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Abstract

We study the asymptotic behavior for asymmetric neuronal dynamics in a network of linear Hopfield neurons. The interaction between the neurons is modeled by random couplings which are centered *i.i.d.* random variables with finite moments of all orders. We prove that if the initial condition of the network is a set of *i.i.d.* random variables and independent of the synaptic weights, each component of the limit system is described as the sum of the corresponding coordinate of the initial condition with a centered Gaussian process whose covariance function can be described in terms of a modified Bessel function. This process is not Markovian. The convergence is in law almost surely with respect to the random weights. Our method is essentially based on the method of moments to obtain a Central Limit Theorem.

1   Introduction

We revisit the problem of characterizing the limit of a network of Hopfield neurons. Hopfield [7] defined a large class of neuronal networks and characterized some of their computational properties [8, 9], i.e. their ability to perform computations. Inspired by his work, Sompolinsky and co-workers studied the thermodynamic limit of these networks when the interaction term is linear [4] using the dynamic mean-field theory developed in [13] for symmetric spin glasses. The method they use is a functional integral formalism used in particle physics and produces the self-consistent mean-field equations of the network. This was later extended to the case of a nonlinear interaction term, the nonlinearity being an odd sigmoidal function [12]. Using the same formalism the authors established the self-consistent mean-field equations of the network and the dynamics of its solutions which featured a chaotic behavior for some values of the network parameters. A little later the problem was picked up again by mathematicians. Ben Arous and Guionnet applied large deviation techniques to study the thermodynamic limit of

a network of spins interacting linearly with *i.i.d.* centered Gaussian weights. The intrinsic spin dynamics (without interactions) is a stochastic differential equation. They prove that the annealed (averaged) law of the empirical measure satisfies a large deviation principle and that the good rate function of this large deviation principle achieves its minimum value at a unique non Markovian measure [5, 2, 6]. They also prove averaged propagation of chaos results. Moynot and Samuelides [10] adapt their work to the case of a network of Hopfield neurons with a nonlinear interaction term, the nonlinearity being a sigmoidal function, and prove similar results in the case of discrete time. The intrinsic neural dynamics is the gradient of a quadratic potential. Our work is in-between that of Ben Arous and Guionnet and Moynot and Samuelides: we consider a network of Hopfield neurons, hence the intrinsic dynamics is simpler than the one in Ben Arous and Guionnet's case, with linear interaction between the neurons, hence simpler than the one in Moynot and Samuelides' work. We do not make the hypothesis that the interaction (synaptic) weights are Gaussian unlike the previous authors. The equations of our network are linear and therefore their solutions can be expressed analytically. As a consequence of this, we are able to use variants of the CLT and the moments method to characterize in a simple way the thermodynamic limit of the network without the tools of the theory of large deviations. Our main result is that the solution to the network equations converges in law toward a non Markovian process, sum of the initial condition and a centered Gaussian process whose covariance is characterized by a modified Bessel function.

Plan of the paper

We introduce the precise model in Section 2. In Section 3, we state and prove our main result (Theorem 3.1) on the asymptotic behavior of the dynamics in the absence of additive white noise. Section 4 is devoted to the general case with additive white noise (Theorem 4.1). Our approach in establishing these results is "syntactic", based on Lemmas 3.3 and 3.8.

2   Network model

We consider a network of N neurons in interaction. Each neuron $i \in \{1, \cdots, N\}$ is characterized by its membrane potential $(V^{i,(N)}(t))_t$ where $t \in \mathbb{R}_+$ represents the time. The membrane potentials evolve according to the system of stochastic differential equations

$$
\begin{cases}
V^{i,(N)}(t) = V_0^i - \lambda \int_0^t V^{i,(N)}(s)\mathrm{d}s + \dfrac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_0^t J_{i,j}^{(N)} V^{j,(N)}(s)\mathrm{d}s + \gamma B^i(t), \quad \forall i \in \{1, \cdots, N\} \\
\mathcal{L}(V_0^{(N)}) = \nu_0^{\otimes N},
\end{cases}
$$

$$(2.1)$$

where $V_0^{(N)} = \left(V_0^1, \cdots, V_0^N\right)$ is the vector of initial conditions. The matrix $J^{(N)}$ is a square matrix of size N and contains the synaptic weights. For $i \neq j$, the coefficient $J_{i,j}^{(N)}/\sqrt{N}$ represents the synaptic weight for pre- synaptic neuron j to post-synaptic neuron i. The coefficient $J_{i,i}^{(N)}/\sqrt{N}$ can be seen as describing the interaction of the neuron i with itself. It turns out that it has no role in defining the mean field limit. The parameters λ and γ are constants. The

$(B^i(t))_t$, $i \in \{1, \cdots, N\}$ are N independent standard Brownian motions modelling the internal noise of each neuron. The initial condition is a random vector with *i.i.d.* coordinates, each of distribution ν_0.

We denote by $V^{(N)}(t)$ the vector $(V^{1,(N)}(t), \cdots, V^{N,(N)}(t))$. Hence, we can write the system (2.1) in matrix form:

$$\begin{cases} V^{(N)}(t) = V_0^{(N)} - \int_0^t \lambda V^{(N)}(s)\mathrm{d}s + \int_0^t \frac{J^{(N)}}{\sqrt{N}} V^{(N)}(s)\mathrm{d}s + \gamma B(t) \\ \mathcal{L}\left(V_0^{(N)}\right) = \nu_0^{\otimes N}. \end{cases} \tag{2.2}$$

System (2.2) can be solved explicitly and its solution $V^{(N)}(t)$ is given by

$$V^{(N)}(t) = e^{-\lambda t}\left[\exp\left(\frac{J^{(N)}}{\sqrt{N}}t\right) V_0^{(N)} + \gamma \int_0^t e^{\lambda s} \exp\left(\frac{J^{(N)}}{\sqrt{N}}(t-s)\right) \mathrm{d}B(s)\right], \quad \forall t \in \mathbb{R}_+. \tag{2.3}$$

For the rest of the paper, we make the following hypotheses on the distributions of $V_0^{(N)}$ and $J^{(N)}$.

(H1) ν_0 is of compact support and we note

$$\mu_0 := \int_{\mathbb{R}} x \, d\nu_0(x) \quad \text{and} \quad \phi_0 = \int_{\mathbb{R}} x^2 \, d\nu_0(x),$$

its first and second order moments.

(H2) The elements of the matrix $J^{(N)}$ are *i.i.d.* centered and bounded random variables of variance σ^2. They are independent of the initial condition.

3 Convergence of the particle system without additive Brownian noise ($\gamma = 0$)

In this section, we consider the model without any additive noise, that is $\gamma = 0$ in (2.1): The unique source of randomness in the dynamics comes from the random matrices $(J^{(N)})$ describing the synaptic weights, and the initial condition.

3.1 Mean field limit

The following result describes the convergence when $N \to +\infty$ of the coordinates of the vector $(V^{(N)}(t))_{t \in \mathbb{R}_+}$ to a Gaussian process whose covariance is determined by a Bessel function. Theorem 3.1 below can be seen as a kind of mean-field description of (2.1) as the number of neurons tends to infinity.

Theorem 3.1. *Under the hypotheses (H1) and (H2), for each $k \in \mathbb{N}^*$, the process $(V^{k,(N)}(t))_{t \in \mathbb{R}_+}$ converges in law to $(V^{k,(\infty)}(t))_{t \in \mathbb{R}_+}$ where,*

$$V^{k,(\infty)}(t) = e^{-\lambda t} \left[V_0^k + Z^k(t) \right], \qquad \forall t \in \mathbb{R}_+.$$

The process $\left(Z^k(t) \right)_{t \in \mathbb{R}_+}$ is a centered Gaussian process starting from 0 ($Z^k(0) = 0$) such that

$$\mathbb{E}\left[Z^k(t) Z^k(s) \right] = \phi_0 \widetilde{I}_0(2\sigma\sqrt{st}), \qquad where \quad \widetilde{I}_0(z) = \sum_{\ell \geq 1} z^{2\ell}/(2^{2\ell}(\ell!)^2). \tag{3.1}$$

Moreover, for all $t \in \mathbb{R}_+$, $Z^k(t)$ is independent of V_0^k.

