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Set Equidistribution of subsets of $(\mathbb{Z}/n\mathbb{Z})^*$

Jaitra Chattopadhyay, Veekesh Kumar and R. Thangadurai

To the memory of S. Srinivasan

Abstract. In 2010, Murty and Thangadurai [MuTh10] provided a criterion for the set equidistribution of residue classes of subgroups in $(\mathbb{Z}/n\mathbb{Z})^*$. In this article, using similar methods, we study set equidistribution for some class of subsets of $(\mathbb{Z}/n\mathbb{Z})^*$. In particular, we study the set equidistribution modulo 1 of cosets, complement of subgroups of the cyclic group $(\mathbb{Z}/n\mathbb{Z})^*$ and the subset of elements of fixed order, whenever the size of the subset is sufficiently large.

Keywords. Set equi-distribution, residue classes mod n

2010 Mathematics Subject Classification. 11K45.

1. Introduction

We say (as defined in [MuSi09]) that a sequence of finite multisets A_n with $A_n \subseteq [0, 1]$ and $|A_n| \rightarrow \infty$ is *set equidistributed mod 1* with respect to a probability measure μ , if for every continuous function f on $[0, 1]$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{|A_n|} \sum_{t \in A_n} f(t) = \int_0^1 f(x) d\mu. \quad (1.1)$$

In order to verify this condition, it suffices to check that this limit exists on a dense family of functions f in $C[0, 1]$. Here, we shall make use of the family of Bernoulli polynomials.

Murty and Thangadurai [MuTh10] proved that the elements of the subgroup H_n of $(\mathbb{Z}/n\mathbb{Z})^*$, are set equidistributed modulo 1, whenever $|H_n|/\sqrt{n} \rightarrow \infty$ as $n \rightarrow \infty$.

Motivated from this, one may ask the following natural question: If S_n is a subset of $(\mathbb{Z}/n\mathbb{Z})^*$ such that $|S_n| > n^{\frac{1}{2}+\epsilon}$, are the elements of the subset S_n of $(\mathbb{Z}/n\mathbb{Z})^*$ set equidistributed modulo 1, as $n \rightarrow \infty$? In other words, does the result of [MuTh10] apply for subsets and not just subgroups?

In general, the answer is not affirmative. For instance, if $S'_n = \{a_1, a_2, \dots, a_m\} \subset (\mathbb{Z}/n\mathbb{Z})^*$ where $m = \lfloor n^{\frac{1}{2}+\epsilon} \rfloor + 1$ and a_i 's are the first m integers $\leq n$ with $(a_i, n) = 1$, then the elements of $S_n := S'_n/n$ are close to 0 in $[0, 1]$ for all integers $n \rightarrow \infty$ and hence these sets are not set equidistributed mod 1. However, for many arithmetical subsets like the set of all quadratic non-residues modulo p (which is not a subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$), and the set of all generators of $(\mathbb{Z}/n\mathbb{Z})^*$, whenever it is cyclic, the above question makes sense.

In this article, we give a partial answer to the above question. More precisely, we prove the following theorems:

Theorem 1.1. *Let ϵ be a given number with $0 < \epsilon < 1/12$. Consider an integer $n = p^k$ or $2p^k$ for some odd prime p , some integer $k \geq 1$ and a positive divisor f of n satisfying $\phi(n)/f \geq n^{1/2+3\epsilon}$. Let $S_{n,f}$ be a subset of $(\mathbb{Z}/n\mathbb{Z})^*$ which consists precisely of those elements whose index is f in $(\mathbb{Z}/n\mathbb{Z})^*$ and take the representatives $\mathcal{S}_{f,n}$ as integers, say, s_n with $1 < s_n \leq n-1$ and $(s_n, n) = 1$. Let $S'_{f,n} = \{s/(n-1) : s \in \mathcal{S}_{f,n}\} \subset [0, 1]$. Then the sets $S'_{f,n}$'s are set equidistributed in $[0, 1]$ with respect to the Lebesgue measure.*

In Theorem 1.1, when we take $f = 1$, then trivially the hypothesis is true. Hence, when n runs through numbers of the form $n = p^k$ or $2p^k$ for an odd prime p and for some integer $k \geq 1$, we find that the sets of generators of $(\mathbb{Z}/n\mathbb{Z})^*$ are set equidistributed modulo 1.

