to some vertex  $v \in V(H)$  by an edge of multiplicity  
655 2. Let  $e'$  and  $e''$  be the two edges joining  $z_0$  and  $v$  in  $G^*$ . Claim 6 implies that not both of  $e', e''$   
656 are blue; say  $e'$  is white. If  $e''$  is blue, then by Claim 8, the third edge  $e'''$  incident to  $v$  in  $G^*$   
657 must be white. If  $e''$  is white, then Claim 6 implies that  $e'''$  is white, so the edge  $e''' = vu$  must be  
658 white. We can assume  $v \in X$  and hence  $u \in Y$ . Let  $Z$  denote the set of vertices in  $G - v$  which are  
659 incident to exactly one edge incident to  $u$ . Note that either  $Z$  is empty or  $Z$  has size 2. Note that  
660  $X' = (X \setminus (Z \cup \{v\})) \cup (Y \cap Z)$  has even size, so by Lemma 2.4, there is an  $(X', V(G - v - u) \setminus X')$ -  
661  $a$ -parity  $\{a, b\}$ -edge-weighting of  $G - v - u$ . We now extend this edge-weighting to all of  $G$  by  
662 assigning weight  $b$  to all edges in  $E(v)$  and weight  $a$  to the two edges incident to  $u$  distinct from  $uv$ .  
663 Thus, we have obtained an  $(X \setminus \{v\}, Y \cup \{v\})$ - $a$ -parity  $\{a, b\}$ -edge-weighting of  $G$  where all edges  
664 incident to  $v$  have weight  $b$  and  $u$  is incident to exactly one  $b$ -edge. Observation 2.8 implies that the  
665 only potential conflict is  $uz_0$  in the case where  $z_0$  has degree at least 5. We can also assume that  
666 there is no  $z_0$ -changing cycle in  $G$  avoiding  $v$  and  $u$ . Hence,  $z_0$  must have degree 2 in  $G - H$  and be  
667 joined by an edge in  $G^*$  to at least one vertex  $v'$  in  $H$  distinct from  $v$ . We can now find a desired  
668  $z_0$ -changing cycle unless the only neighbours of  $v'$  in  $B$  are  $z_0$  and  $u$ . But in this case  $G - u - N(u)$   
669 is connected, contradicting Claim 6.

670 Thus, we can assume that  $z_0$ , and similarly  $y_0$ , is not joined to any vertex in  $H$  by a multiple  
671 edge in  $G^*$ . Claim 7 now implies that any vertex in  $H$  has three distinct neighbours in  $G^*$ . Let  
672  $v \in X$  be any vertex in  $H$  incident to  $z_0$ . The graph  $G - v - N(v)$  is disconnected by Claim 6, so

673 there must be a vertex  $v'$  in  $H$  which in  $G^*$  has the same neighbourhood as  $v$ . Since all vertices in  $H$   
674 have degree 3 (Claim 5) this implies that  $H$  only has four vertices: two joined to  $z_0$  and two joined  
675 to  $y_0$ . The graph  $G - v - (N(v) \setminus \{z_0\})$  is connected, so as above, there is an  $(X \setminus \{v\}, Y \cup \{v\})$ - $a$ -  
676 parity  $\{a, b\}$ -edge-weighting of  $G$  where all edges incident to  $v$  have weight  $b$ , and the neighbours of  
677  $v$  distinct from  $z_0$  which have degree 3 are incident to exactly one  $b$ -edge. Again, Observation 2.8  
678 implies that the only possible conflict is  $vz_0$  in the case where  $z_0$  has degree at least 5. In this case,  
679 it is easy to see that there is a  $z_0$ -changing cycle avoiding  $H$ , thus, we can get rid of this conflict  
680 and obtain a proper  $\{a, b\}$ -edge-weighting of  $G$ .  $\square$

### 681 3 Proof of Theorem 1.2

682 Before describing the structure of trees without the  $\{-1, 1\}$ -property, we first introduce the following  
683 two general lemmas which will be used in the proof. Some of the tools and results used here were  
684 introduced in Section 2.

