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▶ To cite this version:

Julien Bensmail, Fionn Mc Inerney, Kasper Lyngsie. On a, b-edge-weightings of bipartite graphs with odd a, b. Discussiones Mathematicae Graph Theory, 2022, 42 (1), pp.159-185. $10.7151/\rm dmgt.2250$. hal-01988399v2

HAL Id: hal-01988399 https://hal.science/hal-01988399v2

Submitted on 28 Aug 2019

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| 1 | On $\{a, b\}$ -edge-weightings of bipartite graphs with odd a, b^* |
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Abstract

For any $S \subset \mathbb{Z}$ we say that a graph G has the S-property if there exists an S-edge-weighting $w : E(G) \to S$ such that for any pair of adjacent vertices u, v we have $\sum_{e \in E(v)} w(e) \neq \sum_{e \in E(u)} w(e)$, where E(v) and E(u) are the sets of edges incident to v and u, respectively. This work focuses on $\{a, a + 2\}$ -edge-weightings where $a \in \mathbb{Z}$ is odd. We show that a 2-connected bipartite graph has the $\{a, a + 2\}$ -property if and only if it is not a so-called odd multi-cactus. In the case of trees, we show that only one case is pathological. That is, we show that all trees have the $\{a, a + 2\}$ -property for odd $a \neq -1$, while there is an easy characterization of trees without the $\{-1, 1\}$ -property.

15 1 Introduction

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Let G be an undirected graph. For an S-edge-weighting $w: E(G) \to S$ of G, where $S \subset \mathbb{Z}$, each 16 vertex $v \in V(G)$ has weighted degree equal to the sum of the weights of its incident edges. We call 17 w neighbour sum-distinguishing if no two adjacent vertices of G have the same weighted degree. For 18 a set S of weights, we say that G has the S-property if it admits neighbour sum-distinguishing S-19 edge-weightings. The study of graphs having or not having the S-property for some sets S is highly 20 related to the well-known 1-2-3 Conjecture raised by Karonski, Luczak, and Thomason in 2004 [6]. 21 That conjecture states that every connected graph different from K_2^1 has the $\{1, 2, 3\}$ -property. A 22 particular case of a list version of the 1-2-3 Conjecture (introduced by Bartnicki, Grytczuk, and 23 Niwczyk [2]), even states that every graph should have the $\{a, b, c\}$ -property for every distinct 24 $a, b, c \in \mathbb{N}$. For more details on the progress towards the 1-2-3 Conjecture (and variants of it), 25 please refer to [11] for a survey on this topic. 26

For any smaller set $S \subset \mathbb{Z}$ of weights, *i.e.*, with |S| = 2, one can easily come up with examples showing that there do exist graphs not having the S-property (complete graphs are such examples).

^{*}The first author was supported by PEPS grant POCODIS.

¹This requirement is mandatory for any graph to be weightable; throughout this work, it is thus implicit, unless stated otherwise, that every considered graph does not have K_2 as a connected component.

A natural question that has been investigated is about the existence of a good characterization 29 of graphs that have the S-property for such smaller sets S. Here and further on, by a "good 30 characterization" we mean a description in terms of a graph class whose members can be recognized 31 in polynomial time. Dudek and Wajc [5] settled the question in the negative, as they proved that, 32 unless P=NP, there is no good characterization of graphs with the $\{1,2\}$ -property, and similarly 33 for the $\{0,1\}$ -property. Later on, noticing that, for any two distinct sets $S, S' \subset \mathbb{Z}$ of weights 34 with |S|, |S'| = 2, any neighbour sum-distinguishing S-edge-weighting of a regular graph yields a 35 neighbour sum-distinguishing S'-edge-weighting, Ahadi, Dehghan, and Sadeghi [1] proved that there 36 is no good characterization of graphs with the $\{a, b\}$ -property for any two distinct $a, b \in \mathbb{Z}$. 37

From this point on, it thus made sense investigating, for any two distinct $a, b \in \mathbb{Z}$, sufficient 38 conditions for graphs to have the $\{a, b\}$ -property. A special focus has been dedicated to bipartite 39 graphs, as 1) the aforementioned NP-completeness results were not known to hold in the bipartite 40 context, and 2) bipartite graphs form one of the rare graph classes for which the 1-2-3 Conjecture is 41 relatively well understood (see [6]). As a first step, several works [3, 4, 7, 8, 9] investigated whether 42 there is a good characterization of bipartite graphs with the $\{1,2\}$ -property. Back then, it was 43 believed that such a good characterization should exist, as, notably, all 3-connected bipartite graphs 44 were proved to have the $\{1, 2\}$ -property [9]. It was not until quite recently that Thomassen, Wu, and 45 Zhang proved that, indeed, bipartite graphs without the $\{1,2\}$ -property are easy to describe [13]. 46 Namely, only so-called **odd multi-cacti** are bipartite and do not have the $\{1, 2\}$ -property. These 47 graphs are defined as follows (the comprehensive definition is from [10]; refer to Figure 2 later on 48 for an illustration): 49

⁵⁰ "Take a collection of cycles of length 2 modulo 4, each of which has edges coloured alternately ⁵¹ red and green. Then form a connected simple graph by pasting the cycles together, one by one, in a ⁵² tree-like fashion along green edges; the resulting graph is an odd multi-cactus. The graph with one ⁵³ green edge and two vertices (K_2) is also an odd multi-cactus. When replacing a green edge of an ⁵⁴ odd multi-cactus by a green edge of any multiplicity, we again obtain an odd multi-cactus."

One main ingredient behind Thomassen *et al.*'s result is the nice observation, already made back in [3], that, when *a* and *b* are integers with distinct parity, every bipartite graph *G* with bipartition (X, Y) such that at least one of *X* and *Y* has even cardinality has the $\{a, b\}$ -property. This is because, in such a case, one can easily construct $\{a, b\}$ -edge-weightings of *G* where all vertices in *X* have odd weighted degree while those in *Y* have even weighted degree. These observations also imply that, for *a* and *b* with distinct parity, bipartite graphs without the $\{a, b\}$ -property have their two partite sets of odd cardinality, and they thus have even order.

Reusing some of Thomassen *et al.*'s ideas, Lyngsie later considered the $\{0, 1\}$ -property for bipartite graphs [10]. His main result is a good characterization of 2-edge-connected bipartite graphs without the $\{0, 1\}$ -property, which turns out to be nothing but the class of odd multi-cacti. This result was established, in particular, through aforementioned tools and results for cases where *a* and *b* have different parities. However, both Thomassen *et al.* and Lyngsie observed that there exist infinitely many separable (*i.e.*, with cut-vertices) bipartite graphs without the $\{0, 1\}$ -property.

Although they are far from covering all the cases of a and b, the previous series of results show two things. First, that, when considering 2-connected bipartite graphs without the $\{a, b\}$ -property, one should pay attention to odd multi-cacti. Second, that separable bipartite graphs without the



Figure 1: Constructing graphs without the $\{-1, 1\}$ -property from graphs without that property.

⁷¹ {a, b}-property and those without the {a', b'}-property may differ for different pairs a, b and a', b'. ⁷² This is already well illustrated by the class of trees: while they all have the {1, 2}-property [3], ⁷³ infinitely many of them do not have the {0, 1}-property [10].

This paper is mainly devoted to studying $\{a, b\}$ -properties where both a and b are odd. As a first step, we focus on the cases where b = a + 2. We introduce mechanisms that are reminiscent of the ones mentioned above (for a and b with distinct parity), which allow us to study the $\{a, a + 2\}$ property for bipartite graphs and odd $a \in \mathbb{Z}$. One of the main results we get from these is that, for any odd a, 2-connected bipartite graphs without the $\{a, a+2\}$ -property are precisely odd multi-cacti again.

Theorem 1.1. Let $a, b \in \mathbb{Z}$ be odd integers with b = a + 2. A 2-connected bipartite graph G does not have the $\{a, b\}$ -property if and only if G is an odd multi-cactus.

Similarly as for the $\{0, 1\}$ -property, the structure of separable bipartite graphs without the $\{a, b\}$ property for odd a and b does not appear obvious. As a first step, we give a special focus on the case $\{a, b\} = \{-1, 1\}$. In that case, we can already point out two operations that, given bipartite graphs without the $\{-1, 1\}$ -property, clearly provide more separable bipartite graphs without the $\{-1, 1\}$ -property (see Figure 1):

• Let G_1, G_2, G_3, G_4 be four bipartite graphs without the $\{-1, 1\}$ -property, and let v_1, v_2, v_3, v_4 be any four degree-1 vertices of G_1, G_2, G_3, G_4 , respectively. The operation (see Figure 1 (a) and (b)) consists in considering the disjoint union $G_1 \cup G_2 \cup G_3 \cup G_4$, identifying the vertices v_1 and v_2 , identifying the vertices v_3 and v_4 , and adding an edge joining the two vertices resulting from these identifications (*i.e.*, $v_1 \sim v_2$ and $v_3 \sim v_4$).

• Let G_1, G_2 be two bipartite graphs without the $\{-1, 1\}$ -property, and let v_1, v_2 be any two vertices of G_1, G_2 , respectively. The operation (see Figure 1, (c) and (d)) consists in considering the disjoint union $G_1 + G_2$, adding the edge v_1, v_2 , and further joining v_1, v_2 by a path with odd length at least 3.

In the case of trees, when a and b are any two non-zero integers that are both positive (or 96 negative), it is easy to see that K_2 is the only tree without the $\{a, b\}$ -property: consider a vertex v 97 whose all neighbours $u_1, ..., u_{d-1}$ but one u_d (if any) are leaves, remove $u_1, ..., u_{d-1}$, apply induction 98 to deduce a neighbour sum-distinguishing $\{a, b\}$ -edge-weighting, and extend the weighting to the 99 edges $vu_1, ..., vu_{d-1}$ so that the conflict vu_d is avoided. Thus, when b = a + 2 and a, b are odd, 100 only the case a = -1, b = 1 is potentially non-trivial. In Section 3, we show that trees without the 101 $\{-1,1\}$ -property can all be constructed through the first operation above (illustrated in Figure 1, 102 (a) and (b)) performed on K_2 's. 103

Theorem 1.2. A tree does not have the $\{-1, 1\}$ -property if and only if it can be constructed from a disjoint union of K_2 's through repeated applications of the first operation above.