Theorem 1.2. *For an integer $n = p^k$ or $2p^k$ for some odd prime p and for some integer $k \geq 1$, let S_n be a subset of $(\mathbb{Z}/n\mathbb{Z})^*$ such that its complement is a subgroup of $(\mathbb{Z}/n\mathbb{Z})^*$ and we take the representatives S_n as integers, say, s_n with $1 < s_n \leq n - 1$ and $(s_n, n) = 1$. Let $S'_n = \{s/(n - 1) : s \in S_n\} \subset [0, 1]$. For a given $\epsilon > 0$, if $|S_n|/n^{\frac{1}{2}+2\epsilon} \rightarrow \infty$ as $n \rightarrow \infty$, then the S'_n s are set equidistributed in $[0, 1]$ with respect to the Lebesgue measure.*

As an application of Theorem 1.2, we have the following corollary.

Corollary 1.3. *Let $r \geq 2$ be an integer. For any prime number p such that $p \equiv 1 \pmod{r}$, let $H_p = \{a \in (\mathbb{Z}/p\mathbb{Z})^* : a^{\frac{p-1}{r}} \equiv 1 \pmod{p}\} \subset (\mathbb{Z}/p\mathbb{Z})^*$ and let the representatives of H_p be $\{h_1, \dots, h_{(p-1)/r}\}$ as a subset of $\{1, 2, \dots, p - 1\}$. Let*

$$S_p = \{a/p : a \in \{1, 2, \dots, p - 1\} \text{ and } a \neq h_i \text{ for any } i\}.$$

Then, as $p \rightarrow \infty$ such that $p \equiv 1 \pmod{r}$, the sets S_p 's are set equidistributed in $[0, 1]$ with respect to Lebesgue measure. In particular, when $r = 2$, we get the set of all quadratic non-residues modulo p , are set equidistributed in $[0, 1]$.

Theorem 1.4. *For any integer $n \geq 2$, let H'_n be a subgroup of $(\mathbb{Z}/n\mathbb{Z})^*$ and take the representatives of H'_n as integers, say, h such that $1 \leq h < n$ and $(n, h) = 1$. Let $H_n = \{h/n : h \in H'_n\}$ be a finite subset of $[0, 1]$. If $|H_n|/\sqrt{n} \rightarrow \infty$ as $n \rightarrow \infty$, then for any given $g_n \in (\mathbb{Z}/n\mathbb{Z})^*$, the cosets $g_n H_n$'s are set equidistributed in $[0, 1]$ with respect to the Lebesgue measure in $[0, 1]$.*

2. Preliminaries

In order to prove the sets S_n are set equidistributed, it suffices to determine the behaviour of sums of the form

$$\sum_{k=1}^{|S_n|} f_m(g_k),$$

for any suitable family of polynomials f_m of degree m for each integer $m \geq 1$, with $g_k \in S_n$. It is convenient to take the Bernoulli polynomials which are defined as

$$B_m(X) = \sum_{k=0}^m \binom{m}{k} B_k X^{m-k},$$

for each integer $m \geq 1$ where B_k denotes the k th-Bernoulli number, because the set of all finite \mathbb{Q} -linear combinations of $\{B_m(X)\}$ is a dense subset of $C[0, 1]$ (see [Apo76]). Therefore, we consider the sum

$$\sum_{k=1}^{|S_n|} B_m\left(\frac{g_k}{n}\right)$$

and we would like to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{|S_n|} \sum_{k=1}^{|S_n|} B_m\left(\frac{g_k}{n}\right) = \int_0^1 B_m(t) dt.$$

A well-known result states that (for instance, see [Mu08], page 19)

