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ADVANCED TOOLS FOR DESIGN AND ANALYSIS FOR ASSEMBLIES OF STRUCTURES WITH UNCERTAINTIES

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Abstract. *The numerical obtention of stochastic responses of assemblies or of response surfaces needs to carry out a large number of costly computations. This paper propose efficient techniques for obtention of such informations.*

A decomposition of the assemblies into substructures and interfaces is defined and associated with a dedicated computational strategy which leads to a local/global algorithm enabling the treatments of the substructure and of the interface problems to be uncoupled: the LATIN method [8].

The first proposed approach is a point by point calculation of response surfaces: the calculation of the solution for a new set of parameters is accelerated by using the solution of a previous one as an initialization. This procedure can be easily set up in the LATIN method [3]. The applications concern complex assemblies of 3D structures with uncertain frictional contact zones.

The second proposed technique is a dedicated approach to the calculation of the random response of assemblies with uncertain interface characteristics[1]. The random response is constructed using a Polynomial Chaos Expansion (PCE)[7]. Since the only uncertain parameters are those which appear in the interface equations, this approach results in a drastic reduction of the computational costs.

1 INTRODUCTION

The solutions to deterministic problems are often calculated by finite element analysis (FEA). For structural engineers, the incorporation of parametric uncertainties of a system into a mechanical model represents a challenge; however, without this information, the structural response could not be analyzed accurately, particularly in terms of reliability. These parametric system uncertainties may affect the mechanical and geometric properties, the boundary conditions. . . In the case of structural assemblies, ones knowledge of the friction coefficients or of the stiffness of bonded joints is especially poor. In order to take such uncertainty into account, it is necessary to calculate the response of the structure for all possible sets of values of the design parameters or to use a probabilistic structural analysis approach [2].

Our approach is based on a decomposition of the assembly into substructures and interfaces. The interfaces play the vital role of enabling local nonlinearities, such as contact and friction, to be modeled easily and accurately. The problem is solved in each substructure by the finite element method, and an iterative scheme based on the LATIN method is used for the global resolution.

The first approach proposed consists in calculating response surfaces such that each point of a surface is associated with a design configuration. Each design configuration corresponds to a set of values of all the variable parameters (friction coefficients, prestress) which are introduced into the mechanical analysis. A full calculation is needed for each point. Here, as an alternative to carrying out these full calculations, we propose to take advantage of the capability of the LATIN method to reuse the solution of a given problem (for one set of parameters) to solve similar problems (for the other sets of parameters) [3]. The numerical examples presented in the paper concern 3D assemblies. For some of these examples, over a thousand different calculations had to be carried out for the parametric study. The comparison of the computation costs with those of classical industrial codes shows the algorithm is very efficient when applied to the case of multiple resolutions for the analysis of static problems with contact and friction.

The second technique proposed addresses the random response of assemblies whose interfaces have uncertain characteristics. Thus, the randomness of the response comes from the random behavior of the connections. In this context, the random response is developed using a Polynomial Chaos Expansion (PCE) [4] coupled with the LATIN approach. A dedicated computational strategy to determine the random response of assemblies with probabilistic interface characteristics is presented [5].

The numerical example concerns an assembly of structures connected by screws with elastic joint. The stiffness characteristics of the screws and the joint are random. The results of the two approaches are compared.

Since the uncertain parameters are present only in the joints, the use of the LATIN method results in a drastic reduction of the computation costs.

2 THE REFERENCE PROBLEM

2.1 Decomposition of an assembly

An assembly is composed of a set of *substructures* (each substructure is a component of the assembly) which communicate with one another through *interfaces* (each interface represents a connection), see Fig. 1. Each interface is a mechanical entity with its own variables and its specific behavior which depends on the type of connection. Many different connection types can be modeled by this approach, but in this paper we consider only classical contact connections. Two connected substructures are denoted V_E and $V_{E'}$ and the associated interface is designated by $\Gamma^{EE'}$.