In particular, the structure of trees without the $\{-1, 1\}$ -property is very different and simpler than that of trees without the $\{0, 1\}$ -property (for more on the structure of these trees, see [10]). Recall that all trees have the $\{1, 2\}$ -property, as was shown, e.g., in [3].

Terminology and notation. Let G be a connected graph. For a given vertex v of G we denote by E(v) the set of edges incident to v. A bridge in G is an edge whose removal results in two components. Let w be an edge-weighting of G. Abusing the notation, the weighted degree of v in G by w will sometimes be denoted w(v) for convenience. We say that an edge uv of G is a conflict by w if w(u) = w(v). In other words, w is neighbour sum-distinguishing if no edge is a conflict. In what follows, we will instead use the term proper in place of neighbour sum-distinguishing to lighten the writing. By an x-edge of G (by w), we mean an edge assigned weight x by w.

¹¹⁶ 2 Proof of Theorem 1.1

In this section, we prove that for every odd integer $a \in \mathbb{Z}$, the class of 2-connected bipartite graphs without the $\{a, a + 2\}$ -property is exactly that of odd multi-cacti. Another way to define these graphs is as follows. Start from K_2 , the simple connected graph on two vertices, having its only edge coloured green. Then, repeatedly apply an arbitrary number of the following operation (see Figure 2 for an illustration). Consider any green edge uv of the current graph, and join u, v by a new path P of length $\ell \geq 1$ congruent to 1 modulo 4 whose edges are coloured red and green as follows:

• if $\ell = 1$, *i.e.*, *P* has a unique edge, then this edge is green;

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• if $\ell \geq 5$, then the edges of P are coloured red and green properly (*i.e.*, no two subsequent edges have the same colour) so that the two end-edges are red.

Figure 2 notably shows that performing this operation multiple times for a same green edge is allowed, and that adding paths of length 1 is similar to increasing the multiplicity of a green edge. Note also that it is not possible to get two adjacent green edges with distinct ends at any point of the process. Furthermore, every obtained graph is bipartite. An odd multi-cactus is any graph



Figure 2: Constructing an odd multi-cactus through several steps, from K_2 (a). Red-green paths with length at least 5 congruent to 1 modulo 4 are being attached onto the green edge uv through steps (b) to (d). In step (e), (green) paths of length 1 are added, which corresponds to increasing the multiplicity of some green edges.

that can be obtained during this process, no matter how many times the operation is applied. In particular, K_2 itself is regarded as an odd multi-cactus.

¹³³ In this section, we will implicitly use several properties of odd multi-cacti, such as:

Observation 2.1. Let M be an odd multi-cactus with its edges being coloured red and green as described above. Then:

• M is 2-connected;

• when replacing every (green) edge of M by an edge with multiplicity 1, a 2-degenerate graph (i.e., a graph in which every subgraph has a vertex of degree at most 2) is obtained;

• for every green edge uv of M, we have $d_M(u) = d_M(v)$.

Having the structure of odd multi-cacti in mind, it can be proved that the following holds true.

Lemma 2.2. If G is not an odd multi-cactus and was obtained from an odd multi-cactus M by replacing a red edge with an edge of multiplicity at least 2 or by replacing a green edge by a path of length $k \ge 5$ with $k \equiv 1 \mod 4$, then G has the $\{a, b\}$ -property for any two distinct integers $a, b \in \mathbb{Z}$.

Proof. The proof is by induction on the order of M. So suppose that G is obtained from an odd multi-cactus M by replacing an edge e with either an edge of multiplicity at least 2 (if e is red in

M) or a path of length $k \ge 5$ with $k \equiv 1 \mod 4$ (if e is green in M). It is easy to check that the 147 statement is true if M is just a cycle with some multiple green edges. So we can focus on cases 148 where M was obtained in the general way, *i.e.*, by pasting together cycles with length 2 modulo 4 149 having possibly green edges of any multiplicity. Furthermore, it is easy to check that the statement 150 is true if M was constructed by only pasting cycles together along one single green edge e' as in 151 Figures 2 (c), (d) and (e) where there have only been pasted cycles together along the edge uv: 152 in this case, since G is not an odd multi-cactus, G is either obtained from M by replacing a red 153 edge with an edge of multiplicity at least 2, or e = e' must be a simple edge and G is obtained by 154 replacing that edge e with a path of length $k \ge 5$ with $k \equiv 1 \mod 4$ and $k \ge 5$. In both cases it is 155 easy to check that G has the $\{a, b\}$ -property. 156

Thus, we can assume that M was obtained by pasting together at least three cycles and that 157 there are at least two disjoint edges to which other cycles have been pasted to. This implies that 158 there are at least two disjoint cycles of length congruent to 2 modulo 4 where all vertices except 159 two which are adjacent have exactly two neighbours. One of these cycles $C = v_1 v_2 \dots v_n v_1$ does not 160 contain e. By possibly relabelling the vertices we can assume that all vertices of C except v_1 and 161 v_2 only have two distinct neighbours. By induction the graph G' obtained from G by replacing the 162 path $v_2v_3...v_n$ with an edge e'' has the $\{a, b\}$ -property, but any proper $\{a, b\}$ -edge-weighting of G' 163 can be converted to a proper $\{a, b\}$ -edge-weighting of G by assigning the same weight to an edge in 164 G as in G' and assigning the weight assigned to e'' in G' to the edges v_2v_3 and $v_{n-1}v_n$, and finally 165 assigning the weights of the remaining edges $v_2v_3, ..., v_{n-2}v_{n-1}$ in a way avoiding conflicts inside 166 C. 167

We now introduce or recall results that will be needed during the course of our main proof below. The following first observation is obvious and implies that studying the $\{a, b\}$ -property only makes sense when gcd(a, b) = 1.

Observation 2.3. Let w be a proper $\{a, b\}$ -edge-weighting of a graph G. If we multiply all edge weights of w by a non-zero integer α , then we get a proper $\{a\alpha, b\alpha\}$ -edge-weighting of G.

In what follows, given a graph G and a mapping $f: V(G) \to \mathbb{Z}_k$, by an *f*-factor modulo k we mean a spanning subgraph H of G such that, for every $v \in V(G)$, we have $d_H(v) \equiv f(v) \pmod{k}$.

Lemma 2.4 (Thomassen [12]). Let G be a connected graph. If $f : V(G) \to \mathbb{Z}_2$ is a mapping satisfying $\sum_{v \in V(G)} f(v) \equiv 0 \mod 2$, then G contains an f-factor modulo 2.

When dealing with bipartite graphs with a bipartition set of even cardinality, and when a and b177 have distinct parity, f-factors modulo 2 can be employed as a convenient tool to deduce proper $\{a, b\}$ -178 edge-weightings in quite an easy way (see [10, 13]). More precisely, let (X, Y) be the bipartition of a 179 bipartite graph G where |X| is even. Lemma 2.4 (when applied onto the function f where f(x) = 1180 for $x \in X$ and f(y) = 0 for $y \in Y$ implies that G has a spanning subgraph H where all of the 181 vertices in X have odd degree, while all of the vertices in Y have even degree. From this, it is easy 182 to see that, assuming a is odd and b is even, assigning weight a to all of the edges in E(H) and 183 weight b to all of the edges in $E(G) \setminus E(H)$ yields a proper $\{a, b\}$ -edge-weighting of G. 184

The upcoming new tools and concepts (in particular that of *mod-4 vertex-colourings*) are the key to generalize this approach to odd $a, b \in \mathbb{Z}$ when |a - b| = 2. **Definition 2.5.** A mod-4 vertex-colouring of a graph G is a vertex-colouring $c : V(G) \to \{1, 2\}$ of satisfying the following conditions for any $uv \in E(G)$ where d(u) and d(v) have the same parity:

189 1.
$$d(u) \equiv d(v) \mod 4 \Rightarrow c(u) \neq c(v)$$
.

190 2. $d(u) \not\equiv d(v) \mod 4 \Rightarrow c(u) = c(v)$.

In the next result, we prove that every bipartite graph G admits a mod-4 vertex-colouring c. It is important to point out that, in general, c might be far from fitting with the bipartition of G. Actually, G might have many edges whose two ends have the same colour by c.

¹⁹⁴ Lemma 2.6. Every bipartite graph has a mod-4 vertex-colouring.

Proof. It suffices to prove the lemma for connected bipartite graphs where all vertices have odd 195 degree or where all vertices have even degree (as otherwise we can consider, still in the whole graph, 196 the vertices with even degree first, and then those with odd degree). So let G be a connected 197 bipartite graph where all vertex degrees have the same parity. Let v be a vertex in G and let 198 $D_0, D_1, ..., D_m$ denote the distance classes of G from $v \in D_0$. Since G is bipartite, each D_i is an 199 independent set. Now give v colour 1 and colour the distance classes in the given order starting 200 with D_1 , then D_2 and so on until we reach a vertex $v' \in D_{i'}$ we cannot assign a colour without 201 violating conditions 1 or 2 in Definition 2.5. If this happens one or both of the following two cases 202 have occurred: 203

1. there are two neighbours $v_1, v_2 \in D_{i'-1}$ of v' with $d(v_1) \equiv d(v_2) \mod 4$ and $c(v_1) \neq c(v_2)$;

205 2. there are two neighbours $v_1, v_2 \in D_{i'-1}$ of v' with $d(v_1) \not\equiv d(v_2) \mod 4$ and $c(v_1) = c(v_2)$.