Lemma 2.1. *For any integer $m \geq 1$, we have*

$$\int_0^1 B_m(t) dt = 0.$$

Thus, by Lemma 2.1, in order to prove that the sequence of sets $\{S_n\}$ are set equidistributed mod 1, it is enough to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{|S_n|} \sum_{k=1}^{|S_n|} B_m\left(\frac{g_k}{n}\right) = 0.$$

The way to understand this sum, $\sum_{k=1}^{|S_n|} B_m\left(\frac{g_k}{n}\right)$, is through the generalized Bernoulli numbers (see for instance [Wa97]) which are defined as follows. For any Dirichlet character $\chi : (\mathbb{Z}/n\mathbb{Z})^* \rightarrow \mathbb{C}^*$ and for any integer $m \geq 1$, we define the m -th generalized Bernoulli number $B_{m,\chi}$ as

$$B_{m,\chi} = n^{m-1} \sum_{a=1}^n \chi(a) B_m\left(\frac{a}{n}\right).$$

Then we get the connection between $B_{m,\chi}$ and the Dirichlet L -function with character χ at $s = m$ and use the estimates of the special values of L -functions. For more information, we refer to Murty [Mu08]. Indeed, we need the following Lemma which can be found in [Mu08], pp 122.

Lemma 2.2. *We have the following;*

1. *For any character χ on $(\mathbb{Z}/n\mathbb{Z})^*$ and for any integer $m \geq 1$, we have*

$$L(1 - m, \chi) = -\frac{B_{m,\chi}}{m}.$$

2. *If χ is any character on $(\mathbb{Z}/n\mathbb{Z})^*$, then, there exists a positive constant $C(m)$, depending only on m such that*

$$|L(1 - m, \chi)| \leq C(m) n^{m-\frac{1}{2}}$$

for all integers $m \geq 1$ and for all $n > e^{17}$. (Proof of this fact can be seen in the proof of Theorem 2 in [MuTh10]).

The following lemma is standard and we shall state as follows.

Lemma 2.3. *Let $\sigma_0(n)$ denote the number of positive divisors n . Then, we have*

$$\sigma_0(n) \leq n^\epsilon \text{ for all large enough integers } n,$$

for any given $\epsilon > 0$. Also, we know that

$$\phi(n) \gg n^{1-\epsilon}$$

for any given $\epsilon > 0$, where ϕ stands for the Euler's totient function.

We need the following two crucial lemmas for the proof of Theorems 1.1 and 1.2 (see Lemma 3 in [Jo73]).

Lemma 2.4. *Let R be a finite ring such that R^* is the cyclic group of order n for some integer $n \geq 2$ and let f be a positive divisor of n . For any $a \in R$, we define*

$$I_f(a) = \begin{cases} 1 & \text{if } a \in R^* \text{ and } a \text{ is of index } f \text{ in } R^*; \\ 0 & \text{otherwise,} \end{cases}$$

where the index of an element $a \in R^*$ means the index of the subgroup generated by a in R^* . Then, for any $a \in R^*$, we have,

$$I_f(a) = \frac{1}{f} \sum_{d|(n/f)} \frac{\mu(d)}{d} \sum_{\chi^{fd} = \chi_0} \chi(a),$$

where μ is the Möbius function and the inner summation runs over all the multiplicative characters χ of R of order at most fd .

The following lemma computes the characteristic function for a given subset \mathcal{S} of a cyclic group G such that its complement is a subgroup.

Lemma 2.5. *Let G be a cyclic group of order n for some integer $n \geq 2$. Let \mathcal{S} be a finite subset of G such that $G \setminus \mathcal{S}$ is a subgroup of G . Let*

$$R = \{r \in \mathbb{N} : r \text{ is the index of } a \in \mathcal{S} \text{ for some } a\} = \{r_1, \dots, r_\ell\}$$

be the finite subset of \mathbb{N} . Then

$$\sum_{i=1}^{\ell} \left(\frac{1}{r_i} \sum_{d|\frac{n}{r_i}} \frac{\mu(d)}{d} \sum_{\chi^{r_i d} = \chi_0} \chi(a) \right) = \begin{cases} 1 & \text{if } a \in \mathcal{S}; \\ 0 & \text{otherwise,} \end{cases}$$

where μ is the Möbius function and the inner sum runs over the multiplicative characters χ of G of order at most $r_i d$.

