BLOW UP DYNAMICS FOR THE HYPERBOLIC VANISHING MEAN CURVATURE FLOW OF SURFACES ASYMPTOTIC TO SIMONS CONE

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BLOW UP DYNAMICS FOR THE HYPERBOLIC VANISHING MEAN CURVATURE FLOW OF SURFACES ASYMPTOTIC TO SIMONS CONE

HAJER BAHOURI, ALAA MARACHLI, AND GALINA PERELMAN

Abstract. In this article, we establish the existence of a family of hypersurfaces $(\Gamma(t))_{0<t\leq T}$ which evolve by the vanishing mean curvature flow in Minkowski space and which as t tends to 0 blow up towards a hypersurface which behaves like the Simons cone at infinity. This issue amounts to investigate the singularity formation for a second order quasilinear wave equation. Our constructive approach consists in proving the existence of finite time blow up solutions of this hyperbolic equation under the form $u(t,x) \sim t^{\nu+1} Q\Big(\frac{x}{t^{\nu+1}}\Big)$, where Q is a stationary solution and ν an arbitrary large positive irrational number. Our approach roughly follows that of Krieger, Schlag and Tataru in [19, 20, 21]. However contrary to these works, the equation to be handled in this article is quasilinear. This induces a number of difficulties to face.

Contents

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1. Introduction

1.1. **Setting of the problem.** In this article we address the question of singularity formation for the hyperbolic vanishing mean curvature flow of surfaces that are asymptotic to Simons cones at infinity.

In [6], Bombieri, De Giorgi and Giusti proved that the Simons cone defined as follows

$$C_n = \big\{X = (x_1, \cdots, x_{2n}) \in \mathbb{R}^{2n}\,, x_1^2 + \cdots + x_n^2 = x_{n+1}^2 + \cdots + x_{2n}^2\big\}, \tag{1.1}$$

is a globally minimizing surface if and only if $n \geq 4$. It is clear that the Simons cone which has dimension $d = 2n-1$ is invariant under the action of the group $O(n) \times O(n)$, where $O(n)$ is the orthogonal group of $\mathbb{R}^n$, and that can be parametrized in the following way:

$$\begin{aligned} \mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} &\longrightarrow C_n \subset \mathbb{R}^{2n} \\ (\rho, \omega_1, \omega_2) &\longrightarrow (\rho\omega_1, \rho\omega_2)\,. \end{aligned} \tag{1.2}$$

The Simons cones are linked to Bernstein's problem which states as follows: if the graph of a C^2 function u on $\mathbb{R}^{m-1}$ is a minimal surface in $\mathbb{R}^m$, does this imply that this graph is an hyperplane? Such issue amounts to ask if the solution u to the following equation known as the minimal surface equation

$$\sum_{j=1}^{m-1} \partial_{x_j}\Big(\frac{u_{x_j}}{\sqrt{1+|\nabla u|^2}}\Big) = 0\,,$$

is linear. This problem due to Sergei Natanovich Bernstein who solved the case $m = 3$ in 1914 admits only an affirmative answer in the case of dimension $m \leq 8$. Actually in [10], De Giorgi shows that the falsity of the extension Bernstein's theorem to the case of $\mathbb{R}^m$ would imply the existence of a minimizing cone in $\mathbb{R}^{m-1}$. We refer for instance to [2, 3, 4, 6, 10, 11, 23, 25, 30, 31] and the references therein for further details on Bernstein's problem and related issues.

By the works [6, 33], it is known that for $n \geq 4$ the complementary of the Simons cone (which has two connected components $|x| < |y|$ and $|y| < |x|$) is foliated by two families of smooth minimal surfaces $(M_a)_{a>0}$ and $(\widetilde{M}_a)_{a>0}$ asymptotic to the Simons cone at infinity. These families are the scaling invariant: $M_a = aM$ and $\widetilde{M}_a = a\widetilde{M}$ with M and $\widetilde{M}$ admitting respectively the parametrization:

$$\mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \ni (\rho, \omega_1, \omega_2) \mapsto (\rho\omega_1, Q(\rho)\omega_2) \in \mathbb{R}^{2n}\,, \tag{1.3}$$

$$\mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \ni (\rho, \omega_1, \omega_2) \mapsto (Q(\rho)\omega_1, \rho\omega_2) \in \mathbb{R}^{2n}, \tag{1.4}$$

where Q is a positive radial function which belongs to $\mathcal{C}^\infty(\mathbb{R}^n)$ and satisfies $Q(0) = 1$, $Q(\rho) > \rho$ for any positive real number ρ, and

$$Q(\rho) = \rho + \frac{d_\alpha}{\rho^\alpha}\big(1 + \circ(1)\big)\,,$$

as ρ tends to infinity, with d_α some positive constant and

$$\alpha = -1 + \frac{1}{2}\Big((2n-1) - \sqrt{(2n-1)^2 - 16(n-1)}\Big)\,.$$

The minimal surface equation in Riemannian geometry has a natural hyperbolic analogue in the Lorentzian framework. In particular working in the Minkowski space $\mathbb{R}^{1,2n}$ equipped with the standard metric: $dg = -dt^2 + \sum_{j=1}^n dx_j^2$, and considering the time-like surfaces that for fixed t can be parametrized under the form

$$\mathbb{R}^n \times \mathbb{S}^{n-1} \ni (x,\omega) \to \Gamma(t) = (x, u(t,x)\omega) \in \mathbb{R}^{2n}\,, \tag{1.5}$$

with some positive function u, lead to the following quasilinear second order wave equation (see Appendix A for the corresponding computations)

$$\partial_t\Big(\frac{u_t}{\sqrt{1-(u_t)^2+|\nabla u|^2}}\Big) - \sum_{j=1}^n \partial_{x_j}\Big(\frac{u_{x_j}}{\sqrt{1-(u_t)^2+|\nabla u|^2}}\Big) + \frac{n-1}{u\sqrt{1-(u_t)^2+|\nabla u|^2}} = 0\,,$$

that can be also rewritten as

$$\begin{aligned}(\mathrm{NW})\, u &:= (1+|\nabla u|^2)\, u_{tt} - (1-(u_t)^2+|\nabla u|^2)\,\Delta u \\ &\quad + \sum_{j,k=1}^n u_{x_j}u_{x_k}u_{x_jx_k} - 2u_t(\nabla u\cdot\nabla u_t) + \frac{(n-1)}{u}\,(1-(u_t)^2+|\nabla u|^2) = 0\,\cdot\end{aligned} \tag{1.6}$$

Note that this equation is invariant by the scaling

$$u_a(t,x) = a\, u\Big(\frac{t}{a}, \frac{x}{a}\Big)\,, \tag{1.7}$$

in the sense that if u solves (1.6) then u_a is also a solution to (1.6). In the framework of Sobolev spaces[1], $\dot H^{\frac{n+2}{2}}(\mathbb{R}^n)$ is invariant under the scaling (1.7).

In this paper, we shall consider the case when $n=4$ and assume that u is radial which implies that for fixed t the surfaces we are considering are invariant under the action of the group $O(4)\times O(4)$. We readily check that in that case the function u satisfies the following equation:

$$(1+u_\rho^2)\, u_{tt} - (1-u_t^2)\, u_{\rho\rho} - 2u_tu_\rho u_{\rho t} + 3(1+u_\rho^2-u_t^2)\Big(\frac{1}{u} - \frac{u_\rho}{\rho}\Big) = 0\,\cdot \tag{1.8}$$

Note that the Simons cone and the minimal surfaces M_a are stationary solutions of our model with $u(t,\rho)=\rho$ in the case of Simons cone and $u(t,\rho) = Q_a(\rho)$, $Q_a(\rho) = aQ\Big(\frac{\rho}{a}\Big)$ in the case of M_a. Let us also emphasize that in that case, we have

$$Q(\rho) = \rho + \frac{d_2}{\rho^2}\,\big(1+\circ(1)\big)\,, \tag{1.9}$$

as ρ tends to infinity, with d_2 some positive constant.

We shall be interested in time-like surfaces of the form (1.5) that are asymptotic to the Simons cone as $|x|\to\infty$. To take care of this behavior we introduce the spaces X_L, with L an integer sufficiently large, that we define as being the set of functions (u_0,u_1) such that $\nabla(u_0-Q)$ and u_1 belong to $H^{L-1}(\mathbb{R}^4)$, and which satisfy

$$\inf u_0 > 0 \quad \text{and} \quad \inf\,(1+|\nabla u_0|^2-(u_1)^2) > 0\,. \tag{1.10}$$

[1] All along this article, we shall denote by $H^s(\mathbb{R}^n)$ the non homogeneous Sobolev space and by $\dot H^s(\mathbb{R}^n)$ the homogeneous Sobolev space. We refer to [5] and the references therein for all necessary definitions and properties of those spaces.

The Cauchy problem for the quasilinear wave equation (1.6) is locally well posed in X_L provided that L is sufficiently large. More precisely one has the following theorem the proof of which is given in Appendix C:

Theorem 1.1. *Consider the Cauchy problem*

$$\begin{cases} (1.6)\, u = 0 \\ u_{|t=0} = u_0 \\ (\partial_t u)_{|t=0} = u_1\,. \end{cases} \tag{1.11}$$

Assume that the Cauchy data (u_0, u_1) belongs to the functions class X_L with $L > 4$, then there exists a unique maximal solution u of (1.11) on $[0, T^*[$ such that

$$(u, \partial_t u) \in C([0, T^*[, X_L)\,. \tag{1.12}$$

Besides if the maximal time T^* of such a solution is finite (we then say that it blows up), then

$$\limsup_{t\nearrow T^*} \Big(\Big\| \frac{1}{u(t,\cdot)} \Big\|_{L^\infty} + \Big\| \frac{1}{(1+|\nabla u|^2 - (\partial_t u)^2)(t,\cdot)} \Big\|_{L^\infty} + \sup_{|\gamma|\le 1} \|\partial_x^\gamma \nabla_{t,x} u\|_{L^\infty} \Big) = \infty\cdot \tag{1.13}$$

The question we would like to address in this paper is that of blow up i.e., the description of possible singularities that smooth hypersurfaces may develop as they evolve by the Minkowski zero mean curvature flow, which amounts to investigate blow up dynamics for quasilinear wave equations. There is by now a considerable literature dealing with the construction of type II blow up solutions for semilinear heat, wave and Schrödinger type equations both in critical and supercritical cases (see for instance the articles [7, 9, 14, 15, 17, 18, 19, 20, 21, 22, 24, 26, 27, 33, 34] and the references therein). At the most basic level, the strategy in these works is to construct solutions in a two step process, first building an approximate solution, and then completing it to an exact solution, by controlling the remaining error via well-established arguments. The viewpoint we shall adopt here is the one which has been initiated by Krieger, Schlag and Tataru in [21], where for the energy critical focusing wave equation they constructed type II blow up solutions, with a continuum of blow up rates, that become singular via a concentration of a stationary state profile. The goal of the present paper is to show that this blow up mechanism exists as well for the quasilinear wave equation defined by (1.8).