Let us first assume that we are in the first case and let P_1, P_2 be two internally disjoint shortest 206 paths towards v starting with $v'v_1$ and $v'v_2$ respectively and ending in a common vertex $v'' \in D_{i''}$. 207 That is, v'' is the first vertex on both P_1 and P_2 that is encountered when going from v_1 towards 208 v along P_1 ; possibly v'' = v. All the vertices of P_1 and P_2 except v' are coloured without violating 209 conditions 1 and 2 in Definition 2.5, and P_1 and P_2 have the same length. The parity of the number 210 of times the degree modulo 4 changes when walking from v'' to v_1 on P_1 is the same as the parity 211 of the number of times the degree modulo 4 changes when walking from v'' to v_2 on P_2 . Thus, the 212 parity of the number of times the degree modulo 4 does not change when walking from v'' to v_1 213 on P_1 is the same as the parity of the number of times the degree modulo 4 does not change when 214 walking from v'' to v_2 on P_2 . Since conditions 1 and 2 in Definition 2.5 are not violated, this implies 215 that the parity of the number of times the colour changes when walking from v'' towards v' is the 216 same when walking along P_1 as when walking along P_2 . Thus, $c(v_1) = c(v_2)$, a contradiction. The 217 second case above can be dealt with in a similar way. 218

Let $a, b \in \mathbb{Z}$ be two odd integers with b = a + 2. Let G be a graph and X, Y be two disjoint subsets of its vertices. By an (X, Y)-*a*-parity $\{a, b\}$ -edge-weighting of G, we mean an $\{a, b\}$ -edgeweighting where all vertices in X are incident to an odd number of *a*-edges and all vertices in Y are incident to an even number of *a*-edges. (X, Y)-*b*-parity $\{a, b\}$ -edge-weightings are defined similarly, but with respect to the incident *b*-edges. In the following result, we establish a crucial connection between mod-4 vertex-colourings and (X, Y)-parity $\{a, b\}$ -edge-weightings, leading to the existence of proper $\{a, b\}$ -edge-weightings. **Lemma 2.7.** Let G be a connected bipartite graph and let $a, b \in \mathbb{Z}$ be odd integers with b = a + 2. If G has a mod-4 vertex-colouring where at least one of the two colour classes has even size, then G has the $\{a, b\}$ -property. Consequently, if G does not have the $\{a, b\}$ -property, then, in every mod-4 vertex-colouring, the two colour classes have odd size.

Proof. Let G be a connected bipartite graph, and c a mod-4 vertex-colouring of G. We denote by 230 X and Y the sets of vertices with colour 1 and 2, respectively. Assume |X| is even. By Lemma 2.4 231 there is an $\{a, b\}$ -edge-weighting $w: E(G) \to \{a, b\}$ such that all vertices in X are incident to an odd 232 number of b-edges and all vertices in Y are incident to an even number of b-edges. This corresponds 233 to our notion of an (X, Y)-b-parity $\{a, b\}$ -edge-weighting. The possible weighted degrees of a vertex 234 v of even degree and colour 1 induced by such an edge-weighting are $\{a(d(v)-1)+b, a(d(v)-1)+b\}$ 235 $b+4, a(d(v)-1)+b+8, \dots, a+b(d(v)-1)$ and the possible weighted degrees of a vertex v' of even 236 degree and colour 2 induced by such an edge-weighting are $\{ad(v'), ad(v') + 4, ad(v') + 8, \dots, bd(v')\}$. 237 The possible weighted degrees of a vertex u of odd degree and colour 1 induced by such an edge-238 weighting are $\{a(d(u) - 1) + b, a(d(u) - 1) + b + 4, a(d(u) - 1) + b + 8, ..., bd(u)\}$ and the possible 239 weighted degrees of a vertex u' of odd degree and colour 2 induced by such an edge-weighting are 240 $\{ad(u'), ad(u') + 4, ad(u') + 8, ..., a + b(d(u') - 1)\}$. Let $xy \in E(G)$. We will show that $w(x) \neq w(y)$. 241 To do this we distinguish two distinct cases (note that we can assume that x and y have the same 242 degree parity, as otherwise w(x) cannot be equal to w(y): 243

1. x and y have the same colour by c.

245 2. x and y have distinct colours by c.

First assume that x and y have the same colour. Since c is a mod-4 vertex-colouring we have that $d(x) \not\equiv d(y) \mod 4$. Note that by the above it suffices to show that $ad(x) \not\equiv ad(y) \mod 4$ and this is trivially true since gcd(a, 4) = 1. Now assume that x and y have distinct colours. Since c is a mod-4 vertex-colouring we have that $d(x) \equiv d(y) \mod 4$. Note that by the above it suffices to show that $ad(x) \equiv ad(y) \mod 4$, and as mentioned above this follows since gcd(a, 4) = 1.

²⁵¹ From the previous proof, we can also extract the following:

Observation 2.8. Let $a, b \in \mathbb{Z}$ be odd integers with b = a + 2 and let uv be an edge in a graph Gwhose edges are weighted with a and b. If either

²⁵⁴ 1. d(u) and d(v) have distinct parity, or

255 2. $d(u) \equiv d(v) \mod 4$ and v is incident to an odd number of a-edges while u is incident to an 256 even number of a-edges, or

257 3. $d(u) \neq d(v) \mod 4$ and both v and u are incident to an odd or even number of a-edges,

then u and v have distinct weighted degrees. This is also true if one considers the parity of the numbers of incident b-edges instead of the parity of the numbers of incident a-edges.

Let G be a graph and w an $\{a, b\}$ -edge-weighting of G. By *swapping* (the weight of) an edge, we mean changing its weight to a if it is a b-edge, or changing its weight to b otherwise. By swapping a path or a cycle, we mean swapping all of its edges. For a vertex v in a cycle C of G, it can be observed that the parity of the number of *a*-edges (and similarly *b*-edges) incident to v is not altered upon swapping *C*. In the proof of our main result below, this fact will be used a lot to get rid of conflicts in the following way.

Let X, Y be the two colour classes of a mod-4 vertex-colouring of G and assume that, for some 266 vertex $v \in X$, w is an $(X \setminus \{v\}, Y \cup \{v\})$ -a-parity $\{a, b\}$ -edge-weighting, *i.e.*, all vertices in $X \setminus \{v\}$ 267 are incident to an odd number of a-edges and all vertices in $Y\{v\}$ are incident to an even number of 268 a-edges. According to Observation 2.8, all conflicts (if any) involve v. So let uv be a conflict. To get 269 rid of this conflict while controlling the possible creation of new conflicts, we will swap particular 270 cycles of G. Let C be a cycle of G going through u using two edges e, e' incident to u. If e, e' are 271 assigned the same weight by w, then C is called *u*-changing. C will be called *v*-avoiding if it does 272 not go through v. 273

Observation 2.9. Let G be a graph, X,Y be the two colour classes of a mod-4 vertex-colouring of G, and let w be an $(X \setminus \{v\}, Y, \cup \{v\})$ -a-parity $\{a, b\}$ -edge-weighting for some vertex v, where $a, b \in \mathbb{Z}$ are odd integers with b = a + 2. If uv is a conflict, then, by swapping a u-changing v-avoiding cycle C of G, we get rid of this conflict. Furthermore, any remaining/arising conflicts involve v.

Proof. According to Observation 2.8, all original conflicts by w must involve v. When swapping C, the weighted degree of u is altered since C is u-changing, while the weighted degree of v is unaltered since C is v-avoiding. So we get rid of the conflict uv. Furthermore, it can be noticed that, upon swapping any cycle of G, the parities of the number of a-edges (and similarly b-edges) incident to the vertices are unaltered. Therefore, we get another $(X \setminus \{v\}, Y \cup \{v\})$ -a-parity $\{a, b\}$ -edge-weighting, and Observation 2.8 indicates that, after the swapping of C, all conflicts (if any) in the resulting $\{a, b\}$ -edge-weighting must involve v.

We finish with a few general lemmas to be used in particular cases of our upcoming main proof.

Lemma 2.10. Let G be a 2-connected bipartite graph, X, Y be the two colour classes of a mod-4 vertex-colouring of G, and let $a, b \in \mathbb{Z}$ be odd integers with b = a + 2. If both X and Y have odd size and $v \in X$ is such that G - v - N(v) is connected, then there is an $(X \setminus \{v\}, Y \cup \{v\})$ -a-parity $\{a, b\}$ -edge-weighting of G where all edges incident to v have weight b and every vertex $u \in N(v)$ is incident to at most 1 + M(uv) b-edges, where M(uv) denotes the multiplicity of the edge uv.

Proof. Suppose G' = G - v - N(v) is connected. Let G'' be obtained from G - v by, for every vertex 291 $u \in N(v)$, removing all edges but one incident to u in G - v. For each $u \in N(v)$, let e_u be the 292 unique edge incident to u in G'' and let n(u) denote the unique neighbour of u in G''. Note that 293 since G' is connected, then so is G''. Let S denote the set of edges in G not incident to v and not in 294 G''. That is, S is the set of edges removed from G-v to obtain G''. Let G[S] denote the subgraph 295 of G induced by the edges in S and let Z denote the vertices of odd degree in G[S]. Clearly |Z| is 296 even, so, since $X \setminus \{v\}$ has even size, the set $X' = (X \setminus (Z \cup \{v\})) \cup Z \cap Y$ also has even size. Thus, 297 Lemma 2.4 implies that there is an $(X', V(G'') \setminus X')$ -a-parity $\{a, b\}$ -edge-weighting of G''. We now 298 extend this weighting to G by assigning weight a to all edges in S and weight b to all edges in E(v); 299 this results in a desired edge-weighting of G. 300

Lemma 2.11. Let G be a 2-connected bipartite graph. If there is a vertex $v \in V(G)$ of degree at least 4 and with $|N(v)| \ge 3$ such that G - v - N(v) is connected, then G has the $\{-1, 1\}$ -property. Proof. By Lemma 2.10, there is an $(X \setminus \{v\}, Y \cup \{v\})$ -(-1)-parity $\{-1, 1\}$ -edge-weighting of G, where all edges incident to v have weight 1 and any vertex $u \in N(v)$ is incident to at most 1 + M(uv)1-edges, where M(uv) denotes the multiplicity of the edge uv. Observation 2.8 implies that the only potential conflicts are between v and its neighbours. But the weighted degree of v is d(v) and since $|N(v)| \geq 3$, we have for any $u \in N(u)$ that the multiplicity of uv is less than d(v) - 1. Thus, the weighted degree of any $u \in N(v)$ is less than d(v) and therefore, there can be no conflicts.