Proof. Suppose $a \in \mathcal{S}$ and let r_j be the index of a for some integer $j \in \{1, \dots, \ell\}$. Then by Lemma 2.4, we get

$$\frac{1}{r_i} \sum_{d|\frac{n}{r_i}} \frac{\mu(d)}{d} \sum_{\chi^{r_i d} = \chi_0} \chi(a) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\sum_{i=1}^{\ell} \left(\frac{1}{r_i} \sum_{d|(n/r_i)} \frac{\mu(d)}{d} \sum_{\chi^{r_i d} = \chi_0} \chi(a) \right) = 1.$$

Now, let $b \in G \setminus \mathcal{S}$ and let q be the index of b . Then, we shall show that

$$\frac{1}{r_i} \sum_{d|\frac{n}{r_i}} \frac{\mu(d)}{d} \sum_{\chi^{r_i d} = \chi_0} \chi(b) = 0$$

for all $1 \leq i \leq \ell$.

To prove this, it suffices to show that $q \notin \{r_1, r_2, \dots, r_\ell\}$. Since G is a finite cyclic group, there exists a unique subgroup H_q of index q . Since the index of b is q , we conclude that the subgroup generated by b is equal to H_q . Also, note that any element in G , which is of index q , is a generator of H_q . Since $b \in G \setminus \mathcal{S}$ and by hypothesis $G \setminus \mathcal{S}$ is a subgroup, we conclude that $b \in H_q \subset G \setminus \mathcal{S}$. Since b is arbitrary, we conclude that any element of index q lies in $G \setminus \mathcal{S}$. Therefore, $q \notin \{r_1, r_2, \dots, r_\ell\}$ and proves the lemma.

3. Proof of Theorem 1.1

By Lemma 2.4, we have

$$\frac{1}{f} \sum_{d|\frac{\phi(n)}{f}} \frac{\mu(d)}{d} \sum_{\chi^{fd}=\chi_0} \chi(a) = \begin{cases} 1 & \text{if } a \in \mathcal{S}_{f,n} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{S}_{f,n} = \{g_1, \dots, g_{|\mathcal{S}_{f,n}|}\}$ and $m \geq 1$ be a given integer. Then consider

$$\begin{aligned} \sum_{k=1}^{|\mathcal{S}_{f,n}|} B_m\left(\frac{g_k}{n}\right) &= \sum_{k=1}^n B_m\left(\frac{k}{n}\right) \left(\frac{1}{f} \sum_{d|\frac{\phi(n)}{f}} \frac{\mu(d)}{d} \sum_{\chi^{fd}=\chi_0} \chi(k) \right) \\ &= \frac{1}{f} \sum_{d|\frac{\phi(n)}{f}} \frac{\mu(d)}{d} \left(\sum_{k=1}^n B_m\left(\frac{k}{n}\right) \sum_{\chi^{fd}=\chi_0} \chi(k) \right) \\ &= \frac{1}{f} \sum_{d|\frac{\phi(n)}{f}} \frac{\mu(d)}{d} \left(\sum_{\chi^{fd}=\chi_0} \sum_{k=1}^n \chi(k) B_m\left(\frac{k}{n}\right) \right) \\ &= \frac{1}{f} \sum_{d|\frac{\phi(n)}{f}} \frac{\mu(d)}{d} \left(\frac{1}{n^{m-1}} \sum_{\chi^{fd}=\chi_0} B_{m,\chi} \right). \end{aligned}$$