The interface variables are two force fields \vec{f}^E and $\vec{f}^{E'}$ and two dual velocity fields

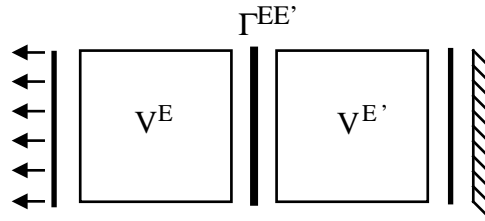


Figure 1: Decomposition of an assembly

\vec{w}^E et $\vec{w}^{E'}$ (Fig. 2). By convention, \vec{f}^E and $\vec{f}^{E'}$ are the actions of the interface on the substructures and \vec{w}^E et $\vec{w}^{E'}$ are the velocities of the substructures seen from the interface.

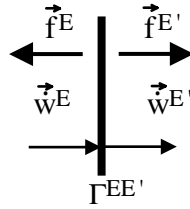


Figure 2: Interface variables

2.2 The problem in the substructures

The displacement field at any point M of V_E and at any time t of $[0, T]$ is $\vec{u}^E(M, t)$; the associated space is $\mathcal{U}^{[0, T]}$. ϵ is the strain field and the current state of the structure is characterized by the stress field σ^E .

The mechanical problem to be solved in each substructure is:

Find the histories of the displacement field $\vec{u}^E(M, t)$ and stress field $\sigma^E(M, t)$ such that:

- Kinematic admissibility:

$$\epsilon = \epsilon(\vec{u}^E); \quad \vec{u}^E(M, t)|_{\partial V_E} = \vec{w}^E(M, t); \quad \vec{u}^E \in \mathcal{U}^{[0, T]} \quad (1)$$

- Equilibrium: $\forall \vec{u}^* \in \mathcal{U}^{[0, T]}$

$$\int_{V_E} \text{Tr}(\sigma^E \epsilon(\vec{u}^*)) dV_E - \int_{V_E} \vec{f}_d \cdot \vec{u}^* dV - \int_{\partial V_E} \vec{f}^E \cdot \vec{u}^* dS = 0 \quad (2)$$

- Elastic behavior: $\forall M \in V_E$ and $\forall t \in [0, T]$,

$$\sigma^E(M, t) = \mathbf{D}\epsilon(\vec{u}^E(M, t)) \quad (3)$$

where \mathbf{D} is Hooke's operator.

2.3 The problem on the interfaces

The mechanical problem to be solved on each interface is:

Find the histories of the force fields ($\vec{f}^E(M, t)$ and $\vec{f}^{E'}(M, t)$) and of the velocity fields ($\vec{w}^E(M, t)$ and $\vec{w}^{E'}(M, t)$) such that:

- Equilibrium: $\forall M \in \Gamma^{EE'}$ and $\forall t \in [0, T]$,

$$\vec{f}^E(M, t) + \vec{f}^{E'}(M, t) = 0 \quad (4)$$

- Behavior: $\forall M \in \Gamma^{EE'}$ and $\forall t \in [0, T]$,

$$\vec{f}^E(M, t) = \mathcal{R}(\vec{w}^{EE'}(M, \tau), \tau \in [0, t]) \quad (5)$$

where the behavior is expressed as a nonlinear evolution law \mathcal{R} between the forces and the rate $\vec{w}^{EE'}$ of jump in displacement across the interface which is defined by:

$$\vec{w}^{EE'} = \vec{w}^{E'} - \vec{w}^E \quad (6)$$

For example, a perfect connection between two substructures would be modeled by the following behavior:

$$\vec{w}^{EE'}(M, t) = 0 \quad (7)$$

The form of the evolution law \mathcal{R} in the case of frictional contact conditions is described in [2].