Lemma 2.12 (Thomassen, Wu, Zhang [13]). Let q be a natural number such that $q \ge 4$. Let G be a connected graph and let A be an independent set of at most q vertices such that each vertex in A has degree at least q-1, or, each vertex in A, except possibly one, has degree at least q. Assume that no vertex in A is adjacent to a bridge in G. Then, for each vertex a of A, there is an edge e_a incident with a such that the deletion of all e_a , $a \in A$, results in a connected graph unless |A| = q = 4, all vertices of A have degree 3, and G - A has six components each of which is joined to two distinct vertices of A.

Let a and b be two odd integers with b = a + 2. In some cases Lemmas 2.4 and 2.12 work well 316 together when trying to construct a proper $\{a, b\}$ -edge-weighting of a connected bipartite graph 317 G: suppose $c: V(G) \to \{1,2\}$ is a mod-4 vertex-colouring of G and let X and Y denote the sets 318 of vertices in G of colour 1 and 2 respectively and assume that both X and Y have odd size. 319 Furthermore, suppose that the degree of a vertex $v \in X$ is at least 4 and no vertex in N(v) has 320 degree strictly larger than 4. Let A be the vertices in N(v) with the same degree as v and suppose 321 that no vertex in A is incident to a bridge in G - v, the graph G - v is connected, and we are 322 not in the exceptional case of Lemma 2.10, that is, for each $u \in A$ there is an edge e_u such that 323 $G - \bigcup_{u \in A} \{e_u\}$ is connected. Define $S = \bigcup_{u \in A} \{e_u\}$ and let Z denote the set of vertices in G which 324 have odd degree in the subgraph of G induced by S (note that $A \subset Z$). Since $X \setminus \{v\}$ and Z 325 have even size, the set $X' = (X \setminus (Z \cup \{v\})) \cup Z \cap Y$ also has even size. Thus, Lemma 2.4 implies 326 that there is an $(X', V(G-v) \setminus X')$ -a-parity $\{a, b\}$ -edge-weighting of G-v. We can extend this 327 edge-weighting to G by assigning weight a to all edges in S and weight b to all edges in E(v) to 328 obtain an $(X \setminus \{v\}, Y \cup \{v\})$ -a-parity $\{a, b\}$ -edge-weighting of G where all edges incident to v have 329 weight b and every vertex $u \in N(v)$ is incident to at least one a-edge. Thus, the weighted degree 330 of v is greater than that of its neighbours and Observation 2.8 implies that the edge-weighting is 331 proper. 332

We are now ready to prove the main result of this section. Let us emphasize that the main steps in the proof follow the lines of those in the proofs of the main results in [10] and [13]. In particular, Claims 2, 3, 4, 5, as stated below, can also be found in [10]. The proofs are rather different, though, as most arguments used to deal with the $\{0, 1\}$ -property do not apply immediately for the $\{a, b\}$ property when $a, b \in \mathbb{Z}$ are odd and b = a + 2. Instead, some of the tools and results we have introduced earlier are used. Also, the end of the proof in our case is more straightforward than those for the $\{1, 2\}$ - and $\{0, 1\}$ -properties.

Proof of Theorem 1.1. Suppose the theorem is false and, for some odd $a \in \mathbb{Z}$ and b = a + 2, let Gbe a counterexample which has smallest order possible. By possibly multiplying the weights by -1we can assume b > 0. Let c be a mod-4 vertex-colouring of G (such a colouring exists by Lemma ³⁴³ 2.6), and let X denote the set of vertices with colour 1 and Y denote the set of vertices with colour ³⁴⁴ 2. By Lemma 2.7, we can assume that both X and Y have odd size.

³⁴⁵ Claim 1. G has no multiple edge uv where both u and v have only two distinct neighbours.

Proof of the claim. Suppose uv is a multiple edge and both u and v have only two distinct neighbours. By the minimality of G, Lemma 2.2, and the fact that G is not an odd multi-cactus, the graph obtained from G by replacing uv with one non-multiple edge has a proper $\{a, b\}$ -edge-weighting w. But since the multiplicity of uv in G is at least 2 and since u and v can each be in only one conflict distinct from uv, we can obtain a proper $\{a, b\}$ -edge-weighting of G from w by weighting the edges joining u and v to avoid the conflicts involving u and v (such an edge-weighting exists because there are at least three possible sums for the weight of edges joining u and v).

Since G is 2-connected, it has minimum degree at least 2. In what follows, by a suspended path of G, we mean a path $v_1x_1...x_kv_2$ where all internal vertices $x_1, ..., x_k$ have degree 2 and v_1 and v_2 have degree at least 3.

Claim 2. G has no suspended path of length 2.

Proof of the claim. Suppose the claim is false and let v_1xv_2 be a suspended path in G, where d(x) = 2 and $d(v_1), d(v_2) \ge 3$. We can assume $x \in X$. Define G' = G - x. Recall that G' is connected since G is 2-connected. Lemma 2.4 implies that there is an $(X \setminus \{x\}, Y \cup \{x\})$ -a-parity $\{a, b\}$ -edge-weighting w of G, where $w(v_1x) = w(v_2x) = a$. More precisely, w can be obtained as follows (recall that $|V(G') \cap X|$ is even):

- If $v_1, v_2 \in X$, then, according to Lemma 2.4, there is an $(X \setminus \{v_1, v_2, x\}, Y \cup \{v_1, v_2\})$ -aparity $\{a, b\}$ -edge-weighting of G'. Then, assigning weight a to v_1x and v_2x gives the desired weighting of G.
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• If $v_1, v_2 \in Y$, then the same conclusion can be reached when applying Lemma 2.4 so that we start from an $(X - \{x\} \cup \{v_1, v_2\}, Y \setminus \{v_1, v_2\})$ -a-parity $\{a, b\}$ -edge-weighting of G'.

• If $v_1 \in X$ and $v_2 \in Y$ (resp. $v_2 \in X$ and $v_1 \in Y$), then, again, we can get the same conclusion after applying Lemma 2.4 from an $(X \setminus \{v_1, x\} \cup \{v_2\}, Y \cup \{v_1\} \setminus \{v_2\})$ -*a*-parity $\{a, b\}$ -edge-weighting (resp. $(X \setminus \{v_2\} \cup \{v_1\}, Y \cup \{v_2\} \setminus \{v_1\})$ -*a*-parity $\{a, b\}$ -edge-weighting) of G'.

Observation 2.8 implies that the only conflicts that can arise are xv_1 and xv_2 . So we can assume 371 that x and v_1 are in conflict, *i.e.*, both x and v_1 have weighted degree 2a. This implies that v_1 has 372 even degree at least 4 (by the definition of a suspended path, and only vertices with degree of the 373 same parity can be in conflict) and a < 0 < b and hence a = -1 and b = 1. Let u_1, u_2 be two 374 neighbours of v_1 in G' such that u_1v_1 and u_2v_1 have weight -1 and let C be a cycle in G' using the 375 edges u_1v_1 and v_1u_2 . Such a cycle exists as, because G is 2-connected, there is a path from u_1 to u_2 376 in $G - v_1$. Because C is v_1 -changing and x-avoiding, if we swap all weights on C we do not create 377 new conflicts in G' and we lose the conflict xv_1 (recall Observation 2.9). In particular, x remains of 378 weighted degree -2 while v_1 becomes of weighted degree 2. 379

Thus, we can now assume that xv_2 is a conflict. This implies that v_2 also has even degree at least 4. We can now get rid of this conflict in the same way as we got rid of the conflict v_1x , unless all v_2 -changing cycles in G' (that thus use two edges in $E(v_2)$ having the same weight) all use two edges in $E(v_1)$ both having weight 1 (in this case we only move the conflict from xv_2 to xv_1). Since xv_2 is a conflict, v_2 must be incident to at least two -1-edges in G' and at least one 1-edge. Furthermore, as mentioned above, we can assume that any v_2 -changing cycle in G' contains two edges incident to v_1 having weight 1.

Assume that v_1 is incident to a -1-edge e in G'. Since G is 2-connected, there is, in $G - v_1$, a 387 path from v_2 to the end of e different from v_1 . From the existence of that path, we get that there is 388 a path P in G' from v_1 to v_2 using e. If the weight on the last edge e' of P (the one incident to v_2) 389 is 1, then swapping the weights on the cycle $P \cup v_1 x \cup xv_2$ yields a proper edge-weighting; so we can 390 assume e' has weight -1. Since xv_2 is a conflict and v_2 has even degree at least 4, vertex v_2 must 391 be incident to a -1-edge $e'' \neq e'$ in G'. Now, because G is 2-connected, the graph $G - v_2$ has a path 392 P' joining the end of e' different from v_2 and the end of e'' different from v_2 . Note that if P' does 393 not contain v_1 , then we would get a cycle whose weights can be swapped to immediately deduce a 394 proper edge-weighting of G. The same conclusion holds if P' and P intersect for the first time on 395 a vertex different from v_1 . So v_1 is the first intersection point between P and P', in which case we 396 deduce a cycle of G containing v_2 as well as all of e, e', e'' (first go from v_2 to v_1 along P', before 397 going back to v_2 along P); when swapping the weights along that cycle, we get rid of all conflicts 398 between x and v_1, v_2 . 399

We are left with the case where v_1 is not incident to a -1-edge e in G'. Thus, we deduce that 400 v_1 has degree 4 and, by symmetry, v_2 also has degree 4. Note that this implies that $v_1, v_2 \in X$ 401 and furthermore, that all four edges incident to v_1 except $v_1 x$ have weight 1. Also, according to all 402 hypotheses so far, v_1 has weighted degree 2, v_2 has weighted degree -2, and so two edges incident 403 to v_2 in G' have weight -1 while the last edge incident to v_2 in G' has weight 1. We now consider 404 the graph $G'' = G' - v_1 - v_2$. If G'' is connected, then we can find in G' a cycle including the two 405 -1-edges incident to v_2 (thus v_2 -changing), and not passing through v_1 ; then, as earlier, we can 406 swap the weights along this cycle to get rid of the conflict $v_2 x$. So we may assume the two -1-edges 407 incident to v_2 in G' are incident to two distinct components of G''. This leaves us with the following 408 three cases to consider: 409

⁴¹⁰ **Case 1:** G'' has two components C_1, C_2 , such that v_1 has two neighbours in C_1 and one neighbour ⁴¹¹ in C_2 , and v_2 has two neighbours in C_2 and one neighbour in C_1 .

