

On $\{a, b\}$ -edge-weightings of bipartite graphs with odd a, b^*

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Abstract

For any $S \subset \mathbb{Z}$ we say that a graph G has the S -property if there exists an S -edge-weighting $w : E(G) \rightarrow S$ such that for any pair of adjacent vertices u, v we have $\sum_{e \in E(v)} w(e) \neq \sum_{e \in E(u)} w(e)$, where $E(v)$ and $E(u)$ are the sets of edges incident to v and u , respectively. This work focuses on $\{a, a + 2\}$ -edge-weightings where $a \in \mathbb{Z}$ is odd. We show that a 2-connected bipartite graph has the $\{a, a + 2\}$ -property if and only if it is not a so-called odd multi-cactus. In the case of trees, we show that only one case is pathological. That is, we show that all trees have the $\{a, a + 2\}$ -property for odd $a \neq -1$, while there is an easy characterization of trees without the $\{-1, 1\}$ -property.

1 Introduction

Let G be an undirected graph. For an S -edge-weighting $w : E(G) \rightarrow S$ of G , where $S \subset \mathbb{Z}$, each vertex $v \in V(G)$ has *weighted degree* equal to the sum of the weights of its incident edges. We call w *neighbour sum-distinguishing* if no two adjacent vertices of G have the same weighted degree. For a set S of weights, we say that G has the S -property if it admits neighbour sum-distinguishing S -edge-weightings. The study of graphs having or not having the S -property for some sets S is highly related to the well-known **1-2-3 Conjecture** raised by Karonski, Luczak, and Thomason in 2004 [6]. That conjecture states that every connected graph different from K_2 ¹ has the $\{1, 2, 3\}$ -property. A particular case of a list version of the 1-2-3 Conjecture (introduced by Bartnicki, Grytczuk, and Niwczyk [2]), even states that every graph should have the $\{a, b, c\}$ -property for every distinct $a, b, c \in \mathbb{N}$. For more details on the progress towards the 1-2-3 Conjecture (and variants of it), please refer to [11] for a survey on this topic.

For any smaller set $S \subset \mathbb{Z}$ of weights, *i.e.*, with $|S| = 2$, one can easily come up with examples showing that there do exist graphs not having the S -property (complete graphs are such examples).

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¹This requirement is mandatory for any graph to be weightable; throughout this work, it is thus implicit, unless stated otherwise, that every considered graph does not have K_2 as a connected component.

29 A natural question that has been investigated is about the existence of a good characterization
30 of graphs that have the S -property for such smaller sets S . Here and further on, by a “good
31 characterization” we mean a description in terms of a graph class whose members can be recognized
32 in polynomial time. Dudek and Wajc [5] settled the question in the negative, as they proved that,
33 unless $P=NP$, there is no good characterization of graphs with the $\{1, 2\}$ -property, and similarly
34 for the $\{0, 1\}$ -property. Later on, noticing that, for any two distinct sets $S, S' \subset \mathbb{Z}$ of weights
35 with $|S|, |S'| = 2$, any neighbour sum-distinguishing S -edge-weighting of a regular graph yields a
36 neighbour sum-distinguishing S' -edge-weighting, Ahadi, Dehghan, and Sadeghi [1] proved that there
37 is no good characterization of graphs with the $\{a, b\}$ -property for any two distinct $a, b \in \mathbb{Z}$.

38 From this point on, it thus made sense investigating, for any two distinct $a, b \in \mathbb{Z}$, sufficient
39 conditions for graphs to have the $\{a, b\}$ -property. A special focus has been dedicated to bipartite
40 graphs, as 1) the aforementioned NP-completeness results were not known to hold in the bipartite
41 context, and 2) bipartite graphs form one of the rare graph classes for which the 1-2-3 Conjecture is
42 relatively well understood (see [6]). As a first step, several works [3, 4, 7, 8, 9] investigated whether
43 there is a good characterization of bipartite graphs with the $\{1, 2\}$ -property. Back then, it was
44 believed that such a good characterization should exist, as, notably, all 3-connected bipartite graphs
45 were proved to have the $\{1, 2\}$ -property [9]. It was not until quite recently that Thomassen, Wu, and
46 Zhang proved that, indeed, bipartite graphs without the $\{1, 2\}$ -property are easy to describe [13].
47 Namely, only so-called **odd multi-cacti** are bipartite and do not have the $\{1, 2\}$ -property. These
48 graphs are defined as follows (the comprehensive definition is from [10]; refer to Figure 2 later on
49 for an illustration):

50 *“Take a collection of cycles of length 2 modulo 4, each of which has edges coloured alternately*
51 *red and green. Then form a connected simple graph by pasting the cycles together, one by one, in a*
52 *tree-like fashion along green edges; the resulting graph is an odd multi-cactus. The graph with one*
53 *green edge and two vertices (K_2) is also an odd multi-cactus. When replacing a green edge of an*
54 *odd multi-cactus by a green edge of any multiplicity, we again obtain an odd multi-cactus.”*

55 One main ingredient behind Thomassen *et al.*'s result is the nice observation, already made back
56 in [3], that, when a and b are integers with distinct parity, every bipartite graph G with bipartition
57 (X, Y) such that at least one of X and Y has even cardinality has the $\{a, b\}$ -property. This is
58 because, in such a case, one can easily construct $\{a, b\}$ -edge-weightings of G where all vertices in
59 X have odd weighted degree while those in Y have even weighted degree. These observations also
60 imply that, for a and b with distinct parity, bipartite graphs without the $\{a, b\}$ -property have their
61 two partite sets of odd cardinality, and they thus have even order.

62 Reusing some of Thomassen *et al.*'s ideas, Lyngsie later considered the $\{0, 1\}$ -property for bi-
63 partite graphs [10]. His main result is a good characterization of 2-edge-connected bipartite graphs
64 without the $\{0, 1\}$ -property, which turns out to be nothing but the class of odd multi-cacti. This
65 result was established, in particular, through aforementioned tools and results for cases where a and
66 b have different parities. However, both Thomassen *et al.* and Lyngsie observed that there exist
67 infinitely many separable (*i.e.*, with cut-vertices) bipartite graphs without the $\{0, 1\}$ -property.

68 Although they are far from covering all the cases of a and b , the previous series of results show
69 two things. First, that, when considering 2-connected bipartite graphs without the $\{a, b\}$ -property,
70 one should pay attention to odd multi-cacti. Second, that separable bipartite graphs without the

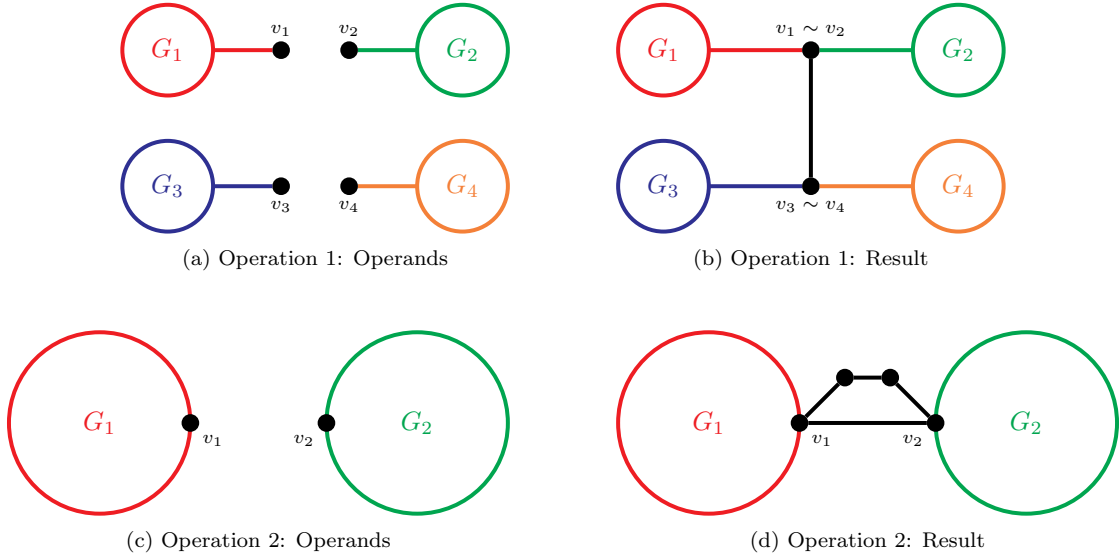


Figure 1: Constructing graphs without the $\{-1, 1\}$ -property from graphs without that property.

71 $\{a, b\}$ -property and those without the $\{a', b'\}$ -property may differ for different pairs a, b and a', b' .
 72 This is already well illustrated by the class of trees: while they all have the $\{1, 2\}$ -property [3],
 73 infinitely many of them do not have the $\{0, 1\}$ -property [10].

74 This paper is mainly devoted to studying $\{a, b\}$ -properties where both a and b are odd. As a
 75 first step, we focus on the cases where $b = a + 2$. We introduce mechanisms that are reminiscent of
 76 the ones mentioned above (for a and b with distinct parity), which allow us to study the $\{a, a + 2\}$ -
 77 property for bipartite graphs and odd $a \in \mathbb{Z}$. One of the main results we get from these is that, for
 78 any odd a , 2-connected bipartite graphs without the $\{a, a + 2\}$ -property are precisely odd multi-cacti
 79 again.

80 **Theorem 1.1.** *Let $a, b \in \mathbb{Z}$ be odd integers with $b = a + 2$. A 2-connected bipartite graph G does*
 81 *not have the $\{a, b\}$ -property if and only if G is an odd multi-cactus.*

82 Similarly as for the $\{0, 1\}$ -property, the structure of separable bipartite graphs without the $\{a, b\}$ -
 83 property for odd a and b does not appear obvious. As a first step, we give a special focus on the
 84 case $\{a, b\} = \{-1, 1\}$. In that case, we can already point out two operations that, given bipartite
 85 graphs without the $\{-1, 1\}$ -property, clearly provide more separable bipartite graphs without the
 86 $\{-1, 1\}$ -property (see Figure 1):

- 87 • Let G_1, G_2, G_3, G_4 be four bipartite graphs without the $\{-1, 1\}$ -property, and let v_1, v_2, v_3, v_4
 88 be any four degree-1 vertices of G_1, G_2, G_3, G_4 , respectively. The operation (see Figure 1 (a)
 89 and (b)) consists in considering the disjoint union $G_1 \cup G_2 \cup G_3 \cup G_4$, identifying the vertices
 90 v_1 and v_2 , identifying the vertices v_3 and v_4 , and adding an edge joining the two vertices
 91 resulting from these identifications (*i.e.*, $v_1 \sim v_2$ and $v_3 \sim v_4$).
- 92 • Let G_1, G_2 be two bipartite graphs without the $\{-1, 1\}$ -property, and let v_1, v_2 be any two
 93 vertices of G_1, G_2 , respectively. The operation (see Figure 1, (c) and (d)) consists in consid-

94 ering the disjoint union $G_1 + G_2$, adding the edge v_1, v_2 , and further joining v_1, v_2 by a path
 95 with odd length at least 3.