1.2. **Statement of the result.** Our main result is given by the following theorem.

Theorem 1.2. *For any irrational number $\nu > \dfrac{1}{2}$ and any positive real number δ sufficiently small, there exist a positive time T and a radial solution $u(t,\cdot)$ to (1.8) on the interval $(0,T]$ such that*[2] *for any time t in $(0,T]$*

$$(u, \partial_t u) \in C((0,T], X_{L_0}) \quad \textit{with} \quad L_0 := 2M+1\,,\ M = \Big[\frac{3}{2}\nu + \frac{5}{4}\Big], \tag{1.14}$$

and such that it blows up at $t = 0$ by concentrating the soliton profile: there exist two radial functions $\mathfrak{g}_0 \in \dot H^{s+1}(\mathbb{R}^4)$ and $\mathfrak{g}_1 \in \dot H^{s}(\mathbb{R}^4)$, for any $0 \le s < 3\nu + 2$, such that one has

$$\begin{aligned} u(t,x) &= t^{\nu+1} Q\Big(\frac{x}{t^{\nu+1}}\Big) + \mathfrak{g}_0(x) + \eta(t,x)\,, \\ u_t(t,x) &= \mathfrak{g}_1(x) + \eta_1(t,x), \end{aligned}$$

with

$$\|\nabla \eta(t,\cdot)\|_{H^2(\mathbb{R}^4)} + \|\eta_1(t,\cdot)\|_{H^2(\mathbb{R}^4)} \overset{t\to 0}{\longrightarrow} 0\,.$$

[2] where all along this paper, $[x]$ denotes the entire part of x.

Moreover, writing

$$\begin{aligned} u(t,x) &= t^{\nu+1}\Big(Q\Big(\frac{x}{t^{\nu+1}}\Big)+\zeta\Big(t,\frac{x}{t^{\nu+1}}\Big)\Big)\,, \\ u_t(t,x) &= \zeta_1\Big(t,\frac{x}{t^{\nu+1}}\Big), \end{aligned}$$

we have

$$\|\nabla\zeta(t,\cdot)\|_{\dot H^s(\mathbb{R}^4)}+\|\zeta_1(t,\cdot)\|_{\dot H^s(\mathbb{R}^4)}\overset{t\to 0}{\longrightarrow}0\,,\forall\, 2<s\leq L_0-1\,.$$

Besides, $\mathfrak{g}_0$, $\mathfrak{g}_1$ *are compactly supported, belong to* $\mathcal{C}^\infty(\mathbb{R}^4\setminus\{0\})$ *and verify for all* $0\leq s<3\nu+2$

$$\|\nabla\mathfrak{g}_0\|_{H^s(\mathbb{R}^4)}+\|\mathfrak{g}_1\|_{H^s(\mathbb{R}^4)}\leq C_s\,\delta^{3\nu+2-s}\,,$$

$$\mathfrak{g}_0(x)\sim\frac{d_2}{3\nu+4}|\sqrt{2}\,x|^{3\nu+1}\,,\,\mathfrak{g}_1(x)\sim d_2|\sqrt{2}\,x|^{3\nu}\,,\ as\, x\to 0\,,$$

where d_2 *denotes the constant involved in (1.9).*

Corollary 1.1. *There exists a family of hypersurfaces* $(\Gamma(t))_{0<t\leq T}$ *in* $\mathbb{R}^8$ *which evolve by the hyperbolic vanishing mean curvature flow, and which as* t *tends to* 0 *blow up towards a hypersurface which behaves asymptotically as the Simons cone at infinity. Moreover*

$$t^{-(\nu+1)}\Gamma(t)\overset{t\to 0}{\longrightarrow}M\,,$$

uniformly on compact sets, where M *denotes the hypersurface defined by* (1.3).

Remark 1.1.

- *A similar result was established in the case of parabolic vanishing mean curvature flow by Velázquez in* [33].
- *Combining Theorem 1.2 with the asymptotic (1.9), we readily gather that the blow up solution* u *to* (1.8) *given by Theorem 1.2 satisfies*
  (1) $\|\nabla(u(t,\cdot)-Q)\|_{L^\infty((0,T],\dot H^s(\mathbb{R}^4))}\lesssim 1\,,\forall\,0\leq s<2,$
  (2) $\|\nabla(u(t,\cdot)-|x|-\mathfrak{g}_0)\|_{\dot H^s(\mathbb{R}^4)}\overset{t\to 0}{\longrightarrow}0\,,\forall\,0\leq s<2,$
  (3) $\|\nabla(u(t,\cdot)-Q)\|_{\dot H^s(\mathbb{R}^4)}\overset{t\to 0}{\longrightarrow}\infty\,,\forall\,2\leq s\leq L_0-1\cdot$
- *The parameter* ν *is restricted to the irrationals just to avoid the formation of additional logarithms in the construction of an approximate solution to* (1.8)*. Its limitation to* $\nu>\frac{1}{2}$ *is technical.*

1.3. **Strategy of the proof.** Let us outline our strategy that is concisely implemented in Sections 3, 4, 5 and 7. Roughly speaking, the proof of Theorem 1.2 is done in two main steps. The first step is dedicated to the construction of an approximate solution to (1.8) as a perturbation of the concentrating soliton profile $t^{\nu+1}Q\Big(\frac{x}{t^{\nu+1}}\Big)$, where $\nu>\dfrac{1}{2}$ is a fixed irrational number. The second step which will be the subject of Section 7 is to complement this approximate solution to an actual solution u by a perturbative argument. In that step, the properties of the linearized operator of the quasilinear wave equation (1.8) around Q, which are studied in Appendix B, are essential.

As we shall see, the blow up result we establish in this article heavily relies on the asymptotic behavior of the soliton Q. Thus we shall focus in Section 2 on its analysis.

To built a good approximate solution, we shall analyze separately the three regions that correspond to three different space scales: the inner region corresponding to $\frac{\rho}{t} \leq t^{\epsilon_1}$, the self-similar region where $\frac{1}{10} t^{\epsilon_1} \leq \frac{\rho}{t} \leq 10\, t^{-\epsilon_2}$, and finally the remote region defined by $\frac{\rho}{t} \geq t^{-\epsilon_2}$, where $0 < \epsilon_1 < \nu$ and $0 < \epsilon_2 < 1$ are two fixed positive real numbers. The inner region is the region where the blow up concentrates. In this region the solution will be constructed as a perturbation of the profile $t^{\nu+1} Q\Big(\frac{x}{t^{\nu+1}}\Big)\cdot$ In the self-similar region, the profile of the solution is determined uniquely by the matching conditions coming out of the inner region, while in the remote region the profile remains essentially a free parameter of the construction.

In Section 3, we investigate the equation in the inner region $\frac{\rho}{t} \leq t^{\epsilon_1}\cdot$ In that region, we shall look for an approximate solution as a power expansion in $t^{2\nu}$ of the form:

$$u_{\rm in}^{(N)}(t,\rho) = t^{\nu+1} \sum_{k=0}^{N} t^{2\nu k} V_k\Big(\frac{\rho}{t^{\nu+1}}\Big)\,, \tag{1.15}$$

where V_0 is nothing else than the soliton Q, and where the functions V_k, for $1 \leq k \leq N$, are obtained recursively, by solving a recurrent system of the form:

$$\begin{cases} \mathcal{L} V_k = F_k(V_0, \cdots, V_{k-1}) \\ V_k(0) = 0 \quad \text{and} \quad V_k'(0) = 0\,, \end{cases}$$

where $\mathcal{L}$ is the operator defined by:

$$\mathcal{L} = \partial_y^2 + \Big(\frac{3}{y} + B_1\Big)\partial_y + B_0\,, \tag{1.16}$$

with

$$\begin{cases} B_1(y) = 9\dfrac{Q_y^2}{y} - 6\dfrac{Q_y}{Q}\,, \\[2ex] B_0(y) = 3\dfrac{1+Q_y^2}{Q^2}\,\cdot \end{cases} \tag{1.17}$$

As it will be established in Paragraph 3.2, these functions V_k grow at infinity as follows:

$$V_k(y) = \sum_{\ell=0}^{k} (\log y)^\ell \sum_{n \geq 2-2(k-\ell)} d_{n,k,\ell}\, y^{-n}\,.$$

To obtain a good approximate solution, we are then constrained to restrict the construction to the region $\frac{\rho}{t} \leq t^{\epsilon_1}\cdot$

The aim of Section 4 is to extend the approximate solution built in Section 3 to the self-similar region $\frac{1}{10} t^{\epsilon_1} \leq \frac{\rho}{t} \leq 10\, t^{-\epsilon_2}$. Taking into account the matching conditions coming out from the inner region, we seek to this extension under the form:

$$u_{\rm ss}^{(N)}(t,\rho) = \rho + \lambda(t) \sum_{k=3}^{N} t^{\nu k} \sum_{\ell=0}^{\ell(k)} \big(\log t\big)^\ell\, w_{k,\ell}\Big(\frac{\rho}{\lambda(t)}\Big)\,, \tag{1.18}$$

where $\lambda(t)$ is a suitable function which behaves like t near $t=0$, and where the functions $w_{k,\ell}$, for $3 \leq k \leq N$ and $0 \leq \ell \leq \ell(k) = [\frac{k-3}{2}]$, are determined by induction using again (1.8). Actually, the natural idea is rather to look for $u_{\rm ss}^{(N)}$ under the form (1.18), where $\lambda(t)$ is replaced by t. However, it turns out that the operator coming to play in this region that is the linearized of (1.8) around ρ written with respect to the variable $\Big(t, \dfrac{\rho}{t}\Big)$, is degenerate on the light cone $\dfrac{\rho}{t} = \dfrac{1}{\sqrt{2}}$, which induces a loss of regularity at each step.

Actually, the light cone associated to the quasilinear equation to be dealt with in this region is different from $\dfrac{\rho}{t} = \dfrac{1}{\sqrt{2}}$, and is rather given by $\dfrac{\rho}{\lambda(t)} = \dfrac{1}{\sqrt{2}}\cdot$ It will be constructed concurrently with the solution. As it will be established in Paragraph 4.2.1, the functions $w_{k,\ell}$ which are determined successively by solving a recurrent system admits an asymptotic behavior under the form:

$$w_{k,\ell}(z) \sim c_{k,\ell}\, z^{k\nu+1}(\log z)^{\frac{k-3}{2}-\ell}\,,$$

as z tends to infinity, which imposes to restrict the self-similar region to $\dfrac{\rho}{t} \lesssim t^{-\epsilon_2}$, with $0 < \epsilon_2 < 1$.

