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# Integral points on circles

A. Schinzel and M. Skalba

*In memory of S. Srinivasan*

**Abstract.** Sixty years ago the first named author gave an example [Sch58] of a circle passing through an arbitrary number of integral points. Now we shall prove: *The number  $N$  of integral points on the circle  $(x-a)^2 + (y-b)^2 = r^2$  with radius  $r = \frac{1}{n}\sqrt{m}$ , where  $m, n \in \mathbb{Z}$ ,  $m, n > 0$ ,  $\gcd(m, n^2)$  squarefree and  $a, b \in \mathbb{Q}$  does not exceed  $r(m)/4$ , where  $r(m)$  is the number of representations of  $m$  as the sum of two squares, unless  $n|2$  and  $n \cdot (a, b) \in \mathbb{Z}^2$ ; then  $N \leq r(m)$ .*

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Sixty years ago the first named author gave an example [Sch58] of a circle passing through an arbitrary number of integral points. If the center of a circle is not a rational point (i.e. not both coordinates are rational numbers) then it passes through no more than 2 rational points. In fact, the equation of the perpendicular bisector of a segment joining two rational points has rational coefficients, hence the circumcenter of a triangle with rational vertices has to be rational as well. From now on we will consider only circles

$$(x-a)^2 + (y-b)^2 = r^2, \quad (0.1)$$

with  $a, b \in \mathbb{Q}$  and we shall prove

**Theorem 0.1.** *The number  $N$  of integral points on the circle (0.1) with radius  $r = \frac{1}{n}\sqrt{m}$ , where  $m, n \in \mathbb{Z}$ ,  $m, n > 0$ ,  $\gcd(m, n^2)$  squarefree does not exceed  $r(m)/4$ , where  $r(m)$  is the number of representations of  $m$  as the sum of two squares, unless  $n|2$  and  $n \cdot (a, b) \in \mathbb{Z}^2$ ; then  $N \leq r(m)$ .*

**Lemma 0.2.** *Assume that  $\beta, \gamma_1, \gamma_2 \in \mathbb{Z}[i]$  and  $c \in \mathbb{N}$  satisfy*

$$N(\gamma_1) = N(\gamma_2) = c^2, \quad (0.2)$$

$$\beta\gamma_1 \equiv \beta\gamma_2 \pmod{c}, \quad (0.3)$$

$$\text{if a rational prime } t \text{ divides } c \text{ then } t \nmid \beta\gamma_1 \text{ and } t \nmid \beta\gamma_2. \quad (0.4)$$

*Then  $\gamma_1 \sim \gamma_2$  in  $\mathbb{Z}[i]$ .*

**Proof.** We assume from the beginning that  $\gamma_1 \neq \gamma_2$ .

**1. Case  $\gcd(\beta, c) \sim 1$ :** We can divide the congruence (0.3) by  $\beta$  and obtain

$$\gamma_1 - \gamma_2 = c\delta \quad \text{with } \delta \in \mathbb{Z}[i], \delta \neq 0.$$

Further

$$N(\gamma_1) + N(\gamma_2) - \gamma_1\overline{\gamma_2} - \gamma_2\overline{\gamma_1} = c^2N(\delta).$$

If we put  $\gamma_1\overline{\gamma_2} = f + gi$  with  $f, g \in \mathbb{Z}$  then by equation (0.2) we obtain

$$2f = (2 - N(\delta))c^2.$$

Hence

$$f = \frac{u}{2} \cdot c^2 \quad \text{with } u \in \mathbb{Z}, u \leq 1.$$

Because

$$f^2 + g^2 = N(\gamma_1 \overline{\gamma_2}) = c^4 \quad \text{by (0.2)}$$

one obtains

$$g^2 = c^4 - f^2 = c^4 \left(1 - \frac{u^2}{4}\right).$$

It follows  $u \in \{-2, -1, 0, 1\}$  but  $u \in \{-1, 1\}$  would lead to  $g \notin \mathbb{Q}$ . Hence  $u \in \{0, -2\}$ .

If  $u = 0$  then  $f = 0, g = \pm c^2$  hence

$$\gamma_1 \overline{\gamma_2} = \pm c^2 i \quad \text{what gives } \gamma_1 c^2 = \pm c^2 i \gamma_2,$$

and finally  $\gamma_1 = \pm i \gamma_2$ .

If  $u = -2$  then  $f = -c^2, g = 0$  hence  $\gamma_1 \overline{\gamma_2} = -c^2$  and  $\gamma_1 = -\gamma_2$ .

**2. Case  $N(\gcd(\beta, c)) = d > 1$ :** We adopt inductive method and assume that the assertion of lemma holds for  $N(\gcd(\beta, c)) < d$ . Let  $\pi$  be a prime element of the ring  $\mathbb{Z}[i]$  satisfying  $\pi|\beta$  and  $\pi|c$ . By condition (0.4) (and (0.2))  $N(\pi) = p$  is a rational prime of the form  $4k + 1$ .