Remark 3.2. *The function $\widetilde{I}_0$ is closely connected to the modified Bessel function of the first kind I_0, defined as a solution of the ordinary differential equation $z^2 y'' + zy' - z^2 y = 0$, $y' = dy/dz$. This function is the sum of the series $(z^{2\ell}/(2^{2\ell}(\ell!)^2))_{\ell \geq 0}$ which is absolutely convergent for all $z \in \mathbb{C}$, i.e.: $I_0(z) = \sum_{\ell \geq 0}(z/2)^{2\ell}/(\ell!)^2$, so that we have*

$$\widetilde{I}_0(z) = I_0(z) - 1.$$

The proof of Theorem 3.1 requires the following lemma.

Lemma 3.3. *Let J be an infinite matrix such that its finite restrictions $J^{(N)}$ satisfy (H2) and consider a sequence of bounded (not necessarily identically distributed) random variables $(Y_j)_{j \in \mathbb{N}^*}$, independent of J.*

Assume that, almost surely,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} Y_j^2 := \phi < +\infty. \tag{3.2}$$

For all $\ell \in \mathbb{N}^*$ and for all $1 \leq k \leq N$, define

$$U_\ell^{k,(N)} := {}^t e_k \left((J^{(N)})^\ell Y^{(N)} \right),$$

where e_k is the k-th vector of the standard basis of $\mathbb{R}^N$. Then, for all $1 \leq \ell_1 < \cdots < \ell_m$, $m \in \mathbb{N}^*$, the vector

$$\left(\frac{1}{\sqrt{N}^{\ell_1}} U_{\ell_1}^{k,(N)}, \cdots, \frac{1}{\sqrt{N}^{\ell_m}} U_{\ell_m}^{k,(N)} \right)$$

converges in law as $N \to +\infty$, to an m-dimensional Gaussian random vector of diagonal covariance matrix $\operatorname{diag}(\sigma^{2\ell_i}\phi)$ independent of any finite subset of the sequence $(Y_j)_{j \in \mathbb{N}^*}$.

Remark 3.4. *The hypothesis that the Y_js are bounded is not the only possible one. The Lemma is also true for independent Y_js with finite moments of all orders.*

We give one corollary of this Lemma.

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Corollary 3.5. *Under the same assumptions as in Lemma 3.3, for all integers $p > 1$ and $1 \le k_1 < k_2 < \cdots < k_p$ the vector*

$$\left(\frac{1}{\sqrt{N}^{\ell_1}} U_{\ell_1}^{k_1,(N)}, \cdots, \frac{1}{\sqrt{N}^{\ell_m}} U_{\ell_m}^{k_1,(N)}, \right.$$

$$\left. \frac{1}{\sqrt{N}^{\ell_1}} U_{\ell_1}^{k_2,(N)}, \cdots, \frac{1}{\sqrt{N}^{\ell_m}} U_{\ell_m}^{k_2,(N)}, \cdots, \frac{1}{\sqrt{N}^{\ell_1}} U_{\ell_1}^{k_p,(N)}, \cdots, \frac{1}{\sqrt{N}^{\ell_m}} U_{\ell_m}^{k_p,(N)} \right)$$

converges in law as $N \to +\infty$, to an mp-dimensional Gaussian random vector of diagonal covariance matrix $\mathrm{diag}(\sigma^{2\ell_i}\phi)$, $i = 1, \cdots, m$ repeated p times, independent of any finite subset of the sequence $(Y_j)_{j \in \mathbb{N}^*}$.

Proof. It is easy to adapt the proof of Lemma 3.3. $\qquad\square$

Proof of Lemma 3.3.
W.l.o.g, we consider the case $k = 1$ and do not show the index 1 in the proof, i.e. we write $U_{\ell}^{(N)}$ for $U_{\ell}^{1,(N)}$. We first prove by the method of moments that $\frac{1}{N^{\ell/2}} U_{\ell}^{(N)}$ converges in law when $N \to \infty$ toward a centered Gaussian random variable of variance $\sigma^{2\ell}\phi$, i.e. the case $m = 1$ of the Lemma. We then sketch the generalization of the proof to the case $m > 1$.

To do this we expand $U_{\ell}^{(N)}$ and write

$$\frac{1}{N^{n\ell/2}} \left(U_{\ell}^{(N)} \right)^n = \frac{1}{N^{n\ell/2}} \sum_{j^1, \cdots, j^n} J_{1,j_1^1}^{(N)} J_{j_1^1,j_2^1}^{(N)} \cdots J_{j_{\ell-1}^1, j_\ell^1}^{(N)} \cdots J_{1,j_1^n}^{(N)} J_{j_1^n, j_2^n}^{(N)} \cdots J_{j_{\ell-1}^n, j_\ell^n}^{(N)} Y_{j_\ell^1} \cdots Y_{j_\ell^n},$$

where $\sum_{j^1, \cdots, j^n}$ means $\sum_{\substack{j_1^1, \cdots, j_\ell^1 = 1 \\ \vdots \\ j_1^n, \cdots, j_\ell^n = 1}}^{N}$.

We follow and recall the notations of [1]: we denote j^r the sequence (word) of $\ell + 1$ indexes $(1, j_1^r, \cdots, j_\ell^r)$. To each word j^r we associate its length $\ell + 1$, its support $supp(j^r)$, the set of different integers in $\{1, \cdots, N\}$ in j^r, and its weight $wt(j^r)$, the cardinality of $supp(j^r)$. We also associate the graph $G_{j^r} = (V_{j^r}, E_{j^r})$ where $V_{j^r} = supp(j^r)$ has by definition $wt(j^r)$ vertices and the edges are constructed by walking through j^r from left to right. Edges are oriented if they connect two different indexes. In detail we have

$$E_{j^r} = \{(1, j_1^r) \cup (j_i^r, j_{i+1}^r), i = 1, \cdots \ell - 1\}$$

Each edge $e \in E_{j^r}$ has a weight noted $N_e^{j^r}$ which is the number of times it is traversed by the sequence j^r.

We note $\mathbf{j}$ the sentence $(j^1, \cdots, j^n)$ of n words j^r, $r = 1, \cdots, n$, of length $\ell + 1$ and we associate to it the graph $G_{\mathbf{j}} = (V_{\mathbf{j}}, E_{\mathbf{j}})$ obtained by piecing together the n graphs G_{j^r}. In detail, as with words, we define for a sentence $\mathbf{j}$ its support $supp(\mathbf{j}) = \cup_{r=1}^n supp(j^r)$ and its weight $wt(\mathbf{j})$ as the cardinality of $supp(\mathbf{j})$. We then set $V_{\mathbf{j}} = supp(\mathbf{j})$ and $E_{\mathbf{j}}$ the set of edges. Edges are directed if they connect two different vertices and we have

$$E_{\mathbf{j}} = \{(1, j_1^r) \cup (j_i^r, j_{i+1}^r), i = 1, \cdots \ell - 1, r = 1, \cdots, n\}$$

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Two sentences $\mathbf{j}_1$ and $\mathbf{j}_2$ are equivalent, noted $\mathbf{j}_1 \simeq \mathbf{j}_2$ if there exists a bijection on $\{1, \cdots, N\}$ that maps one into the other.

For $e \in E_{\mathbf{j}}$ we note $N_e^{\mathbf{j}}$ the number of times e is traversed by the union of the sequences j^r.