By Lemma 2.1, it is enough to show that for each integer $m \geq 1$, we have

$$\frac{1}{|\mathcal{S}_{f,n}|} \sum_{k=1}^{|\mathcal{S}_{f,n}|} B_m\left(\frac{g_k}{n}\right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Also, by Lemma 2.2 (1), for any character χ , we have $L(1-m, \chi) = -\frac{B_{m,\chi}}{m}$. Therefore, we get,

$$\begin{aligned} \frac{1}{|\mathcal{S}_{f,n}|} \left| \sum_{k=1}^{|\mathcal{S}_{f,n}|} B_m\left(\frac{g_k}{n}\right) \right| &= \frac{1}{|\mathcal{S}_{f,n}|} \left| \frac{1}{f} \sum_{d|\frac{\phi(n)}{f}} \frac{\mu(d)}{d} \left(\frac{1}{n^{m-1}} \sum_{\chi^{fd}=\chi_0} (-m)L(1-m, \chi) \right) \right| \\ &\leq \frac{1}{|\mathcal{S}_{f,n}|} \frac{1}{f} \sum_{d|\frac{\phi(n)}{f}} \frac{|\mu(d)|}{d} \left(\frac{m}{n^{m-1}} \sum_{\chi^{fd}=\chi_0} |L(1-m, \chi)| \right) \\ &= \frac{m}{|\mathcal{S}_{f,n}| n^{m-1}} \frac{1}{f} \sum_{d|\frac{\phi(n)}{f}} \frac{|\mu(d)|}{d} \left(\sum_{\chi^{fd}=\chi_0} |L(1-m, \chi)| \right) \\ &\leq \frac{C'(m)}{|\mathcal{S}_{f,n}| n^{m-1}} \frac{1}{f} \sum_{d|\frac{\phi(n)}{f}} \frac{1}{d} \left(\sum_{\chi^{fd}=\chi_0} n^{m-\frac{1}{2}} \right), \end{aligned}$$

for some positive constant $C'(m)$ that depends only on m by Lemma 2.2 (2). Therefore, we get,

$$\begin{aligned} \left| \frac{1}{|\mathcal{S}_{f,n}|} \sum_{k=1}^{|\mathcal{S}_{f,n}|} B_m \left(\frac{g_k}{n} \right) \right| &\leq \frac{C'(m)\sqrt{n}}{|\mathcal{S}_{f,n}|} \frac{1}{f} \sum_{d|\frac{\phi(n)}{f}} \frac{1}{d} \left(\sum_{\chi^{fd}=\chi_0} 1 \right) \\ &\leq \frac{C'(m)\sqrt{n}}{|\mathcal{S}_{f,n}|} \frac{1}{f} \sum_{d|\frac{\phi(n)}{f}} \frac{1}{d} (fd) = \frac{C'(m)\sqrt{n}}{|\mathcal{S}_{f,n}|} \left(\sum_{d|\frac{\phi(n)}{f}} 1 \right) \\ &= \frac{C'(m)\sqrt{n}}{|\mathcal{S}_{f,n}|} \sigma_0 \left(\frac{\phi(n)}{f} \right). \end{aligned}$$

Since the set $\mathcal{S}_{f,n}$ precisely contains the generators of the cyclic subgroup of order $\frac{\phi(n)}{f}$, the cardinality of the set $\mathcal{S}_{f,n}$ is $\phi \left(\frac{\phi(n)}{f} \right)$. Therefore, we have

$$\begin{aligned} \left| \frac{1}{|\mathcal{S}_{f,n}|} \sum_{k=1}^{|\mathcal{S}_{f,n}|} B_m \left(\frac{g_k}{n} \right) \right| &\leq \frac{C'(m)\sqrt{n}}{|\mathcal{S}_{f,n}|} \sigma_0 \left(\frac{\phi(n)}{f} \right) \\ &= \frac{C'(m)\sqrt{n}}{\phi \left(\frac{\phi(n)}{f} \right)} \sigma_0 \left(\frac{\phi(n)}{f} \right). \end{aligned}$$