In Section 5, we construct an approximate solution $u^{(N)}_{\rm out}$ which extends $u^{(N)}_{\rm ss}$ to the whole space, by solving the quasilinear wave equation (1.8) associated to an adapted Cauchy data in the remote region $\dfrac{\rho}{t} \geq t^{-\epsilon_2}\cdot$ We shall look for the approximate solution in that region under the following form:

$$u^{(N)}_{\rm out}(t,\rho) = \rho + \mathfrak{g}_0(\rho) + t\mathfrak{g}_1(\rho) + \sum_{k=2}^{N} t^k \mathfrak{g}_k(\rho)\,, \tag{1.19}$$

where the Cauchy data $(\cdot + \mathfrak{g}_0, \mathfrak{g}_1)$ is determined by the the matching conditions coming out of the self-similar region, and where for $k \geq 2$ the functions $\mathfrak{g}_k$ are determined successively by a recurrent relation under the form

$$\mathfrak{g}_k = \mathcal{G}_k\big(\mathfrak{g}_j, j \leq k-1\big)\cdot$$

As it will be seen in Paragraph 5.2, these functions $\mathfrak{g}_k$, $k \geq 0$, are compactly supported and behaves as $\rho^{1-k+3\nu}$ close to 0, which ensures that (1.19) provides us with a good approximate solution in the remote region.

Section 7 is dedicated to the end of the proof of the blow up result by constructing an exact solution to (1.8), thanks to a perturbative argument. For that purpose, we firstly write

$$u = u^{(N)} + \varepsilon^{(N)}\,,$$

where $u^{(N)}$ is the approximate solution to (1.8) constructed in Sections 3, 4, 5 , and then taking into account (1.8), we derive the equation satisfied by the remainder term $\varepsilon^{(N)}$ with respect to the variable $\Big(t, \dfrac{x}{t^{1+\nu}}\Big)\cdot$

The study of the equation for $\varepsilon^{(N)}$ is based on continuity arguments coupled with suitable energy estimates. These energy estimates are established by combining the spectral properties of the operator $\mathcal{L}$ together with the estimates of the approximate solution $u^{(N)}$.

Finally, we deal in appendix with several complements for the sake of completeness and the convenience of the reader. It is organized as follows. Section A is devoted to the derivation of the quasilinear wave equations (1.6). In Section B, we analyze the spectral properties of the operator $\mathcal{L}$. In Section C, we give the proof of the local well-posedness result for the Cauchy problem (1.11), namely Theorem 1.1, and in Section D, we collect some useful ordinary differential equations results that we use in the self-similar region.

To avoid heaviness, we shall omit in this text the dependence of all the functions on the parameter ν. All along this article, T and C will denote respectively positive time and constant depending on several parameters, and which may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some absolute constant C.

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2. Analysis of the stationary solution

2.1. **Asymptotic behavior of the stationary solution.** Our analysis in this paper is intimately connected to the behavior at infinity of the stationary solution to the quasilinear wave equation (1.8). In this subsection, we collect the properties of Q that we will use through out this paper.

Lemma 2.1. *The Cauchy problem*

$$\begin{cases} -Q_{\rho\rho} + 3(1+Q_\rho^2)\Big(\dfrac{1}{Q} - \dfrac{Q_\rho}{\rho}\Big) = 0 \\ Q(0) = 1 \quad \text{and} \quad Q_\rho(0) = 0 \end{cases} \tag{2.1}$$

has a unique solution [3] $Q \in \mathcal{C}^\infty(\mathbb{R}_+)$ *which satisfies the following properties:*

- *Q has an even Taylor expansion* [4] *at 0:*

$$Q(\rho) = \sum_{n\geq 0} \gamma_{2n}\, \rho^{2n}, \tag{2.2}$$

  with some constants γ_{2n} such that $\gamma_0 = 1$,
- *Q enjoys the following bounds for any ρ in $\mathbb{R}_+$*

$$Q(\rho) > \rho \quad \text{and} \quad Q''(\rho) > 0\,, \tag{2.3}$$

- *Q has the following asymptotic expansion as ρ tends to infinity:*

$$Q(\rho) = \rho + \sum_{n\geq 2} d_n \rho^{-n}\,, \tag{2.4}$$

with some constants d_n such that $d_2 > 0$ and $d_4 = 0$.

Proof. It is well-known (see for instance [6, 33] and the references therein) that the Cauchy problem (2.1) admits a unique solution Q in $\mathcal{C}^\infty(\mathbb{R}_+)$ satisfying (2.3), and which behaves as

$$Q(\rho) = \rho + \frac{d_2}{\rho^2}\,(1 + \circ(1)) \text{ when } \rho \to \infty\,,$$

with $d_2 > 0$.

In order to determine the asymptotic formula (2.4), let us for $\rho \geq 1$ set

$$Q(\rho) = \rho\, v(\log \rho)\,.$$

According to (2.1), this ensures that the function v satisfies

$$-\,(v_{yy} + v_y) + 3(1 + (v+v_y)^2)(\frac{1}{v} - v - v_y) = 0\,. \tag{2.5}$$

Observe that the function $v \equiv 1$ solves (2.5) and that the linearization of (2.5) around $v \equiv 1$ takes the following form:

$$-\,w_{yy} - 7w_y - 12w = 0\,. \tag{2.6}$$

[3] All along this paper, we identify the radial functions on $\mathbb{R}^n$ with the functions on $\mathbb{R}_+$.

[4] All the asymptotic expansions through this paper can be differentiated any number of times.

The characteristic equation of the above linear differential equation (2.6) admits two real distinct roots $r_1 = -3$ and $r_2 = -4$. This ensures that Q has the following asymptotic development as ρ tends to infinity:

$$Q(\rho) = \rho + \sum_{n \geq 2} d_n \rho^{-n}\,,$$

for some constants d_n, with $d_4 = 0$.

Finally one can check that the formulae (2.2) and (2.4) can be differentiated at any order with respect to the variable ρ, which completes the proof of the lemma. $\square$

2.2. **Properties of the linearized operator of the quasilinear wave equation around the ground state.** The blow up solution we construct in this paper is a small perturbation of the profile

$$t^{\nu+1} Q\Big(\frac{\rho}{t^{\nu+1}}\Big),$$

and thus the linearization of the quasilinear wave equation (1.8) around Q will play an important role in our approach. This linearized equation has the form:

$$(1 + Q_\rho^2) w_{tt} - \mathcal{L} w = 0\,, \tag{2.7}$$

where $\mathcal{L}$ denotes the operator introduced in (1.16) that will be extensively analyzed in Appendix B. It will be useful later on to emphasize that the function ΛQ, where $\Lambda Q = Q - \rho\, Q_\rho$ is a particular solution of the homogeneous equation $\mathcal{L} w = 0$ which is positive. Indeed by virtue of Lemma 2.1, ΛQ is positive on $\rho = 0$, tends to 0 at infinity and satisfies $(\Lambda Q)_\rho = -\rho\, Q_{\rho\rho}$. Recalling that $Q_{\rho\rho}(\rho) > 0$, we end up with the claim.

The strategy we shall adopt in this article is based on the fact that, up to the change of function $w = H\, g$, with

$$H := \frac{(1 + Q_\rho^2)^{\frac{1}{4}}}{Q^{\frac{3}{2}}}\,, \tag{2.8}$$

the above equation (2.7) rewrites on the following way:

$$g_{tt} + \mathfrak{L} g = 0\,, \tag{2.9}$$

where $\mathfrak{L}$ is the positive self-adjoint operator on $L^2(\mathbb{R}^4)$ defined by (see Appendix B for the proof of this fact):

$$\mathfrak{L} = -q\, \Delta\, q + \mathcal{P}\,, \tag{2.10}$$

with $q = \dfrac{1}{(1 + Q_\rho^2)^{\frac{1}{2}}}$, and where the potential $\mathcal{P}$ belongs to $\mathcal{C}^\infty_{rad}(\mathbb{R}^4)$ and satisfies

$$\mathcal{P}(\rho) = -\frac{3}{8\rho^2}(1 + \circ(1)), \text{ as } \rho \to \infty\cdot \tag{2.11}$$

The spectral properties of the operator $\mathfrak{L}$ which are investigated in Appendix B rely on the asymptotic behavior of the potential $\mathcal{P}$ at infinity given by (2.11). It comes out of this spectral analysis that the operator $\mathfrak{L}$ is positive. Furthermore, there is a positive constant c such that we have

$$\big(\mathfrak{L} f | f\big)_{L^2(\mathbb{R}^4)} \geq c\, \|\nabla f\|^2_{L^2(\mathbb{R}^4)}\,, \quad \forall f \in \dot{H}^1_{rad}(\mathbb{R}^4)\,. \tag{2.12}$$

3. Approximate solution in the inner region

3.1. **General scheme of the construction of the approximate solution in the inner region.** In this section, we shall built in the region $\frac{\rho}{t} \leq t^{\epsilon_1}$ (where $0 < \epsilon_1 < \nu$ is a fixed positive real number) a family of approximate solutions $u_{\rm in}^{(N)}$ to the quasilinear wave equation (1.8) as a perturbation of the profile $t^{\nu+1} Q\Big(\frac{x}{t^{\nu+1}}\Big)\cdot$

Writing

$$u(t,\rho) = t^{\nu+1} V\Big(t, \frac{\rho}{t^{\nu+1}}\Big)\,, \tag{3.1}$$

we get by straightforward computations

$$\begin{aligned}
u_\rho(t,\rho) &= V_y\Big(t, \frac{\rho}{t^{\nu+1}}\Big)\,,\\
u_{\rho\rho}(t,\rho) &= \frac{1}{t^{\nu+1}}\, V_{yy}\Big(t, \frac{\rho}{t^{\nu+1}}\Big)\,,\\
u_t(t,\rho) &= t^{\nu+1}\, V_t\Big(t, \frac{\rho}{t^{\nu+1}}\Big) + (\nu+1)\, t^{\nu}\, \Lambda V\Big(t, \frac{\rho}{t^{\nu+1}}\Big) := t^{\nu} (\Gamma V)\Big(t, \frac{\rho}{t^{\nu+1}}\Big)\,,\\
u_{t\rho}(t,\rho) &= t^{-1}\, (\Gamma V)_y\Big(t, \frac{\rho}{t^{\nu+1}}\Big) \quad \text{and}\\
t^{\nu+1}\, u_{tt}(t,\rho) &= t^{2\nu}[\Gamma^2 V - \Gamma V]\Big(t, \frac{\rho}{t^{\nu+1}}\Big)\,,
\end{aligned}$$

where we denote

$$\Gamma V := t\partial_t V + (\nu+1)\Lambda V \quad \text{with} \quad \Lambda V = V - yV_y \quad \text{and} \quad y = \frac{\rho}{t^{\nu+1}}\,\cdot \tag{3.2}$$

Thus replacing u by means of (3.1) into (1.8) and multiplying by $t^{\nu+1}$, we get the following equation

$$\begin{aligned}
&(1+V_y^2)t^{2\nu}\,[\Gamma^2 V - \Gamma V] - \big(1 - t^{2\nu}(\Gamma V)^2\big)V_{yy}\\
&- 2\, t^{2\nu}\, V_y\, (\Gamma V)\, (\Gamma V)_y + 3\,(1 + V_y^2 - t^{2\nu}(\Gamma V)^2)\Big(\frac{1}{V} - \frac{V_y}{y}\Big) = 0\,\cdot
\end{aligned} \tag{3.3}$$

It will be useful later on to point out that the above equation (3.3) multiplied by $\frac{V}{Q}$ is polynomial of order four with respect to $(V, V_y, V_{yy}, \Gamma V, (\Gamma V)_y, \Gamma^2 V)$.