96 In the case of trees, when a and b are any two non-zero integers that are both positive (or
 97 negative), it is easy to see that K_2 is the only tree without the $\{a, b\}$ -property: consider a vertex v
 98 whose all neighbours u_1, \dots, u_{d-1} but one u_d (if any) are leaves, remove u_1, \dots, u_{d-1} , apply induction
 99 to deduce a neighbour sum-distinguishing $\{a, b\}$ -edge-weighting, and extend the weighting to the
 100 edges vu_1, \dots, vu_{d-1} so that the conflict vu_d is avoided. Thus, when $b = a + 2$ and a, b are odd,
 101 only the case $a = -1, b = 1$ is potentially non-trivial. In Section 3, we show that trees without the
 102 $\{-1, 1\}$ -property can all be constructed through the first operation above (illustrated in Figure 1,
 103 (a) and (b)) performed on K_2 's.

104 **Theorem 1.2.** *A tree does not have the $\{-1, 1\}$ -property if and only if it can be constructed from
 105 a disjoint union of K_2 's through repeated applications of the first operation above.*

106 In particular, the structure of trees without the $\{-1, 1\}$ -property is very different and simpler
 107 than that of trees without the $\{0, 1\}$ -property (for more on the structure of these trees, see [10]).
 108 Recall that all trees have the $\{1, 2\}$ -property, as was shown, e.g., in [3].

109 **Terminology and notation.** Let G be a connected graph. For a given vertex v of G we denote
 110 by $E(v)$ the set of edges incident to v . A *bridge* in G is an edge whose removal results in two
 111 components. Let w be an edge-weighting of G . Abusing the notation, the weighted degree of v in
 112 G by w will sometimes be denoted $w(v)$ for convenience. We say that an edge uv of G is a *conflict*
 113 by w if $w(u) = w(v)$. In other words, w is neighbour sum-distinguishing if no edge is a conflict.
 114 In what follows, we will instead use the term *proper* in place of neighbour sum-distinguishing to
 115 lighten the writing. By an *x -edge* of G (by w), we mean an edge assigned weight x by w .

116 2 Proof of Theorem 1.1

117 In this section, we prove that for every odd integer $a \in \mathbb{Z}$, the class of 2-connected bipartite graphs
 118 without the $\{a, a + 2\}$ -property is exactly that of odd multi-cacti. Another way to define these
 119 graphs is as follows. Start from K_2 , the simple connected graph on two vertices, having its only
 120 edge coloured green. Then, repeatedly apply an arbitrary number of the following operation (see
 121 Figure 2 for an illustration). Consider any green edge uv of the current graph, and join u, v by a
 122 new path P of length $\ell \geq 1$ congruent to 1 modulo 4 whose edges are coloured red and green as
 123 follows:

- 124 • if $\ell = 1$, *i.e.*, P has a unique edge, then this edge is green;
- 125 • if $\ell \geq 5$, then the edges of P are coloured red and green properly (*i.e.*, no two subsequent
 126 edges have the same colour) so that the two end-edges are red.

127 Figure 2 notably shows that performing this operation multiple times for a same green edge is
 128 allowed, and that adding paths of length 1 is similar to increasing the multiplicity of a green edge.
 129 Note also that it is not possible to get two adjacent green edges with distinct ends at any point
 130 of the process. Furthermore, every obtained graph is bipartite. An odd multi-cactus is any graph

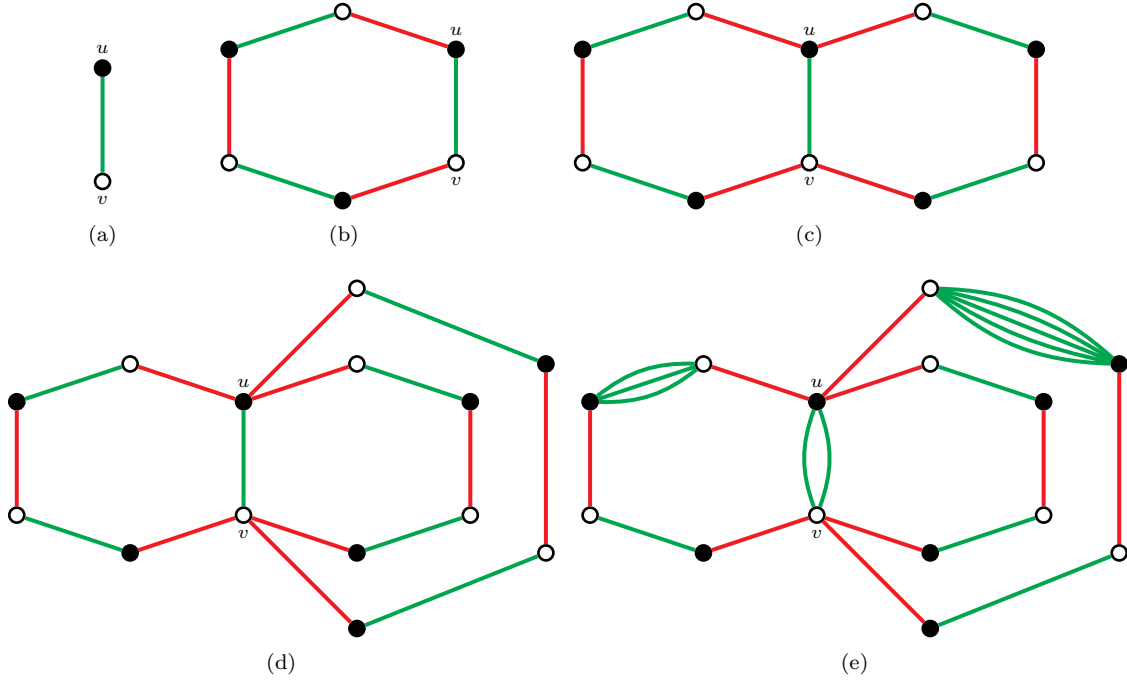


Figure 2: Constructing an odd multi-cactus through several steps, from K_2 (a). Red-green paths with length at least 5 congruent to 1 modulo 4 are being attached onto the green edge uv through steps (b) to (d). In step (e), (green) paths of length 1 are added, which corresponds to increasing the multiplicity of some green edges.

131 that can be obtained during this process, no matter how many times the operation is applied. In
132 particular, K_2 itself is regarded as an odd multi-cactus.

133 In this section, we will implicitly use several properties of odd multi-cacti, such as:

134 **Observation 2.1.** *Let M be an odd multi-cactus with its edges being coloured red and green as*
135 *described above. Then:*

- 136 • M is 2-connected;
- 137 • when replacing every (green) edge of M by an edge with multiplicity 1, a 2-degenerate graph
138 (i.e., a graph in which every subgraph has a vertex of degree at most 2) is obtained;
- 139 • for every green edge uv of M , we have $d_M(u) = d_M(v)$.

140 Having the structure of odd multi-cacti in mind, it can be proved that the following holds true.

141 **Lemma 2.2.** *If G is not an odd multi-cactus and was obtained from an odd multi-cactus M by*
142 *replacing a red edge with an edge of multiplicity at least 2 or by replacing a green edge by a path*
143 *of length $k \geq 5$ with $k \equiv 1 \pmod{4}$, then G has the $\{a, b\}$ -property for any two distinct integers*
144 *$a, b \in \mathbb{Z}$.*

145 *Proof.* The proof is by induction on the order of M . So suppose that G is obtained from an odd
146 multi-cactus M by replacing an edge e with either an edge of multiplicity at least 2 (if e is red in

147 M) or a path of length $k \geq 5$ with $k \equiv 1 \pmod{4}$ (if e is green in M). It is easy to check that the
148 statement is true if M is just a cycle with some multiple green edges. So we can focus on cases
149 where M was obtained in the general way, *i.e.*, by pasting together cycles with length 2 modulo 4
150 having possibly green edges of any multiplicity. Furthermore, it is easy to check that the statement
151 is true if M was constructed by only pasting cycles together along one single green edge e' as in
152 Figures 2 (c), (d) and (e) where there have only been pasted cycles together along the edge uv :
153 in this case, since G is not an odd multi-cactus, G is either obtained from M by replacing a red
154 edge with an edge of multiplicity at least 2, or $e = e'$ must be a simple edge and G is obtained by
155 replacing that edge e with a path of length $k \geq 5$ with $k \equiv 1 \pmod{4}$ and $k \geq 5$. In both cases it is
156 easy to check that G has the $\{a, b\}$ -property.

157 Thus, we can assume that M was obtained by pasting together at least three cycles and that
158 there are at least two disjoint edges to which other cycles have been pasted to. This implies that
159 there are at least two disjoint cycles of length congruent to 2 modulo 4 where all vertices except
160 two which are adjacent have exactly two neighbours. One of these cycles $C = v_1v_2\dots v_nv_1$ does not
161 contain e . By possibly relabelling the vertices we can assume that all vertices of C except v_1 and
162 v_2 only have two distinct neighbours. By induction the graph G' obtained from G by replacing the
163 path $v_2v_3\dots v_n$ with an edge e'' has the $\{a, b\}$ -property, but any proper $\{a, b\}$ -edge-weighting of G'
164 can be converted to a proper $\{a, b\}$ -edge-weighting of G by assigning the same weight to an edge in
165 G as in G' and assigning the weight assigned to e'' in G' to the edges v_2v_3 and $v_{n-1}v_n$, and finally
166 assigning the weights of the remaining edges $v_2v_3, \dots, v_{n-2}v_{n-1}$ in a way avoiding conflicts inside
167 C . □

168 We now introduce or recall results that will be needed during the course of our main proof below.
169 The following first observation is obvious and implies that studying the $\{a, b\}$ -property only makes
170 sense when $\gcd(a, b) = 1$.

171 **Observation 2.3.** *Let w be a proper $\{a, b\}$ -edge-weighting of a graph G . If we multiply all edge*
172 *weights of w by a non-zero integer α , then we get a proper $\{\alpha a, \alpha b\}$ -edge-weighting of G .*

173 In what follows, given a graph G and a mapping $f : V(G) \rightarrow \mathbb{Z}_k$, by an f -factor modulo k we
174 mean a spanning subgraph H of G such that, for every $v \in V(G)$, we have $d_H(v) \equiv f(v) \pmod{k}$.

175 **Lemma 2.4** (Thomassen [12]). *Let G be a connected graph. If $f : V(G) \rightarrow \mathbb{Z}_2$ is a mapping*
176 *satisfying $\sum_{v \in V(G)} f(v) \equiv 0 \pmod{2}$, then G contains an f -factor modulo 2.*

177 When dealing with bipartite graphs with a bipartition set of even cardinality, and when a and b
178 have distinct parity, f -factors modulo 2 can be employed as a convenient tool to deduce proper $\{a, b\}$ -
179 edge-weightings in quite an easy way (see [10, 13]). More precisely, let (X, Y) be the bipartition of a
180 bipartite graph G where $|X|$ is even. Lemma 2.4 (when applied onto the function f where $f(x) = 1$
181 for $x \in X$ and $f(y) = 0$ for $y \in Y$) implies that G has a spanning subgraph H where all of the
182 vertices in X have odd degree, while all of the vertices in Y have even degree. From this, it is easy
183 to see that, assuming a is odd and b is even, assigning weight a to all of the edges in $E(H)$ and
184 weight b to all of the edges in $E(G) \setminus E(H)$ yields a proper $\{a, b\}$ -edge-weighting of G .