Let $e_{1,a}$ and $e_{1,b}$ denote the two edges incident to v_1 going to C_1 . Recall that $e_{1,a}, e_{1,b}$ have weight 1. Since C_1 is connected, there is a path from the end of $e_{1,a}$ different from v_1 to the end of $e_{1,b}$ different from v_1 . We swap all weights along the cycle formed by this path and $e_{1,a}, e_{1,b}$ to get another edge-weighting of G where v_1 has weighted degree -2; so, now, both xv_1 and xv_2 are conflicts.

Now let $e_{1,c}$ denote the edge incident to v_1 going to C_2 , and $e_{2,a}$ denote the 1-edge incident to v_2 going to C_2 . Both these edges are weighted 1. Since C_2 is connected, there is a path P from the end of $e_{1,c}$ different from v_1 to the end of $e_{2,a}$ different from v_2 . Now consider the cycle of Gstarting in x, going through xv_2 and $e_{2,a}$, then going along P, and finally going through $e_{1,c}$ and v_1x . When swapping all weights along this cycle, note that v_1, v_2 remain of weighted degree -2, while x becomes of weighted degree 2. So the $\{-1,1\}$ -edge-weighting of G becomes proper according to Observation 2.9. ⁴²⁴ **Case 2:** G'' has two components C_1, C_2 , such that both v_1 and v_2 have two neighbours in C_1 and ⁴²⁵ one neighbour in C_2 .

Let $v_{1,a}, v_{1,b}$ denote the two neighbours of v_1 in C_1 , and let $v_{1,c}$ denote the neighbour of v_1 in C_2 . Note that for one of $v_{1,a}, v_{1,b}$, say $v_{1,a}$, the graph $G''' = G - v_1 - v_{1,a} - v_{1,c}$ is connected, for if G' is disconnected, then it must be the case that $v_{1,a}$ is a cut-vertex in $G - v_1$ and then it is easy to see that $G - v_1 - v_{1,b} - v_{1,c}$ is connected, and we can just rename $v_{1,a}$ and $v_{1,b}$ accordingly. Note that, by Lemma 2.11, we can assume that $G''' - v_{1,b}$ is disconnected. Let $L_1, ..., L_n$ denote the components of $G''' - v_{1,b}$, where $v_2 \in V(L_1)$. Since $v_{1,b}$ is not a cut-vertex in G, it follows that $v_{1,a}$ has a neighbour in each of the components L_i for $i \geq 2$.

Let us now consider the graph obtained from G by removing the vertex v_1 , and, for each of 433 $v_{1,a}, v_{1,c}$, removing all remaining incident edges but one. Note that, in that graph, x also has de-434 gree 1. Lemma 2.4 implies that this graph has an $(X \setminus \{v_1\}, Y)$ -(-1)-parity $\{-1, 1\}$ -edge-weighting. 435 By then assigning weight -1 to all removed edges incident to v_1 and weight 1 to all remaining edges 436 (incident to one of $v_{1,a}, v_{1,c}$), we get that G has an $(X \setminus \{v_1\}, Y)$ -(-1)-parity $\{-1, 1\}$ -edge-weighting 437 where all four edges incident to v_1 are weighted -1, and each of $v_{1,a}, v_{1,c}$ is incident to at most two 438 -1-edges. Now the only possible conflict is $v_1v_{1,b}$, so we can assume this is indeed a conflict, and 439 hence, both v_1 and $v_{1,b}$ have weighted degree -4. We can also assume that we cannot swap the 440 weights in a cycle in G''' containing two edges incident to $v_{1,b}$ having the same weight, and hence, 441 $v_{1,b}$ has at most two neighbours in each of L_i for i = 1, ..., n. Since the weighted degree of $v_{1,b}$ is 442 -4, there must be some components in $G''' - v_{1,b}$ which are incident to strictly more -1-edges in 443 $E(v_{1,b})$ than 1-edges in $E(v_{1,b})$. Again, since we can assume that there is no cycle in G''' containing 444 two edges incident to $v_{1,b}$ having the same weight, we can also deduce that no two edges incident to 445 $v_{1,b}$ having the same weight go to the same component in $G''' - v_{1,b}$. Thus, there are at least three 446 components L'_1, L'_2, L'_3 in $G''' - v_{1,b}$, each of which is incident to only one edge in $E(v_{1,b})$ and each 447 of these edges has weight -1. We can assume that L'_1 and L'_2 are distinct from L_1 , and, since $v_{1,b}$ 448 is not a cut-vertex in G, the vertex $v_{1,a}$ has a neighbour u'_i in each L'_i for i = 1, 2. There is now a 449 cycle C in $G''' + v_{1,a}$ containing two edges incident to $v_{1,b}$ having weight -1 and containing the two 450 edges $v_{1,a}u'_1$ and $v_{1,a}u'_2$. If we swap the weights on C, then the only possible conflict is $v_1v_{1,a}$ in the 451 case where $v_{1,a}$ is a vertex of degree 4 and $v_1v_{1,a}$ has weight -1, both $v_{1,a}u'_1$ and $v_{1,a}u'_2$ have weight 452 1, and $v_{1,a}$ is incident to some fourth -1-edge $v_{1,a}u'$. We can assume that the component L' to 453 which u' belongs in $G''' - v_{1,b}$ is not incident to a -1-edge of $v_{1,b}$, since otherwise, we could modify 454 C to contain the edge $v_{1,a}u'$. Note that this also implies that $L'_3 = L_1$. Since $v_{1,b}$ had weighted 455 degree -4 this implies that there is another component L'_4 distinct from all of L'_1, L'_2, L'_3 , which is 456 incident to an edge in $E(v_{1,b})$ having weight -1. The vertex $v_{1,a}$ must have a neighbour u'' in this 457 component L'_4 and, since we can assume that we cannot modify C to contain $v_{1,a}u'$, we must have 458 $u'' \neq u'$. This contradicts $v_{1,a}$ having degree 4. 459

Case 3: G'' has three components C_1, C_2, C_3 , such that v_1 and v_2 have one neighbour in each of these three components.

In that case, G has the $\{-1, 1\}$ -property according to Lemma 2.11 as v_1 has even degree 4, and it can be checked that $G - v_1 - N_G(v_1)$ remains connected due to the 2-connectedness of G. In particular, for i = 1, 2, 3, note that the edge incident to v_1 going to C_i and the edge incident to v_2 going to C_i cannot share an end. A contradiction.

466 Claim 3. G has no suspended path of length 4.

Proof of the claim. Suppose the claim is false and let $v_1x_1x_2x_3v_2$ be a suspended path in G, where $d(x_1) = d(x_2) = d(x_3) = 2$ and $d(v_1), d(v_2) \ge 3$. We can assume $x_2 \in X$, which implies that $x_1, x_3 \in Y$. Define $G' = G - x_1 - x_2 - x_3$. Using Lemma 2.4 similarly as in the proof of Claim 2, we can come up with an $(X \setminus \{x_1\}, Y \setminus \{x_2, x_3\})$ -a-parity $\{a, b\}$ -edge-weighting w of Gwhere $w(v_1x_1) = w(x_3v_2) = a$ and $w(x_1x_2) = w(x_2x_3) = b$. One can check that, by slightly modifying the exact same arguments used in the proof of Claim 2, we can eventually remove all conflicts from w, or deduce another proper $\{a, b\}$ -edge-weighting of G.

474 Claim 4. G has no suspended path of length at least 5.

Proof of the claim. Suppose the claim is false and let $v_1x_1x_2x_3x_4v_2$ be a path in G, where 475 x_1, x_2, x_3, x_4 all have degree 2, and v_1, v_2 here might be of degree 2. Let G' be obtained from 476 G by replacing $v_1x_1x_2x_3x_4v_2$ by an edge $e = v_1v_2$ even if that edge is already there. If G' has the 477 $\{a, b\}$ -property, then so does G. Indeed, assume there is a proper $\{a, b\}$ -edge-weighting of G where 478 the weight of e is, say a, and consider that weighting back in G. We start the extension to the five 479 edges by assigning weight a to v_1x_1 and x_4v_2 , so that v_1 and v_2 keep the same weighted degree as 480 in G'. Since v_1v_2 is an edge in G', note that v_1 and v_2 have different weighted degrees. From this, 481 we deduce that either v_1 has weighted degree different from 2a and v_2 has weighted degree different 482 from a + b, or conversely. Assume the first situation holds. Then, we can achieve the weighting of 483 G by assigning weight a to x_1x_2 and weight b to x_2x_3 and x_3x_4 . 484

So we can assume that G' does not have the $\{a, b\}$ -property and is thus an odd multi-cactus by the minimality of G. The edge e cannot be red in G', since then G would also be an odd multicactus. Thus e is green and Lemma 2.2 implies that G has the $\{a, b\}$ -property.

⁴⁸⁹ By Claims 1, 2, 3, 4, all degree-2 vertices in G (if any) lie on suspended paths of length 3. In G⁴⁹⁰ we replace all suspended paths of length 3 by edges (even if the two ends were already adjacent) to ⁴⁹¹ form a bipartite multigraph G^* . Edges arising from suspended paths of length 3, we call *blue edges*. ⁴⁹² Every other edge of G^* , *i.e.*, which was already present in G, we call a *white edge*.

Note that G^* is bipartite, 2-connected, has minimum degree at least 3, and it may have more multiple edges than G has. Also, note that for every vertex v in G^* , we have $d_{G^*}(v) = d_G(v)$. In general, it is not easy to deduce a proper $\{a, b\}$ -edge-weighting of G from one of G^* (typically because of blue edges); however, information on the structure of G can be deduced from that of G^* . In particular, we will study the existence of paths or cycles in G^* to deduce that of corresponding paths or cycles in G (where any traversed blue edge in G^* is replaced by the corresponding path of length 3 in G).