2.4 Discretization in the geometric space

Standard finite element discretization is used for the displacement field within the substructures and at the interfaces:

$$\vec{u}^E(M, t) = \mathbf{N}\mathbf{u}^E(t) \quad \text{and} \quad \epsilon(\vec{u}^E) = \mathbf{B}\mathbf{u}^E(t) \quad (8)$$

where \mathbf{u}^E is the vector of nodal displacements and \mathbf{N} is the vector of classical finite element basis functions.

A LATIN (LArge Time INcrement) approach [8] is used to solve the problem. The procedure is iterative and creates, at each iteration, an approximation of the displacement and strain variables over the entire domain. The equations are split into two groups:

- linear equations related to the substructures
- local equations (which may be nonlinear) related to the interfaces

The search for the overall solution (i.e. the intersection of the two sets) is conducted iteratively by constructing approximate solutions which verify the two groups of equations alternatively over the whole time history. An error indicator is used to control the convergence of the algorithm.

3 FIRST APPROACH PROPOSED

The first approach proposed consists in calculating response surfaces such that each point of a surface is associated with a set of values of all the variable parameters. At each iteration, the LATIN method leads to an approximate solution to the problem over the whole time interval. Therefore, the trick is to reuse this approximation (associated to one set of values of all the design parameters) to find the solution to another design configuration (another set of the design parameters) similar to the one for which it was calculated in the first place. Our multiple solution method uses the fact that the LATIN algorithm can be initialized with any solution (usually an elastic solution) provided that it verifies the admissibility conditions. Therefore, the key to our technique is to initialize the process associated with a new design configuration using the results of the calculation carried out on the first set of values of the design parameter. In this manner, a first approximation of the solution to the new design with a strong mechanical content is immediately available from the start. In this particular case of elastic structures in contact, the interfaces play a vital role: they enable one to initiate the calculation on the new design configuration without having to save all data on the substructures as well as to search for the solution of the new design configuration with an initial solution well-suited to the target problem. In the best-case scenario, only a few iterations are necessary: the solution to the problem is obtained at low cost. If the solutions to the design configurations are close enough, the latter can still be derived at a significantly lower cost than by using a full calculation. For the parametric study presented herein, we just change the

parameters between iterations. Thus, the new computation is initialized by the solution to the previous one. If the parameters change slowly, the two solutions are close and only a few iterations are needed to reach convergence in the new calculation.

4 SECOND STRATEGY PROPOSED

Here, we are dealing with structural assemblies problems in which the behavior of the components and the external loads are deterministic: the randomness of the response comes from the random behavior of the connections. The systems parameters, calibrated using experimental data, are modeled as random variables or processes which are assumed to be properly represented by a set of random variables $\{\xi(\theta)\}$, where θ belongs to the space of random events Ω . Considering static loading cases, the main differences from the initial problem are the interfaces equations:

For each interface, we seek the force fields ($\vec{f}^E(M, \theta)$ and $\vec{f}^{E'}(M, \theta)$) and the displacement fields ($\vec{w}^E(M, \theta)$ and $\vec{w}^{E'}(M, \theta)$) such that:

- Equilibrium: $\forall M \in \Gamma^{EE'}, \forall \theta \in \Omega$

$$\vec{f}^E(M, \theta) + \vec{f}^{E'}(M, \theta) = 0 \quad (9)$$

- Behavior: $\forall M \in \Gamma^{EE'}, \forall \theta \in \Omega$

$$\vec{f}^E(M, \theta) = \mathbf{k}(\alpha(\theta))\vec{w}^{EE'}(M, \theta) = \mathbf{k}(\alpha(\theta))(\vec{w}^{E'} - \vec{w}^E) \quad (10)$$

where \mathbf{k} is the interface stiffness operator, which depends on a random material parameter $\alpha(\theta)$.