In what follows, we shall look for solutions V to Equation (3.3) under the form

$$V(t,y) = \sum_{k\geq 0} t^{2\nu k} V_k(y)\,, \tag{3.4}$$

with $V_0 = Q$, where Q is the stationary solution introduced in Lemma 2.1.

Substituting this ansatz into (3.3) multiplied by $\frac{V}{Q}$, we deduce the following recurrent equation for $k \geq 1$

$$\mathcal{L} V_k = F_k(V_0, \cdots, V_{k-1}) \tag{3.5}$$

subject to the initial conditions

$$V_k(0) = 0 \quad \text{and} \quad V_k'(0) = 0\,, \tag{3.6}$$

where F_k depends on V_j, $j = 0, \cdots, k-1$ only.

Here $\mathcal{L}$ is defined by (1.16). Taking advantage of the asymptotic formula (2.4), this easily leads for y large to the following asymptotic expansions

$$\begin{cases} B_1(y) = \dfrac{3}{y} + \displaystyle\sum_{n\geq 4} \beta_n y^{-n} \\ B_0(y) = \dfrac{6}{y^2} + \displaystyle\sum_{n\geq 5} \alpha_n y^{-n} \,, \end{cases} \tag{3.7}$$

with some constants β_n and α_n that can be computed by means of the coefficients d_n involved in the asymptotic formula (2.4).

Along the same lines, in view of (2.2) we find the following asymptotic formulae when y is close to 0

$$\begin{cases} B_1(y) = \displaystyle\sum_{n\geq 0} a_{2n+1}\, y^{2n+1} \\ B_0(y) = 3 + \displaystyle\sum_{n\geq 1} b_{2n}\, y^{2n} \,, \end{cases} \tag{3.8}$$

with some constants (a_{2n+1}) and (b_{2n}) that can be expressed in terms of the coefficients (γ_{2n}) that arise in (2.2).

Besides the source term F_k can be splitted on two parts as follows:

$$F_k = F_k^{(1)} + F_k^{(2)} \,,$$

with $F_1^{(1)} = 0$ and $F_k^{(1)}$ for $k \geq 2$ determined by the following equation

$$-\frac{V}{Q} V_{yy} + 3\,(1 + V_y^2)\Big(\frac{1}{Q} - \frac{V\,V_y}{y\,Q}\Big) = \sum_{k\geq 1} \Big(-\mathcal{L}V_k + F_k^{(1)}\Big) t^{2\nu k} \,. \tag{3.9}$$

According to (3.4), this gives explicitly [5]

$$\begin{aligned} F_k^{(1)} = -\frac{1}{Q} \sum_{\substack{j_1+j_2=k\\ j_i\geq 1}} V_{j_1} \Big((V_{j_2})_{yy} + 3\frac{(V_{j_2})_y}{y}\Big) - \frac{3}{y\,Q} \sum_{\substack{j_1+j_2+j_3+j_4=k\\ j_i\leq k-1}} (V_{j_1})_y (V_{j_2})_y (V_{j_3})_y V_{j_4} \\ + \frac{3}{Q} \sum_{\substack{j_1+j_2=k\\ j_i\geq 1}} (V_{j_1})_y (V_{j_2})_y \,. \end{aligned} \tag{3.10}$$

Finally combining (3.3) together with (3.9) and using the fact that

$$\Gamma(t^{2\nu k} V_k) = t^{2\nu k} \Gamma_k V_k \,, \tag{3.11}$$

where $\Gamma_k = 2\nu k + (1+\nu)\Lambda$, we readily gather that

$$\begin{aligned} F_k^{(2)} = & \sum_{\substack{j_1+j_2+j_3+j_4=k-1\\ j_i\geq 0}} \frac{V_{j_1}}{Q} (\Gamma_{j_2} V_{j_2})(\Gamma_{j_3} V_{j_3}) \Big((V_{j_4})_{yy} + \frac{3\,(V_{j_4})_y}{y}\Big) \\ & + \sum_{\substack{j_1+j_2=k-1\\ j_i\geq 0}} \frac{V_{j_1}}{Q} \Big(\Gamma_{j_2}^2 - \Gamma_{j_2}\Big) V_{j_2} + \sum_{\substack{j_1+j_2+j_3+j_4=k-1\\ j_i\geq 0}} \frac{V_{j_1} (V_{j_2})_y (V_{j_3})_y}{Q} \Big(\Gamma_{j_4}^2 - \Gamma_{j_4}\Big) V_{j_2} \\ & - 2 \sum_{\substack{j_1+j_2+j_3+j_4=k-1\\ j_i\geq 0}} \frac{V_{j_1} (V_{j_2})_y}{Q} (\Gamma_{j_3} V_{j_3})\, (\Gamma_{j_4} V_{j_4})_y - \sum_{\substack{j_1+j_2=k-1\\ j_i\geq 0}} \frac{3}{Q} (\Gamma_{j_1} V_{j_1})(\Gamma_{j_2} V_{j_2}) \,. \end{aligned} \tag{3.12}$$

[5] Here and below, we use the convention that the sum is null if it is over an empty set.

3.2. **Analysis of the functions V_k.** The goal of the present paragraph is to prove the following result:

Lemma 3.1. *For any integer $k \geq 1$, the Cauchy problem* (3.5)-(3.6) *has a unique solution V_k in $\mathcal{C}^\infty(\mathbb{R}_+)$ with the following asymptotic behaviors:*

$$V_k(y) = \sum_{n\geq 1} c_{2n,k}\, y^{2n}\,, \ \text{ as } y \sim 0\,, \tag{3.13}$$

and

$$V_k(y) = \sum_{\ell=0}^{k} (\log y)^\ell \sum_{n\geq 2-2(k-\ell)} d_{n,k,\ell}\, y^{-n}\,, \ \text{ as } y \sim \infty\,, \tag{3.14}$$

with

$$d_{-2(k-2),k,1} = 0\,. \tag{3.15}$$

Proof. Let us firstly emphasize that by classical techniques of ordinary differential equations, for any regular function g, the solution to the Cauchy problem

$$\begin{cases} \mathcal{L}f = g \\ f(0) = 0 \quad \text{and} \quad f'(0) = 0\,, \end{cases} \tag{3.16}$$

writes under the following form (see Appendix D for the proof of this formula)

$$f(y) = -(\Lambda Q)(y) \int_0^y \frac{(1+(Q_r(r))^2)^{\frac{3}{2}}}{Q^3(r)\, r^3\, (\Lambda Q)^2(r)} \int_0^r \frac{Q^3(s)\, s^3\, (\Lambda Q)(s)}{(1+(Q_s(s))^2)^{\frac{3}{2}}}\, g(s)\, ds\, dr\,\cdot \tag{3.17}$$

Let us start by considering the case when $k = 1$. Under notations (3.2) and in light of (3.10) and (3.12), we have

$$\begin{aligned} F_1(Q) = F_1^{(2)}(Q) &= (1+Q_y^2)\Big((1+\nu)^2\Lambda^2 - (1+\nu)\Lambda\Big)Q \\ &\quad - 2(1+\nu)^2 Q_y(\Lambda Q)(\Lambda Q)_y + (1+\nu)^2(\Lambda Q)^2\,\frac{Q_{yy}\,(Q_y)^2}{(1+Q_y^2)}\,\cdot \end{aligned} \tag{3.18}$$

According to (2.2), this implies that for y close to 0 the following asymptotic formula holds

$$F_1(Q) = \sum_{n\geq 0} g_{2n,1}\, y^{2n}. \tag{3.19}$$

Besides in view of (2.4), we get for y sufficiently large the following expansion

$$F_1(Q) = \sum_{n\geq 2} c_{n,1,0}\, y^{-n}\,. \tag{3.20}$$

By virtue of Lemma 2.1 which asserts that [6] $d_{4,0,0} := d_4 = 0$, we find that $c_{4,1,0} = 0$. Indeed invoking (2.4) together with (3.18), we easily check that

$$c_{4,1,0} = 10\,(1+\nu)\,(4+5\nu)\, d_{4,0,0}\,,$$

which implies that the coefficient $c_{4,1,0}$ is null.

This ensures in view of Duhamel formula (3.17) that in that case, the Cauchy problem (3.5)-(3.6) admits a unique solution V_1 in $\mathcal{C}^\infty(\mathbb{R}_+)$ satisfying the asymptotic expansions (3.13) and (3.14) respectively close to 0 and at infinity.

[6] We shall designate in what follows the coefficients d_p involved in Formula (2.4) by $d_{p,0,0}$.

Regarding to the expansion coefficients $d_{n,1,\ell}$ of V_1 at infinity, we can find them by substituting

$$V_1(y) = \sum_{\ell=0}^{1} (\log y)^\ell \sum_{n\geq 2\ell} d_{n,1,\ell}\, y^{-n}$$

into (3.5) and taking into account (3.7) and (3.20). This gives rise to

$$\begin{aligned}
&\frac{2\,d_{1,1,0}}{y^3} - \sum_{n\geq 2}((2n+1)d_{n,1,1} - n(n+1)\,d_{n,1,0})\,y^{-n-2}\\
&+\Big(\frac{6}{y} + \sum_{n\geq 4}\beta_n y^{-n}\Big)\Big(-\frac{d_{1,1,0}}{y^2} + \sum_{n\geq 2}(d_{n,1,1} - n\,d_{n,1,0})y^{-n-1} - \sum_{n\geq 2} n\,d_{n,1,1}\,(\log y)\,y^{-n-1}\Big)\\
&+\Big(\frac{6}{y^2} + \sum_{n\geq 5}\alpha_n y^{-n}\Big)\Big(d_{0,1,0} + \frac{d_{1,1,0}}{y} + \sum_{n\geq 2} d_{n,1,0}y^{-n} + \sum_{n\geq 2} d_{n,1,1}\,(\log y)\,y^{-n}\Big)\\
&+\sum_{n\geq 2} n(n+1)\,d_{n,1,1}\,(\log y)\,y^{-n-2} = \sum_{n\geq 2} c_{n,1,0}y^{-n}\,.
\end{aligned} \tag{3.21}$$

In particular, the identification of the coefficient of y^{-4} in (3.21) gives

$$d_{2,1,1} = c_{4,1,0} = 0\,, \tag{3.22}$$

which proves that Condition (3.15) is fulfiled for $k = 1$.