If the deletion of some pair of adjacent vertices u, v disconnects G^* , then let $z_0y_0 \in E(G^*)$ be such that $G - z_0 - y_0$ is disconnected and such that some component H of $G^* - z_0 - y_0$ has smallest possible order. The union of that component H and z_0, y_0 together with all edges connecting them is denoted B. In case G has no pair of adjacent vertices whose removal disconnects the graph, we define $H = B = G^*$, and y_0, z_0 do not exist.

⁵⁰⁵ Claim 5. For every vertex v of H, we have $d_{G^*}(v) = 3$.

Proof of the claim. Suppose the claim is false and let w_0 be a vertex in H of maximum degree 506 $d = d(w_0)$ at least 4. Without loss of generality, we can suppose that $w_0 \in X$. Assume first that 507 w_0 is adjacent to none of z_0, y_0 (this is the case if these two vertices do not exist). By the remark 508 following Lemma 2.12, we can assume that we are in the exceptional case when considering $G - w_0$ 509 and defining A to be the set of vertices in N(v) which have the same degree as w_0 . Thus, w_0 and 510 all vertices in $N(w_0)$ have degree 4 and $G - w_0 - N(w_0)$ has exactly six components. We can now 511 choose another vertex of degree d as w_0 by choosing w_0 such that the order of the component of 512 $G - w_0 - N(w_0)$ containing z_0 is maximum and avoid the exceptional case in Lemma 2.12. Again, 513 the remark following Lemma 2.12 shows how to find a proper $\{a, b\}$ -edge-weighting of G. 514

So we can assume w_0 is adjacent to, say, z_0 and that all the neighbours of w_0 in H which have the same degree as w_0 are adjacent to y_0 (due to the bipartiteness of G^*). Since y_0, z_0 thus exist, we have $G \neq H$. We can also assume that all vertices in H having maximum degree are adjacent to z_0 or y_0 (as otherwise, the previous situation would apply). Note that this implies that we can never be in the exceptional case in Lemma 2.12 when we delete a vertex v in H of maximum degree and define A to be the neighbours of v with the same degree as v.

As pointed out earlier, there is an $(X \setminus \{w_0\}, Y)$ -a-parity $\{a, b\}$ -edge-weighting of G where all 521 edges incident to w_0 are weighted b and every neighbour of w_0 with degree d is incident to at least 522 one a-edge. This edge-weighting is proper unless $z_0 w_0$ is a conflict, which occurs only if the degree 523 of z_0 is strictly greater than that of w_0 . Note that we can assume that z_0 is incident to exactly 524 one edge going to each component other than H in $G - y_0 - z_0$, since otherwise, we could deduce 525 an $\{a, b\}$ -edge-weighting of G as above with the extra condition that two edges e, e' incident to z_0 526 going to a component C of $G - y_0 - z_0$ other than H are weighted b. Then, if $z_0 w_0$ is a conflict, we 527 could get rid of it by swapping the weights along a cycle going through z_0 and C via e, e' (so that 528 it is z_0 -changing) and not going through H (so that it is w_0 -avoiding). So z_0 is incident to exactly 529 one edge going to each component other than H in $G - y_0 - z_0$. We can also assume that there is at 530 most one component C other than H in $G - y_0 - z_0$, since otherwise, we could reach the exact same 531 conclusion by deducing an $\{a, b\}$ -edge-weighting of G as before with the extra condition that two 532 edges e, e' incident to z_0 going to two different components C, C' distinct from H are weighted b. 533 In case $z_0 w_0$ is a conflict, we could again get rid of it by swapping the weights along a cycle going 534 through z_0 , in C via e, back to y_0 , in C', and back to z_0 via e'. This would be correct since such a 535 cycle would not go through H, and thus, would be w_0 -avoiding. Similarly, we can assume that the 536 multiplicity of $z_0 y_0$ is 1. 537

Let us denote by z_1 the unique neighbour of z_0 in C. By swapping the weights along a cycle through C containing z_0, y_0 and the edge $z_0 z_1$, and not going through H, we can further assume that the edge $z_0 z_1$ is weighted a. For a similar reason, we can assume that the edge $y_0 z_0$ is weighted b. Recall that w_0 and all the neighbours of w_0 with degree d are all incident to an even number of a-edges; thus, each of the neighbours of w_0 with degree d is incident to at least two a-edges.

To get rid of the conflict $z_0 w_0$, we would like to swap the weights along a z_0 -changing cycle in $G - w_0$ (thus, w_0 -avoiding). According to Observation 2.9, recall that this would not alter the parity of the number of incident *a*'s of any vertex in $V(G) \setminus \{w_0\}$. Furthermore, this would get rid of the conflict $z_0 w_0$. However, this swapping process can create a conflict between w_0 and a neighbour vof w_0 with degree *d*; but such a conflict can only arise when the cycle goes through the only two *a*-edges incident to v. For a neighbour v of w_0 with degree *d* that is incident to only two *a*-edges, we call this pair of edges a *forbidden pair*. Our goal in what follows is to show that $G - w_0$ has a z_0 -changing cycle not containing any forbidden pair of edges.

Let us denote by $v_1, ..., v_m$ the neighbours of w_0 with degree d. As mentioned earlier, recall that 551 the v_i 's are all adjacent to y_0 . Since $z_0 w_0$ is a conflict, recall that $d(z_0)$ and $d(w_0)$ have the same 552 parity. Furthermore, since $d(z_0) > d(w_0) > 3$, it follows that $d(z_0) \ge 6$, and, because the only 553 neighbours of z_0 outside H are y_0 and z_1 , there is a vertex $z_2 \neq w_0$ in $N(z_0) \cap V(H)$. Note that, 554 to find the desired cycle through z_0 in $G - w_0$, it suffices to find a path P from y_0 to a vertex z' in 555 $N(z_0) \cap V(H)$ in the connected graph $G - z_0 - w_0$ (which is connected by the minimality of H) not 556 using any forbidden pair of edges. Indeed, if the weight on $z_0 z'$ is b, then we can define our cycle 557 to be $P \cup \{z_0y_0, z'z_0\}$, while, if the weight on z_0z' is a, then we can define our cycle to be $P \cup P_c$, 558 where P_c is a path from z_0 to y_0 in $G - H - z_0 y_0$ (thus, through C). Since the graph $G - z_0 - w_0$ is 559 connected, there is a path P_1 from z_2 to y_0 . We can assume that P_1 uses forbidden pairs of edges. 560 Without loss of generality, let pv_1 and v_1q be the first forbidden pair of edges P_1 used when going 561 from z_2 to y_0 . Since v_1 is adjacent to y_0 , it follows that $q = y_0$, since otherwise, we have found a 562 path from y_0 to z_2 not using any forbidden pair of edges. Thus, we can assume that all paths from 563 y_0 to a vertex in $N(z_0) \cap V(H)$ use exactly one forbidden pair of edges. Now we look at all such 564 paths using only one pair of forbidden edges y_0v_i and v_ip (for $i \in \{1, ..., m\}$) and consider one such 565 path P that goes through the most neighbours of w_0 . Let $y_0 v_i$ and $v_i p$ be the pair of forbidden 566 edges that P contains. 567

First suppose that v_i has a neighbour v'_i distinct from y_0, p, w_0 . The edge $v_i v'_i$ must have weight 568 b. Since $G - w_0 - v_i$ is connected, it has a path P' from v'_i to a vertex in $N(z_0) \cap V(H)$. The path 569 P' must use a forbidden pair of edges, as otherwise, the graph induced by $E(P) \cup E(P')$ would 570 contain a desired path from y_0 to a vertex in $N(z_0) \cap V(H)$ avoiding forbidden pairs of edges. Let 571 the first pair of forbidden edges P' used when starting from v'_i be qv and vr. The subpath P'_1 of P'572 from v'_i to v must be disjoint from P, since otherwise, the graph induced by $E(P) \cup E(P'_1)$ contains 573 a desired path from y_0 to $N(z_0) \cap V(H)$ avoiding forbidden pairs of edges. Furthermore, we must 574 have that $r = y_0$, since otherwise, the path P'' defined to be $y_0 v$ together with the subpath of P'575 from v to v'_i followed by $v'_i v_i$ and the subpath of P from v_i to $N(z_0) \cap V(H)$ is a desired path from 576 y_0 to $N(z_0) \cap V(H)$ avoiding forbidden pairs of edges. Now the path P'' contradicts the maximality 577 of P. 578

So we can assume that $N(v_i) = \{y_0, p, w_0\}$, which means, because v_i has degree d > 3, that 579 some of the edges $v_i y_0$, $v_i p$, $v_i w_0$ have multiplicity more than 1. The multiplicity of both $y_0 v_i$ and 580 $v_i p$ must be 1, since otherwise, we would get a desired path from z_2 to y_0 avoiding the forbidden 581 pair of edges $y_0 v_i, v_i p$. So the multiplicity of $v_i w_0$ is at least 2. If the only neighbours of w_0 are v_i 582 and z_0 , then we can swap the weights on two edges between w_0 and v_i to avoid the conflict $z_0 w_0$ 583 and obtain a proper $\{a, b\}$ -edge-weighting of G; so we can assume that w_0 is incident to a vertex 584 v' in H distinct from v_i . Since $d(v_i) = d(w_0)$, the edges $w_0 z_0$ and $w_0 v'$ are not multiple and since 585 z_0 has degree at least 6, there are two edges $z_0 z'_2$, $z_0 z''_2$ incident with z_0 having the same weight 586 and where $z'_2, z''_2 \in H$. Possibly $z'_2 = z''_2$ or $z'_2 = z_2$. The graph $B - w_0 - z_0$ is connected, so it 587 contains a path P'_1 from z'_2 to y_0 and a path P'_2 from from z''_2 to y_0 . These paths P'_1 and P'_2 must 588 be internally disjoint, since otherwise, there would be a z_0 -changing cycle in B not containing any 589 pair of forbidden edges. We can assume that both P'_1 and P'_2 contain a pair of forbidden edges, 590 since otherwise, there is a desired path from y_0 to $(N(z_0) - w_0) \cap V(H)$. Hence, we can assume that 591

⁵⁹² P'_1 contains y_0v_i and v_ip and P'_2 contains a pair of forbidden edges qv', v'r incident to v'. Since ⁵⁹³ this implies that y_0 and v' are adjacent, we can assume that $y_0 = q$. The vertex v' must have a ⁵⁹⁴ neighbour s distinct from y_0, w_0, r and since the graph $B - w_0 - v'$ is connected, there is a path P''⁵⁹⁵ in $B - w_0 - v'$ from s to y_0 . If P'' contains the forbidden pair of edges y_0v_i and v_ip , then the graph ⁵⁹⁶ $P'_1 \cup P'_2 \cup P''$ contains a z_0 -changing cycle in B containing no forbidden pair of edges. Thus, we can ⁵⁹⁷ assume that P'' contains no pair of forbidden edges. Now $P'_2 \cup P''$ contains a desired path from w_0 ⁵⁹⁸ to $(N(z_0) - w_0) \cap V(H)$.