Standard finite element discretization is used for the displacement field within the substructures and at the interfaces:

$$\vec{u}^E(M, \theta) = \mathbf{N}\mathbf{u}^E(\theta) \quad \text{and} \quad \epsilon(\vec{u}^E) = \mathbf{B}\mathbf{u}^E(\theta) \quad (11)$$

where \mathbf{u}^E is the vector of nodal displacements and \mathbf{N} is the vector of classical finite element basis functions.

4.1 Discretization in Random space: Polynomial chaos expansion (PCE)

The nodal variables $\mathbf{u}(\theta)$ can be formally expressed as a nonlinear functional of the set $\{\xi_j(\theta)\}$ used to represent the material stochastic property. It has been shown [4] that this functional dependence can be expanded in terms of polynomial chaoses. Then, the truncated PCE of the response takes the form:

$$\mathbf{u}(\theta) = \sum_{i=0}^P \mathbf{u}_i \Psi_i(\theta) \quad (12)$$

where $\{\Psi_i(\theta)\}$ are polynomials in the Gaussian random variables $\{\xi_i\}$. These can be shown to form a complete basis of the Hilbert space of second-order random variables. The number of polynomials P depends on the order p of the PCE and on the number L of stochastic parameters. Let assume that the material parameters are constant along an interface. Should this not be the case, a Karhunen-Loeve expansion could easily be used to represent the spatial randomness of the interface characteristics [6]. The stochastic material property $\alpha(\theta)$ is represented by

$$\alpha(\theta) = \bar{\alpha}(1 + \delta\xi(\theta)) \quad (13)$$

where $\bar{\alpha}$ is the mathematical expectation of $\alpha(\theta)$, δ the coefficient of variation (standard deviation versus expectation) and $\xi(\theta)$ a standard normal random variable: $\xi(\theta) \sim N(0, 1)$. For non-Gaussian material properties, the PCE is used to represent the material property: the case of a Gaussian process is a particular case chosen for simplicity's sake [5].

4.2 Direct Spectral Stochastic Finite Element Method (SSFEM)

Following the traditional FE assembly procedure, this leads to the corresponding expansion of the stiffness matrix:

$$\mathbf{K}(\theta) = \sum_{i=0}^L \xi_i(\theta) \mathbf{K}_i \quad (14)$$

where \mathbf{K}_0 denotes the stiffness matrix for the mean material properties and the other terms correspond to the random fluctuations about the mean. The number of stochastic parameters is L ($\xi_0 = 1$). Expanding the nodal solution $\mathbf{u}(\theta)$ with respect to the polynomial chaos basis (Eq. (12)) and using Eq. (14), the stochastic equation leads to:

$$\sum_{j=0}^P \sum_{i=0}^L \xi_i \Psi_j(\theta) \mathbf{K}_i \mathbf{u}_j = \mathbf{f} \quad (15)$$

An equality, in a weak sense, can be derived by projecting Eq. (15) onto the subspace spanned by the polynomial chaos subset used in the approximation; this process results in the following equations:

$$\sum_{j=0}^P \sum_{i=0}^L \langle \xi_i \Psi_j(\theta) \Psi_k(\theta) \rangle \mathbf{K}_i \mathbf{u}_j = \langle \Psi_k(\theta) \rangle \mathbf{f} \quad k = 0, 1, \dots, P, \quad (16)$$

The last equation can be rewritten as:

$$\sum_{j=0}^P \sum_{i=0}^L c_{ijk} \mathbf{K}_i \mathbf{u}_j = \delta_{0k} \mathbf{f} \quad k = 0, 1, \dots, P, \quad (17)$$

where the coefficients c_{ijk} denote $\langle \xi_i \Psi_j(\theta) \Psi_k(\theta) \rangle$ and can be calculated only once. This system of linear equations must be solved for the unknown \mathbf{u}_j of the PCE. The details of the above procedure were published in [7]. The implementation issues were addressed in a number of other references [5, 6]. These equations can be assembled into a matrix of size $(P + 1).n \times (P + 1).n$ (n being the number of degrees of freedom) of the form:

$$\begin{pmatrix} \mathbf{K}^{(00)} & \mathbf{K}^{(01)} & \dots & \dots & \mathbf{K}^{(0P)} \\ \cdot & \cdot & \dots & \dots & \cdot \\ \cdot & \cdot & \mathbf{K}^{(jk)} & \dots & \cdot \\ \cdot & \cdot & \dots & \dots & \cdot \\ \mathbf{K}^{(P0)} & \mathbf{K}^{(P1)} & \dots & \dots & \mathbf{K}^{(PP)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_0 \\ \cdot \\ \mathbf{u}_k \\ \cdot \\ \mathbf{u}_P \end{pmatrix} = \begin{pmatrix} \mathbf{f}_0 \\ \cdot \\ 0 \\ \cdot \\ 0 \end{pmatrix} \quad (18)$$

where

$$\mathbf{K}^{(jk)} = \sum_{i=0}^L \langle \xi_i \Psi_j(\theta) \Psi_k(\theta) \rangle \mathbf{K}_i \quad (19)$$

In summary, this approach consist of expanding the random response process about a basis of the Hilbert space of random variables and of calculating the coefficients of the expansion. The result is a convergent expansion of the response in terms of multidimensional orthogonal polynomials. Although the methodology used is becoming widespread, serious obstacles have been encountered, from a computational point of view, in practical implementations. In large and realistic problems, the methodology is either cumbersome or computationally intensive. Some numerical strategies, such as iterative algorithms, have been devised to overcome the numerical difficulties arising in this context [9].

4.3 An adapted approach

The proposed approach uses the localization of the random characteristics in order to solve these types of problems more efficiently. Using the decomposition presented, the resolution of the two stages of the LATIN method can be summarized as follows:

- *Resolution for the local stage:* we assumed the stiffness to be independent of the space coordinate at the interface. Therefore, the solution is achieved through the resolution of small independent systems (size $(P + 1)$) at each node and in each direction.
- *Resolution for the global stage:* this consists of solving $(P+1)$ independent linear systems for each substructure. It is important to note that the matrices which appear in the system remain constant during the iterations and, therefore, need to be factorized only once before the first iteration. An even more important observation is that the problems on the substructures are completely independent of one another and could be solved in parallel very efficiently.

5 EXAMPLE

This example deals with a cylinder submitted to internal pressure. The cylinder is closed by a cover maintained by height prestressed screws (fig. 3). The etancheity is assumed by an elastic joint between the cylinder and the cover (fig. 4).

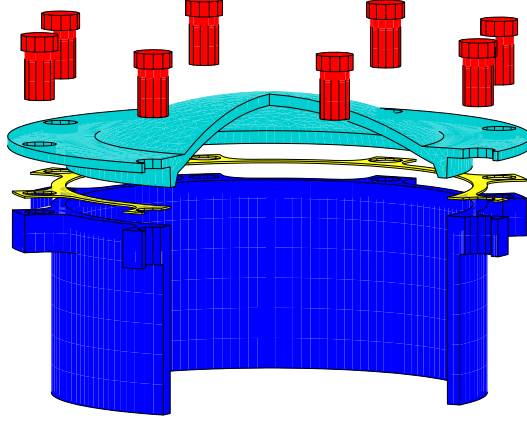


Figure 3: View of the complete assembly

The stiffness of the joint is supposed to be non deterministic. It is modelled by an elastic interface between the cover and the cylinder. The contact conditions between the screws and the cylinder are not modelled in the present study. They would change the way the screws are deformed. Thus, in this computation, the stiffness of the screw is also supposed to be non deterministic. It is taken into account by an interface between the body and the head of the screws. Only the upper part of the cylinder is modelled. For symetry reasons, only 1/16th of the assembly is studied. Figure 4 presents the different interfaces of the model.

The cylinder, the cover and the screws are made with steel:

Young's Modulus $E = 200,000 \text{ MPa}$ and Poisson ration $\nu = 0.3$.