Now using the fact that the coefficient of $(\log y)\,y^{-n-2}$ in (3.21) is null, we find that for any integer $n \geq 2$

$$d_{n,1,1}(n^2 - 5n + 6) + \sum_{\substack{k_1+k_2=n+2\\ k_1\geq 5,\, k_2\geq 2}} d_{k_2,1,1}\,\alpha_{k_1} - \sum_{\substack{k_1+k_2=n+1\\ k_1\geq 4,\, k_2\geq 2}} k_2\,d_{k_2,1,1}\,\beta_{k_1} = 0\,.$$

Along the same lines, by computing the coefficients of y^{-n-2} we get

$$d_{n,1,0}(n^2-5n+6)+(5-2n)d_{n,1,1}+\sum_{\substack{k_1+k_2=n+1\\ k_1\geq 4,\, k_2\geq 2}} \beta_{k_1}(d_{k_2,1,1}-k_2 d_{k_2,1,0})+\sum_{\substack{k_1+k_2=n+2\\ k_1\geq 5,\, k_2\geq 2}} \alpha_{k_1} d_{k_2,1,0} = c_{n+2,1,0}.$$

This implies that all the coefficients $d_{n,1,\ell}$ can be determined successively in terms of the coefficients of $F_1(Q)$ involved in (3.20) and the coefficients $d_{2,1,0}$ and $d_{3,1,0}$ that are fixed by the initial data.

We next turn our attention to the general case of any index $k \geq 2$. To this end, we shall proceed by induction assuming that, for any integer $1 \leq j \leq k-1$, the Cauchy problem (3.5)-(3.6) admits a unique solution V_j in $\mathcal{C}^\infty(\mathbb{R}_+)$ satisfying formulae (3.13) and (3.14) as well as Condition (3.15).

Invoking (3.10) together with (3.13) and (3.14), one can easily check that

$$F_k^{(1)}(V_0, \cdots, V_{k-1}) = \sum_{n\geq 0} g_{2n,k}^{(1)}\, y^{2n}\,, \text{ as } y \sim 0\,, \tag{3.23}$$

$$F_k^{(1)}(V_0, \cdots, V_{k-1})(y) = \sum_{\ell=0}^{k} (\log y)^\ell \sum_{n\geq 7-2(k-\ell)} c_{n,k,\ell}^{(1)}\, y^{-n}\,, \text{ as } y \sim \infty\,. \tag{3.24}$$

Similarly from (3.12), (3.13) and (3.14), we deduce that

$$F_k^{(2)}(V_0, \cdots, V_{k-1})(y) = \sum_{n\geq 0} g_{2n,k}^{(2)}\, y^{2n}\,, \text{ as } y \sim 0\,, \tag{3.25}$$

$$F_k^{(2)}(V_0, \cdots, V_{k-1}) = (1 + Q_y^2)\,\big(\Gamma_{k-1}^2 - \Gamma_{k-1}\big)\,V_{k-1} + \widetilde{F}_k^{(2)}(V_0, \cdots, V_{k-1})\,, \tag{3.26}$$

where $\widetilde{F}_k^{(2)}$ admits the following expansion at infinity

$$\widetilde{F}_k^{(2)}(V_0,\cdots,V_{k-1})(y)=\sum_{\ell=0}^{k-1}(\log y)^\ell\sum_{n\geq 7-2(k-\ell)}\widetilde{c}^{(2)}_{n,k,\ell}\,y^{-n}\,. \tag{3.27}$$

Recall that by definition

$$\Gamma_{k-1}=2\nu(k-1)+(1+\nu)\Lambda:=\alpha(\nu,k-1)+(1+\nu)\Lambda\,,$$

which by straightforward computations gives rise to [7]

$$\left(\Gamma_{k-1}^2-\Gamma_{k-1}\right)V_{k-1}=\alpha(\alpha-1)\,V_{k-1}+(1+\nu)(2\alpha-1)\,\Lambda V_{k-1}+(1+\nu)^2\Lambda^2 V_{k-1}\,.$$

Setting

$$\beta:=\alpha(\alpha-1)+(1+\nu)(2\alpha-1)+(1+\nu)^2\,,$$

we easily gather that

$$\left(\Gamma_{k-1}^2-\Gamma_{k-1}\right)V_{k-1}=\beta\,V_{k-1}-(1+\nu)(2\alpha-1)\,y\partial_y V_{k-1}+(1+\nu)^2y^2\partial_y^2 V_{k-1}\,.$$

It follows therefore from (2.4) and (3.14) that the following expansion holds at infinity

$$(1+Q_y^2)\left(\Gamma_{k-1}^2-\Gamma_{k-1}\right)V_{k-1}(y)=\sum_{\ell=0}^{k-1}(\log y)^\ell\sum_{n\geq 2-2(k-1-\ell)}c^{(2)}_{n,k,\ell}\,y^{-n}\,, \tag{3.28}$$

where, under the above notations, for any integer $0\leq\ell\leq k-1$

$$2\,c^{(2)}_{2-2(k-1-\ell),k,\ell}=\left(\beta+n(1+\nu)(2\alpha-1)+(1+\nu)^2n(n+1)\right)d_{2-2(k-1-\ell),k-1,\ell}\,.$$

In view of the induction assumption (3.15) for the index $k-1$, we get

$$c^{(2)}_{2-2(k-2),k,1}=0\,. \tag{3.29}$$

Combining (3.23) together with (3.24), (3.25), (3.27), (3.28) and (3.29), we deduce that

$$F_k(V_0,\cdots,V_{k-1})=F_k^{(1)}(V_0,\cdots,V_{k-1})+F_k^{(2)}(V_0,\cdots,V_{k-1})$$

admits the following asymptotic expansions:

$$F_k(V_0,\cdots,V_{k-1})(y)=\sum_{n\geq 0}g_{2n,k}\,y^{2n}\,,\ \text{ as } y\sim 0\,, \tag{3.30}$$

$$F_k(V_0,\cdots,V_{k-1})(y)=\sum_{\ell=0}^{k}(\log y)^\ell\sum_{n\geq 4-2(k-\ell)}c_{n,k,\ell}\,y^{-n}\,,\ \text{ as } y\sim\infty\,, \tag{3.31}$$

which can be differentiated any number of times with respect to y, and with

$$c_{2-2(k-2),k,1}=0\,. \tag{3.32}$$

Therefore Duhamel formula (3.17) implies that the Cauchy problem (3.5)-(3.6) admits a unique solution V_k in $\mathcal{C}^\infty(\mathbb{R}_+)$ satisfying the asymptotic formulae (3.13) and (3.14) respectively close to 0 and at infinity. As for V_1 we can determine all the coefficients $d_{n,k,\ell}$ in terms of F_k and $d_{2,k,0}$ and $d_{3,k,0}$ that are fixed by the initial data, by substituting the expansion

$$V_k(y)=\sum_{\ell=0}^{k}(\log y)^\ell\sum_{n\geq 2-2(k-\ell)}d_{n,k,\ell}\,y^{-n}$$

[7] In order to make notations as light as possible, we shall omit all along this proof the dependence of the function α on the parameters ν and k.

into (3.5). In particular, we get for $0 \leq \ell \leq k-1$ and $n = 2 - 2(k-\ell)$

$$(n^2 - 5n + 6) d_{n,k,\ell} = c_{n+2,k,\ell}\,,$$

which by virtue of (3.32) ensures that $d_{-2(k-2),k,1} = 0$ and proves (3.15).

Clearly, the asymptotic expansions (3.13) and (3.14) can be differentiated any number of times with respect to the variable y. This concludes the proof of the lemma. $\square$

3.3. **Estimate of the approximate solution in the inner region.** Under the above notations, set for any integer $N \geq 2$

$$u_{\rm in}^{(N)}(t,\rho) = t^{\nu+1} V_{\rm in}^{(N)}\Big(t, \frac{\rho}{t^{\nu+1}}\Big) \quad \text{with} \quad V_{\rm in}^{(N)}(t,y) = \sum_{k=0}^{N} t^{2\nu k} V_k\big(y\big)\,\cdot \tag{3.33}$$

Our aim in this paragraph is to investigate the properties of $V_{\rm in}^{(N)}$ in the inner region, namely in the region of $\mathbb{R}^4$ defined as follows:

$$\Omega_{\rm in} := \big\{ Y \in \mathbb{R}^4,\, y = |Y| \leq t^{\epsilon_1 - \nu} \big\}\,. \tag{3.34}$$

Thanks to Lemma 3.1, we easily gather that $V_{\rm in}^{(N)}$ satisfies the following L^∞ estimates on $\Omega_{\rm in}$:

Lemma 3.2. *For any multi-index α in $\mathbb{N}^4$ and any integer $\beta \leq |\alpha|$, there exist a positive constant $C_{\alpha,\beta}$ and a small positive time $T = T(\alpha,\beta,N)$ such that for all $0 < t \leq T$, the following estimates hold:*

$$\|\langle\cdot\rangle^\beta \nabla^\alpha (V_{\rm in}^{(N)}(t,\cdot) - Q)\|_{L^\infty(\Omega_{\rm in})} \leq C_{\alpha,\beta}\, t^{2\nu}\,, \tag{3.35}$$

$$\|\nabla^\alpha \partial_t V_{\rm in}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\rm in})} \leq C_\alpha\, t^{2\nu-1}\,, \tag{3.36}$$

$$\|\langle\cdot\rangle^\beta \nabla^\alpha (\Gamma V_{\rm in}^{(N)})(t,\cdot)\|_{L^\infty(\Omega_{\rm in})} \leq C_{\alpha,\beta}\,, \tag{3.37}$$

$$\|\partial_t (\Gamma V_{\rm in}^{(N)})(t,\cdot)\|_{L^\infty(\Omega_{\rm in})} \leq C\, t^{2\nu-1}\,, \tag{3.38}$$

$$\|\langle\cdot\rangle^\beta \nabla^\alpha ((\Gamma^2 - \Gamma) V_{\rm in}^{(N)})(t,\cdot)\|_{L^\infty(\Omega_{\rm in})} \leq C_{\alpha,\beta}\,, \tag{3.39}$$

where as above $\Gamma = t\partial_t + (\nu+1)\Lambda$.