By independence of the elements of the matrix J, the independence of the Js and the Ys, and by construction of the graph $G_{\mathbf{j}}$ we have

$$\mathbb{E}\left[\frac{1}{N^{n\ell/2}}\left(U_\ell^{(N)}\right)^n\right] = \sum_{\mathbf{j}} \frac{1}{N^{n\ell/2}} \prod_{e \in E_{\mathbf{j}}} \mathbb{E}\left[\left(J_{1,1}^{(N)}\right)^{N_e^{\mathbf{j}}}\right] \mathbb{E}[Y_{j_\ell^1} \cdots Y_{j_\ell^n}] := \sum_{\mathbf{j}} T_{\mathbf{j}} \qquad (3.3)$$

In order for $T_{\mathbf{j}}$ to be non zero we need to enforce $N_e^{\mathbf{j}} \geq 2$ for all $e \in E_{\mathbf{j}}$. This implies

$$n\ell = \sum_{e \in E_{\mathbf{j}}} N_e^{\mathbf{j}} \geq 2\,|E_{\mathbf{j}}| \geq 2(wt(\mathbf{j}) - 1),$$

i.e.

$$wt(\mathbf{j}) \leq \lfloor n\ell/2 \rfloor + 1.$$

We now make the following definitions:

Definition 3.6. *Let $\mathcal{W}_{\ell,n,t}$ be the set of representatives of equivalent classes of sentences $\mathbf{j}$, $\mathbf{j} = (j^1, \cdots, j^n)$ of n words of length $\ell + 1$ starting with 1, such that $t = wt(\mathbf{j})$ and $N_e^{\mathbf{j}} \geq 2$ for all $e \in E_{\mathbf{j}}$.*

Definition 3.7. *Let $\mathcal{A}_{\ell,n,t}$ be the set of sentences $\mathbf{j}$, $\mathbf{j} = (j^1, \cdots, j^n)$ of n words of length $\ell + 1$ starting with 1, such that $t = wt(\mathbf{j})$ and $N_e^{\mathbf{j}} \geq 2$ for all $e \in E_{\mathbf{j}}$.*

We now rewrite (3.3)

$$\mathbb{E}\left[\frac{1}{N^{n\ell/2}}\left(U_\ell^{(N)}\right)^n\right] = \frac{1}{N^{n\ell/2}} \sum_{t=1}^{\lfloor n\ell/2 \rfloor+1} \sum_{\mathbf{j} \in \mathcal{A}_{\ell,n,t}} \prod_{e \in E_{\mathbf{j}}} \mathbb{E}\left[\left(J_{1,1}^{(N)}\right)^{N_e^{\mathbf{j}}}\right] \mathbb{E}[Y_{j_\ell^1} \cdots Y_{j_\ell^n}]$$

$$= \sum_{t=1}^{\lfloor n\ell/2 \rfloor+1} \sum_{\mathbf{j}' \in \mathcal{W}_{\ell,n,t}} \sum_{\mathbf{j} \simeq \mathbf{j}'} \frac{1}{N^{n\ell/2}} \prod_{e \in E_{\mathbf{j}}} \mathbb{E}\left[\left(J_{1,1}^{(N)}\right)^{N_e^{\mathbf{j}}}\right] \mathbb{E}[Y_{j_\ell^1} \cdots Y_{j_\ell^n}]. \quad (3.4)$$

Our assumption on Y ensures that

$$\left|\mathbb{E}[Y_{j_\ell^1} \cdots Y_{j_\ell^n}]\right| \leq K$$

for a constant K independent of $\mathbf{j}$ and N. In addition, for $\mathbf{j} \simeq \mathbf{j}'$, we have

$$\prod_{e \in E_{\mathbf{j}}} \mathbb{E}\left[(J_{1,1})^{N_e^{\mathbf{j}}}\right] = \prod_{e \in E_{\mathbf{j}'}} \mathbb{E}\left[(J_{1,1})^{N_e^{\mathbf{j}'}}\right].$$

We deduce that for any $\mathbf{j}' \in \mathcal{W}_{\ell,n,t}$ hence such that $wt(\mathbf{j}') = t$,

$$\left|\sum_{\mathbf{j} \simeq \mathbf{j}'} \frac{1}{N^{n\ell/2}} \prod_{e \in E_{\mathbf{j}}} \mathbb{E}\left[(J_{1,1})^{N_e^{\mathbf{j}}}\right] \mathbb{E}[Y_{j_\ell^1} \cdots Y_{j_\ell^n}]\right| \leq K \frac{C_{N,t}}{N^{n\ell/2}} \left|\prod_{e \in E_{\mathbf{j}'}} \mathbb{E}\left[(J_{1,1})^{N_e^{\mathbf{j}'}}\right]\right|,$$

where
$$C_{N,t} = (N-1)(N-2)\cdots(N-t+1) \simeq N^{t-1} \tag{3.5}$$
is the number of sentences that are equivalent to a given sentence $\mathbf{j}'$, with weight t (remember that the first element of each word is equal to 1). Since the cardinality of $\mathcal{W}_{\ell,n,t}$ is independent of N, we conclude that each term in the right hand side of (3.4) is upper bounded by a constant independent of N times the ratio $C_{N,t}/N^{n\ell/2}$. According to (3.5), this is equivalent to $N^{t-n\ell/2-1}$. Therefore, asymptotically in N, the only relevant term in the right hand side of (3.4) is the one corresponding to $t = \lfloor n\ell/2 \rfloor + 1$.

In order to proceed we use the following Lemma whose proof is postponed.

Lemma 3.8. *If $\mathcal{W}_{\ell,n,t} \neq \emptyset$ and n is odd, then $t \leq \lfloor n\ell/2 \rfloor$.*

We are ready to complete the proof of Lemma 3.3.
If n is odd, Lemma 3.8 shows that $t \leq \lfloor n\ell/2 \rfloor$, so that the maximum value of $C_{N,t}$ in (3.4) is $\mathcal{O}(N^{\lfloor n\ell/2 \rfloor - 1})$ and we have $\lim_{N\to\infty} \frac{1}{N^{n\ell/2}} \mathbb{E}[(U_\ell^{(N)})^n] = 0$. If $n = 2p$ is even, (3.4) commands

$$\mathbb{E}\left[\frac{1}{N^{n\ell/2}} \left(U_\ell^{(N)} \right)^n \right] \simeq \frac{1}{N^{p\ell}} \sum_{\mathbf{j} \in \mathcal{A}_{\ell,2p,p\ell+1}} \prod_{e \in E_\mathbf{j}} \mathbb{E}\left[\left(J_{1,1}^{(N)} \right)^{N_e^\mathbf{j}} \right] \mathbb{E}[Y_{j_\ell^1} \cdots Y_{j_\ell^{2p}}] \tag{3.6}$$

An element of $\mathcal{A}_{\ell,2p,p\ell+1}$ is a set of p pairs of identical words of length $\ell+1$. Each word in a pair contains $\ell + 1$ different symbols and the intersection of their p supports is equal to $\{1\}$. Thus we have $\mathbb{E}\left[(J_{1,1})^{N_e^\mathbf{j}} \right] = \sigma^{2p\ell}$ for all $\mathbf{j} \in \mathcal{A}_{\ell,2p,p\ell+1}$.