The stiffness of the bold is :

$$K = \bar{K}(1 + \delta_1 \xi_1) \quad \text{with } \bar{K} = 5. 10^4 \text{ N.m}^{-1} \text{ and } \delta_1 = 0.1$$

The stiffness of the joint is

$$k = \bar{k}(1 + \delta_2 \xi_2) \quad \text{with } \bar{k} = 1. 10^2 \text{ N.m}^{-1} . \text{m}^{-2} \text{ and } \delta_2 = 0.2$$

The cylinder is submitted to an internal pressure of 16bars. The screws are prestrained by prescribing a relative axial displacement $\Delta = 0.2\text{mm}$.

The structure is decomposed in 6 substrutures and 20 interfaces. Figure 4 presents the different interfaces (blue: perfect connection, white: symmetry, rose: pressure, turquoise: elastic). The mesh is composed of 1,836 quadratic elements with a total of 7,920 nodes (23,760 dof).

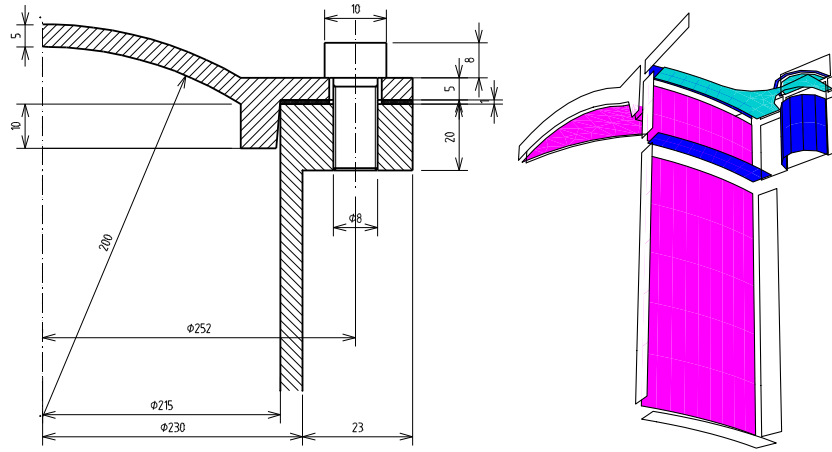


Figure 4: Dimension of the assembly

As the contact conditions between the joint and the cylinder and between the joint and the cover are not considered, the criteria for etancheity is the fact that the joint stays under compression when the internal pressure is applied. Thus in the following study, we investigate whenever the maximal normal force on the joint can become positive.

The computations have been carried out on a PC with a 1.5Gb RAM and an AMD Athlon XP 1800+ processor (1.5GHz).

5.1 Multiresolution

For the multiresolution study, 13 values are taken for the parameter ξ_1 (from -3 to $+3$ by steps of 0.5) and 7 values for the parameter ξ_2 (from -3 to $+3$ by steps of 1). This corresponds to 91 computations that have to be carry out.

Figure 5 presents the evolution of the error indicator during the iterations. Whenever a parameter is changed, the error raise because the interfaces equation are not satisfied anymore. The high pics correspond to the change in parameter ξ_1 (joint's stiffness) and the small ones correspond to the change in parameter ξ_2 (screw's stiffness). One can see that only the first set of parameters leads to a computational effort. The re-utilisation of the previous solution leads to a great reduction of the numerical cost.

Figure 6 presents the repartition of normal force on the joint for the mean value of the coefficients ($\xi_1 = 0$ and $\xi_2 = 0$). One can see that, in that particular case, the joint is in compression. There is no risk of loss of etancheity.

It takes 150s (CPU) to achieve the first computation (350 iterations). Then, it take only 253s (CPU) to achieve the 90 following computations (for a total of 585 iterations).