685 **Lemma 3.1.** *If  $G$  is a simple connected bipartite graph without the  $\{-1, 1\}$ -property and  $e$  is a*  
686 *bridge in  $G$ , then the deletion of  $e$  results in two components each containing an odd number of*  
687 *vertices.*

688 *Proof.* Suppose the lemma is false. Let  $G$  be a connected bipartite graph without the  $\{-1, 1\}$ -  
689 property and let  $e = uv \in E(G)$  be a bridge in  $G$ . Let  $c : V(G) \rightarrow \{1, 2\}$  be a mod-4 vertex-  
690 colouring of  $G$  (such a colouring exists by Lemma 2.6) and let  $X, Y$  denote the sets of vertices  
691 coloured 1 and 2 respectively. By Lemma 2.7, both colour classes of  $c$  have odd size. Let  $C_1, C_2$   
692 denote the two components of  $G - e$  with  $u \in V(C_1)$  and  $v \in V(C_2)$ . For a contradiction, assume  
693 that both  $|V(C_1) \cap X|$  and  $|V(C_1) \cap Y|$  are odd and both  $|V(C_2) \cap X|$  and  $|V(C_2) \cap Y|$  are even.  
694 Recall that two vertices with degree of distinct parity cannot have the same weighted degree by a  
695  $\{-1, 1\}$ -edge-weighting (Observation 2.8). There are thus four cases to be considered:

696 **Case 1:** *Both  $u$  and  $v$  have odd degree and colour 1.*

697 By Lemma 2.4, there is an  $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$ - $(-1)$ -parity  $\{-1, 1\}$ -edge-weighting  
698 of  $C_1$  and a  $(Y \cap V(C_2), X \cap V(C_2))$ -1-parity  $\{-1, 1\}$ -edge-weighting of  $C_2$ . It follows from Obser-  
699 vation 2.8 that these two edge-weightings, together with assigning weight  $-1$  to  $e$ , form a proper  
700 edge-weighting of the whole  $G$ .

701 **Case 2:** *Both  $u$  and  $v$  have even degree and colour 1.*

702 By Lemma 2.4, there is an  $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$ - $(-1)$ -parity  $\{-1, 1\}$ -edge-weighting  
703 of  $C_1$  and an  $(X \cap V(C_2), Y \cap V(C_2))$ -1-parity  $\{-1, 1\}$ -edge-weighting of  $C_2$ . It follows from Ob-  
704 servation 2.8 that these two edge-weightings, together with assigning weight  $-1$  to  $e$ , form a proper  
705 edge-weighting of the whole  $G$ .

706 **Case 3:** *Both  $u$  and  $v$  have odd degree,  $u$  has colour 1 and  $v$  has colour 2.*

707 By Lemma 2.4, there is an  $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$ - $(-1)$ -parity  $\{-1, 1\}$ -edge-weighting  
708 of  $C_1$  and a  $(Y \cap V(C_2), X \cap V(C_2))$ -1-parity  $\{-1, 1\}$ -edge-weighting of  $C_2$ . It follows from Obser-  
709 vation 2.8 that these two edge-weightings, together with assigning weight  $-1$  to  $e$ , form a proper  
710 edge-weighting of the whole  $G$ .

711 **Case 4:** *Both  $u$  and  $v$  have even degree,  $u$  has colour 1 and  $v$  has colour 2.*

712 By Lemma 2.4, there is an  $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$ - $(-1)$ -parity  $\{-1, 1\}$ -edge-weighting  
713 of  $C_1$  and an  $(X \cap V(C_2), Y \cap V(C_2))$ -1-parity  $\{-1, 1\}$ -edge-weighting of  $C_2$ . It follows from Ob-  
714 servation 2.8 that these two edge-weightings, together with assigning weight  $-1$  to  $e$ , form a proper  
715 edge-weighting of the whole  $G$ .  $\square$

716 **Lemma 3.2.** *If  $G$  is a connected bipartite graph without the  $\{-1, 1\}$ -property and  $e$  is a bridge in*  
717  *$G$ , then there is a  $\{-1, 1\}$ -edge-weighting of  $G$  such that  $e$  is the only conflict.*