We have
$$\mathbb{E}\left[\frac{1}{N^{n\ell/2}} \left(U_\ell^{(N)} \right)^n \right] \simeq \frac{\sigma^{2p\ell}}{N^{p\ell}} \sum_{\mathbf{j} \in \mathcal{A}_{\ell,2p,p\ell+1}} \mathbb{E}[Y_{j_\ell^1} \cdots Y_{j_\ell^{2p}}]$$

There are $(2p-1)!!(= 1.3.\cdots(2p-1))$ ways to group the $2p$ words in pairs. Indeed, given a sequence of $p\ell + 1$ different symbols, we set $q_1 = 1$ and pick the first ℓ symbols in the sequence to obtain j^{q_1}. We then choose one, say j^{r_1}, $r_1 \neq q_1$, among the $n - 1 = 2p - 1$ remaining words to form the first pair of identical words. We next go to the first unpaired word after j^{q_1}, say j^{q_2}, $q_2 \notin \{q_1, r_1\}$ and choose one among the $n - 3 = 2p - 3$ remaining words to form the second pair of identical words (with the next ℓ symbols in the sequence). And this goes on until we reach the end, having exhausted the sequence of $p\ell + 1$ different symbols. It follows that there are $(2p-1)!!$ ways to group the $2p$ indexes in pairs.

For each such grouping and for each p-tuple of different indexes $(j_\ell^1, \cdots, j_\ell^p)$ there are $(N - (p+1))(N - (2p+1)) \cdots (N - ((\ell-1)p+1))$ ways of choosing the remaining $\ell - 1$ p-tuples of

different indexes $(j_k^1, \cdots, j_k^p)$, $k = 1, \cdots, \ell - 1$. Putting all this together we obtain

$$\mathbb{E}\left[\frac{1}{N^{n\ell/2}}\left(U_\ell^{(N)}\right)^n\right] \simeq (2p-1)!!\,\sigma^{2p\ell}\frac{1}{N^p}\sum_{\substack{j^1,\cdots,j^p=2,\text{ all indexes different}}}^{N}\mathbb{E}[Y_{j^1}^2\cdots Y_{j^p}^2]$$

$$\simeq (2p-1)!!\sigma^{2p\ell}\frac{1}{N^p}\sum_{j^1,\cdots,j^p=1}^{N}\mathbb{E}[Y_{j^1}^2\cdots Y_{j^p}^2]$$

$$\simeq (2p-1)!!\,\sigma^{2p\ell}\mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^{N}Y_j^2\right)^p\right].$$

By (3.2) and dominated convergence we obtain

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N^{n\ell/2}}\left(U_\ell^{(N)}\right)^n\right] = \begin{cases} 0 & \text{if } n \text{ odd} \\ \sigma^{2\ell p}(2p-1)!!\phi^p & \text{if } n = 2p. \end{cases}$$

At this point we have proved that $\frac{1}{N^{\ell/2}}U_\ell^{(N)}$ converges in law when $N \to \infty$ to a centered Gaussian of variance $\sigma^{2l}\phi$.

We go on to sketch the proof that for all $m \in \mathbb{N}^*$ and integers $1 \le \ell_1 < \ell_2 < \cdots < \ell_m$, the m-dimensional vector $(\frac{1}{N^{\ell_1/2}}U_{\ell_1}^{(N)}, \frac{1}{N^{\ell_2/2}}U_{\ell_2}^{(N)}, \cdots, \frac{1}{N^{\ell_m/2}}U_{\ell_m}^{(N)})$ converges in law toward an m-dimensional centered Gaussian vector with covariance matrix $\mathrm{diag}(\sigma^{2\ell_k}\phi)$.

The proof is essentially the same as in the case $m = 1$ but the notations become much heavier. The major ingredients are

1. To each $U_{\ell_k}^{(N),}$, $1 \le k \le m$ we associate the graphs $G_{\mathbf{j^k}}$, with $\mathbf{j^k} = (j^{k,1}, \cdots, j^{k,n_k})$ and $j^{k,r} = \{1, j_1^{k,r}, \cdots, j_{\ell_k}^{k,r}\}$, $r = 1, \cdots, n_k$.

2. We piece these graphs together to obtain the graph $G_{\mathbf{j^1},\cdots,\mathbf{j^m}}$. This allows us to write a formula similar to (3.3) for $\mathbb{E}\left[\frac{1}{N^{(n_1\ell_1+\cdots+n_m\ell_m)/2}}\left(U_{\ell_1}^{(N)}\right)^{n_1}\cdots\left(U_{\ell_m}^{(N)}\right)^{n_m}\right]$.

3. We enforce the condition that edges in $G_{\mathbf{j^1},\cdots,\mathbf{j^m}}$ must have a weight larger than or equal to 2.

4. We generalize the definition 3.6 to the set noted $\mathcal{W}_{\ell_1,n_1,\cdots,\ell_m,n_m,t}$ and prove an analog to Lemma 3.8.

5. These last two steps allow us to write a formula analog to (3.4).

6. Lemma 3.8 can be easily generalized to the following

    Lemma 3.9. *If $\mathcal{W}_{\ell_1,n_1,\cdots,\ell_m,n_m,t} \ne \emptyset$ and if any of $n_1, \cdots, n_m$ is odd, then $t \le \lfloor(n_1\ell_1 + \cdots + n_m\ell_m)/2\rfloor$.*

7. Combining all this yields the result

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N^{(n_1\ell_1+\cdots+n_m\ell_m)/2}}\left(U_{\ell_1}^{(N)}\right)^{n_1}\cdots\left(U_{\ell_m}^{(N)}\right)^{n_m}\right] =$$

$$\begin{cases} 0 & \text{if any of } n_1,\cdots,n_m \text{ is odd} \\ \prod_{k=1}^m \sigma^{2\ell_k p_k}(2p_k-1)!!\phi^{p_k} & \text{if } n_1 = 2p_1,\cdots,n_m = 2p_m \end{cases}$$

which shows that the m-dimensional vector $\left(\frac{1}{N^{\ell_1/2}}U_{\ell_1}^{(N)}, \frac{1}{N^{\ell_2/2}}U_{\ell_2}^{(N)}, \cdots, \frac{1}{N^{\ell_m/2}}U_{\ell_m}^{(N)}\right)$ converges in law toward an m-dimensional centered Gaussian vector with covariance matrix $\text{diag}(\sigma^{2\ell_k}\phi)$.

This ends the proof of Lemma 3.3. It is central to our approach and allows us to establish Theorems 3.1 and 4.1 in a "syntactic" manner by connecting the stochastic properties of the matrices $J^{(N)}$ and the structure of the sequences of indexes that appear when raising them to integer powers.

Note that the proof also shows that this Gaussian vector is independent of any finite subset of the sequence $(Y_j)_{j\in\mathbb{N}^*}$. Indeed, given k distinct integers $j_1,\cdots,j_k$, if we eliminate from the previous construction all words ending with any of these integers, the limits will not change and will be, by construction, independent of $Y_{j_1},\cdots,Y_{j_k}$. $\qquad\square$

Proof of Lemma 3.8. For any $r \in 1,\cdots,n$, consider the number of times the letter j_1^r appears at this position in the set of n words

$$c_r := \sum_{s=1}^n \mathbb{1}_{\{j_1^s = j_1^r\}}.$$

Since the number n of words is odd, at least one of the c_r is also odd. Indeed, assume $c_1 = 2k_1$ is even. Consider the $n - 2k_1$ indexes which are not equal to j_1^1 and assume w.l.o.g. that one of them is $j_1^{2k_1+1}$. If c_{2k_1+1} is odd, we are done, else let $c_{2k_1+1} = 2k_2$ and consider the $n - 2k_1 - 2k_2$ indexes which are not equal to $j_1^{2k_1+1}$ and to j_1^1. This process terminates in a finite number of steps and since n is odd we must necessarily either find an odd c_r or end up with a single remaining element corresponding also to an odd c_r.

Assume (w.l.o.g,) c_1 is odd.