5.2 Polynomial chaos expansion

For this problem which has two random variables, a polynomial chaos of order 3 is used. This means that $P = 9$, thus 10 functions are used in the expansion. There is

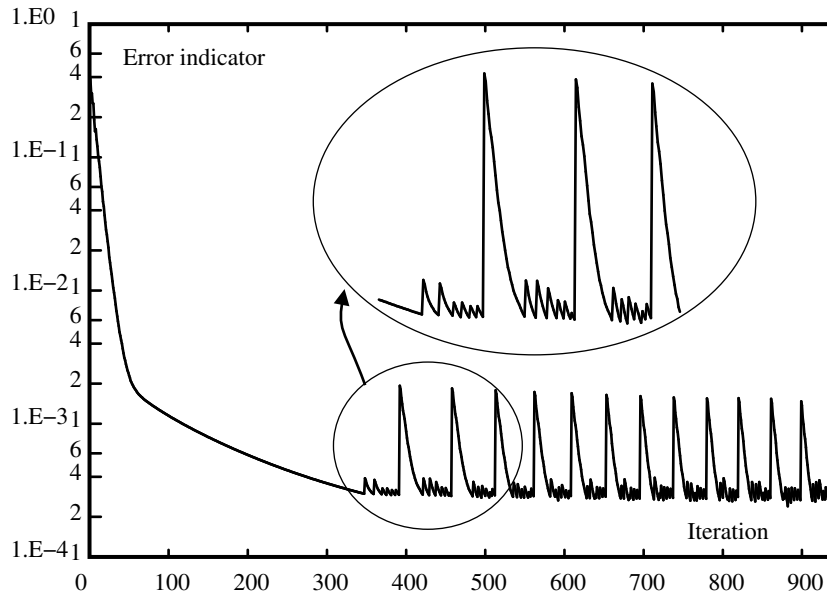


Figure 5: Multiresolution: error indicator during the iterations

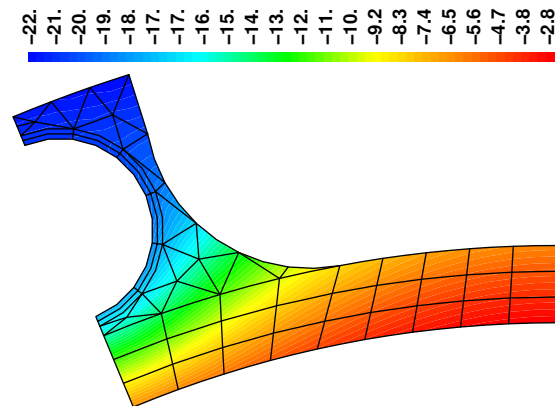


Figure 6: Normal forces on the joint for the mean values of the coefficients

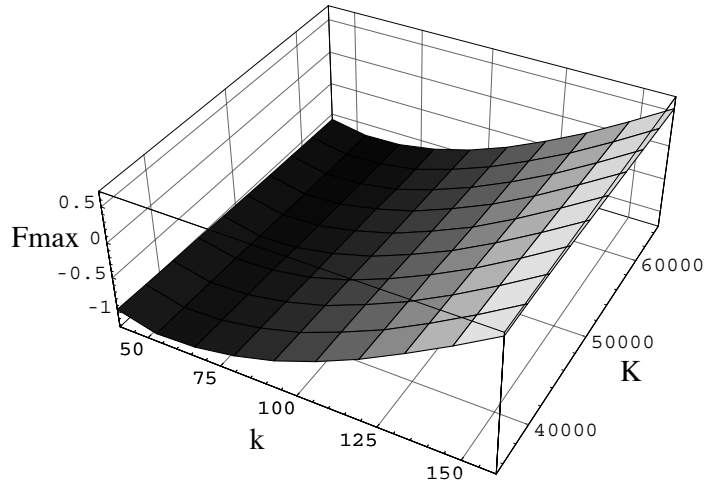


Figure 7: Response surface for the maximum pressure on the joint

237,600 unknowns for this problem.