718 *Proof.* Let  $G$  be a connected bipartite graph without the  $\{-1, 1\}$ -property,  $e$  be a bridge in  $G$ , and  
719  $C_1, C_2$  be the two components of  $G - e$ . Let  $c$  be a mod-4 vertex-colouring of  $G$  (such a colouring  
720 exists by Lemma 2.6), and let  $X$  denote the set of vertices with colour 1 and  $Y$  denote the set of  
721 vertices with colour 2. By Lemma 2.7, both  $X$  and  $Y$  have odd size and by Lemma 3.1, we can  
722 assume that  $|X \cap V(C_1)|$  and  $|Y \cap V(C_2)|$  are even and  $|Y \cap V(C_1)|$  and  $|X \cap V(C_2)|$  are odd. Now  
723 Lemma 2.4 implies that there is an  $(X \cap V(C_1), Y \cap V(C_1))$ -1-parity  $\{-1, 1\}$ -edge-weighting of  $C_1$  and  
724 an  $(X \cap V(C_2), Y \cap V(C_2))$ -1-parity  $\{-1, 1\}$ -edge-weighting of  $C_2$ . Observation 2.8 implies that these  
725 two edge-weightings, together with assigning weight  $-1$  to the edge  $e$ , is a  $\{-1, 1\}$ -edge-weighting  
726 of  $G$  where  $e$  is the only potential conflict.  $\square$

727 We can now prove Theorem 1.2. When referring to Operation 1, we mean the first operation  
728 described at the end of Section 1 (illustrated in Figure 1, (a) and (b)).

729 *Proof of Theorem 1.2.* As mentioned in the introduction, it is straightforward to check that any  
730 graph constructed with Operation 1 from four graphs without the  $\{-1, 1\}$ -property does not have  
731 the  $\{-1, 1\}$ -property itself. An easy argument is that all five edges incident to the vertices  $v_1 \sim v_2$   
732 and  $v_3 \sim v_4$  should have the same weight (in which case a conflict arises), as otherwise, the proper  
733  $\{-1, 1\}$ -edge-weighting would yield one of at least one of the four graphs used in the construction,  
734 a contradiction. Thus, it suffices to prove that any tree without the  $\{-1, 1\}$ -property is constructed  
735 from a disjoint union of  $K_2$ 's through repeated (possibly none) applications of Operation 1. Suppose  
736 this is false and let  $T$  be a minimum counterexample. Note that Lemma 3.1 implies that, for any  
737 vertex  $v \in V(T)$  and any edge  $e \in E(v)$ , the component  $C_e$  not containing  $v$  in  $T - e$  has an odd  
738 number of vertices. Since we can write  $|V(T)| = 1 + \sum_{e \in E(v)} |V(C_e)|$  for any vertex  $v \in V(T)$  and  
739 since  $|V(T)|$  is even, this implies that all vertices in  $T$  have odd degree. A consequence of this is  
740 that if  $S \subset T$  is a subtree of  $T$ , then  $T - S$  has no components isomorphic to  $K_2$  as otherwise  $T$   
741 would have vertices with degree 2.

742 Let  $P = v_1 \dots v_m$  be a longest path in  $T$ . Clearly,  $v_m$  is a leaf and, since all vertices have odd  
743 degree,  $v_{m-1}$  is incident to an even number of leaves. Suppose  $v_{m-1}$  is incident to an even number  
744  $n \geq 4$  of leaves  $u_1, \dots, u_n$ , with  $u_1 = v_m$ . Recall that Lemma 2.7 implies that  $|V(T)|$  is even; now,  
745 since  $T' = T - \{u_1, \dots, u_{n-1}\}$  has an odd number of vertices, Lemma 2.7 implies that  $T'$  has a proper  
746  $\{-1, 1\}$ -edge-weighting. We can now extend this edge-weighting to a proper  $\{-1, 1\}$ -edge-weighting  
747 of  $T$  by assigning the same weight to all the edges  $v_{m-1}u_1, \dots, v_{m-1}u_n$  (we choose whether it is 1  
748 or  $-1$  so that we avoid the conflict  $v_{m-2}v_{m-1}$ ). Thus,  $v_{m-1}$  has degree exactly 3 and, from these  
749 arguments and the maximality of  $P$ , any neighbour of  $v_{m-2}$  distinct from  $v_{m-3}$  and  $v_{m-1}$  is either a  
750 leaf or a vertex of degree 3 adjacent to two leaves. Let  $U' = \{u'_1, \dots, u'_p\}$  be the set of leaves adjacent  
751 to  $v_{m-2}$  and let  $U'' = \{u''_1, \dots, u''_q\}$  be the set of neighbours of  $v_{m-2}$  distinct from  $v_{m-1}$  and  $v_{m-3}$   
752 which have degree 3. Possibly  $p = 0$  or  $q = 0$ , but  $p + q > 0$  since  $v_{m-2}$  has odd degree and hence