Case 1 If $c_1 = 1$, there must be another oriented edge $(1, j_1^1)$ in the sentence and it cannot be the first edge of any of the remaining $n-1$ words. It means that at least one of the j_k^s is equal to 1. Hence there is an oriented edge $(j, 1)$, and it has to appear at least twice so that there are at least two j_k^s equal to 1. The total number of letters is $n(\ell+1)$ and we have shown that the letter 1 appeared at least $n + 2$ times.

Now, since

$$supp(\mathbf{j}) = \{1\} \cup (supp(\mathbf{j})\setminus\{1\}),$$

and every letter in $supp(\mathbf{j})\setminus\{1\}$ has to appear at least twice, we conclude that

$$2|supp(\mathbf{j})\setminus\{1\}| \leq n\ell - 2,$$

and therefore $wt(\mathbf{j}) - 1 \leq (n\ell - 2)/2$.

Case 2 If $c_1 = 3$, w.l.o.g, we assume that $j_1^1 = j_1^2 = j_1^3$

2a If $j_2^1 = j_2^2 = j_2^3$, then we have

$$2|supp(\mathbf{j})\backslash\{1, j_1^1, j_2^1\}| \leq n(\ell + 1) - \underbrace{n}_{1} - \underbrace{3}_{j_1^1} - \underbrace{3}_{j_2^1},$$

and hence $wt(\mathbf{j}) - 3 \leq (n\ell - 6)/2$.

2b Otherwise, w.l.o.g., $j_2^1 \neq j_2^2$ and $j_2^1 \neq j_2^3$. The edge (j_1^1, j_2^1) must appear twice and hence the letter j_1^1 appears at least 4 times, i.e.

$$2(wt(\mathbf{j}) - 2) \leq n\ell - 4$$

Case 3 If $c_1 \geq 5$, then $2(wt(\mathbf{j}) - 2) \leq n\ell - 5$.

This ends the proof of Lemma 3.8. It is a good example of the use of our "syntactic" approach: it provides an upper bound on the number of different symbols in a sentence of n words of indexes when n is odd which is key in establishing the convergence properties of the powers of the matrices $J^{(N)}$ when $N \to \infty$. $\qquad\square$

Proof of Theorem 3.1. Without loss of generality we assume $k = 1$. We first expand the exponential of the matrix $J^{(N)}$ in (2.3) (with $\gamma = 0$) and express the first coordinate of $V^{(N)}(t)$ as

$$V^{1,(N)}(t) = e^{-\lambda t} \left(V_0^1 + \sum_{\ell \geq 1} \frac{t^\ell}{\ell!} \frac{1}{\sqrt{N}^\ell} \sum_{j_1, \cdots, j_\ell = 1}^{N} J_{1,j_1}^{(N)} J_{j_1,j_2}^{(N)} \cdots J_{j_{\ell-1}, j_\ell}^{(N)} V_0^{j_\ell} \right). \tag{3.7}$$

The main idea of the proof is to truncate the infinite sum in the right hand side of (3.7) to order n, establish the limit of the truncated term when $N \to \infty$, obtain the limit of the result when $n \to \infty$, and show that it is the limit of the non truncated term when $N \to \infty$.

Let $n \in \mathbb{N}^*$ and define the partial sum $Z_n^{(N)}(t)$ of order n of (3.7)

$$Z_n^{(N)}(t) = \sum_{\ell=1}^{n} \frac{t^\ell}{\ell!} \frac{1}{\sqrt{N}^\ell} U_\ell^{(N)} \qquad \text{with} \quad U_\ell^{(N)} = {}^{t}e_1 \left(J^{(N)} \right)^\ell V_0. \tag{3.8}$$

Lemma 3.3 with $Y = V_0$ dictates the convergence in law, for all $n \in \mathbb{N}^*$, of the vector

$$\left(\frac{1}{\sqrt{N}} U_1^{(N)}, \cdots, \frac{1}{\sqrt{N}^n} U_n^{(N)} \right)_{N \in \mathbb{N}^*}$$

to an n-dimensional centered Gaussian random vector, independent of V_0^1, with a diagonal covariance matrix Σ, such that $\forall 1 \leq i \leq n$, $\Sigma_{i,i} = \sigma^{2i}\phi_0$. It follows that we have the independence between the limits in law of each $\frac{1}{\sqrt{N}^\ell} U_\ell^{(N)}$, $1 \leq \ell \leq n$ and also the convergence in law of

the sum $Z_n^{(N)}(t)$ to a centered Gaussian random variable $Z_n(t)$, independent of V_0^1, of variance $\phi_0 \sum_{\ell=1}^n \frac{(\sigma t)^{2\ell}}{(\ell!)^2} =: \widetilde{I}_{0,n}(2\sigma t)$.

Moreover, since the function $\widetilde{I}_{0,n}$ converges pointwise to $\widetilde{I}_0$ as n goes to infinity, the Kolmogorov–Khinchin Theorem (see e.g. [11, Th. 1 p.6]) gives the convergence of $Z_n(t)$.

$$Z_n(t) \xrightarrow[n \to +\infty]{\mathcal{L}} Z^1(t) \sim \mathcal{N}(0, \phi_0 \widetilde{I}_0(2\sigma t))$$

It is clear that $Z_n^{(N)}(t) \xrightarrow[n \to +\infty]{\mathcal{L}} Z^{(N)}(t)$ where $Z^{(N)}(t) = \sum_{\ell \geq 1} \frac{t^\ell}{\ell!} U_\ell^{(N)}$, so that we have

$$
\begin{array}{ccc}
Z_n^{(N)}(t) & \xrightarrow{\mathcal{L}}{N \to \infty} & Z_n(t) \\
\mathcal{L} \downarrow n \to \infty & & \mathcal{L} \downarrow n \to \infty \\
Z^{(N)}(t) & & Z^1(t)
\end{array}
$$

It remains to show that this diagram is commutative i.e. that $Z^{(N)}(t) \xrightarrow[N \to +\infty]{\mathcal{L}} Z^1(t)$.

According to [3, Th. 25.5], to obtain the convergence in law of $Z^{(N)}(t)$ to $Z^1(t)$ as $N \to +\infty$, it is sufficient to show that for all $t \in \mathbb{R}^*$

$$\lim_{n \to +\infty} \limsup_{N \to +\infty} \mathbb{P}\left[|(Z_n^{(N)}(t)) - Z^{(N)}(t)| \geq \varepsilon \right] = 0, \qquad \forall \varepsilon > 0. \tag{3.9}$$

By Markov inequality, we have

$$\mathbb{P}\left[|Z_n^{(N)}(t) - Z^{(N)}(t)| \geq \varepsilon \right] \leq \frac{1}{\varepsilon} \mathbb{E}\left[\left| Z_n^{(N)}(t) - Z^{(N)}(t) \right| \right], \tag{3.10}$$

and

$$\mathbb{E}\left[|Z_n^{(N)}(t) - Z^{(N)}(t)| \right] = \mathbb{E}\left[\left| \sum_{\ell \geq n+1} \frac{t^\ell}{\ell!} U_\ell^{(N)} \right| \right]$$
$$\leq \sum_{\ell \geq n+1} \frac{t^\ell}{\ell!} \mathbb{E}\left[\left| U_\ell^{(N)} \right| \right].$$

Moreover, by step 1, $\left(U_\ell^{(N)} \right)_N$ converges in law as $N \to \infty$ to a centered Gaussian random variable U_ℓ of variance $\phi_0 \sigma^{2\ell}$. Then the law of $|U_\ell|$ is a half normal distribution, and hence

$$\mathbb{E}[|U_\ell|] = \frac{\sqrt{2\phi_0}}{\sqrt{\pi}} \sigma^\ell.$$

Then,

$$\limsup_{N \to +\infty} \mathbb{E}\left[|Z_n^{(N)}(t) - Z^{(N)}(t)| \right] \leq \frac{\sqrt{2\phi_0}}{\sqrt{\pi}} \sum_{\ell \geq n+1} \frac{t^\ell}{\ell!} \sigma^\ell.$$

The right hand side goes to zero as $n \to \infty$ and (3.9) follows, hence we have obtained the convergence in law of $V^{1,(N)}(t)$ to $e^{-\lambda t}\left[V_0^1 + Z^1(t)\right]$, $Z^1(t)$ being independent of V_0^1.