185 The upcoming new tools and concepts (in particular that of *mod-4 vertex-colourings*) are the
186 key to generalize this approach to odd $a, b \in \mathbb{Z}$ when $|a - b| = 2$.

187 **Definition 2.5.** A mod-4 vertex-colouring of a graph G is a vertex-colouring $c : V(G) \rightarrow \{1, 2\}$ of
 188 G satisfying the following conditions for any $uv \in E(G)$ where $d(u)$ and $d(v)$ have the same parity:

189 1. $d(u) \equiv d(v) \pmod{4} \Rightarrow c(u) \neq c(v)$.

190 2. $d(u) \not\equiv d(v) \pmod{4} \Rightarrow c(u) = c(v)$.

191 In the next result, we prove that every bipartite graph G admits a mod-4 vertex-colouring c .
 192 It is important to point out that, in general, c might be far from fitting with the bipartition of G .
 193 Actually, G might have many edges whose two ends have the same colour by c .

194 **Lemma 2.6.** Every bipartite graph has a mod-4 vertex-colouring.

195 *Proof.* It suffices to prove the lemma for connected bipartite graphs where all vertices have odd
 196 degree or where all vertices have even degree (as otherwise we can consider, still in the whole graph,
 197 the vertices with even degree first, and then those with odd degree). So let G be a connected
 198 bipartite graph where all vertex degrees have the same parity. Let v be a vertex in G and let
 199 D_0, D_1, \dots, D_m denote the distance classes of G from $v \in D_0$. Since G is bipartite, each D_i is an
 200 independent set. Now give v colour 1 and colour the distance classes in the given order starting
 201 with D_1 , then D_2 and so on until we reach a vertex $v' \in D_{i'}$ we cannot assign a colour without
 202 violating conditions 1 or 2 in Definition 2.5. If this happens one or both of the following two cases
 203 have occurred:

204 1. there are two neighbours $v_1, v_2 \in D_{i'-1}$ of v' with $d(v_1) \equiv d(v_2) \pmod{4}$ and $c(v_1) \neq c(v_2)$;

205 2. there are two neighbours $v_1, v_2 \in D_{i'-1}$ of v' with $d(v_1) \not\equiv d(v_2) \pmod{4}$ and $c(v_1) = c(v_2)$.

206 Let us first assume that we are in the first case and let P_1, P_2 be two internally disjoint shortest
 207 paths towards v starting with $v'v_1$ and $v'v_2$ respectively and ending in a common vertex $v'' \in D_{i''}$.
 208 That is, v'' is the first vertex on both P_1 and P_2 that is encountered when going from v_1 towards
 209 v along P_1 ; possibly $v'' = v$. All the vertices of P_1 and P_2 except v' are coloured without violating
 210 conditions 1 and 2 in Definition 2.5, and P_1 and P_2 have the same length. The parity of the number
 211 of times the degree modulo 4 changes when walking from v'' to v_1 on P_1 is the same as the parity
 212 of the number of times the degree modulo 4 changes when walking from v'' to v_2 on P_2 . Thus, the
 213 parity of the number of times the degree modulo 4 does not change when walking from v'' to v_1
 214 on P_1 is the same as the parity of the number of times the degree modulo 4 does not change when
 215 walking from v'' to v_2 on P_2 . Since conditions 1 and 2 in Definition 2.5 are not violated, this implies
 216 that the parity of the number of times the colour changes when walking from v'' towards v' is the
 217 same when walking along P_1 as when walking along P_2 . Thus, $c(v_1) = c(v_2)$, a contradiction. The
 218 second case above can be dealt with in a similar way. \square

219 Let $a, b \in \mathbb{Z}$ be two odd integers with $b = a + 2$. Let G be a graph and X, Y be two disjoint
 220 subsets of its vertices. By an (X, Y) - a -parity $\{a, b\}$ -edge-weighting of G , we mean an $\{a, b\}$ -edge-
 221 weighting where all vertices in X are incident to an odd number of a -edges and all vertices in Y are
 222 incident to an even number of a -edges. (X, Y) - b -parity $\{a, b\}$ -edge-weightings are defined similarly,
 223 but with respect to the incident b -edges. In the following result, we establish a crucial connection
 224 between mod-4 vertex-colourings and (X, Y) -parity $\{a, b\}$ -edge-weightings, leading to the existence
 225 of proper $\{a, b\}$ -edge-weightings.

226 **Lemma 2.7.** *Let G be a connected bipartite graph and let $a, b \in \mathbb{Z}$ be odd integers with $b = a + 2$.
 227 If G has a mod-4 vertex-colouring where at least one of the two colour classes has even size, then G
 228 has the $\{a, b\}$ -property. Consequently, if G does not have the $\{a, b\}$ -property, then, in every mod-4
 229 vertex-colouring, the two colour classes have odd size.*

230 *Proof.* Let G be a connected bipartite graph, and c a mod-4 vertex-colouring of G . We denote by
 231 X and Y the sets of vertices with colour 1 and 2, respectively. Assume $|X|$ is even. By Lemma 2.4
 232 there is an $\{a, b\}$ -edge-weighting $w : E(G) \rightarrow \{a, b\}$ such that all vertices in X are incident to an odd
 233 number of b -edges and all vertices in Y are incident to an even number of b -edges. This corresponds
 234 to our notion of an (X, Y) - b -parity $\{a, b\}$ -edge-weighting. The possible weighted degrees of a vertex
 235 v of even degree and colour 1 induced by such an edge-weighting are $\{a(d(v) - 1) + b, a(d(v) - 1) +$
 236 $b + 4, a(d(v) - 1) + b + 8, \dots, a + b(d(v) - 1)\}$ and the possible weighted degrees of a vertex v' of even
 237 degree and colour 2 induced by such an edge-weighting are $\{ad(v'), ad(v') + 4, ad(v') + 8, \dots, bd(v')\}$.
 238 The possible weighted degrees of a vertex u of odd degree and colour 1 induced by such an edge-
 239 weighting are $\{a(d(u) - 1) + b, a(d(u) - 1) + b + 4, a(d(u) - 1) + b + 8, \dots, bd(u)\}$ and the possible
 240 weighted degrees of a vertex u' of odd degree and colour 2 induced by such an edge-weighting are
 241 $\{ad(u'), ad(u') + 4, ad(u') + 8, \dots, a + b(d(u') - 1)\}$. Let $xy \in E(G)$. We will show that $w(x) \neq w(y)$.
 242 To do this we distinguish two distinct cases (note that we can assume that x and y have the same
 243 degree parity, as otherwise $w(x)$ cannot be equal to $w(y)$):

- 244 1. x and y have the same colour by c .
- 245 2. x and y have distinct colours by c .

246 First assume that x and y have the same colour. Since c is a mod-4 vertex-colouring we have
 247 that $d(x) \not\equiv d(y) \pmod{4}$. Note that by the above it suffices to show that $ad(x) \not\equiv ad(y) \pmod{4}$ and
 248 this is trivially true since $\gcd(a, 4) = 1$. Now assume that x and y have distinct colours. Since c is
 249 a mod-4 vertex-colouring we have that $d(x) \equiv d(y) \pmod{4}$. Note that by the above it suffices to
 250 show that $ad(x) \equiv ad(y) \pmod{4}$, and as mentioned above this follows since $\gcd(a, 4) = 1$. \square

251 From the previous proof, we can also extract the following:

252 **Observation 2.8.** *Let $a, b \in \mathbb{Z}$ be odd integers with $b = a + 2$ and let uv be an edge in a graph G
 253 whose edges are weighted with a and b . If either*

- 254 1. $d(u)$ and $d(v)$ have distinct parity, or
- 255 2. $d(u) \equiv d(v) \pmod{4}$ and v is incident to an odd number of a -edges while u is incident to an
 256 even number of a -edges, or
- 257 3. $d(u) \not\equiv d(v) \pmod{4}$ and both v and u are incident to an odd or even number of a -edges,

258 then u and v have distinct weighted degrees. This is also true if one considers the parity of the
 259 numbers of incident b -edges instead of the parity of the numbers of incident a -edges.

260 Let G be a graph and w an $\{a, b\}$ -edge-weighting of G . By *swapping* (the weight of) an edge, we
 261 mean changing its weight to a if it is a b -edge, or changing its weight to b otherwise. By swapping
 262 a path or a cycle, we mean swapping all of its edges. For a vertex v in a cycle C of G , it can be

263 observed that the parity of the number of a -edges (and similarly b -edges) incident to v is not altered
 264 upon swapping C . In the proof of our main result below, this fact will be used a lot to get rid of
 265 conflicts in the following way.

266 Let X, Y be the two colour classes of a mod-4 vertex-colouring of G and assume that, for some
 267 vertex $v \in X$, w is an $(X \setminus \{v\}, Y \cup \{v\})$ - a -parity $\{a, b\}$ -edge-weighting, *i.e.*, all vertices in $X \setminus \{v\}$
 268 are incident to an odd number of a -edges and all vertices in $Y \cup \{v\}$ are incident to an even number of
 269 a -edges. According to Observation 2.8, all conflicts (if any) involve v . So let uv be a conflict. To get
 270 rid of this conflict while controlling the possible creation of new conflicts, we will swap particular
 271 cycles of G . Let C be a cycle of G going through u using two edges e, e' incident to u . If e, e' are
 272 assigned the same weight by w , then C is called u -changing. C will be called v -avoiding if it does
 273 not go through v .