Along the same lines taking advantage of Lemma 3.1, we get the following L^2 estimates:

Lemma 3.3. *Under the above notations, we have for all $0 < t \leq T$:*

$$\|\nabla (V_{\rm in}^{(N)}(t,\cdot) - Q)\|_{L^2(\Omega_{\rm in})} \leq C\, t^{\nu}\,, \tag{3.40}$$

$$\|\nabla^\alpha (V_{\rm in}^{(N)}(t,\cdot) - Q)\|_{L^2(\Omega_{\rm in})} \leq C_\alpha\, t^{2\nu}\,,\ \forall\, |\alpha| \geq 2\,, \tag{3.41}$$

$$\|(\Gamma^\ell V_{\rm in}^{(N)})(t,\cdot)\|_{L^2(\Omega_{\rm in})} \leq C\, \log t\,,\ \forall\, \ell = 1,2\,, \tag{3.42}$$

$$\|\nabla^\alpha (\Gamma^\ell V_{\rm in}^{(N)})(t,\cdot)\|_{L^2(\Omega_{\rm in})} \leq C_\alpha\,,\ \forall\, |\alpha| \geq 1\,, \forall\, \ell = 1,2\,. \tag{3.43}$$

Remark 3.1. *Denoting by*

$$\Omega_{\rm in}^x := \big\{ x \in \mathbb{R}^4,\, |x| \leq t^{1+\epsilon_1} \big\}\,,$$

and combining (3.33) together with the above lemma, we infer that the following estimates hold for the radial function $u_{\rm in}^{(N)}$ on $\Omega_{\rm in}^x$:

$$\|\nabla^\alpha (u_{\rm in}^{(N)}(t,\cdot) - t^{\nu+1} Q\big(\tfrac{\cdot}{t^{\nu+1}}\big))\|_{L^2(\Omega_{\rm in}^x)} \leq C_\alpha\, t^{\nu + (|\alpha|-3)(\nu+1)}\,,\ \forall\, |\alpha| \geq 1\,, \tag{3.44}$$

$$\|\nabla^\alpha \partial_t u_{\rm in}^{(N)}(t,\cdot)\|_{L^2(\Omega_{\rm in}^x)} \leq C_\alpha\, t^{\nu + (|\alpha|-3)(\nu+1)}\,,\ \forall\, |\alpha| \geq 1\,, \tag{3.45}$$

$$\|\partial_t u_{\rm in}^{(N)}(t,\cdot)\|_{L^2(\Omega_{\rm in}^x)} \leq C\, t^{\nu - 3(\nu+1)}\, \log t\,, \tag{3.46}$$

for all $0 < t \leq T$.

Let us end this section by estimating the remainder term:

$$\mathcal{R}_{\text{in}}^{(N)} := (3.3)\, V_{\text{in}}^{(N)} \,.$$

One has:

Lemma 3.4. *For any multi-index* α*, there exist a positive constant* $C_{\alpha,N}$ *and a small positive time* $T = T(\alpha, N)$ *such that for all* $0 < t \leq T$*, the remainder term* $\mathcal{R}_{\text{in}}^{(N)}$ *satisfies*

$$\|\langle\cdot\rangle^{\frac{3}{2}}\, \nabla^\alpha \mathcal{R}_{\text{in}}^{(N)}(t,\cdot)\|_{L^2(\Omega_{\text{in}})} \leq C_{\alpha,N}\, t^{2\nu+2N\epsilon_1-\frac{3}{2}(\nu-\epsilon_1)} \,. \tag{3.47}$$

Proof. In view of computations carried out in Section 3.2 and particularly on page 11, we have

$$\frac{V}{Q}\, \Big[(3.3)\Big(\sum_{k\geq 0} t^{2\nu k} V_k\Big)\Big] = \sum_{k\geq 1} \Big(-\mathcal{L}V_k + F_k\Big) t^{2\nu k} \,.$$

Thus recalling that $\dfrac{V}{Q}\,\big[(3.3)V\big]$ is a polynomial of order four and taking into account Lemma 3.1, we deduce that

$$\widetilde{\mathcal{R}}_{\text{in}}^{(N)} = \frac{V_{\text{in}}^{(N)}}{Q}\, \mathcal{R}_{\text{in}}^{(N)} = \sum_{N+1\leq k\leq 4N} t^{2\nu k} G_k \,, \tag{3.48}$$

where G_k depending on V_j, $j = 0, \cdots, N$, is defined as the function F_k by formulae similar to (3.10) and (3.12), where we assume in addition that the involved indices j_i range from 0 to N.

This of course implies that the function G_k, $N+1 \leq k \leq 4N$ admits the following expansions that can be differentiated any number of times with respect to the variable y, respectively close to 0 and at infinity:

$$G_k(y) = \sum_{n\geq 0} \widetilde{g}_{2n,k}\, y^{2n} \,, \tag{3.49}$$

$$G_k(y) = \sum_{\ell=0}^{k} (\log y)^\ell \sum_{n\geq 4-2(k-\ell)} \widetilde{c}_{n,k,\ell}\, y^{-n} \,, \tag{3.50}$$

with some constants $\widetilde{g}_{2n,k}$ and $\widetilde{c}_{n,k,\ell}$ that can be determined recursively in terms of the functions V_j, for $j = 0, \cdots, N$.

Recalling that by definition

$$\mathcal{R}_{\text{in}}^{(N)} = \frac{Q}{V_{\text{in}}^{(N)}}\, \widetilde{\mathcal{R}}_{\text{in}}^{(N)} \,,$$

we deduce taking into account Lemma 3.2 and Formula (3.34) that for any multi-index α, there exist a positive constant $C_{\alpha,N}$ and a positive time $T = T(\alpha, N)$ such that for any time $0 < t \leq T$, we have

$$\|\langle\cdot\rangle^{\frac{3}{2}}\, \nabla^\alpha \mathcal{R}_{\text{in}}^{(N)}(t,\cdot)\|_{L^2(\Omega_{\text{in}})} \leq C_{\alpha,N}\, t^{2\nu+2N\epsilon_1-\frac{3}{2}(\nu-\epsilon_1)} \,.$$

This ends the proof of the lemma.

□

4. Approximate solution in the self-similar region

4.1. General scheme of the construction of the approximate solution in the self-similar region. Our aim in this section is to built in the region $\frac{1}{10}t^{\epsilon_1} \leq \frac{\rho}{t} \leq 10\,t^{-\epsilon_2}$ an approximate solution $u_{\rm ss}^{(N)}$ to (1.8) which extends the approximate solution $u_{\rm in}^{(N)}$ constructed in the inner region $\frac{\rho}{t} \leq t^{\epsilon_1}\cdot$ Here $0 < \epsilon_2 < 1$ is fixed.

We shall look for this solution under the following form:

$$u(t,\rho) = \lambda(t)\,(z + W(t,z)) \quad \text{with} \quad z = \frac{\rho}{\lambda(t)}, \tag{4.1}$$

and where $\lambda(t)$ is a function which behaves like t, for t close to 0, and that will be constructed at the same time as the profile W. In fact, $\lambda(t)$ will be given by an expression of the form:

$$\lambda(t) = t\Big(1 + \sum_{k\geq 3}\sum_{\ell=0}^{\ell(k)} \lambda_{k,\ell}\, t^{\nu k}\,(\log t)^{\ell}\Big) \quad \text{with} \quad \ell(k) = [\frac{k-3}{2}]\cdot \tag{4.2}$$

By straightforward computations, we find that

$$\begin{aligned}
u_\rho(t,\rho) &= 1 + W_z\Big(t, \frac{\rho}{\lambda(t)}\Big)\,,\\
u_{\rho\rho}(t,\rho) &= \lambda(t)^{-1} W_{zz}\Big(t, \frac{\rho}{\lambda(t)}\Big)\,,\\
u_t(t,\rho) &= \lambda(t)\, W_t\Big(t, \frac{\rho}{\lambda(t)}\Big) + \lambda'(t)\,\Lambda W\Big(t, \frac{\rho}{\lambda(t)}\Big) := W_1\Big(t, \frac{\rho}{\lambda(t)}\Big)\,, \qquad (4.3)\\
u_{t\rho}(t,\rho) &= \lambda(t)^{-1}(\partial_z W_1)\Big(t, \frac{\rho}{\lambda(t)}\Big)\,,\\
\lambda(t)\, u_{tt}(t,\rho) &= W_2\Big(t, \frac{\rho}{\lambda(t)}\Big)\,, \qquad (4.4)
\end{aligned}$$

with

$$\begin{aligned}
W_2(t,z) &:= \lambda(t)\lambda''(t)\,\Lambda W + 2\lambda(t)\lambda'(t)\,\Lambda W_t + \lambda^2(t)\, W_{tt} + (\lambda'(t))^2 z^2 W_{zz}\\
&= z^2\, W_{zz}(t,z) + t^2 W_{tt}(t,z) + 2t\Lambda W_t(t,z) + \widetilde{W}_2(t,z)\,,
\end{aligned} \tag{4.5}$$

where

$$\widetilde{W}_2(t,z) = ((\lambda'(t))^2 - 1)z^2 W_{zz} + (\lambda^2(t) - t^2)W_{tt} + 2(\lambda(t)\lambda'(t) - t)\Lambda W_t + \lambda(t)\lambda''(t)\,\Lambda W\,, \tag{4.6}$$

and where as above $\Lambda W = W - zW_z$.

Thus substituting u by means of (4.1) into (1.8) multiplied by $\lambda(t)$, we find that the function W solves the following equation:

$$\begin{aligned}
&(1 + (1+W_z)^2)W_2 - (1 - (W_1)^2)W_{zz} - 2(1+W_z)W_1(W_1)_z\\
&\qquad - 3\big(1 + (1+W_z)^2 - (W_1)^2\big)\Big(\frac{\check{W}}{z^2} + \frac{W_z}{z}\Big) = 0\,,
\end{aligned} \tag{4.7}$$

where $\check{W} = \dfrac{W}{(1+\frac{W}{z})}\cdot$

Introducing the notations

$$W_2' := W_2 - (\lambda')^2 z^2 W_{zz} = \lambda^2 W_{tt} + 2\lambda\lambda'\Lambda W_t + \lambda\lambda''\Lambda W\,,\ W_3 := \lambda\, W_{tz}\,, \tag{4.8}$$

we readily gather that the above equation (4.7) rewrites in the following way:

$$(2z^2 - 1 + A_0)\, W_{zz} + A_1 = 0\,, \tag{4.9}$$

with

$$\begin{aligned}(4.10)\quad A_0 &= (\lambda' W + \lambda W_t)^2 + 2\lambda' z(\lambda' W + \lambda W_t) + 2((\lambda')^2 - 1)z^2\,,\\ A_1 &= (1 + (1 + W_z)^2)W_2' - 2(1 + W_z)W_1W_3 - 3\big(1 + (1 + W_z)^2 - (W_1)^2\big)\Big(\frac{\check W}{z^2} + \frac{W_z}{z}\Big)\cdot\end{aligned}$$