For a given $\epsilon > 0$, we know that $\sigma_0(n) = O(n^\epsilon)$ and $\phi(n) > n^{1-\epsilon}$ for all sufficiently large integers n . Hence, since $\sigma_0 \left(\frac{\phi(n)}{f} \right) \leq C \left(\frac{\phi(n)}{f} \right)^\epsilon$ for some positive constant C and $\phi \left(\frac{\phi(n)}{f} \right) > \left(\frac{\phi(n)}{f} \right)^{1-\epsilon}$. Thus, we get,

$$\left| \frac{1}{|\mathcal{S}_{f,n}|} \sum_{k=1}^{|\mathcal{S}_{f,n}|} B_m \left(\frac{g_k}{n} \right) \right| < \frac{C'(m)C\sqrt{n}f^{1-2\epsilon}}{\phi(n)^{1-2\epsilon}}.$$

By hypothesis, we know that $\frac{\phi(n)}{f} \geq n^{1/2+3\epsilon}$, we see that

$$\left| \frac{1}{|\mathcal{S}_{f,n}|} \sum_{k=1}^{|\mathcal{S}_{f,n}|} B_m \left(\frac{g_k}{n} \right) \right| < \frac{C'(m)C}{n^{2\epsilon-6\epsilon^2}}$$

and hence as $n \rightarrow \infty$, we get the desired result, as the given ϵ satisfies $0 < \epsilon < \frac{1}{12}$. □

4. Proof of Theorem 1.2

For each integer $n = p^k$ or $2p^k$, where p is an odd prime and $k \geq 1$ is an integer, we let \mathcal{S}_n be a given subset of $(\mathbb{Z}/n\mathbb{Z})^*$ such that its complement is a subgroup of $(\mathbb{Z}/n\mathbb{Z})^*$. Note that for these values of n , the group of coprime residue classes modulo n is cyclic.

Let n be one such natural number and we consider \mathcal{S}_n . Suppose r_1, r_2, \dots, r_ℓ be the indices of the elements of \mathcal{S}_n . By lemma 2.4, we have

$$\sum_{i=1}^{\ell} \left(\frac{1}{r_i} \sum_{d|\frac{n}{r_i}} \frac{\mu(d)}{d} \sum_{\chi^{r_i d}=\chi_0} \chi(a) \right) = \begin{cases} 1 & \text{if } a \in \mathcal{S}_n \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{S}_n = \{g_1, \dots, g_{|\mathcal{S}_n|}\}$ and $m \geq 1$ be a given integer. Then consider

$$\begin{aligned}
\sum_{k=1}^{|\mathcal{S}_n|} B_m \left(\frac{g_k}{n} \right) &= \sum_{k=1}^n B_m \left(\frac{k}{n} \right) \sum_{i=1}^{\ell} \left(\frac{1}{r_i} \sum_{d|\frac{\phi(n)}{r_i}} \frac{\mu(d)}{d} \sum_{\chi^{r_i d}=\chi_0} \chi(k) \right) \\
&= \sum_{i=1}^{\ell} \frac{1}{r_i} \sum_{d|\frac{\phi(n)}{r_i}} \frac{\mu(d)}{d} \left(\sum_{k=1}^n B_m \left(\frac{k}{n} \right) \sum_{\chi^{r_i d}=\chi_0} \chi(k) \right) \\
&= \sum_{i=1}^{\ell} \frac{1}{r_i} \sum_{d|\frac{\phi(n)}{r_i}} \frac{\mu(d)}{d} \left(\sum_{\chi^{r_i d}=\chi_0} \sum_{k=1}^n \chi(k) B_m \left(\frac{k}{n} \right) \right) \\
&= \sum_{i=1}^{\ell} \frac{1}{r_i} \sum_{d|\frac{\phi(n)}{r_i}} \frac{\mu(d)}{d} \left(\frac{1}{n^{m-1}} \sum_{\chi^{r_i d}=\chi_0} B_{m,\chi} \right).
\end{aligned}$$