274 **Observation 2.9.** *Let G be a graph, X, Y be the two colour classes of a mod-4 vertex-colouring of*
 275 *G , and let w be an $(X \setminus \{v\}, Y \cup \{v\})$ - a -parity $\{a, b\}$ -edge-weighting for some vertex v , where $a, b \in \mathbb{Z}$*
 276 *are odd integers with $b = a + 2$. If uv is a conflict, then, by swapping a u -changing v -avoiding cycle*
 277 *C of G , we get rid of this conflict. Furthermore, any remaining/arising conflicts involve v .*

278 *Proof.* According to Observation 2.8, all original conflicts by w must involve v . When swapping C ,
 279 the weighted degree of u is altered since C is u -changing, while the weighted degree of v is unaltered
 280 since C is v -avoiding. So we get rid of the conflict uv . Furthermore, it can be noticed that, upon
 281 swapping any cycle of G , the parities of the number of a -edges (and similarly b -edges) incident to the
 282 vertices are unaltered. Therefore, we get another $(X \setminus \{v\}, Y \cup \{v\})$ - a -parity $\{a, b\}$ -edge-weighting,
 283 and Observation 2.8 indicates that, after the swapping of C , all conflicts (if any) in the resulting
 284 $\{a, b\}$ -edge-weighting must involve v . □

285 We finish with a few general lemmas to be used in particular cases of our upcoming main proof.

286 **Lemma 2.10.** *Let G be a 2-connected bipartite graph, X, Y be the two colour classes of a mod-4*
 287 *vertex-colouring of G , and let $a, b \in \mathbb{Z}$ be odd integers with $b = a + 2$. If both X and Y have odd*
 288 *size and $v \in X$ is such that $G - v - N(v)$ is connected, then there is an $(X \setminus \{v\}, Y \cup \{v\})$ - a -parity*
 289 *$\{a, b\}$ -edge-weighting of G where all edges incident to v have weight b and every vertex $u \in N(v)$ is*
 290 *incident to at most $1 + M(uv)$ b -edges, where $M(uv)$ denotes the multiplicity of the edge uv .*

291 *Proof.* Suppose $G' = G - v - N(v)$ is connected. Let G'' be obtained from $G - v$ by, for every vertex
 292 $u \in N(v)$, removing all edges but one incident to u in $G - v$. For each $u \in N(v)$, let e_u be the
 293 unique edge incident to u in G'' and let $n(u)$ denote the unique neighbour of u in G'' . Note that
 294 since G' is connected, then so is G'' . Let S denote the set of edges in G not incident to v and not in
 295 G'' . That is, S is the set of edges removed from $G - v$ to obtain G'' . Let $G[S]$ denote the subgraph
 296 of G induced by the edges in S and let Z denote the vertices of odd degree in $G[S]$. Clearly $|Z|$ is
 297 even, so, since $X \setminus \{v\}$ has even size, the set $X' = (X \setminus (Z \cup \{v\})) \cup Z \cap Y$ also has even size. Thus,
 298 Lemma 2.4 implies that there is an $(X', V(G'') \setminus X')$ - a -parity $\{a, b\}$ -edge-weighting of G'' . We now
 299 extend this weighting to G by assigning weight a to all edges in S and weight b to all edges in $E(v)$;
 300 this results in a desired edge-weighting of G . □

301 **Lemma 2.11.** *Let G be a 2-connected bipartite graph. If there is a vertex $v \in V(G)$ of degree at*
 302 *least 4 and with $|N(v)| \geq 3$ such that $G - v - N(v)$ is connected, then G has the $\{-1, 1\}$ -property.*

303 *Proof.* By Lemma 2.10, there is an $(X \setminus \{v\}, Y \cup \{v\})$ - (-1) -parity $\{-1, 1\}$ -edge-weighting of G , where
304 all edges incident to v have weight 1 and any vertex $u \in N(v)$ is incident to at most $1 + M(uv)$
305 1-edges, where $M(uv)$ denotes the multiplicity of the edge uv . Observation 2.8 implies that the only
306 potential conflicts are between v and its neighbours. But the weighted degree of v is $d(v)$ and since
307 $|N(v)| \geq 3$, we have for any $u \in N(v)$ that the multiplicity of uv is less than $d(v) - 1$. Thus, the
308 weighted degree of any $u \in N(v)$ is less than $d(v)$ and therefore, there can be no conflicts. \square

309 **Lemma 2.12** (Thomassen, Wu, Zhang [13]). *Let q be a natural number such that $q \geq 4$. Let G be a*
310 *connected graph and let A be an independent set of at most q vertices such that each vertex in A has*
311 *degree at least $q - 1$, or, each vertex in A , except possibly one, has degree at least q . Assume that no*
312 *vertex in A is adjacent to a bridge in G . Then, for each vertex a of A , there is an edge e_a incident*
313 *with a such that the deletion of all e_a , $a \in A$, results in a connected graph unless $|A| = q = 4$, all*
314 *vertices of A have degree 3, and $G - A$ has six components each of which is joined to two distinct*
315 *vertices of A .*

316 Let a and b be two odd integers with $b = a + 2$. In some cases Lemmas 2.4 and 2.12 work well
317 together when trying to construct a proper $\{a, b\}$ -edge-weighting of a connected bipartite graph
318 G : suppose $c : V(G) \rightarrow \{1, 2\}$ is a mod-4 vertex-colouring of G and let X and Y denote the sets
319 of vertices in G of colour 1 and 2 respectively and assume that both X and Y have odd size.
320 Furthermore, suppose that the degree of a vertex $v \in X$ is at least 4 and no vertex in $N(v)$ has
321 degree strictly larger than 4. Let A be the vertices in $N(v)$ with the same degree as v and suppose
322 that no vertex in A is incident to a bridge in $G - v$, the graph $G - v$ is connected, and we are
323 not in the exceptional case of Lemma 2.10, that is, for each $u \in A$ there is an edge e_u such that
324 $G - \cup_{u \in A} \{e_u\}$ is connected. Define $S = \cup_{u \in A} \{e_u\}$ and let Z denote the set of vertices in G which
325 have odd degree in the subgraph of G induced by S (note that $A \subset Z$). Since $X \setminus \{v\}$ and Z
326 have even size, the set $X' = (X \setminus (Z \cup \{v\})) \cup Z \cap Y$ also has even size. Thus, Lemma 2.4 implies
327 that there is an $(X', V(G - v) \setminus X')$ - a -parity $\{a, b\}$ -edge-weighting of $G - v$. We can extend this
328 edge-weighting to G by assigning weight a to all edges in S and weight b to all edges in $E(v)$ to
329 obtain an $(X \setminus \{v\}, Y \cup \{v\})$ - a -parity $\{a, b\}$ -edge-weighting of G where all edges incident to v have
330 weight b and every vertex $u \in N(v)$ is incident to at least one a -edge. Thus, the weighted degree
331 of v is greater than that of its neighbours and Observation 2.8 implies that the edge-weighting is
332 proper.

333 We are now ready to prove the main result of this section. Let us emphasize that the main steps
334 in the proof follow the lines of those in the proofs of the main results in [10] and [13]. In particular,
335 Claims 2, 3, 4, 5, as stated below, can also be found in [10]. The proofs are rather different, though,
336 as most arguments used to deal with the $\{0, 1\}$ -property do not apply immediately for the $\{a, b\}$ -
337 property when $a, b \in \mathbb{Z}$ are odd and $b = a + 2$. Instead, some of the tools and results we have
338 introduced earlier are used. Also, the end of the proof in our case is more straightforward than
339 those for the $\{1, 2\}$ - and $\{0, 1\}$ -properties.

340 *Proof of Theorem 1.1.* Suppose the theorem is false and, for some odd $a \in \mathbb{Z}$ and $b = a + 2$, let G
341 be a counterexample which has smallest order possible. By possibly multiplying the weights by -1
342 we can assume $b > 0$. Let c be a mod-4 vertex-colouring of G (such a colouring exists by Lemma

343 2.6), and let X denote the set of vertices with colour 1 and Y denote the set of vertices with colour
 344 2. By Lemma 2.7, we can assume that both X and Y have odd size.

345 **Claim 1.** G has no multiple edge uv where both u and v have only two distinct neighbours.

346 *Proof of the claim.* Suppose uv is a multiple edge and both u and v have only two distinct neighbours.
 347 By the minimality of G , Lemma 2.2, and the fact that G is not an odd multi-cactus, the graph
 348 obtained from G by replacing uv with one non-multiple edge has a proper $\{a, b\}$ -edge-weighting w .
 349 But since the multiplicity of uv in G is at least 2 and since u and v can each be in only one conflict
 350 distinct from uv , we can obtain a proper $\{a, b\}$ -edge-weighting of G from w by weighting the edges
 351 joining u and v to avoid the conflicts involving u and v (such an edge-weighting exists because there
 352 are at least three possible sums for the weight of edges joining u and v). \diamond

353 Since G is 2-connected, it has minimum degree at least 2. In what follows, by a *suspended path*
 354 of G , we mean a path $v_1x_1\dots x_kv_2$ where all internal vertices x_1, \dots, x_k have degree 2 and v_1 and v_2
 355 have degree at least 3.

356 **Claim 2.** G has no suspended path of length 2.

357 *Proof of the claim.* Suppose the claim is false and let v_1xv_2 be a suspended path in G , where
 358 $d(x) = 2$ and $d(v_1), d(v_2) \geq 3$. We can assume $x \in X$. Define $G' = G - x$. Recall that G' is
 359 connected since G is 2-connected. Lemma 2.4 implies that there is an $(X \setminus \{x\}, Y \cup \{x\})$ - a -parity
 360 $\{a, b\}$ -edge-weighting w of G , where $w(v_1x) = w(v_2x) = a$. More precisely, w can be obtained as
 361 follows (recall that $|V(G') \cap X|$ is even):

- 362 • If $v_1, v_2 \in X$, then, according to Lemma 2.4, there is an $(X \setminus \{v_1, v_2, x\}, Y \cup \{v_1, v_2\})$ - a -
 363 parity $\{a, b\}$ -edge-weighting of G' . Then, assigning weight a to v_1x and v_2x gives the desired
 364 weighting of G .
- 365 • If $v_1, v_2 \in Y$, then the same conclusion can be reached when applying Lemma 2.4 so that we
 366 start from an $(X - \{x\} \cup \{v_1, v_2\}, Y \setminus \{v_1, v_2\})$ - a -parity $\{a, b\}$ -edge-weighting of G' .
- 367 • If $v_1 \in X$ and $v_2 \in Y$ (resp. $v_2 \in X$ and $v_1 \in Y$), then, again, we can get the same
 368 conclusion after applying Lemma 2.4 from an $(X \setminus \{v_1, x\} \cup \{v_2\}, Y \cup \{v_1\} \setminus \{v_2\})$ - a -parity
 369 $\{a, b\}$ -edge-weighting (resp. $(X \setminus \{v_2\} \cup \{v_1\}, Y \cup \{v_2\} \setminus \{v_1\})$ - a -parity $\{a, b\}$ -edge-weighting)
 370 of G' .

371 Observation 2.8 implies that the only conflicts that can arise are xv_1 and xv_2 . So we can assume
 372 that x and v_1 are in conflict, *i.e.*, both x and v_1 have weighted degree $2a$. This implies that v_1 has
 373 even degree at least 4 (by the definition of a suspended path, and only vertices with degree of the
 374 same parity can be in conflict) and $a < 0 < b$ and hence $a = -1$ and $b = 1$. Let u_1, u_2 be two
 375 neighbours of v_1 in G' such that u_1v_1 and u_2v_1 have weight -1 and let C be a cycle in G' using the
 376 edges u_1v_1 and v_1u_2 . Such a cycle exists as, because G is 2-connected, there is a path from u_1 to u_2
 377 in $G - v_1$. Because C is v_1 -changing and x -avoiding, if we swap all weights on C we do not create
 378 new conflicts in G' and we lose the conflict xv_1 (recall Observation 2.9). In particular, x remains of
 379 weighted degree -2 while v_1 becomes of weighted degree 2.

380 Thus, we can now assume that xv_2 is a conflict. This implies that v_2 also has even degree at least
 381 4. We can now get rid of this conflict in the same way as we got rid of the conflict v_1x , unless all

382 v_2 -changing cycles in G' (that thus use two edges in $E(v_2)$ having the same weight) all use two edges
383 in $E(v_1)$ both having weight 1 (in this case we only move the conflict from xv_2 to xv_1). Since xv_2 is
384 a conflict, v_2 must be incident to at least two -1 -edges in G' and at least one 1 -edge. Furthermore,
385 as mentioned above, we can assume that any v_2 -changing cycle in G' contains two edges incident
386 to v_1 having weight 1.