In order to prove that the process $(Z^1(t))_{t \in \mathbb{R}_+}$ is Gaussian we show that $aZ^1(t) + bZ^1(s)$ is Gaussian for all reals a and b and all $s, t \in \mathbb{R}_+$. But this is clear from (3.8) which shows that

$$aZ_n^{(N)}(t) + bZ_n^{(N)}(s) = \sum_{\ell=1}^{n} \frac{at^\ell + bs^\ell}{\ell!} \frac{1}{\sqrt{N}^\ell} U_\ell^{(N)},$$

and the previous proof commands that $aZ^{(N)}(t) + bZ^{(N)}(s)$ converges in law when $N \to \infty$ toward a centered Gaussian random variable of variance $\phi_0(a^2 \widetilde{I}_0(2\sigma t) + 2ab \widetilde{I}_0(2\sigma\sqrt{ts}) + b^2 \widetilde{I}_0(2\sigma s))$. $\quad\square$

We now make a few remarks concerning the properties of the mean field limit.

Remark 3.10 (Decomposition as an infinite sum of independent standard Gaussian variables)**.** *A consequence of the proof is that $Z^1(t)$ is equal in law to the sum of the following series*

$$Z^k(t) \overset{\mathcal{L}}{=} \sqrt{\phi_0} \sum_{\ell=1}^{\infty} \frac{t^\ell \sigma^\ell}{\ell!} G_\ell^k, \qquad \forall t \in \mathbb{R}_+. \tag{3.11}$$

where $(G_\ell^k)_{\ell \geq 1, k \geq 1}$ are independent standard Gaussian random variables, independent of the initial condition V_0^k.

This decomposition yields the covariance of $V^{k,(\infty)}$:

$$\mathrm{Cov}\left[V^{k,(\infty)}(t), V^{k,(\infty)}(s)\right] = \phi_0 e^{-\lambda(t+s)} I_0(2\sigma\sqrt{ts}) - e^{-\lambda(t+s)}\mathbb{E}(V_0)^2 \tag{3.12}$$

It also shows that the only sources of randomness in the solution to (2.1) (when $\gamma = 0$) are the initial condition and the family $(G_\ell^k)_{\ell \geq 1, k \geq 1}$ which does not depend on time. The limiting process is thus an $\mathcal{F}_{0^+}$-measurable process.

Remark 3.11 (Non independent increments)**.** *It follows from the above that the increments of the process $\left(V^{k,(\infty)}(t)\right)_{t \in \mathbb{R}_+}$ are not independent since for all $0 \leq t_1 < t_2 \leq t_3 < t_4$,*

$$\mathrm{Cov}\left[V^{k,(\infty)}(t_2) - V^{k,(\infty)}(t_1), V^{k,(\infty)}(t_4) - V^{k,(\infty)}(t_3)\right] \neq 0.$$

Remark 3.12. *Using Theorem 3.1 and (3.11) we obtain the SDE satisfied by the process $(V^{k,(\infty)}(t))_{t \in \mathbb{R}_+}$:*

$$\begin{cases} \mathrm{d}V^{k,(\infty)}(t) = -\lambda V^{k,(\infty)}(t)\mathrm{d}t + H^k(t)\mathrm{d}t \\ \mathcal{L}(V_0^{k,(\infty)}) = \nu_0. \end{cases} \tag{3.13}$$

where $\left(H^k(t)\right)_{t \in \mathbb{R}_+}$ is the centered Gaussian process

$$\forall t \geq 0, \quad H^k(t) = \sqrt{\phi_0}\, \sigma \sum_{\ell \geq 0} \frac{(\sigma t)^\ell}{\ell!} G_{\ell+1}^k.$$

The standard Gaussian random variables G_ℓ^k have been introduced in (3.11). It is easily verified that

$$\mathbb{E}\left[H^k(t)H^k(s)\right] = \phi_0 \sigma^2 I_0\left(2\sigma\sqrt{ts}\right) \left(= \phi_0 \sigma^2(1 + \widetilde{I}_0(2\sigma\sqrt{ts}))\right).$$

Remark 3.13 (Long time behavior). *The function $\widetilde{I}_0$ behaves as an $\mathcal{O}\left(e^z/\sqrt{2\pi z}\right)$ as $z \to +\infty$. As a consequence of (3.12) $\sigma = \lambda$ is a critical value for the solution: if $\sigma > \lambda$ the solution V^1 blows up when $t \to \infty$ while if $\sigma < \lambda$ it converges to its mean.*

Remark 3.14 (Non Markov property). *The hypothesis of independence between J and V_0 is crucial in the proof of Theorem 3.1. Therefore the proof is not valid if we start the system at a time $t_1 > 0$, in other words, we cannot establish the existence of a process $\left(\bar{Z}^k(t)\right)_{t\in\mathbb{R}_+}$ with the same law as $\left(Z^k(t)\right)_{t\in\mathbb{R}_+}$, independent of $V^{k,(\infty)}(t_1)$, and such that*

$$V^{k,(\infty)}(t_1 + t) = e^{-\lambda t}\left(V^{k,(\infty)}(t_1) + \bar{Z}^k(t)\right). \tag{3.14}$$

3.2 Propagation of chaos

An important consequence of Lemma 3.3 and its Corollary 3.5 is that the propagation of chaos property is satisfied by the mean field limit.

Theorem 3.15. *For any finite number of labels $1 \le k_1 < k_2 < \cdots < k_p$, and for all $t \in \mathbb{R}_+$, the random processes $V^{k_1,(\infty)}(t)$, $V^{k_2,(\infty)}(t)$, $\cdots$, $V^{k_p,(\infty)}(t)$ are **independent** and identically distributed.*

Proof. The proof follows directly from Corollary 3.5 and the proof of Theorem 3.1. $\qquad\square$

4 Convergence of the particle system with an additive Brownian noise $(\gamma \neq 0)$

In this section, we consider the general equation (2.2) with $\gamma \neq 0$.

4.1 Mean field limit

Here, the randomness is not entirely determined at time $t = 0$, so we expect that the $\mathcal{F}_{0^+}$-measurable property of the limit which is satisfied by the mean field limit of Section 3 is no longer true. Considering $\gamma \neq 0$, the explicit solution of (2.2) is

$$V^{(N)}(t) = e^{-\lambda t}\exp\left(\frac{J^{(N)}t}{\sqrt{N}}\right)\left[V_0^{(N)} + \gamma \int_0^t e^{\lambda s}\exp\left(-\frac{J^{(N)}s}{\sqrt{N}}\right)\mathrm{d}B(s)\right]. \tag{4.1}$$

Theorem 4.1. *Under hypotheses* (H1) *and* (H2) *on the matrix J and the initial random vector $V_0^{(N)}$, for each $k \in \mathbb{N}^*$, $V^{k,(N)}(t)$ converges in law as $N \to +\infty$ to*

$$V^{k,(\infty)}(t) := e^{-\lambda t}\left[V_0^k + \gamma \int_0^t e^{\lambda s}\mathrm{d}B^k(s) + Z^k(t) + \gamma A^k(t)\right]. \tag{4.2}$$

The process $(Z^k(t))_{t\in\mathbb{R}_+}$ is the same centered Gaussian process as in Theorem 3.1. It is independent of the initial condition V_0^k, of the Brownian motion $(B^k(t))_{t\in\mathbb{R}_+}$ and of the process

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$A^k(t)$. *The process $(A^k(t))_{t\in\mathbb{R}_+}$ is a centered Gaussian process, also independent of V_0^k and of $(B^k(t))_{t\in\mathbb{R}_+}$. Its covariance writes*

$$\mathbb{E}\left[A^1(t)A^1(s)\right] = \int_0^s e^{2\lambda u}\widetilde{I}_0(2\sigma\sqrt{(t-u)(s-u)})\,du \quad 0 \le s \le t$$

Proof. Without loss of generality we assume $k = 1$.