⁵⁹⁹ Claim 6. There is no vertex $v \in V(H)$ such that G - v - N(v) is connected.

Proof of the claim. Suppose $v \in V(H)$ and that G' = G - v - N(v) is connected. We can assume 600 $v \in X$. By Lemma 2.10, there is an $(X \setminus \{v\}, Y \cup \{v\})$ -a-parity $\{a, b\}$ -edge-weighting of G, where 601 all edges incident to v have weight b and any vertex $u \in N(v)$ is incident to at most 1 + M(uv)602 b-edges, where M(uv) denotes the multiplicity of the edge uv. Note that Claim 1 implies that if 603 vv' is a multiple edge, then $v' \in \{z_0, y_0\}$. Thus, the only potential conflict is between v and one 604 of z_0, y_0 , say y_0 . This implies that y_0 must have odd degree at least 5. But since G' is connected 605 there must then be a y_0 -changing cycle in $G - (N(v) \setminus \{y_0\})$ and swapping the weights on this cycle 606 yields a proper $\{a, b\}$ -edge-weighting of G. \diamond 607

⁶⁰⁸ Claim 7. There are no multiple edges between two vertices in H.

Proof of the claim. Suppose uv is a multiple edge in H. We can assume $v \in X$. Since u and vhave degree 3 in G^* (by Claim 5), the multiplicity of uv is exactly 2. Let e and e' be the two edges between u and v. By Claim 1, e, e' are not both white. Thus, at least one of e, e', say e, is a blue edge in H. Let v' denote, in G^* , the unique neighbour of v different from u.

Let G' be obtained from G - v by removing all edges but one edge e'' incident to v'. Clearly G' 613 is connected since G - v - v' is connected by the minimality of H. Let $S = E(v') \setminus \{vv', e''\}$. Let 614 X' denote the set of vertices in G - v which are incident to an odd number of edges in S. Note 615 that S has even size. Thus, $X' = (X \setminus S \cup (S \cap Y))$ has even size and Lemma 2.4 implies that 616 there is an $(X', V(G') \setminus X')$ -a-parity $\{a, b\}$ -edge-weighting of G'. We now extend this weighting to 617 G by assigning weight a to all edges in S and weight b to all edges incident to v. This gives an 618 $(X \setminus \{v\}, Y \cup \{v\})$ -a-parity $\{a, b\}$ -edge-weighting of G where all vertices in N(v) are incident to at 619 most two b-edges (the edge in the suspended path of length 3 joining u and v incident to u must 620 have weight a). Observation 2.8 implies that the only conflict can be between v and its neighbours, 621 so the only possible conflict is vv' in the case where $v' \in \{z_0, y_0\}$, say $v' = y_0$, and y_0 has degree at 622 least 5. But since G - v is connected there must then be a y_0 -changing cycle avoiding v and u and 623 swapping the weights on this cycle yields a proper $\{a, b\}$ -edge-weighting of G. 624 \diamond

⁶²⁵ Claim 8. Every vertex of H is incident to at most one blue edge.

Proof of the claim. Recall that every vertex v of H has degree 3, by Claim 5. If v is incident to three blue edges, then G - v - N(v) is connected, which contradicts Claim 6. So now assume v is incident to two blue edges. Let uv denote the white (third) edge incident to v. Still by Claim 6, the graph G - v - N(v) cannot be connected, which means that u, v is a pair contradicting the choice of y_0, z_0 . We now have all the tools in hand for finishing the proof. Two cases must be considered:

Case 1: There is a vertex $v \in V(H)$ not adjacent to any of z_0, y_0 .

Recall that, according to Claim 6, whenever removing from G a vertex of H and its neighbourhood, we get a disconnected graph. Let v be a vertex not adjacent to z_0, y_0 such that the component K of G' = G - v - N(v) containing z_0 and y_0 has maximum order. We can assume that $v \in X$. Note that there must be a vertex $v' \in V(H)$ distinct from v with N(v') = N(v) such that the components of G' are exactly K and the isolated vertex v'. This is because otherwise there would be a vertex $v'' \neq v$ in H such that G - v'' - N(v'') has a bigger K, a contradiction to our choice of v.

Let $e_1 = vv_1, e_2 = vv_2, e_3 = vv_3$ denote the three edges incident to v. Since v, v', v_1, v_2, v_3 all 640 belong to H, by Claim 7, all these vertices are distinct. Furthermore, since N(v) = N(v'), we can 641 also assume that none of vv_1, vv_2, vv_3 are blue. Hence, $v, v' \in X$ and $v_1, v_2, v_3 \in Y$. The graph 642 G-v is connected and so is the graph G' obtained from it by removing the edges $v'v_1, v'v_2$. Lemma 643 2.4 implies that there is an $\{a, b\}$ -edge-weighting of G' where the vertices in $X \setminus \{v\} \cup \{v_2, v_3\}$ are 644 incident to an odd number of a-edges and the vertices in $Y \setminus \{v_2, v_3\}$ are incident to an even number 645 of a-edges. In particular, in G' the only a-edge incident to v' is $v'v_1$. We extend this weighting 646 to G by assigning weight a to $v'v_2, v'v_3$, and weight b to all three edges incident to v. That way, 647 we get an $(X \setminus \{v\}, Y \cup \{v\})$ -a-parity $\{a, b\}$ -edge-weighting of G where all three edges incident to 648 v have weight b and all three edges incident to v' have weight a. Observation 2.8 implies that the 649 only potential conflicts are between v and its neighbours, but since all vertices in N(v) are incident 650 to at least one a-edge (the one incident to v') this edge-weighting is proper. 651

652 Case 2: All vertices in H are adjacent to z_0 or y_0 .

By Claim 6, we can assume that, for every vertex $v \in V(H)$, the graph G' = G - v - N(v) is 653 disconnected. First, suppose z_0 is joined in G^* to some vertex $v \in V(H)$ by an edge of multiplicity 654 2. Let e' and e'' be the two edges joining z_0 and v in G^* . Claim 6 implies that not both of e', e''655 are blue; say e' is white. If e'' is blue, then by Claim 8, the third edge e''' incident to v in G^* 656 must be white. If e'' is white, then Claim 6 implies that e''' is white, so the edge e''' = vu must be 657 white. We can assume $v \in X$ and hence $u \in Y$. Let Z denote the set of vertices in G - v which are 658 incident to exactly one edge incident to u. Note that either Z is empty or Z has size 2. Note that 659 $X' = (X \setminus (Z \cup \{v\})) \cup (Y \cap Z)$ has even size, so by Lemma 2.4, there is an $(X', V(G - v - u) \setminus X')$ -660 a-parity $\{a, b\}$ -edge-weighting of G - v - u. We now extend this edge-weighting to all of G by 661 assigning weight b to all edges in E(v) and weight a to the two edges incident to u distinct from uv. 662 Thus, we have obtained an $(X \setminus \{v\}, Y \cup \{v\})$ -a-parity $\{a, b\}$ -edge-weighting of G where all edges 663 incident to v have weight b and u is incident to exactly one b-edge. Observation 2.8 implies that the 664 only potential conflict is vz_0 in the case where z_0 has degree at least 5. We can also assume that 665 there is no z_0 -changing cycle in G avoiding v and u. Hence, z_0 must have degree 2 in G - H and be 666 joined by an edge in G^* to at least one vertex v' in H distinct from v. We can now find a desired 667 z_0 -changing cycle unless the only neighbours of v' in B are z_0 and u. But in this case G - u - N(u)668 is connected, contradicting Claim 6. 669

Thus, we can assume that z_0 , and similarly y_0 , is not joined to any vertex in H by a multiple edge in G^* . Claim 7 now implies that any vertex in H has three distinct neighbours in G^* . Let $v \in X$ be any vertex in H incident to z_0 . The graph G - v - N(v) is disconnected by Claim 6, so

there must be a vertex v' in H which in G^* has the same neighbourhood as v. Since all vertices in H 673 have degree 3 (Claim 5) this implies that H only has four vertices: two joined to z_0 and two joined 674 to y_0 . The graph $G - v - (N(v) \setminus \{z_0\})$ is connected, so as above, there is an $(X \setminus \{v\}, Y \cup \{v\})$ -a-675 parity $\{a, b\}$ -edge-weighting of G where all edges incident to v have weight b, and the neighbours of 676 v distinct from z_0 which have degree 3 are incident to exactly one b-edge. Again, Observation 2.8 677 implies that the only possible conflict is vz_0 in the case where z_0 has degree at least 5. In this case, 678 it is easy to see that there is a z_0 -changing cycle avoiding H, thus, we can get rid of this conflict 679 and obtain a proper $\{a, b\}$ -edge-weighting of G. 680

⁶⁸¹ 3 Proof of Theorem 1.2

Before describing the structure of trees without the $\{-1, 1\}$ -property, we first introduce the following two general lemmas which will be used in the proof. Some of the tools and results used here were introduced in Section 2.

Lemma 3.1. If G is a simple connected bipartite graph without the $\{-1,1\}$ -property and e is a bridge in G, then the deletion of e results in two components each containing an odd number of vertices.