By (0.2)  $\pi|\gamma_1$  or  $\overline{\pi}|\gamma_1$ , but the latter is excluded by (0.4), hence  $\pi^{2l}||\gamma_1$  where  $p^l|c$ . In the same way  $\pi^{2l}||\gamma_2$ . Rewrite the initial equality

$$\beta \gamma_1 - \beta \gamma_2 = \delta c \quad \text{with } \delta \in \mathbb{Z}[i], \delta \neq 0$$

in the form

$$\beta \frac{\gamma_1}{\pi^{2l}} - \beta \frac{\gamma_2}{\pi^{2l}} = \frac{\delta \overline{\pi}^{2l}}{p^l} \cdot \frac{c}{p^l}$$

where all fractions are algebraic integers. Using the inductive assumption finishes the proof of lemma.

**Proof of Theorem.** The considered circle (0.1) is given by the equation

$$(x - a)^2 + (y - b)^2 = \frac{m}{n^2}. \tag{0.5}$$

Put  $a = A/C, b = B/C$ , where  $A, B, C \in \mathbb{Z}, C > 0, (A, B, C) = 1$ . It follows that  $n|C$  and hence  $C = nc$  with  $c \in \mathbb{N}$ . The number  $N$  of integral points on the circle (0.5) satisfies

$$N = \text{card}\{(x, y) \in \mathbb{Z}^2 | (Cx - A)^2 + (Cy - B)^2 = c^2 m\}.$$

Each solution  $(x, y) \in \mathbb{Z}^2$  to the equation

$$(Cx - A)^2 + (Cy - B)^2 = c^2 m \tag{0.6}$$

is encoded by the equality

$$(Cx - A) + (Cy - B)i = \beta \cdot \gamma \tag{0.7}$$

with  $\beta, \gamma \in \mathbb{Z}[i]$  and  $N(\beta) = m, N(\gamma) = c^2$ .

Assume now to the contrary that the number of solutions  $(x, y) \in \mathbb{Z}^2$  to the equation (0.6) exceeds  $r(m)/4$ . It follows that there exist  $\beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{Z}[i]$  satisfying

$$\beta_1 \sim \beta_2, N(\beta_1) = N(\beta_2) = m, N(\gamma_1) = N(\gamma_2) = c^2,$$

$$\gamma_1 \beta_1 \equiv \gamma_2 \beta_2 \pmod{c} \quad \text{and} \quad \gamma_1 \beta_1 \neq \gamma_2 \beta_2.$$

Adjusting  $\gamma_1, \gamma_2$  for a unit (if necessary) we may assume that there are  $\beta, \gamma_1, \gamma_2 \in \mathbb{Z}[i]$  satisfying

$$N(\beta) = m, N(\gamma_1) = N(\gamma_2) = c^2, \beta \gamma_1 \equiv \beta \gamma_2 \pmod{c}, \gamma_1 \neq \gamma_2.$$

Now we infer by Lemma that

$$\gamma_2 \in \{-\gamma_1, i\gamma_1, -i\gamma_1\}.$$

((0.4) is fulfilled by the assumption  $(A, B, C) = 1$ .) In all above cases we get

$$2\beta\gamma_1 \equiv 0 \pmod{c}.$$

For  $c > 2$  this contradicts the condition  $(A, B, C) = 1$ . In case  $c = 2$ , for any integers  $A, B$  and  $C \equiv 0 \pmod{2}$  the conditions  $(A, B, C) = 1$  and

$$(Cx - A)^2 + (Cy - B)^2 = 4m$$

are incompatible. Concluding:  $N > r(m)/4$  is possible only for  $c = 1$ . In case  $c = 1$ ,  $C = n$  and by (0.6) one gets  $N \leq r(m)$ . It remains to deduce  $n|2$  from  $N > r(m)/4$ . It follows from the last inequality that there exist integers  $x_1, x_2, y_1, y_2$  and  $k \in \{1, 2, 3\}$  satisfying

$$(nx_2 - A) + (ny_2 - B)i = i^k[(nx_1 - A) + (ny_1 - B)i]$$

hence

$$(1 - i^k)(A + Bi) \equiv 0 \pmod{n}.$$

It follows  $n|(2A, 2B)$  and since  $(A, B, n) = 1$  we infer that  $n|2$ .

**Remark.** The number  $1/4$  in our theorem is optimal and here is an example. Let  $m$  be of the form  $3k + 2$  and satisfying  $r(m) > 0$ . The equality  $m = x^2 + y^2$  implies  $x \equiv \pm 1 \pmod{3}$ ,  $y \equiv \pm 1 \pmod{3}$ . It follows that  $(x - 1/3)^2 + (y - 1/3)^2 = m/9$  has  $r(m)/4$  integer solutions.

## References

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