387 Assume that v_1 is incident to a -1 -edge e in G' . Since G is 2-connected, there is, in $G - v_1$, a
388 path from v_2 to the end of e different from v_1 . From the existence of that path, we get that there is
389 a path P in G' from v_1 to v_2 using e . If the weight on the last edge e' of P (the one incident to v_2)
390 is 1, then swapping the weights on the cycle $P \cup v_1x \cup xv_2$ yields a proper edge-weighting; so we can
391 assume e' has weight -1 . Since xv_2 is a conflict and v_2 has even degree at least 4, vertex v_2 must
392 be incident to a -1 -edge $e'' \neq e'$ in G' . Now, because G is 2-connected, the graph $G - v_2$ has a path
393 P' joining the end of e' different from v_2 and the end of e'' different from v_2 . Note that if P' does
394 not contain v_1 , then we would get a cycle whose weights can be swapped to immediately deduce a
395 proper edge-weighting of G . The same conclusion holds if P' and P intersect for the first time on
396 a vertex different from v_1 . So v_1 is the first intersection point between P and P' , in which case we
397 deduce a cycle of G containing v_2 as well as all of e, e', e'' (first go from v_2 to v_1 along P' , before
398 going back to v_2 along P); when swapping the weights along that cycle, we get rid of all conflicts
399 between x and v_1, v_2 .

400 We are left with the case where v_1 is not incident to a -1 -edge e in G' . Thus, we deduce that
401 v_1 has degree 4 and, by symmetry, v_2 also has degree 4. Note that this implies that $v_1, v_2 \in X$
402 and furthermore, that all four edges incident to v_1 except v_1x have weight 1. Also, according to all
403 hypotheses so far, v_1 has weighted degree 2, v_2 has weighted degree -2 , and so two edges incident
404 to v_2 in G' have weight -1 while the last edge incident to v_2 in G' has weight 1. We now consider
405 the graph $G'' = G' - v_1 - v_2$. If G'' is connected, then we can find in G' a cycle including the two
406 -1 -edges incident to v_2 (thus v_2 -changing), and not passing through v_1 ; then, as earlier, we can
407 swap the weights along this cycle to get rid of the conflict v_2x . So we may assume the two -1 -edges
408 incident to v_2 in G' are incident to two distinct components of G'' . This leaves us with the following
409 three cases to consider:

410 **Case 1:** G'' has two components C_1, C_2 , such that v_1 has two neighbours in C_1 and one neighbour
411 in C_2 , and v_2 has two neighbours in C_2 and one neighbour in C_1 .

412 Let $e_{1,a}$ and $e_{1,b}$ denote the two edges incident to v_1 going to C_1 . Recall that $e_{1,a}, e_{1,b}$ have
413 weight 1. Since C_1 is connected, there is a path from the end of $e_{1,a}$ different from v_1 to the end
414 of $e_{1,b}$ different from v_1 . We swap all weights along the cycle formed by this path and $e_{1,a}, e_{1,b}$ to
415 get another edge-weighting of G where v_1 has weighted degree -2 ; so, now, both xv_1 and xv_2 are
416 conflicts.

417 Now let $e_{1,c}$ denote the edge incident to v_1 going to C_2 , and $e_{2,a}$ denote the 1 -edge incident to
418 v_2 going to C_2 . Both these edges are weighted 1. Since C_2 is connected, there is a path P from
419 the end of $e_{1,c}$ different from v_1 to the end of $e_{2,a}$ different from v_2 . Now consider the cycle of G
420 starting in x , going through xv_2 and $e_{2,a}$, then going along P , and finally going through $e_{1,c}$ and v_1x .
421 When swapping all weights along this cycle, note that v_1, v_2 remain of weighted degree -2 , while
422 x becomes of weighted degree 2. So the $\{-1, 1\}$ -edge-weighting of G becomes proper according to
423 Observation 2.9.

424 **Case 2:** G'' has two components C_1, C_2 , such that both v_1 and v_2 have two neighbours in C_1 and
 425 one neighbour in C_2 .

426 Let $v_{1,a}, v_{1,b}$ denote the two neighbours of v_1 in C_1 , and let $v_{1,c}$ denote the neighbour of v_1 in
 427 C_2 . Note that for one of $v_{1,a}, v_{1,b}$, say $v_{1,a}$, the graph $G''' = G - v_1 - v_{1,a} - v_{1,c}$ is connected, for
 428 if G' is disconnected, then it must be the case that $v_{1,a}$ is a cut-vertex in $G - v_1$ and then it is
 429 easy to see that $G - v_1 - v_{1,b} - v_{1,c}$ is connected, and we can just rename $v_{1,a}$ and $v_{1,b}$ accordingly.
 430 Note that, by Lemma 2.11, we can assume that $G''' - v_{1,b}$ is disconnected. Let L_1, \dots, L_n denote the
 431 components of $G''' - v_{1,b}$, where $v_2 \in V(L_1)$. Since $v_{1,b}$ is not a cut-vertex in G , it follows that $v_{1,a}$
 432 has a neighbour in each of the components L_i for $i \geq 2$.

433 Let us now consider the graph obtained from G by removing the vertex v_1 , and, for each of
 434 $v_{1,a}, v_{1,c}$, removing all remaining incident edges but one. Note that, in that graph, x also has de-
 435 gree 1. Lemma 2.4 implies that this graph has an $(X \setminus \{v_1\}, Y)$ - (-1) -parity $\{-1, 1\}$ -edge-weighting.
 436 By then assigning weight -1 to all removed edges incident to v_1 and weight 1 to all remaining edges
 437 (incident to one of $v_{1,a}, v_{1,c}$), we get that G has an $(X \setminus \{v_1\}, Y)$ - (-1) -parity $\{-1, 1\}$ -edge-weighting
 438 where all four edges incident to v_1 are weighted -1 , and each of $v_{1,a}, v_{1,c}$ is incident to at most two
 439 -1 -edges. Now the only possible conflict is $v_1 v_{1,b}$, so we can assume this is indeed a conflict, and
 440 hence, both v_1 and $v_{1,b}$ have weighted degree -4 . We can also assume that we cannot swap the
 441 weights in a cycle in G''' containing two edges incident to $v_{1,b}$ having the same weight, and hence,
 442 $v_{1,b}$ has at most two neighbours in each of L_i for $i = 1, \dots, n$. Since the weighted degree of $v_{1,b}$ is
 443 -4 , there must be some components in $G''' - v_{1,b}$ which are incident to strictly more -1 -edges in
 444 $E(v_{1,b})$ than 1-edges in $E(v_{1,b})$. Again, since we can assume that there is no cycle in G''' containing
 445 two edges incident to $v_{1,b}$ having the same weight, we can also deduce that no two edges incident to
 446 $v_{1,b}$ having the same weight go to the same component in $G''' - v_{1,b}$. Thus, there are at least three
 447 components L'_1, L'_2, L'_3 in $G''' - v_{1,b}$, each of which is incident to only one edge in $E(v_{1,b})$ and each
 448 of these edges has weight -1 . We can assume that L'_1 and L'_2 are distinct from L_1 , and, since $v_{1,b}$
 449 is not a cut-vertex in G , the vertex $v_{1,a}$ has a neighbour u'_i in each L'_i for $i = 1, 2$. There is now a
 450 cycle C in $G''' + v_{1,a}$ containing two edges incident to $v_{1,b}$ having weight -1 and containing the two
 451 edges $v_{1,a}u'_1$ and $v_{1,a}u'_2$. If we swap the weights on C , then the only possible conflict is $v_1 v_{1,a}$ in the
 452 case where $v_{1,a}$ is a vertex of degree 4 and $v_1 v_{1,a}$ has weight -1 , both $v_{1,a}u'_1$ and $v_{1,a}u'_2$ have weight
 453 1, and $v_{1,a}$ is incident to some fourth -1 -edge $v_{1,a}u'$. We can assume that the component L' to
 454 which u' belongs in $G''' - v_{1,b}$ is not incident to a -1 -edge of $v_{1,b}$, since otherwise, we could modify
 455 C to contain the edge $v_{1,a}u'$. Note that this also implies that $L'_3 = L_1$. Since $v_{1,b}$ had weighted
 456 degree -4 this implies that there is another component L'_4 distinct from all of L'_1, L'_2, L'_3 , which is
 457 incident to an edge in $E(v_{1,b})$ having weight -1 . The vertex $v_{1,a}$ must have a neighbour u'' in this
 458 component L'_4 and, since we can assume that we cannot modify C to contain $v_{1,a}u'$, we must have
 459 $u'' \neq u'$. This contradicts $v_{1,a}$ having degree 4.

460 **Case 3:** G'' has three components C_1, C_2, C_3 , such that v_1 and v_2 have one neighbour in each of
 461 these three components.

462 In that case, G has the $\{-1, 1\}$ -property according to Lemma 2.11 as v_1 has even degree 4, and
 463 it can be checked that $G - v_1 - N_G(v_1)$ remains connected due to the 2-connectedness of G . In
 464 particular, for $i = 1, 2, 3$, note that the edge incident to v_1 going to C_i and the edge incident to v_2
 465 going to C_i cannot share an end. A contradiction. \diamond

466 **Claim 3.** G has no suspended path of length 4.

467 *Proof of the claim.* Suppose the claim is false and let $v_1x_1x_2x_3v_2$ be a suspended path in G ,
 468 where $d(x_1) = d(x_2) = d(x_3) = 2$ and $d(v_1), d(v_2) \geq 3$. We can assume $x_2 \in X$, which implies
 469 that $x_1, x_3 \in Y$. Define $G' = G - x_1 - x_2 - x_3$. Using Lemma 2.4 similarly as in the proof of
 470 Claim 2, we can come up with an $(X \setminus \{x_1\}, Y \setminus \{x_2, x_3\})$ - a -parity $\{a, b\}$ -edge-weighting w of G
 471 where $w(v_1x_1) = w(x_3v_2) = a$ and $w(x_1x_2) = w(x_2x_3) = b$. One can check that, by slightly
 472 modifying the exact same arguments used in the proof of Claim 2, we can eventually remove all
 473 conflicts from w , or deduce another proper $\{a, b\}$ -edge-weighting of G . \diamond

474 **Claim 4.** G has no suspended path of length at least 5.

475 *Proof of the claim.* Suppose the claim is false and let $v_1x_1x_2x_3x_4v_2$ be a path in G , where
 476 x_1, x_2, x_3, x_4 all have degree 2, and v_1, v_2 here might be of degree 2. Let G' be obtained from
 477 G by replacing $v_1x_1x_2x_3x_4v_2$ by an edge $e = v_1v_2$ even if that edge is already there. If G' has the
 478 $\{a, b\}$ -property, then so does G . Indeed, assume there is a proper $\{a, b\}$ -edge-weighting of G where
 479 the weight of e is, say a , and consider that weighting back in G . We start the extension to the five
 480 edges by assigning weight a to v_1x_1 and x_4v_2 , so that v_1 and v_2 keep the same weighted degree as
 481 in G' . Since v_1v_2 is an edge in G' , note that v_1 and v_2 have different weighted degrees. From this,
 482 we deduce that either v_1 has weighted degree different from $2a$ and v_2 has weighted degree different
 483 from $a + b$, or conversely. Assume the first situation holds. Then, we can achieve the weighting of
 484 G by assigning weight a to x_1x_2 and weight b to x_2x_3 and x_3x_4 .