It takes 1,300s (CPU) to achieve the 310 iteration that are needed to reach convergence.

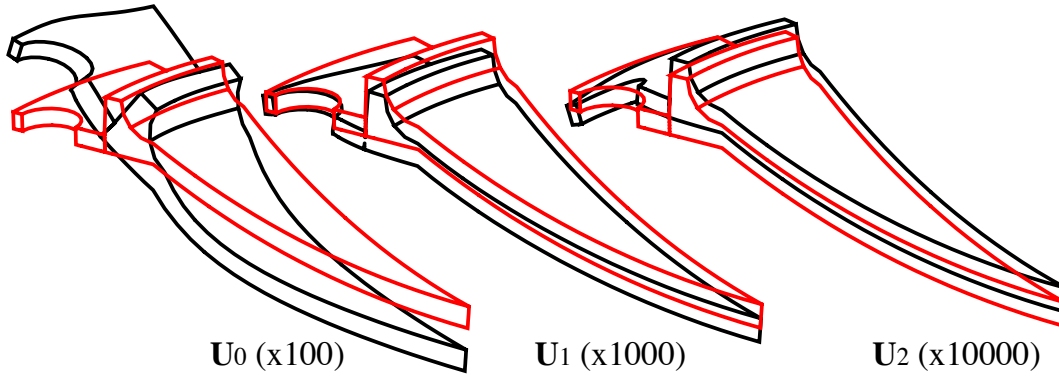


Figure 8: The first three coefficients in the PCE

Figure 8 presents the first three coefficients in the Polynomial Chaos Expansion of order 3. The first coefficient represents the mean displacement of the response \mathbf{u}_0 and the other two (\mathbf{u}_1 and \mathbf{u}_2) the mainly fluctuations around the mean.

Figure 9 presents a comparison between the multiresolution and the Latin Method with Polynomial Chaos : we present the density probability function for the normal force on the joint.

One can see that there is a risk of obtaining traction on the joint in this configuration. Traction would here mean possibility of a loss of etancheity of the cylinder.

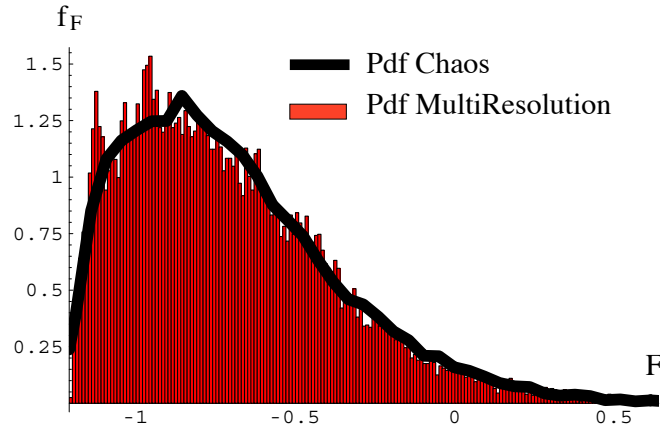


Figure 9: Probability density function of the maximal normal force on the joint

6 CONCLUSIONS

The two proposed approaches based on the LATIN method can be very efficient numerically. The first strategy is based on its capability to reuse the solution to a given problem in order to solve similar problems. Numerical examples showed the very good behavior of the algorithm applied to the case of multiple resolutions in the analysis of 3D assemblies. The solution to the initial problem is a very good starting point for the calculations conducted on other problems provided that these calculations do not exert excessive perturbations on the response. Moreover, the interfaces play a vital role in allowing a considerable reduction of the computation costs.

In the second approach, the uncoupled treatment of the local and global problems leads to a considerable reduction of problemsizes. Another important point is that the linear systems corresponding to the substructures are independent of one another and could be solved in parallel very efficiently. This approach is quite general by nature and should be applicable to a number of other nonlinear problems.

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