753  $p + q$  is odd. Let  $T_1$  and  $T_2$  be the two components of  $T - v_{m-3}v_{m-2}$  such that  $v_{m-2} \in V(T_2)$ .  
 754 By Lemma 3.2, there is a  $\{-1, 1\}$ -edge-weighting  $w$  of  $T$  such that the only potential conflict is  
 755  $v_{m-3}v_{m-2}$ . By possibly multiplying all edge weights by  $-1$ , we can assume that the weight of  
 756  $v_{m-3}v_{m-2}$  is 1. We look at three separate cases:

757 **Case 1:**  $p + q \geq 5$ .

758 By possibly modifying the weights of the edges in  $E(T_2)$  such that they all have weight 1 or  $-1$ ,  
 759 the vertex  $v_{m-2}$  can obtain weighted degree  $2 + p + q$  and  $-p - q$ . Now we simply pick the one of  
 760 these two options such that  $v_{m-3}v_{m-2}$  is not a conflict. Since all vertices in  $T_2$  except  $v_{m-2}$  have  
 761 degree at most 3, this gives a proper  $\{-1, 1\}$ -edge-weighting of  $T$ .

762 **Case 2:**  $p + q = 3$ .

763 As in Case 1, we can modify the edge weights such that the vertex  $v_{m-2}$  can obtain weighted  
 764 degree  $2 + p + q$  and  $-p - q$ . Furthermore, since  $2 + p + q = 5$  and all vertices in  $T_2$  except  $v_{m-2}$   
 765 have degree at most 3, we can in this way find a proper  $\{-1, 1\}$ -edge-weighting of  $T$ , unless  $v_{m-3}$   
 766 has weight 5. So we can assume that  $v_{m-3}$  has weight 5. If  $p \geq 1$  and  $p$  is odd (resp. even), then  
 767 we modify the weights in  $T_2$  such that all edges incident with  $u'_1, \dots, u'_p$  have weight 1 (resp.  $-1$ )  
 768 and all other edges in  $T_2$  have weight  $-1$  (resp. 1). This yields a proper  $\{-1, 1\}$ -edge-weighting  
 769 of  $T$ , so we can assume  $p = 0$  and  $q = 3$ . In this case, we modify the weights in  $T_2$  such that all  
 770 edges incident to  $v_{m-1}$  and  $u''_1$  have weight 1 and all other edges in  $T_2$  have weight  $-1$ . This yields  
 771 a proper  $\{-1, 1\}$ -edge-weighting of  $T$ .

772 **Case 3:**  $p + q = 1$ .

773 First suppose  $q = 1$  and  $p = 0$ . If we modify the edge weights in  $T_2$  such that they all have weight  
 774  $-1$ , then we obtain a proper  $\{-1, 1\}$ -edge-weighting of  $T$ , unless  $v_{m-3}$  has weight  $-1$ . In this case, we  
 775 change the weights of the three edges incident to  $v_{m-1}$  to 1 to obtain a proper  $\{-1, 1\}$ -edge-weighting  
 776 of  $T$ . Thus, we can assume  $p = 1$  and  $q = 0$ . We can assume that  $T''' = T - v_m - v_{m-1} - u_2 - u'_1$  has a  
 777 proper  $\{-1, 1\}$ -edge-weighting  $w$ , since otherwise, the minimality of  $T$  implies that  $T'''$  is constructed  
 778 from a disjoint union of  $K_2$ 's through repeated (possibly none) applications of Operation 1, and  
 779 then so is  $T$ . By possibly multiplying all edge weights of  $w$  by  $-1$ , we can assume that  $v_{m-3}v_{m-2}$   
 780 has weight 1. Now assigning weight 1 to all edges incident to  $v_{m-1}$  and weight  $-1$  to  $v_{m-2}u'_1$  yields  
 781 a proper  $\{-1, 1\}$ -edge-weighting of  $T$ . □

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