485 So we can assume that G' does not have the $\{a, b\}$ -property and is thus an odd multi-cactus by
 486 the minimality of G . The edge e cannot be red in G' , since then G would also be an odd multi-
 487 cactus. Thus e is green and Lemma 2.2 implies that G has the $\{a, b\}$ -property. \diamond

488
 489 By Claims 1, 2, 3, 4, all degree-2 vertices in G (if any) lie on suspended paths of length 3. In G
 490 we replace all suspended paths of length 3 by edges (even if the two ends were already adjacent) to
 491 form a bipartite multigraph G^* . Edges arising from suspended paths of length 3, we call *blue edges*.
 492 Every other edge of G^* , *i.e.*, which was already present in G , we call a *white edge*.

493 Note that G^* is bipartite, 2-connected, has minimum degree at least 3, and it may have more
 494 multiple edges than G has. Also, note that for every vertex v in G^* , we have $d_{G^*}(v) = d_G(v)$.
 495 In general, it is not easy to deduce a proper $\{a, b\}$ -edge-weighting of G from one of G^* (typically
 496 because of blue edges); however, information on the structure of G can be deduced from that of G^* .
 497 In particular, we will study the existence of paths or cycles in G^* to deduce that of corresponding
 498 paths or cycles in G (where any traversed blue edge in G^* is replaced by the corresponding path of
 499 length 3 in G).

500 If the deletion of some pair of adjacent vertices u, v disconnects G^* , then let $z_0y_0 \in E(G^*)$ be
 501 such that $G - z_0 - y_0$ is disconnected and such that some component H of $G^* - z_0 - y_0$ has smallest
 502 possible order. The union of that component H and z_0, y_0 together with all edges connecting them
 503 is denoted B . In case G has no pair of adjacent vertices whose removal disconnects the graph, we
 504 define $H = B = G^*$, and y_0, z_0 do not exist.

505 **Claim 5.** For every vertex v of H , we have $d_{G^*}(v) = 3$.

506 *Proof of the claim.* Suppose the claim is false and let w_0 be a vertex in H of maximum degree
507 $d = d(w_0)$ at least 4. Without loss of generality, we can suppose that $w_0 \in X$. Assume first that
508 w_0 is adjacent to none of z_0, y_0 (this is the case if these two vertices do not exist). By the remark
509 following Lemma 2.12, we can assume that we are in the exceptional case when considering $G - w_0$
510 and defining A to be the set of vertices in $N(w_0)$ which have the same degree as w_0 . Thus, w_0 and
511 all vertices in $N(w_0)$ have degree 4 and $G - w_0 - N(w_0)$ has exactly six components. We can now
512 choose another vertex of degree d as w_0 by choosing w_0 such that the order of the component of
513 $G - w_0 - N(w_0)$ containing z_0 is maximum and avoid the exceptional case in Lemma 2.12. Again,
514 the remark following Lemma 2.12 shows how to find a proper $\{a, b\}$ -edge-weighting of G .

515 So we can assume w_0 is adjacent to, say, z_0 and that all the neighbours of w_0 in H which have
516 the same degree as w_0 are adjacent to y_0 (due to the bipartiteness of G^*). Since y_0, z_0 thus exist,
517 we have $G \neq H$. We can also assume that all vertices in H having maximum degree are adjacent
518 to z_0 or y_0 (as otherwise, the previous situation would apply). Note that this implies that we can
519 never be in the exceptional case in Lemma 2.12 when we delete a vertex v in H of maximum degree
520 and define A to be the neighbours of v with the same degree as v .

521 As pointed out earlier, there is an $(X \setminus \{w_0\}, Y)$ - a -parity $\{a, b\}$ -edge-weighting of G where all
522 edges incident to w_0 are weighted b and every neighbour of w_0 with degree d is incident to at least
523 one a -edge. This edge-weighting is proper unless $z_0 w_0$ is a conflict, which occurs only if the degree
524 of z_0 is strictly greater than that of w_0 . Note that we can assume that z_0 is incident to exactly
525 one edge going to each component other than H in $G - y_0 - z_0$, since otherwise, we could deduce
526 an $\{a, b\}$ -edge-weighting of G as above with the extra condition that two edges e, e' incident to z_0
527 going to a component C of $G - y_0 - z_0$ other than H are weighted b . Then, if $z_0 w_0$ is a conflict, we
528 could get rid of it by swapping the weights along a cycle going through z_0 and C via e, e' (so that
529 it is z_0 -changing) and not going through H (so that it is w_0 -avoiding). So z_0 is incident to exactly
530 one edge going to each component other than H in $G - y_0 - z_0$. We can also assume that there is at
531 most one component C other than H in $G - y_0 - z_0$, since otherwise, we could reach the exact same
532 conclusion by deducing an $\{a, b\}$ -edge-weighting of G as before with the extra condition that two
533 edges e, e' incident to z_0 going to two different components C, C' distinct from H are weighted b .
534 In case $z_0 w_0$ is a conflict, we could again get rid of it by swapping the weights along a cycle going
535 through z_0 , in C via e , back to y_0 , in C' , and back to z_0 via e' . This would be correct since such a
536 cycle would not go through H , and thus, would be w_0 -avoiding. Similarly, we can assume that the
537 multiplicity of $z_0 y_0$ is 1.

538 Let us denote by z_1 the unique neighbour of z_0 in C . By swapping the weights along a cycle
539 through C containing z_0, y_0 and the edge $z_0 z_1$, and not going through H , we can further assume that
540 the edge $z_0 z_1$ is weighted a . For a similar reason, we can assume that the edge $y_0 z_0$ is weighted b .
541 Recall that w_0 and all the neighbours of w_0 with degree d are all incident to an even number of
542 a -edges; thus, each of the neighbours of w_0 with degree d is incident to at least two a -edges.

543 To get rid of the conflict $z_0 w_0$, we would like to swap the weights along a z_0 -changing cycle in
544 $G - w_0$ (thus, w_0 -avoiding). According to Observation 2.9, recall that this would not alter the parity
545 of the number of incident a 's of any vertex in $V(G) \setminus \{w_0\}$. Furthermore, this would get rid of the
546 conflict $z_0 w_0$. However, this swapping process can create a conflict between w_0 and a neighbour v
547 of w_0 with degree d ; but such a conflict can only arise when the cycle goes through the only two
548 a -edges incident to v . For a neighbour v of w_0 with degree d that is incident to only two a -edges,

549 we call this pair of edges a *forbidden pair*. Our goal in what follows is to show that $G - w_0$ has a
550 z_0 -changing cycle not containing any forbidden pair of edges.

551 Let us denote by v_1, \dots, v_m the neighbours of w_0 with degree d . As mentioned earlier, recall that
552 the v_i 's are all adjacent to y_0 . Since $z_0 w_0$ is a conflict, recall that $d(z_0)$ and $d(w_0)$ have the same
553 parity. Furthermore, since $d(z_0) > d(w_0) > 3$, it follows that $d(z_0) \geq 6$, and, because the only
554 neighbours of z_0 outside H are y_0 and z_1 , there is a vertex $z_2 \neq w_0$ in $N(z_0) \cap V(H)$. Note that,
555 to find the desired cycle through z_0 in $G - w_0$, it suffices to find a path P from y_0 to a vertex z' in
556 $N(z_0) \cap V(H)$ in the connected graph $G - z_0 - w_0$ (which is connected by the minimality of H) not
557 using any forbidden pair of edges. Indeed, if the weight on $z_0 z'$ is b , then we can define our cycle
558 to be $P \cup \{z_0 y_0, z' z_0\}$, while, if the weight on $z_0 z'$ is a , then we can define our cycle to be $P \cup P_c$,
559 where P_c is a path from z_0 to y_0 in $G - H - z_0 y_0$ (thus, through C). Since the graph $G - z_0 - w_0$ is
560 connected, there is a path P_1 from z_2 to y_0 . We can assume that P_1 uses forbidden pairs of edges.
561 Without loss of generality, let $p v_1$ and $v_1 q$ be the first forbidden pair of edges P_1 used when going
562 from z_2 to y_0 . Since v_1 is adjacent to y_0 , it follows that $q = y_0$, since otherwise, we have found a
563 path from y_0 to z_2 not using any forbidden pair of edges. Thus, we can assume that all paths from
564 y_0 to a vertex in $N(z_0) \cap V(H)$ use exactly one forbidden pair of edges. Now we look at all such
565 paths using only one pair of forbidden edges $y_0 v_i$ and $v_i p$ (for $i \in \{1, \dots, m\}$) and consider one such
566 path P that goes through the most neighbours of w_0 . Let $y_0 v_i$ and $v_i p$ be the pair of forbidden
567 edges that P contains.

568 First suppose that v_i has a neighbour v'_i distinct from y_0, p, w_0 . The edge $v_i v'_i$ must have weight
569 b . Since $G - w_0 - v_i$ is connected, it has a path P' from v'_i to a vertex in $N(z_0) \cap V(H)$. The path
570 P' must use a forbidden pair of edges, as otherwise, the graph induced by $E(P) \cup E(P')$ would
571 contain a desired path from y_0 to a vertex in $N(z_0) \cap V(H)$ avoiding forbidden pairs of edges. Let
572 the first pair of forbidden edges P' used when starting from v'_i be $q v$ and vr . The subpath P'_1 of P'
573 from v'_i to v must be disjoint from P , since otherwise, the graph induced by $E(P) \cup E(P'_1)$ contains
574 a desired path from y_0 to $N(z_0) \cap V(H)$ avoiding forbidden pairs of edges. Furthermore, we must
575 have that $r = y_0$, since otherwise, the path P'' defined to be $y_0 v$ together with the subpath of P'
576 from v to v'_i followed by $v'_i v_i$ and the subpath of P from v_i to $N(z_0) \cap V(H)$ is a desired path from
577 y_0 to $N(z_0) \cap V(H)$ avoiding forbidden pairs of edges. Now the path P'' contradicts the maximality
578 of P .