By Lemma 2.1, it is enough to show that for each integer $m \geq 1$, we have

$$\frac{1}{|\mathcal{S}_n|} \sum_{k=1}^{|\mathcal{S}_n|} B_m \left(\frac{g_k}{n} \right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Also, by Lemma 2.2 (1), for any character χ , we know that $L(1-m, \chi) = -\frac{B_{m,\chi}}{m}$. Thus, we need to estimate the following

$$\frac{1}{|\mathcal{S}_n|} \sum_{k=1}^{|\mathcal{S}_n|} B_m \left(\frac{g_k}{n} \right) = \frac{1}{|\mathcal{S}_n|} \sum_{i=1}^{\ell} \frac{1}{r_i} \sum_{d|\frac{\phi(n)}{r_i}} \frac{\mu(d)}{d} \left(\frac{1}{n^{m-1}} \sum_{\chi^{r_i d}=\chi_0} (-m)L(1-m, \chi) \right).$$

Therefore, by Lemma 2.2 (2), we get

$$\begin{aligned}
\left| \frac{1}{|\mathcal{S}_n|} \sum_{k=1}^{|\mathcal{S}_n|} B_m \left(\frac{g_k}{n} \right) \right| &\leq \frac{1}{|\mathcal{S}_n|} \sum_{i=1}^{\ell} \frac{1}{r_i} \sum_{d|\frac{\phi(n)}{r_i}} \frac{|\mu(d)|}{d} \left(\frac{m}{n^{m-1}} \sum_{\chi^{r_i d}=\chi_0} |L(1-m, \chi)| \right) \\
&= \frac{m}{|\mathcal{S}_n| n^{m-1}} \sum_{i=1}^{\ell} \frac{1}{r_i} \sum_{d|\frac{\phi(n)}{r_i}} \frac{|\mu(d)|}{d} \left(\sum_{\chi^{r_i d}=\chi_0} |L(1-m, \chi)| \right) \\
&\leq \frac{C'(m)}{|\mathcal{S}_n| n^{m-1}} \sum_{i=1}^{\ell} \frac{1}{r_i} \sum_{d|\frac{\phi(n)}{r_i}} \frac{1}{d} \left(\sum_{\chi^{r_i d}=\chi_0} n^{m-\frac{1}{2}} \right) \\
&= \frac{C'(m)\sqrt{n}}{|\mathcal{S}_n|} \sum_{i=1}^{\ell} \frac{1}{r_i} \sum_{d|\frac{\phi(n)}{r_i}} \frac{1}{d} \left(\sum_{\chi^{r_i d}=\chi_0} 1 \right) \\
&\leq \frac{C'(m)\sqrt{n}}{|\mathcal{S}_n|} \sum_{i=1}^{\ell} \frac{1}{r_i} \sum_{d|\frac{\phi(n)}{r_i}} \frac{1}{d} (r_i d) = \frac{C'(m)\sqrt{n}}{|\mathcal{S}_n|} \sum_{i=1}^{\ell} \left(\sum_{d|\frac{\phi(n)}{r_i}} 1 \right) \\
&= \frac{C'(m)\sqrt{n}}{|\mathcal{S}_n|} \sum_{i=1}^{\ell} \sigma_0 \left(\frac{\phi(n)}{r_i} \right) \leq \frac{C'(m)\sqrt{n}}{|\mathcal{S}_n|} \ell \sigma_0(\phi(n)),
\end{aligned}$$

where $\sigma_0(n)$ stands for the number of divisors of n and $C'(m)$ is a positive constant depending only on m . By Lemma 2.3, for any given $\epsilon > 0$, we have $\sigma_0(n) = O(n^\epsilon)$. Also, since $\phi(n) \leq n$, we get, $\sigma_0(\phi(n)) = O(\phi(n)^\epsilon) = O(n^\epsilon)$.