Denoting by L the linear operator defined by:

$$LW = (2z^2 - 1)\, W_{zz} + 2t^2 W_{tt} + 4t\Lambda W_t - 6\frac{W_z}{z} - 6\frac{W}{z^2}\,, \tag{4.11}$$

we infer that the above equation (4.7) undertakes the following form:

$$\begin{aligned}LW = -A_0 W_{zz} - \,&\big[2W_z + (W_z)^2\big]\, W_2' - 2\widetilde{W}_2' + 2\,(1 + W_z)W_1W_3\\ &- \frac{6}{z^3}\check W\, W + 3\big(2W_z + (W_z)^2 - (W_1)^2\big)\Big(\frac{\check W}{z^2} + \frac{W_z}{z}\Big)\,,\end{aligned} \tag{4.12}$$

where

$$\widetilde{W}_2' := W_2' - t^2 W_{tt} - 2t\Lambda W_t = \widetilde{W}_2 - ((\lambda')^2 - 1)z^2\, W_{zz}\,. \tag{4.13}$$

It will be useful later on to notice that under the above notations, (4.12) also rewrites in the following way:

$$\begin{aligned}LW = -2\,\widetilde{W}_2 - \,&\big[2W_z + (W_z)^2\big]\, W_2 - (W_1)^2\, W_{zz} + 2\,(1 + W_z)W_1(W_1)_z\\ &- \frac{6}{z^3}\check W\, W + 3\big(2W_z + (W_z)^2 - (W_1)^2\big)\Big(\frac{\check W}{z^2} + \frac{W_z}{z}\Big)\cdot\end{aligned} \tag{4.14}$$

The asymptotic of the solution (4.1) at the origin has to be coherent with that of (3.1) at infinity. To determine this asymptotic, we combine the expansion (4.2) together with Formula (3.14), which gives:

$$u_{\rm in}(t,\rho) = \lambda(t)\Big(z + \sum_{k\geq 3} t^{\nu k}\sum_{\ell=0}^{\ell(k)}(\log t)^\ell \sum_{0\leq\alpha\leq\frac{k-3}{2}-\ell}(\log z)^\alpha \sum_{\beta\geq 1-k+2(\alpha+\ell)} c^{k,\ell}_{\alpha,\beta} z^\beta\Big),\ \text{ as } z\to 0\,, \tag{4.15}$$

where the coefficients $c^{k,\ell}_{\alpha,\beta}$ admit the representation:

$$c^{k,\ell}_{\alpha,\beta} = c^{k,\ell,0}_{\alpha,\beta} + c^{k,\ell,1}_{\alpha,\beta}\,,$$

with $c^{k,\ell,0}_{\alpha,\beta}$ independent of λ and given by

$$\begin{cases} c^{k,\ell,0}_{\alpha,\beta} = 0\,,\ \text{ if } \beta + k - 1 \text{ is odd} \quad \text{and}\\ c^{k,\ell,0}_{\alpha,\beta} = (-\nu)^\ell \dbinom{\alpha+\ell}{\alpha}\, d_{-\beta,\frac{\beta+k-1}{2},\alpha+\ell}\,,\ \text{ if } \beta + k - 1 \text{ is even}\,,\end{cases}$$

where $d_{n,k,\ell}$ denotes the coefficient arising in (3.1).

The coefficients $c^{k,\ell,1}_{\alpha,\beta}$ depend only on $\lambda_{p,q}$ involved in (4.2) with $3\leq p\leq k-3$ and are equal to zero if $\beta + k - 1 - 2(\alpha+\ell)\leq 2$ or if $k < 6$ or if $\ell > \dfrac{k-6}{2}\cdot$

Let us point out that taking into account Lemma 2.1 together with Property (3.15), which respectively assert that $d_{4,0,0} = 0$ and $d_{4-2m,m,1} = 0$ for any integer $m\geq 1$, we infer that

$$c^{5,0}_{0,-4} = 0 \quad \text{and} \quad c^{5,1}_{0,\beta} = 0,\ \forall \beta\,. \tag{4.16}$$

Formula (4.15) leads us to look for the approximate solution in the self-similar region under the form:

$$u(t,\rho)=\rho+\lambda(t)\,W\Big(t,\frac{\rho}{\lambda(t)}\Big)\,, \tag{4.17}$$

where

$$W(t,z)=\sum_{k\geq 3}t^{\nu k}\sum_{\ell=0}^{\ell(k)}(\log t)^{\ell}\,w_{k,\ell}(z)\,. \tag{4.18}$$

To fix $\lambda(t)$, we require that the function A_0 defined by (4.10) satisfies

$$\big(A_0\big)_{z=\frac{1}{\sqrt{2}}}=0\cdot \tag{4.19}$$

Actually a difficulty that we face in solving (4.12) is in handling the singularity of the operator L defined by (4.11) on the light cone $z=\dfrac{1}{\sqrt{2}}\cdot$ The above condition (4.19) ensures that the coefficient of W_{zz} involved in the equation we deal with vanishes at $z=\dfrac{1}{\sqrt{2}}\cdot$ This will enable us to determine successively the functions $w_{k,\ell}$ involved in (4.18) without loss of regularity at each step.

Invoking (4.2) together with (4.18), we infer that the functions W_1, W_2, $\widetilde{W}_2$, W_2', $\widetilde{W}_2'$, W_3, $\check{W}$ and A_0 defined above admit expansions of the same form as W. More precisely, one has:

$$\begin{aligned}
W_i(t,z)&=\sum_{k\geq 3}t^{\nu k}\sum_{0\leq\ell\leq\frac{k-3}{2}}(\log t)^{\ell}\,w^i_{k,\ell}(z)\,,i=1,2,3\,,\\
\widetilde{W}_2(t,z)&=\sum_{k\geq 6}t^{\nu k}\sum_{0\leq\ell\leq\frac{k-6}{2}}(\log t)^{\ell}\,\widetilde{w}^2_{k,\ell}(z)\,,\\
W_2'(t,z)&=\sum_{k\geq 3}t^{\nu k}\sum_{0\leq\ell\leq\frac{k-3}{2}}(\log t)^{\ell}\,w^{(2,')}_{k,\ell}(z)\,,\\
\widetilde{W}_2'(t,z)&=\sum_{k\geq 6}t^{\nu k}\sum_{0\leq\ell\leq\frac{k-6}{2}}(\log t)^{\ell}\,\widetilde{w}^{(2,')}_{k,\ell}(z)\,,\\
\check{W}(t,z)&=\sum_{k\geq 3}t^{\nu k}\sum_{0\leq\ell\leq\frac{k-3}{2}}(\log t)^{\ell}\,\check{w}_{k,\ell}(z)\,,\\
A^0(t,z)&=\sum_{k\geq 3}t^{\nu k}\sum_{0\leq\ell\leq\frac{k-3}{2}}(\log t)^{\ell}\,A^0_{k,\ell}(z)\,,
\end{aligned}$$

where $w^i_{k,\ell}$, $i=1,2,3$, and $w^{(2,')}_{k,\ell}$ depend only on $w_{k',\ell'}$, $3\leq k'\leq k$ and $\lambda_{k'',\ell''}$, $3\leq k''\leq k-3$, where $\widetilde{w}^2_{k,\ell}$ and $\widetilde{w}^{(2,')}_{k,\ell}$ depend on $w_{k',\ell'}$ and $\lambda_{k'',\ell''}$, $3\leq k',k''\leq k-3$ and $A^0_{k,\ell}$ on $w_{k',\ell'}$ and $\lambda_{k'',\ell''}$, with $3\leq k',k''\leq k$.

Observe also that [8]

$$\begin{aligned}
&w^1_{k,\ell}=\big(\nu k+\Lambda\big)w_{k,\ell}+(\ell+1)w_{k,\ell+1}+\widetilde{w}^1_{k,\ell}\,,\\
&\widetilde{w}^1_{k,\ell}=\sum_{k_1+k_2=k,\ell_1+\ell_2=\ell}\lambda_{k_2,\ell_2}\,\big(\nu k_1\,w_{k_1,\ell_1}+(\ell_1+1)w_{k_1,\ell_1+1}\big)\\
&\qquad\qquad+\sum_{k_1+k_2=k,\ell_1+\ell_2=\ell}\big[(1+\nu k_2)\lambda_{k_2,\ell_2}++(\ell_2+1)\lambda_{k_2,\ell_2+1}\big]\,\Lambda w_{k_1,\ell_1}\,,
\end{aligned} \tag{4.20}$$

[8] with the convention all along this section that $\lambda_{k,\ell'}=0$ and $w_{k,\ell'}\equiv 0$ if $k<3$ or $\ell'>[\dfrac{k-3}{2}]\cdot$

and that

$$w^3_{k,\ell} = \nu k \partial_z w_{k,\ell} + (\ell+1)\partial_z w_{k,\ell+1} + \widetilde{w}^3_{k,\ell}\,,$$
$$\widetilde{w}^3_{k,\ell} = \sum_{k_1+k_2=k,\ell_1+\ell_2=\ell} \lambda_{k_2,\ell_2}\left(\nu k_1\,\partial_z w_{k_1,\ell_1} + (\ell_1+1)\partial_z w_{k_1,\ell_1+1}\right). \tag{4.21}$$

In addition, one has [9]

$$w^2_{k,\ell}(z) = z^2\partial_z^2 w_{k,\ell} + \nu k\big(\nu k + 1 - 2z\partial_z\big)w_{k,\ell} + (\ell+1)\big(2\nu k + 1 - 2z\partial_z\big)w_{k,\ell+1} + (\ell+1)(\ell+2)w_{k,\ell+2} + \widetilde{w}^2_{k,\ell}\,, \tag{4.22}$$

and

$$w^{(2,')}_{k,\ell} = \nu k\big(\nu k + 1 - 2z\partial_z\big)w_{k,\ell} + (\ell+1)\big(2\nu k + 1 - 2z\partial_z\big)w_{k,\ell+1} + (\ell+1)(\ell+2)w_{k,\ell+2} + \widetilde{w}^{(2,')}_{k,\ell}\,. \tag{4.23}$$

Now substituting expansions (4.2) and (4.18) into (4.12) and (4.19), we deduce the following recurrent system for $k \geq 3$:

$$\begin{cases} \widetilde{\mathcal{L}}_k w_{k,\ell} = F_{k,\ell}\,,\ 0 \leq \ell \leq \ell(k) \\ (1+\nu k)\lambda_{k,\ell} + (\ell+1)\lambda_{k,\ell+1} = -\big((1+\nu k)w_{k,\ell} + (\ell+1)w_{k,\ell+1}\big)_{z=\frac{1}{\sqrt{2}}} + g_{k,\ell}\,. \end{cases} \tag{4.24}$$

Here $\widetilde{\mathcal{L}}_k$ refers to the operator

$$\widetilde{\mathcal{L}}_k w = (2z^2-1)w_{zz} - \Big(4\,z\,\nu\,k + \frac{6}{z}\Big)w_z + \Big(2\,\nu k(1+\nu k) - \frac{6}{z^2}\Big)w\,, \tag{4.25}$$

and the source term $F_{k,\ell}$ can be divided into a linear and a nonlinear parts as follows:

$$F_{k,\ell} = F^{\rm lin}_{k,\ell} + F^{\rm nl}_{k,\ell}\,, \tag{4.26}$$

where

$$F^{\rm lin}_{k,\ell} = -2(2\nu k+1)\,(\ell+1)w_{k,\ell+1} + 4\,z\,(\ell+1)(w_{k,\ell+1})_z - 2(\ell+1)(\ell+2)w_{k,\ell+2}\,, \tag{4.27}$$

and where [10] the nonlinear part $F^{\rm nl}_{k,\ell}$, for $k \geq 6$ and $0 \leq \ell \leq \dfrac{k-6}{2}$, only depends on $w_{k',\ell'}$ and $\lambda_{k'',\ell''}$, for $3 \leq k', k'' \leq k-3$. Similarly, the coefficients $g_{k,\ell}$ only depends on the values of the functions $w_{k',\ell'}$ on $z = \frac{1}{\sqrt{2}}$ and the coefficients $\lambda_{k'',\ell''}$, for $3 \leq k', k'' \leq k-3$.