579 So we can assume that $N(v_i) = \{y_0, p, w_0\}$, which means, because v_i has degree $d > 3$, that
580 some of the edges $v_i y_0, v_i p, v_i w_0$ have multiplicity more than 1. The multiplicity of both $y_0 v_i$ and
581 $v_i p$ must be 1, since otherwise, we would get a desired path from z_2 to y_0 avoiding the forbidden
582 pair of edges $y_0 v_i, v_i p$. So the multiplicity of $v_i w_0$ is at least 2. If the only neighbours of w_0 are v_i
583 and z_0 , then we can swap the weights on two edges between w_0 and v_i to avoid the conflict $z_0 w_0$
584 and obtain a proper $\{a, b\}$ -edge-weighting of G ; so we can assume that w_0 is incident to a vertex
585 v' in H distinct from v_i . Since $d(v_i) = d(w_0)$, the edges $w_0 z_0$ and $w_0 v'$ are not multiple and since
586 z_0 has degree at least 6, there are two edges $z_0 z'_2, z_0 z''_2$ incident with z_0 having the same weight
587 and where $z'_2, z''_2 \in H$. Possibly $z'_2 = z''_2$ or $z'_2 = z_2$. The graph $B - w_0 - z_0$ is connected, so it
588 contains a path P'_1 from z'_2 to y_0 and a path P'_2 from from z''_2 to y_0 . These paths P'_1 and P'_2 must
589 be internally disjoint, since otherwise, there would be a z_0 -changing cycle in B not containing any
590 pair of forbidden edges. We can assume that both P'_1 and P'_2 contain a pair of forbidden edges,
591 since otherwise, there is a desired path from y_0 to $(N(z_0) - w_0) \cap V(H)$. Hence, we can assume that

592 P'_1 contains y_0v_i and $v_i p$ and P'_2 contains a pair of forbidden edges $qv', v'r$ incident to v' . Since
593 this implies that y_0 and v' are adjacent, we can assume that $y_0 = q$. The vertex v' must have a
594 neighbour s distinct from y_0, w_0, r and since the graph $B - w_0 - v'$ is connected, there is a path P''
595 in $B - w_0 - v'$ from s to y_0 . If P'' contains the forbidden pair of edges y_0v_i and $v_i p$, then the graph
596 $P'_1 \cup P'_2 \cup P''$ contains a z_0 -changing cycle in B containing no forbidden pair of edges. Thus, we can
597 assume that P'' contains no pair of forbidden edges. Now $P'_2 \cup P''$ contains a desired path from w_0
598 to $(N(z_0) - w_0) \cap V(H)$. \diamond

599 **Claim 6.** *There is no vertex $v \in V(H)$ such that $G - v - N(v)$ is connected.*

600 *Proof of the claim.* Suppose $v \in V(H)$ and that $G' = G - v - N(v)$ is connected. We can assume
601 $v \in X$. By Lemma 2.10, there is an $(X \setminus \{v\}, Y \cup \{v\})$ - a -parity $\{a, b\}$ -edge-weighting of G , where
602 all edges incident to v have weight b and any vertex $u \in N(v)$ is incident to at most $1 + M(uv)$
603 b -edges, where $M(uv)$ denotes the multiplicity of the edge uv . Note that Claim 1 implies that if
604 vv' is a multiple edge, then $v' \in \{z_0, y_0\}$. Thus, the only potential conflict is between v and one
605 of z_0, y_0 , say y_0 . This implies that y_0 must have odd degree at least 5. But since G' is connected
606 there must then be a y_0 -changing cycle in $G - (N(v) \setminus \{y_0\})$ and swapping the weights on this cycle
607 yields a proper $\{a, b\}$ -edge-weighting of G . \diamond

608 **Claim 7.** *There are no multiple edges between two vertices in H .*

609 *Proof of the claim.* Suppose uv is a multiple edge in H . We can assume $v \in X$. Since u and v
610 have degree 3 in G^* (by Claim 5), the multiplicity of uv is exactly 2. Let e and e' be the two edges
611 between u and v . By Claim 1, e, e' are not both white. Thus, at least one of e, e' , say e , is a blue
612 edge in H . Let v' denote, in G^* , the unique neighbour of v different from u .

613 Let G' be obtained from $G - v$ by removing all edges but one edge e'' incident to v' . Clearly G'
614 is connected since $G - v - v'$ is connected by the minimality of H . Let $S = E(v') \setminus \{vv', e''\}$. Let
615 X' denote the set of vertices in $G - v$ which are incident to an odd number of edges in S . Note
616 that S has even size. Thus, $X' = (X \setminus S) \cup (S \cap Y)$ has even size and Lemma 2.4 implies that
617 there is an $(X', V(G') \setminus X')$ - a -parity $\{a, b\}$ -edge-weighting of G' . We now extend this weighting to
618 G by assigning weight a to all edges in S and weight b to all edges incident to v . This gives an
619 $(X \setminus \{v\}, Y \cup \{v\})$ - a -parity $\{a, b\}$ -edge-weighting of G where all vertices in $N(v)$ are incident to at
620 most two b -edges (the edge in the suspended path of length 3 joining u and v incident to u must
621 have weight a). Observation 2.8 implies that the only conflict can be between v and its neighbours,
622 so the only possible conflict is vv' in the case where $v' \in \{z_0, y_0\}$, say $v' = y_0$, and y_0 has degree at
623 least 5. But since $G - v$ is connected there must then be a y_0 -changing cycle avoiding v and u and
624 swapping the weights on this cycle yields a proper $\{a, b\}$ -edge-weighting of G . \diamond

625 **Claim 8.** *Every vertex of H is incident to at most one blue edge.*

626 *Proof of the claim.* Recall that every vertex v of H has degree 3, by Claim 5. If v is incident to
627 three blue edges, then $G - v - N(v)$ is connected, which contradicts Claim 6. So now assume v is
628 incident to two blue edges. Let uv denote the white (third) edge incident to v . Still by Claim 6, the
629 graph $G - v - N(v)$ cannot be connected, which means that u, v is a pair contradicting the choice
630 of y_0, z_0 . \diamond

631 We now have all the tools in hand for finishing the proof. Two cases must be considered:

632 **Case 1:** *There is a vertex $v \in V(H)$ not adjacent to any of z_0, y_0 .*

633 Recall that, according to Claim 6, whenever removing from G a vertex of H and its neighbour-
634 hood, we get a disconnected graph. Let v be a vertex not adjacent to z_0, y_0 such that the component
635 K of $G' = G - v - N(v)$ containing z_0 and y_0 has maximum order. We can assume that $v \in X$.
636 Note that there must be a vertex $v' \in V(H)$ distinct from v with $N(v') = N(v)$ such that the
637 components of G' are exactly K and the isolated vertex v' . This is because otherwise there would
638 be a vertex $v'' \neq v$ in H such that $G - v'' - N(v'')$ has a bigger K , a contradiction to our choice of
639 v .

640 Let $e_1 = vv_1, e_2 = vv_2, e_3 = vv_3$ denote the three edges incident to v . Since v, v', v_1, v_2, v_3 all
641 belong to H , by Claim 7, all these vertices are distinct. Furthermore, since $N(v) = N(v')$, we can
642 also assume that none of vv_1, vv_2, vv_3 are blue. Hence, $v, v' \in X$ and $v_1, v_2, v_3 \in Y$. The graph
643 $G - v$ is connected and so is the graph G' obtained from it by removing the edges $v'v_1, v'v_2$. Lemma
644 2.4 implies that there is an $\{a, b\}$ -edge-weighting of G' where the vertices in $X \setminus \{v\} \cup \{v_2, v_3\}$ are
645 incident to an odd number of a -edges and the vertices in $Y \setminus \{v_2, v_3\}$ are incident to an even number
646 of a -edges. In particular, in G' the only a -edge incident to v' is $v'v_1$. We extend this weighting
647 to G by assigning weight a to $v'v_2, v'v_3$, and weight b to all three edges incident to v . That way,
648 we get an $(X \setminus \{v\}, Y \cup \{v\})$ - a -parity $\{a, b\}$ -edge-weighting of G where all three edges incident to
649 v have weight b and all three edges incident to v' have weight a . Observation 2.8 implies that the
650 only potential conflicts are between v and its neighbours, but since all vertices in $N(v)$ are incident
651 to at least one a -edge (the one incident to v') this edge-weighting is proper.

652 **Case 2:** *All vertices in H are adjacent to z_0 or y_0 .*

653 By Claim 6, we can assume that, for every vertex $v \in V(H)$, the graph $G' = G - v - N(v)$ is
654 disconnected. First, suppose z_0 is joined in G^* to some vertex $v \in V(H)$ by an edge of multiplicity
655 2. Let e' and e'' be the two edges joining z_0 and v in G^* . Claim 6 implies that not both of e', e''
656 are blue; say e' is white. If e'' is blue, then by Claim 8, the third edge e''' incident to v in G^*
657 must be white. If e'' is white, then Claim 6 implies that e''' is white, so the edge $e''' = vu$ must be
658 white. We can assume $v \in X$ and hence $u \in Y$. Let Z denote the set of vertices in $G - v$ which are
659 incident to exactly one edge incident to u . Note that either Z is empty or Z has size 2. Note that
660 $X' = (X \setminus (Z \cup \{v\})) \cup (Y \cap Z)$ has even size, so by Lemma 2.4, there is an $(X', V(G - v - u) \setminus X')$ -
661 a -parity $\{a, b\}$ -edge-weighting of $G - v - u$. We now extend this edge-weighting to all of G by
662 assigning weight b to all edges in $E(v)$ and weight a to the two edges incident to u distinct from uv .
663 Thus, we have obtained an $(X \setminus \{v\}, Y \cup \{v\})$ - a -parity $\{a, b\}$ -edge-weighting of G where all edges
664 incident to v have weight b and u is incident to exactly one b -edge. Observation 2.8 implies that the
665 only potential conflict is uz_0 in the case where z_0 has degree at least 5. We can also assume that
666 there is no z_0 -changing cycle in G avoiding v and u . Hence, z_0 must have degree 2 in $G - H$ and be
667 joined by an edge in G^* to at least one vertex v' in H distinct from v . We can now find a desired
668 z_0 -changing cycle unless the only neighbours of v' in B are z_0 and u . But in this case $G - u - N(u)$
669 is connected, contradicting Claim 6.

670 Thus, we can assume that z_0 , and similarly y_0 , is not joined to any vertex in H by a multiple
671 edge in G^* . Claim 7 now implies that any vertex in H has three distinct neighbours in G^* . Let
672 $v \in X$ be any vertex in H incident to z_0 . The graph $G - v - N(v)$ is disconnected by Claim 6, so

673 there must be a vertex v' in H which in G^* has the same neighbourhood as v . Since all vertices in H
674 have degree 3 (Claim 5) this implies that H only has four vertices: two joined to z_0 and two joined
675 to y_0 . The graph $G - v - (N(v) \setminus \{z_0\})$ is connected, so as above, there is an $(X \setminus \{v\}, Y \cup \{v\})$ - a -
676 parity $\{a, b\}$ -edge-weighting of G where all edges incident to v have weight b , and the neighbours of
677 v distinct from z_0 which have degree 3 are incident to exactly one b -edge. Again, Observation 2.8
678 implies that the only possible conflict is vz_0 in the case where z_0 has degree at least 5. In this case,
679 it is easy to see that there is a z_0 -changing cycle avoiding H , thus, we can get rid of this conflict
680 and obtain a proper $\{a, b\}$ -edge-weighting of G . \square

681 3 Proof of Theorem 1.2

682 Before describing the structure of trees without the $\{-1, 1\}$ -property, we first introduce the following
683 two general lemmas which will be used in the proof. Some of the tools and results used here were
684 introduced in Section 2.