The proof is in two parts. We first assume $\nu_0 = \delta_0$ and show that, starting from 0 (which implies that only the Brownian part of (2.3) acts) the process converges in law to $\gamma\int_0^t e^{-\lambda(t-s)}\mathrm{d}B^1(s) + \gamma e^{-\lambda t}A^1(t)$. In the second part we remove the condition $\nu_0 = \delta_0$.

Part 1: Let $\nu_0 = \delta_0$, then the explicit solution of (2.2) is given by

$$V^{(N)}(t) = \gamma \int_0^t e^{-\lambda(t-s)} \exp\left(\frac{J^{(N)}}{\sqrt{N}}(t-s)\right)\mathrm{d}B(s). \tag{4.3}$$

As before, we expand the exponential of J and obtain

$$V^{(N)}(t) = \gamma e^{-\lambda t}\left[\int_0^t e^{\lambda s}\mathrm{d}B(s) + \sum_{\ell\ge1}\int_0^t e^{\lambda s}\left(\frac{(J^{(N)})^\ell}{\sqrt{N}^\ell\,\ell!}(t-s)^\ell\right)\mathrm{d}B(s)\right]. \tag{4.4}$$

For all $\ell \ge 1$, we introduce the C^1 function Λ_ℓ defined by

$$\Lambda_\ell(t,s) = \frac{1}{\ell!}e^{\lambda s}(t-s)^\ell, \tag{4.5}$$

and write

$$V^{(N)}(t) = \gamma e^{-\lambda t}\left[\int_0^t e^{\lambda s}\mathrm{d}B(s) + \sum_{\ell\ge1}\int_0^t \frac{(J^{(N)})^\ell}{\sqrt{N}^\ell}\Lambda_\ell(t,s)\mathrm{d}B(s)\right].$$

We focus on the first coordinate of $V^{(N)}(t)$ and we introduce, for all $\ell \ge 1$, the following notations.

$$U_\ell^{(N)}(t) := {}^t e_1 \left(J^{(N)}\right)^\ell \int_0^t \Lambda_\ell(t,s)dB(s). \tag{4.6}$$

Then, we have

$$V^{1,(N)}(t) = \gamma e^{-\lambda t}\left[\int_0^t e^{\lambda s}\mathrm{d}B^1(s) + \sum_{\ell\ge1}\frac{1}{\sqrt{N}^\ell}U_\ell^{(N)}(t)\right].$$

We next define

$$A_n^{(N)}(t) := \sum_{\ell=1}^n \frac{1}{\sqrt{N}^\ell}U_\ell^{(N)}(t)$$

For each $\ell \ge 1$ and $t \in \mathbb{R}_+$ the sequence $(Y_j^\ell(t))_{j\in\mathbb{N}^*}$, $Y_j^\ell(t) = \int_0^t \Lambda_\ell(t,s)dB^j(s)$, satisfies (3.2). Indeed, by the law of large numbers

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^N (Y_j^\ell(t))^2 = \mathbb{E}\left[\left(\int_0^t \Lambda_\ell(t,s)\,dB^1(s)\right)^2\right] = \int_0^t (\Lambda_\ell(t,s))^2 ds := \phi_\ell(t) \tag{4.7}$$

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Moreover, for each $\ell \geq 1$, for each $t \geq 0$, the $Y_j^\ell(t)$ are independent centered Gaussian variables.

It follows from Remark 3.4 and a slight modification of the proof of Lemma 3.3 that the n-dimensional vector $\left(\frac{1}{\sqrt{N}} U_1^{(N)}(t) \cdots \frac{1}{\sqrt{N^n}} U_n^{(N)}(t) \right)$ converges in law to an n-dimensional centered Gaussian process with diagonal covariance matrix $\mathrm{diag}(\sigma^{2\ell}\phi_\ell(t))$, $\ell = 1, \cdots, n$. This process is independent of any finite subset of the Brownians B^j, hence of $\int_0^t e^{\lambda s} \mathrm{d}B^1(s)$. It follows that $A_n^{(N)}(t)$ converges in law toward a centered Gaussian process $A_n(t)$ of variance $\sum_{\ell=1}^n \sigma^{2\ell}\phi_\ell(t)$.

Because $\sum_{\ell=1}^n \sigma^{2\ell}\phi_\ell(t)$ converges to $\sum_{\ell \geq 1} \sigma^{2\ell}\phi_\ell(t)$ and $A_n(t)$ is a centered Gaussian process, the Kolmogorov–Khinchin Theorem commands that $A_n(t)$ converges in law when $n \to \infty$ to a centered Gaussian process $A^1(t)$ independent of the Brownian B^1, with covariance $\Lambda^2(t) := \sum_{\ell \geq 1} \sigma^{2\ell}\phi_\ell(t)$. By (4.5) and (3.1) we have

$$\Lambda^2(t) = \int_0^t e^{2\lambda u} \widetilde{I}_0(2\sigma(t-u))\, du,$$

It remains to prove that $A^{(N)}(t)$ converges in law to $A^1(t)$. According to [3, Th. 25.5], to obtain the weak convergence of $A^{(N)}(t)$ to $A(t)$ as $N \to +\infty$, it is sufficient to show that for all $t \in \mathbb{R}_+$

$$\lim_{n \to +\infty} \limsup_{N \to +\infty} \mathbb{P}\left[\mid A_n^{(N)}(t) - A^{(N)}(t) \mid \geq \varepsilon \right] = 0, \qquad \forall \varepsilon > 0. \tag{4.8}$$

By Markov inequality, we have

$$\mathbb{P}\left[\mid A_n^{(N)}(t) - A^{(N)}(t) \mid \geq \varepsilon \right] \leq \frac{1}{\varepsilon} \mathbb{E}\left[\left| A_n^{(N)}(t) - A^{(N)}(t) \right| \right],$$

and

$$\mathbb{E}\left[\mid A_n^{(N)}(t) - A^{(N)}(t) \mid \right] \leq \sum_{\ell \geq n+1} \mathbb{E}\left[\left| \frac{1}{\sqrt{N^l}} U_\ell^{(N)}(t) \right| \right].$$

We know from the beginning of the proof that for all $t \in \mathbb{R}_+$ and for all $\ell \in \mathbb{N}^*$, ,

$$\lim_{N \to +\infty} \frac{1}{\sqrt{N^l}} U_\ell^{(N)}(t) := U_\ell(t) \overset{\mathcal{L}}{=} \mathcal{N}(0, \sigma^{2\ell}\phi_\ell(t))$$

Then the law of $|U_\ell(t)|$ for all $t \in \mathbb{R}_+$ is a half normal distribution, and

$$\mathbb{E}\left[|U_\ell(t)|\right] = \sqrt{\frac{2}{\pi}} \sigma^\ell \left(\phi_\ell(t)\right)^{1/2}$$

Then,

$$\limsup_{N \to +\infty} \mathbb{E}\left[\left| A_n^{(N)}(t) - A^{(N)}(t) \right| \right] \leq \frac{1}{\varepsilon} \sqrt{\frac{2}{\pi}} \sum_{\ell > n} \sigma^\ell \phi_\ell(t)$$

The right hand side of this inequality goes to zero when $n \to \infty$ and (4.8) follows. This concludes the first part of the proof.