Proof. Suppose the lemma is false. Let G be a connected bipartite graph without the $\{-1, 1\}$ -688 property and let $e = uv \in E(G)$ be a bridge in G. Let $c: V(G) \to \{1,2\}$ be a mod-4 vertex-689 colouring of G (such a colouring exists by Lemma 2.6) and let X, Y denote the sets of vertices 690 coloured 1 and 2 respectively. By Lemma 2.7, both colour classes of c have odd size. Let C_1, C_2 691 denote the two components of G - e with $u \in V(C_1)$ and $v \in V(C_2)$. For a contradiction, assume 692 that both $|V(C_1) \cap X|$ and $|V(C_1) \cap Y|$ are odd and both $|V(C_2) \cap X|$ and $|V(C_2) \cap Y|$ are even. 693 Recall that two vertices with degree of distinct parity cannot have the same weighted degree by a 694 $\{-1,1\}$ -edge-weighting (Observation 2.8). There are thus four cases to be considered: 695

 G_{96} Case 1: Both u and v have odd degree and colour 1.

⁶⁹⁷ By Lemma 2.4, there is an $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$ -(-1)-parity $\{-1, 1\}$ -edge-weighting ⁶⁹⁸ of C_1 and a $(Y \cap V(C_2), X \cap V(C_2))$ -1-parity $\{-1, 1\}$ -edge-weighting of C_2 . It follows from Obser-⁶⁹⁹ vation 2.8 that these two edge-weightings, together with assigning weight -1 to e, form a proper ⁷⁰⁰ edge-weighting of the whole G.

⁷⁰¹ Case 2: Both u and v have even degree and colour 1.

By Lemma 2.4, there is an $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$ -(-1)-parity $\{-1, 1\}$ -edge-weighting of C_1 and an $(X \cap V(C_2), Y \cap V(C_2))$ -1-parity $\{-1, 1\}$ -edge-weighting of C_2 . It follows from Observation 2.8 that these two edge-weightings, together with assigning weight -1 to e, form a proper edge-weighting of the whole G.

Case 3: Both u and v have odd degree, u has colour 1 and v has colour 2.

By Lemma 2.4, there is an $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$ -(-1)-parity $\{-1, 1\}$ -edge-weighting of C_1 and a $(Y \cap V(C_2), X \cap V(C_2))$ -1-parity $\{-1, 1\}$ -edge-weighting of C_2 . It follows from Observation 2.8 that these two edge-weightings, together with assigning weight -1 to e, form a proper edge-weighting of the whole G.

⁷¹¹ Case 4: Both u and v have even degree, u has colour 1 and v has colour 2.

By Lemma 2.4, there is an $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$ -(-1)-parity $\{-1, 1\}$ -edge-weighting of C_1 and an $(X \cap V(C_2), Y \cap V(C_2))$ -1-parity $\{-1, 1\}$ -edge-weighting of C_2 . It follows from Observation 2.8 that these two edge-weightings, together with assigning weight -1 to e, form a proper edge-weighting of the whole G.

Lemma 3.2. If G is a connected bipartite graph without the $\{-1,1\}$ -property and e is a bridge in G, then there is a $\{-1,1\}$ -edge-weighting of G such that e is the only conflict.

Proof. Let G be a connected bipartite graph without the $\{-1, 1\}$ -property, e be a bridge in G, and 718 C_1, C_2 be the two components of G - e. Let c be a mod-4 vertex-colouring of G (such a colouring 719 exists by Lemma 2.6), and let X denote the set of vertices with colour 1 and Y denote the set of 720 vertices with colour 2. By Lemma 2.7, both X and Y have odd size and by Lemma 3.1, we can 721 assume that $|X \cap V(C_1)|$ and $|Y \cap V(C_2)|$ are even and $|Y \cap V(C_1)|$ and $|X \cap V(C_2)|$ are odd. Now 722 Lemma 2.4 implies that there is an $(X \cap V(C_1), Y \cap V(C_1))$ -1-parity $\{-1, 1\}$ -edge-weighting of C_1 and 723 an $(X \cap V(C_2), Y \cap V(C_2))$ -1-parity $\{-1, 1\}$ -edge-weighting of C_2 . Observation 2.8 implies that these 724 two edge-weightings, together with assigning weight -1 to the edge e, is a $\{-1, 1\}$ -edge-weighting 725 of G where e is the only potential conflict. 726

We can now prove Theorem 1.2. When referring to Operation 1, we mean the first operation described at the end of Section 1 (illustrated in Figure 1, (a) and (b)).

Proof of Theorem 1.2. As mentioned in the introduction, it is straightforward to check that any 729 graph constructed with Operation 1 from four graphs without the $\{-1, 1\}$ -property does not have 730 the $\{-1, 1\}$ -property itself. An easy argument is that all five edges incident to the vertices $v_1 \sim v_2$ 731 and $v_3 \sim v_4$ should have the same weight (in which case a conflict arises), as otherwise, the proper 732 $\{-1,1\}$ -edge-weighting would yield one of at least one of the four graphs used in the construction, 733 a contradiction. Thus, it suffices to prove that any tree without the $\{-1, 1\}$ -property is constructed 734 from a disjoint union of K_2 's through repeated (possibly none) applications of Operation 1. Suppose 735 this is false and let T be a minimum counterexample. Note that Lemma 3.1 implies that, for any 736 vertex $v \in V(T)$ and any edge $e \in E(v)$, the component C_e not containing v in T - e has an odd 737 number of vertices. Since we can write $|V(T)| = 1 + \sum_{e \in E(v)} |V(C_e)|$ for any vertex $v \in V(T)$ and 738 since |V(T)| is even, this implies that all vertices in T have odd degree. A consequence of this is 739 that if $S \subset T$ is a subtree of T, then T - S has no components isomorphic to K_2 as otherwise T 740 would have vertices with degree 2. 741

Let $P = v_1...v_m$ be a longest path in T. Clearly, v_m is a leaf and, since all vertices have odd 742 degree, v_{m-1} is incident to an even number of leaves. Suppose v_{m-1} is incident to an even number 743 $n \geq 4$ of leaves u_1, \ldots, u_n , with $u_1 = v_m$. Recall that Lemma 2.7 implies that |V(T)| is even; now, 744 since $T' = T - \{u_1, \dots, u_{n-1}\}$ has an odd number of vertices, Lemma 2.7 implies that T' has a proper 745 $\{-1,1\}$ -edge-weighting. We can now extend this edge-weighting to a proper $\{-1,1\}$ -edge-weighting 746 of T by assigning the same weight to all the edges $v_{m-1}u_1, ..., v_{m-1}u_n$ (we choose whether it is 1 747 or -1 so that we avoid the conflict $v_{m-2}v_{m-1}$). Thus, v_{m-1} has degree exactly 3 and, from these 748 arguments and the maximality of P, any neighbour of v_{m-2} distinct from v_{m-3} and v_{m-1} is either a 749 leaf or a vertex of degree 3 adjacent to two leaves. Let $U' = \{u'_1, ..., u'_n\}$ be the set of leaves adjacent 750 to v_{m-2} and let $U'' = \{u''_1, ..., u''_q\}$ be the set of neighbours of v_{m-2} distinct from v_{m-1} and v_{m-3} 751 which have degree 3. Possibly p = 0 or q = 0, but p + q > 0 since v_{m-2} has odd degree and hence 752

p + q is odd. Let T_1 and T_2 be the two components of $T - v_{m-3}v_{m-2}$ such that $v_{m-2} \in V(T_2)$. By Lemma 3.2, there is a $\{-1, 1\}$ -edge-weighting w of T such that the only potential conflict is $v_{m-3}v_{m-2}$. By possibly multiplying all edge weights by -1, we can assume that the weight of $v_{m-3}v_{m-2}$ is 1. We look at three separate cases:

757 **Case 1:** $p + q \ge 5$.

By possibly modifying the weights of the edges in $E(T_2)$ such that they all have weight 1 or -1, the vertex v_{m-2} can obtain weighted degree 2 + p + q and -p - q. Now we simply pick the one of these two options such that $v_{m-3}v_{m-2}$ is not a conflict. Since all vertices in T_2 except v_{m-2} have degree at most 3, this gives a proper $\{-1, 1\}$ -edge-weighting of T.

⁷⁶² Case 2: p + q = 3.

As in Case 1, we can modify the edge weights such that the vertex v_{m-2} can obtain weighted 763 degree 2 + p + q and -p - q. Furthermore, since 2 + p + q = 5 and all vertices in T_2 except v_{m-2} 764 have degree at most 3, we can in this way find a proper $\{-1,1\}$ -edge-weighting of T, unless v_{m-3} 765 has weight 5. So we can assume that v_{m-3} has weight 5. If $p \ge 1$ and p is odd (resp. even), then 766 we modify the weights in T_2 such that all edges incident with u'_1, \ldots, u'_n have weight 1 (resp. -1) 767 and all other edges in T_2 have weight -1 (resp. 1). This yields a proper $\{-1,1\}$ -edge-weighting 768 of T, so we can assume p = 0 and q = 3. In this case, we modify the weights in T₂ such that all 769 edges incident to v_{m-1} and u''_1 have weight 1 and all other edges in T_2 have weight -1. This yields 770 a proper $\{-1, 1\}$ -edge-weighting of T. 771

772 **Case 3:** p + q = 1.

First suppose q = 1 and p = 0. If we modify the edge weights in T_2 such that they all have weight 773 -1, then we obtain a proper $\{-1, 1\}$ -edge-weighting of T, unless v_{m-3} has weight -1. In this case, we 774 change the weights of the three edges incident to v_{m-1} to 1 to obtain a proper $\{-1, 1\}$ -edge-weighting 775 of T. Thus, we can assume p = 1 and q = 0. We can assume that $T'' = T - v_m - v_{m-1} - u_2 - u'_1$ has a 776 proper $\{-1, 1\}$ -edge-weighting w, since otherwise, the minimality of T implies that T''' is constructed 777 from a disjoint union of K_2 's through repeated (possibly none) applications of Operation 1, and 778 then so is T. By possibly multiplying all edge weights of w by -1, we can assume that $v_{m-3}v_{m-2}$ 779 has weight 1. Now assigning weight 1 to all edges incident to v_{m-1} and weight -1 to $v_{m-2}u'_1$ yields 780 a proper $\{-1, 1\}$ -edge-weighting of T. 781

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