Also, since r_1, r_2, \dots, r_l are the indices of elements of \mathcal{S}_n and each r_i divides $\phi(n)$, we have

$$l \leq \sigma_0(\phi(n)) = O(\phi(n)^\epsilon) = O(n^\epsilon).$$

Thus,

$$\left| \frac{1}{|\mathcal{S}_n|} \sum_{k=1}^{|\mathcal{S}_n|} B_m \left(\frac{g_k}{n} \right) \right| \leq \frac{C'(m)n^{\frac{1}{2}+2\epsilon}}{|\mathcal{S}_n|},$$

which holds for any $\epsilon > 0$. This proves the theorem. □

5. Proof of Corollary 1.3

Let H_p be the given subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ of cardinality $(p-1)/r$ and \mathcal{S}_p is the complement of H_p . Then,

$$|\mathcal{S}_p| = p - 1 - \frac{p-1}{r} \geq \frac{p-1}{2} \geq (p-1)^{\frac{1}{2}+\epsilon},$$

for all sufficiently large p and for any ϵ with $0 < \epsilon < \frac{1}{2}$. Therefore, by Theorem 1.2, the assertion follows. □

6. Proof of Theorem 1.4

For any integer $n \geq 2$, we are given a subgroup H'_n of the group $(\mathbb{Z}/n\mathbb{Z})^*$ and we take the elements of H'_n as integers m such that $1 \leq m \leq n$ and $(m, n) = 1$. Also, it is given that for each integer $n \geq 2$, the element $g_n \in (\mathbb{Z}/n\mathbb{Z})^*$. Then consider the subset $H_n = H'_n/n$ of $[0, 1]$.

We want to prove that the sets $g_n H_n$ are set equidistributed mod 1. For each integer $n \geq 2$, we denote \widehat{H}_n the group of all Dirichlet characters of $(\mathbb{Z}/n\mathbb{Z})^*$ which are trivial on the subgroup H'_n . Therefore, we have a canonical isomorphism

$$\widehat{H}_n \cong (\mathbb{Z}/n\mathbb{Z})^*/H'_n$$

and so,

$$|\widehat{H}_n| = \frac{\phi(n)}{|H'_n|} = \frac{\phi(n)}{|g_n H_n|}.$$

Then, we see that

$$\frac{1}{|\widehat{H}_n|} \sum_{\chi \in \widehat{H}_n} \chi(a)\chi(g_n^{-1}) = \begin{cases} 1 & \text{if } a \in g_n H_n \\ 0 & \text{otherwise.} \end{cases}$$

By letting $H'_n = \{a_1, \dots, a_{|H_n|}\}$, for each integer $m \geq 1$, we see that

$$\begin{aligned} \sum_{k=1}^{|H_n|} B_m \left(\frac{a_k g_n}{n} \right) &= \frac{1}{|\widehat{H}_n|} \sum_{k=1}^n B_m \left(\frac{k}{n} \right) \sum_{\chi \in \widehat{H}_n} \chi(k) \chi(g_n^{-1}) \\ &= \frac{1}{|\widehat{H}_n|} \sum_{k=1}^n B_m \left(\frac{k}{n} \right) \sum_{\chi \in \widehat{H}_n} \chi(k g_n^{-1}) \\ &= \frac{1}{|\widehat{H}_n|} \sum_{\chi \in \widehat{H}_n} \chi(g_n^{-1}) \left(\sum_{k=1}^n B_m \left(\frac{k}{n} \right) \chi(k) \right) \\ &= \frac{1}{n^{m-1} |\widehat{H}_n|} \sum_{\chi \in \widehat{H}_n} \chi(g_n^{-1}) B_{m, \chi}. \end{aligned}$$

By Lemma 2.1, it is enough to show that for each $m \geq 1$

$$\frac{1}{|g_n H_n|} \sum_{k=1}^{|g_n H_n|} B_m \left(\frac{a_k g_n}{n} \right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since $|g_n H_n| = |H_n|$, the rest of the proof goes along the proof of subgroup H_n proved in [MuTh10]. Hence, we omit the proof here. \square

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