In order to prove that the process $(A^1(t))_{t \in \mathbb{R}_+}$ is Gaussian we proceed exactly as in the proof of Theorem 3.1 and show that $aA^1(t) + bA^1(s)$ is Gaussian for all reals a and b and all $s < t \in \mathbb{R}_+$.

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Indeed, the previous proof commands that $aA^{(N)}(t) + bA^{(N)}(s)$ converges in law when $N \to \infty$ toward a centered Gaussian random variable of variance $a^2\Lambda^2(t) + 2ab\Lambda(t,s) + b^2\Lambda^2(s)$, where

$$\Lambda(t,s) = \int_0^s e^{2\lambda u}\widetilde{I}_0(2\sigma\sqrt{(t-u)(s-u)})\,du \quad 0 \leq s \leq t$$

Part 2:

We remove the assumption $\nu_0 = \delta_0$. A slight modification of the proof of Lemma 3.3 shows that the $2n$-dimensional vector

$$\left(\frac{1}{\sqrt{N}}U_1^{(N)}, \cdots, \frac{1}{\sqrt{N}^n}U_n^{(N)}, \frac{1}{\sqrt{N}}U_1^{(N)}(t), \cdots, \frac{1}{\sqrt{N}^n}U_n^{(N)}(t)\right),$$

where $U_\ell^{(N)}$ is defined by (3.8) and $U_\ell^{(N)}(t)$ by (4.6), converges in law when $N \to \infty$ to the $2n$-dimensional centered Gaussian vector with covariance $\mathrm{diag}(\sigma^2\phi_0, \cdots, \sigma^{2n}\phi_0, \sigma^2\phi_1, \cdots, \sigma^{2n}\phi_n)$, where ϕ_ℓ, $\ell \geq 1$ is defined by (4.7).

We conclude that $Z_n^{(N)}(t) + A_n^{(N)}(t)$ converges in law when $N \to \infty$ to $Z_n(t) + A_n(t)$, and that $Z_n(t)$ and $A_n(t)$ are independent. The convergence of $Z_n(t) + A_n(t)$ to $Z^1(t) + A^1(t)$ follows again from the Kolmogorov–Khinchin Theorem. $\qquad\square$

Remark 4.2. *Note that the process $A^1(t)$ does not have independent increments.*

4.2 Propagation of chaos

As in the case without noise, propagation of chaos occurs.

Theorem 4.3. *For any finite number of labels $k_1 < k_2 < \cdots < k_p$, and for all $t \in \mathbb{R}_+$, the random processes $V^{k_1,(\infty)}(t)$, $V^{k_2,(\infty)}(t)$, $\cdots$, $V^{k_p,(\infty)}(t)$ are **independent** and identically distributed.*

Proof. The proof follows directly from the following extension of Corollary 3.5 and the proof of Theorem 3.1.

Corollary 4.4. *Under the same assumptions as in Lemma 3.3 (see also Remark 3.4), for all integers $p > 1$ and $1 \leq k_1 < k_2 < \cdots < k_p$ the $2mp$-dimensional vector obtained by concatenating the two mp-dimensional vectors*

$$\left(\frac{1}{\sqrt{N}^{\ell_1}}U_{\ell_1}^{k_1,(N)}, \cdots, \frac{1}{\sqrt{N}^{\ell_m}}U_{\ell_m}^{k_1,(N)},\right.$$

$$\left.\frac{1}{\sqrt{N}^{\ell_1}}U_{\ell_1}^{k_2,(N)}, \cdots, \frac{1}{\sqrt{N}^{\ell_m}}U_{\ell_m}^{k_2,(N)}, \cdots, \frac{1}{\sqrt{N}^{\ell_1}}U_{\ell_1}^{k_p,(N)}, \cdots, \frac{1}{\sqrt{N}^{\ell_m}}U_{\ell_m}^{k_p,(N)}\right),$$

where $U_{\ell_j}^{k_i,(N)} = {}^t e_{k_i}J^{\ell_j}V_0$, $i = 1, \cdots, p$, $j = 1, \cdots, m$ and

$$\left(\frac{1}{\sqrt{N}^{\ell_1}}U_{\ell_1}^{k_1,(N)}(t), \cdots, \frac{1}{\sqrt{N}^{\ell_m}}U_{\ell_m}^{k_1,(N)}(t),\right.$$

$$\left.\frac{1}{\sqrt{N}^{\ell_1}}U_{\ell_1}^{k_2,(N)}(t), \cdots, \frac{1}{\sqrt{N}^{\ell_m}}U_{\ell_m}^{k_2,(N)}(t), \cdots, \frac{1}{\sqrt{N}^{\ell_1}}U_{\ell_1}^{k_p,(N)}(t), \cdots, \frac{1}{\sqrt{N}^{\ell_m}}U_{\ell_m}^{k_p,(N)}(t)\right),$$

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where $U_{\ell_j}^{k_i,(N)}(t) := {}^t e_{k_i} J^{\ell_j} \int_0^t \Lambda_{\ell_j}(t,s) dB(s)$, $i = 1, \cdots, p$, $j = 1, \cdots, m$, *converges in law as* $N \to +\infty$, *to an 2mp-dimensional Gaussian random vector of diagonal covariance matrix* $\mathrm{diag}(\sigma^{2\ell_i}\phi_0)$, $i = 1, \cdots, m$ *repeated p times, and* $\mathrm{diag}(\sigma^{2\ell_i}\phi_{\ell_i}(t))$, $i = 1, \cdots, m$ *repeated p times.*

Proof. Follows from the one of Lemma 3.3. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

$\hfill\square$

Acknowledgements

References

[1] G. Anderson, A. Guionnet and O. Zeitouni. An Introduction to Random Matrices (Cambridge Studies in Advanced Mathematics). Cambridge University Press `doi:10.1017/CBO9780511801334`, 2009.

[2] G. Ben-Arous and A. Guionnet. Large deviations for Langevin spin glass dynamics. *Probability Theory and Related Fields*, 102(4):455–509, 1995.

[3] P. Billingsley. *Probability and measure. Wiley series in probability and mathematical statistics.* Wiley New York, 19 95.

[4] A. Crisanti and H. Sompolinsky. Dynamics of spin systems with randomly asymmetric bonds: Langevin dynamics and a spherical model. *Physical Review A*, 36(10):4922–4939, 1987.

[5] A. Guionnet. *Dynamique de Langevin d'un verre de spins*. PhD thesis, Université de Paris Sud, 1995.

[6] A. Guionnet. Averaged and quenched propagation of chaos for spin glass dynamics. *Probability Theory and Related Fields*, 109(2):183–215, 1997.

[7] J. J. Hopfield. Neural networks and physical systems with emergent collective computational abilities. *Proc. National Academy of Sciences, USA*, 79:2554–2558, 1982.

[8] J. J. Hopfield. Neurons with graded response have collective computational properties like those of two-state neurons. *Proceedings of the National Academy of Sciences, USA*, 81(10):3088–3092, 1984.

[9] J. J. Hopfield and D. W. Tank. Computing with neural circuits- a model. *Science*, 233(4764):625–633, 1986.

[10] O. Moynot and M. Samuelides. Large deviations and mean-field theory for asymmetric random recurrent neural networks. *Probability Theory and Related Fields*, 123(1):41–75, 2002.

[11] A.N. Shiryaev. Probability. 2, Graduate Texts in Mathematics (95), *Springer, New York*, 2019.

[12] H. Sompolinsky, A. Crisanti, and HJ Sommers. Chaos in Random Neural Networks. *Physical Review Letters*, 61(3):259–262, 1988.

[13] H. Sompolinsky and A. Zippelius. Relaxational dynamics of the Edwards-Anderson model and the mean-field theory of spin-glasses. *Physical Review B*, 25(11):6860–6875, 1982.