685 **Lemma 3.1.** *If G is a simple connected bipartite graph without the $\{-1, 1\}$ -property and e is a*
686 *bridge in G , then the deletion of e results in two components each containing an odd number of*
687 *vertices.*

688 *Proof.* Suppose the lemma is false. Let G be a connected bipartite graph without the $\{-1, 1\}$ -
689 property and let $e = uv \in E(G)$ be a bridge in G . Let $c : V(G) \rightarrow \{1, 2\}$ be a mod-4 vertex-
690 colouring of G (such a colouring exists by Lemma 2.6) and let X, Y denote the sets of vertices
691 coloured 1 and 2 respectively. By Lemma 2.7, both colour classes of c have odd size. Let C_1, C_2
692 denote the two components of $G - e$ with $u \in V(C_1)$ and $v \in V(C_2)$. For a contradiction, assume
693 that both $|V(C_1) \cap X|$ and $|V(C_1) \cap Y|$ are odd and both $|V(C_2) \cap X|$ and $|V(C_2) \cap Y|$ are even.
694 Recall that two vertices with degree of distinct parity cannot have the same weighted degree by a
695 $\{-1, 1\}$ -edge-weighting (Observation 2.8). There are thus four cases to be considered:

696 **Case 1:** *Both u and v have odd degree and colour 1.*

697 By Lemma 2.4, there is an $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$ - (-1) -parity $\{-1, 1\}$ -edge-weighting
698 of C_1 and a $(Y \cap V(C_2), X \cap V(C_2))$ -1-parity $\{-1, 1\}$ -edge-weighting of C_2 . It follows from Obser-
699 vation 2.8 that these two edge-weightings, together with assigning weight -1 to e , form a proper
700 edge-weighting of the whole G .

701 **Case 2:** *Both u and v have even degree and colour 1.*

702 By Lemma 2.4, there is an $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$ - (-1) -parity $\{-1, 1\}$ -edge-weighting
703 of C_1 and an $(X \cap V(C_2), Y \cap V(C_2))$ -1-parity $\{-1, 1\}$ -edge-weighting of C_2 . It follows from Ob-
704 servation 2.8 that these two edge-weightings, together with assigning weight -1 to e , form a proper
705 edge-weighting of the whole G .

706 **Case 3:** *Both u and v have odd degree, u has colour 1 and v has colour 2.*

707 By Lemma 2.4, there is an $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$ - (-1) -parity $\{-1, 1\}$ -edge-weighting
708 of C_1 and a $(Y \cap V(C_2), X \cap V(C_2))$ -1-parity $\{-1, 1\}$ -edge-weighting of C_2 . It follows from Obser-
709 vation 2.8 that these two edge-weightings, together with assigning weight -1 to e , form a proper
710 edge-weighting of the whole G .

711 **Case 4:** *Both u and v have even degree, u has colour 1 and v has colour 2.*

712 By Lemma 2.4, there is an $(X \cap V(C_1) \setminus \{u\}, Y \cap V(C_1) \cup \{u\})$ - (-1) -parity $\{-1, 1\}$ -edge-weighting
713 of C_1 and an $(X \cap V(C_2), Y \cap V(C_2))$ -1-parity $\{-1, 1\}$ -edge-weighting of C_2 . It follows from Ob-
714 servation 2.8 that these two edge-weightings, together with assigning weight -1 to e , form a proper
715 edge-weighting of the whole G . \square

716 **Lemma 3.2.** *If G is a connected bipartite graph without the $\{-1, 1\}$ -property and e is a bridge in*
717 *G , then there is a $\{-1, 1\}$ -edge-weighting of G such that e is the only conflict.*

718 *Proof.* Let G be a connected bipartite graph without the $\{-1, 1\}$ -property, e be a bridge in G , and
719 C_1, C_2 be the two components of $G - e$. Let c be a mod-4 vertex-colouring of G (such a colouring
720 exists by Lemma 2.6), and let X denote the set of vertices with colour 1 and Y denote the set of
721 vertices with colour 2. By Lemma 2.7, both X and Y have odd size and by Lemma 3.1, we can
722 assume that $|X \cap V(C_1)|$ and $|Y \cap V(C_2)|$ are even and $|Y \cap V(C_1)|$ and $|X \cap V(C_2)|$ are odd. Now
723 Lemma 2.4 implies that there is an $(X \cap V(C_1), Y \cap V(C_1))$ -1-parity $\{-1, 1\}$ -edge-weighting of C_1 and
724 an $(X \cap V(C_2), Y \cap V(C_2))$ -1-parity $\{-1, 1\}$ -edge-weighting of C_2 . Observation 2.8 implies that these
725 two edge-weightings, together with assigning weight -1 to the edge e , is a $\{-1, 1\}$ -edge-weighting
726 of G where e is the only potential conflict. \square

727 We can now prove Theorem 1.2. When referring to Operation 1, we mean the first operation
728 described at the end of Section 1 (illustrated in Figure 1, (a) and (b)).

729 *Proof of Theorem 1.2.* As mentioned in the introduction, it is straightforward to check that any
730 graph constructed with Operation 1 from four graphs without the $\{-1, 1\}$ -property does not have
731 the $\{-1, 1\}$ -property itself. An easy argument is that all five edges incident to the vertices $v_1 \sim v_2$
732 and $v_3 \sim v_4$ should have the same weight (in which case a conflict arises), as otherwise, the proper
733 $\{-1, 1\}$ -edge-weighting would yield one of at least one of the four graphs used in the construction,
734 a contradiction. Thus, it suffices to prove that any tree without the $\{-1, 1\}$ -property is constructed
735 from a disjoint union of K_2 's through repeated (possibly none) applications of Operation 1. Suppose
736 this is false and let T be a minimum counterexample. Note that Lemma 3.1 implies that, for any
737 vertex $v \in V(T)$ and any edge $e \in E(v)$, the component C_e not containing v in $T - e$ has an odd
738 number of vertices. Since we can write $|V(T)| = 1 + \sum_{e \in E(v)} |V(C_e)|$ for any vertex $v \in V(T)$ and
739 since $|V(T)|$ is even, this implies that all vertices in T have odd degree. A consequence of this is
740 that if $S \subset T$ is a subtree of T , then $T - S$ has no components isomorphic to K_2 as otherwise T
741 would have vertices with degree 2.

742 Let $P = v_1 \dots v_m$ be a longest path in T . Clearly, v_m is a leaf and, since all vertices have odd
743 degree, v_{m-1} is incident to an even number of leaves. Suppose v_{m-1} is incident to an even number
744 $n \geq 4$ of leaves u_1, \dots, u_n , with $u_1 = v_m$. Recall that Lemma 2.7 implies that $|V(T)|$ is even; now,
745 since $T' = T - \{u_1, \dots, u_{n-1}\}$ has an odd number of vertices, Lemma 2.7 implies that T' has a proper
746 $\{-1, 1\}$ -edge-weighting. We can now extend this edge-weighting to a proper $\{-1, 1\}$ -edge-weighting
747 of T by assigning the same weight to all the edges $v_{m-1}u_1, \dots, v_{m-1}u_n$ (we choose whether it is 1
748 or -1 so that we avoid the conflict $v_{m-2}v_{m-1}$). Thus, v_{m-1} has degree exactly 3 and, from these
749 arguments and the maximality of P , any neighbour of v_{m-2} distinct from v_{m-3} and v_{m-1} is either a
750 leaf or a vertex of degree 3 adjacent to two leaves. Let $U' = \{u'_1, \dots, u'_p\}$ be the set of leaves adjacent
751 to v_{m-2} and let $U'' = \{u''_1, \dots, u''_q\}$ be the set of neighbours of v_{m-2} distinct from v_{m-1} and v_{m-3}
752 which have degree 3. Possibly $p = 0$ or $q = 0$, but $p + q > 0$ since v_{m-2} has odd degree and hence

753 $p + q$ is odd. Let T_1 and T_2 be the two components of $T - v_{m-3}v_{m-2}$ such that $v_{m-2} \in V(T_2)$.
 754 By Lemma 3.2, there is a $\{-1, 1\}$ -edge-weighting w of T such that the only potential conflict is
 755 $v_{m-3}v_{m-2}$. By possibly multiplying all edge weights by -1 , we can assume that the weight of
 756 $v_{m-3}v_{m-2}$ is 1. We look at three separate cases:

757 **Case 1:** $p + q \geq 5$.

758 By possibly modifying the weights of the edges in $E(T_2)$ such that they all have weight 1 or -1 ,
 759 the vertex v_{m-2} can obtain weighted degree $2 + p + q$ and $-p - q$. Now we simply pick the one of
 760 these two options such that $v_{m-3}v_{m-2}$ is not a conflict. Since all vertices in T_2 except v_{m-2} have
 761 degree at most 3, this gives a proper $\{-1, 1\}$ -edge-weighting of T .

762 **Case 2:** $p + q = 3$.

763 As in Case 1, we can modify the edge weights such that the vertex v_{m-2} can obtain weighted
 764 degree $2 + p + q$ and $-p - q$. Furthermore, since $2 + p + q = 5$ and all vertices in T_2 except v_{m-2}
 765 have degree at most 3, we can in this way find a proper $\{-1, 1\}$ -edge-weighting of T , unless v_{m-3}
 766 has weight 5. So we can assume that v_{m-3} has weight 5. If $p \geq 1$ and p is odd (resp. even), then
 767 we modify the weights in T_2 such that all edges incident with u'_1, \dots, u'_p have weight 1 (resp. -1)
 768 and all other edges in T_2 have weight -1 (resp. 1). This yields a proper $\{-1, 1\}$ -edge-weighting
 769 of T , so we can assume $p = 0$ and $q = 3$. In this case, we modify the weights in T_2 such that all
 770 edges incident to v_{m-1} and u''_1 have weight 1 and all other edges in T_2 have weight -1 . This yields
 771 a proper $\{-1, 1\}$ -edge-weighting of T .

772 **Case 3:** $p + q = 1$.

773 First suppose $q = 1$ and $p = 0$. If we modify the edge weights in T_2 such that they all have weight
 774 -1 , then we obtain a proper $\{-1, 1\}$ -edge-weighting of T , unless v_{m-3} has weight -1 . In this case, we
 775 change the weights of the three edges incident to v_{m-1} to 1 to obtain a proper $\{-1, 1\}$ -edge-weighting
 776 of T . Thus, we can assume $p = 1$ and $q = 0$. We can assume that $T''' = T - v_m - v_{m-1} - u_2 - u'_1$ has a
 777 proper $\{-1, 1\}$ -edge-weighting w , since otherwise, the minimality of T implies that T''' is constructed
 778 from a disjoint union of K_2 's through repeated (possibly none) applications of Operation 1, and
 779 then so is T . By possibly multiplying all edge weights of w by -1 , we can assume that $v_{m-3}v_{m-2}$
 780 has weight 1. Now assigning weight 1 to all edges incident to v_{m-1} and weight -1 to $v_{m-2}u'_1$ yields
 781 a proper $\{-1, 1\}$ -edge-weighting of T . □

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