In other words, for any integer $k \geq 3$ the functions $(w_{k,\ell})_{0\leq \ell\leq \ell(k)}$ satisfy:

$$\mathcal{S}_k\,\mathcal{W}_k = \mathcal{F}^{\rm nl}_k \tag{4.28}$$

where $\mathcal{S}_k$ denotes the following matrix operator:

---

[9] One can for $\widetilde{w}^2_{k,\ell}$ and $\widetilde{w}^{(2,')}_{k,\ell}$ give explicit expressions of the same type as for $\widetilde{w}^1_{k,\ell}$ and $\widetilde{w}^3_{k,\ell}$, but to avoid needlessly burdening the text, we will not explicit them.

[10] $F^{\rm nl}_{k,\ell}$ and $g_{k,\ell}$ are identically null if $k < 6$ or $\ell > \frac{k-6}{2}$.

$$\begin{pmatrix} \widetilde{\mathcal{L}}_k & \mathcal{A}_k(0)+\mathcal{B}(0,z)\partial_z & \mathcal{C}(0) & 0 & \dots & \dots & \dots \\ 0 & \widetilde{\mathcal{L}}_k & \mathcal{A}_k(1)+\mathcal{B}(1,z)\partial_z & \mathcal{C}(1) & 0 & \dots & \dots \\ \dots & 0 & \widetilde{\mathcal{L}}_k & \dots & \dots & 0 & \dots \\ \dots & \dots & \dots & \widetilde{\mathcal{L}}_k & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \widetilde{\mathcal{L}}_k & \mathcal{A}_k(\ell(k)-1)+\mathcal{B}(\ell(k)-1,z)\partial_z \\ \dots & \dots & \dots & \dots & \dots & \dots & \widetilde{\mathcal{L}}_k \end{pmatrix}$$

with

$$\begin{aligned} \mathcal{A}_k(\ell) &= -2(2\nu k+1)\,(\ell+1)\,, \\ \mathcal{B}(\ell,z) &= 4z(\ell+1)\,, \\ \mathcal{C}(\ell) &= -2(\ell+1)(\ell+2)\,, \end{aligned}$$

and

$$\mathcal{W}_k = \begin{pmatrix} w_{k,0} \\ \vdots \\ w_{k,\ell} \\ \vdots \\ w_{k,\ell(k)} \end{pmatrix}, \qquad \mathcal{F}_k^{\rm nl} = \begin{pmatrix} F_{k,0}^{\rm nl} \\ \vdots \\ F_{k,\ell}^{\rm nl} \\ \vdots \\ F_{k,\ell(k)}^{\rm nl} \end{pmatrix}.$$

Let us emphasize that we do not subject the above system to any Cauchy data as for the system (3.5) corresponding to the inner region. In order to solve uniquely (4.24), we shall take into account the matching conditions coming out from the inner region, namely we require that

$$w_{k,\ell}(z) = \sum_{\substack{0\le\alpha\le\frac{k-3}{2}-\ell \\ \beta\ge 1-k+2(\alpha+\ell)}} c_{\alpha,\beta}^{k,\ell}\,(\log z)^{\alpha}\, z^{\beta}\,,\ \text{as } z\to 0\,, \tag{4.29}$$

where $c_{\alpha,\beta}^{k,\ell} = c_{\alpha,\beta}^{k,\ell}(\lambda)$ are given by (4.15).

In view of (4.12), one can write $F_{k,\ell}^{\rm nl}$ explicitely as follows:

$$F_{k,\ell}^{\rm nl} = F_{k,\ell}^{\rm nl,1} + F_{k,\ell}^{\rm nl,2} + F_{k,\ell}^{\rm nl,3} + F_{k,\ell}^{\rm nl,4}\,, \tag{4.30}$$

where

$$F_{k,\ell}^{\rm nl,1} = -2\widetilde{w}_{k,\ell}^{(2,')}, \tag{4.31}$$

$$\begin{aligned} F_{k,\ell}^{\rm nl,2} = \sum_{\substack{j_1+j_2=k \\ \ell_1+\ell_2=\ell}} & 6\,(w_{j_1,\ell_1})_z\Big(\frac{1}{z}(w_{j_2,\ell_2})_z + \frac{1}{z^2}\breve{w}_{j_2,\ell_2}\Big) - 2(w_{j_1,\ell_1})_z w_{j_2,\ell_2}^{(2,')} \\ & + \sum_{\substack{j_1+j_2=k \\ \ell_1+\ell_2=\ell}} 2\,w_{j_1,\ell_1}^{1} w_{j_2,\ell_2}^{3} - \frac{6}{z^3} w_{j_1,\ell_1}\breve{w}_{j_2,\ell_2}\,, \end{aligned} \tag{4.32}$$

$$
\begin{aligned}
F^{\mathrm{nl},3}_{k,\ell} = & \sum_{\substack{j_1+j_2+j_3=k\\ \ell_1+\ell_2+\ell_3=\ell}} 2\, w^1_{j_1,\ell_1} w^3_{j_2,\ell_2} (w_{j_3,\ell_3})_z - (w_{j_1,\ell_1})_z (w_{j_2,\ell_2})_z w^{(2,')}_{j_3,\ell_3} \\
& + 3 \sum_{\substack{j_1+j_2+j_3=k\\ \ell_1+\ell_2+\ell_3=\ell}} \big((w_{j_1,\ell_1})_z (w_{j_2,\ell_2})_z - w^1_{j_1,\ell_1} w^1_{j_2,\ell_2}\big)\Big(\frac{\check{w}_{j_3,\ell_3}}{z^2} + \frac{(w_{j_3,\ell_3})_z}{z}\Big),
\end{aligned} \tag{4.33}
$$

and

$$
F^{\mathrm{nl},4}_{k,\ell} = - \sum_{\substack{j_1+j_2=k\\ \ell_1+\ell_2=\ell}} A^0_{j_1,\ell_1} (w_{j_2,\ell_2})_{zz}\,. \tag{4.34}
$$

For our purpose, it will be useful to point out that according to (4.14), one also has:

$$
F^{\mathrm{nl}}_{k,\ell} = \widetilde{F}^{\mathrm{nl},1}_{k,\ell} + \widetilde{F}^{\mathrm{nl},2}_{k,\ell} + \widetilde{F}^{\mathrm{nl},3}_{k,\ell}\,, \tag{4.35}
$$

where

$$
\widetilde{F}^{\mathrm{nl},1}_{k,\ell} = -2\widetilde{w}^2_{k,\ell}, \tag{4.36}
$$

$$
\begin{aligned}
\widetilde{F}^{\mathrm{nl},2}_{k,\ell} = & \sum_{\substack{j_1+j_2=k\\ \ell_1+\ell_2=\ell}} 6\,(w_{j_1,\ell_1})_z \Big(\frac{1}{z}(w_{j_2,\ell_2})_z + \frac{1}{z^2}\check{w}_{j_2,\ell_2}\Big) - 2\,(w_{j_1,\ell_1})_z w^2_{j_2,\ell_2} \\
& + \sum_{\substack{j_1+j_2=k\\ \ell_1+\ell_2=\ell}} 2\, w^1_{j_1,\ell_1} (w^1_{j_2,\ell_2})_z - \frac{6}{z^3} w_{j_1,\ell_1} \check{w}_{j_2,\ell_2}\,,
\end{aligned} \tag{4.37}
$$

$$
\begin{aligned}
\widetilde{F}^{\mathrm{nl},3}_{k,\ell} = & - \sum_{\substack{j_1+j_2+j_3=k\\ \ell_1+\ell_2+\ell_3=\ell}} (w_{j_1,\ell_1})_z (w_{j_2,\ell_2})_z w^2_{j_3,\ell_3} + w^1_{j_1,\ell_1} w^1_{j_2,\ell_2} (w_{j_3,\ell_3})_{zz} \\
& + 2 \sum_{\substack{j_1+j_2+j_3=k\\ \ell_1+\ell_2+\ell_3=\ell}} w^1_{j_1,\ell_1} (w^1_{j_2,\ell_2})_z (w_{j_3,\ell_3})_z \\
& + 3 \sum_{\substack{j_1+j_2+j_3=k\\ \ell_1+\ell_2+\ell_3=\ell}} \big((w_{j_1,\ell_1})_z (w_{j_2,\ell_2})_z - w^1_{j_1,\ell_1} w^1_{j_2,\ell_2}\big)\Big(\frac{\check{w}_{j_3,\ell_3}}{z^2} + \frac{(w_{j_3,\ell_3})_z}{z}\Big).
\end{aligned} \tag{4.38}
$$

4.2. **Analysis of the vector functions $\mathcal{W}_k$.**

4.2.1. *Study of the linear system $\mathcal{S}_k$.* In order to determine successively the solutions $w_{k,\ell}$ of the recurrent system (4.28), let us under the above notations, start by investigating the homogeneous equation:

$$
\mathcal{S}_k\, X = 0\,. \tag{4.39}
$$

We infer that the following lemma holds:

Lemma 4.1. *For j in $\big\{0,\cdots,\ell(k)\big\}$, define $(f^{j,\pm}_{k,\ell})_{0\le\ell\le\ell(k)}$ by*

$$
\begin{aligned}
f^{j,\pm}_{k,\ell}(z) &= \binom{j}{\ell} \Big(\log\Big|\frac{1}{\sqrt{2}} \pm z\Big|\Big)^{j-\ell} \frac{\big|\frac{1}{\sqrt{2}} \pm z\big|^{\alpha(\nu,k)}}{z^3}\,, \\
f^{j,\pm}_{k,\ell} &= 0\,,\ \textit{for } j+1 \le \ell \le \ell(k)\,,
\end{aligned} \tag{4.40}
$$

where $\alpha(\nu,k) = \nu k + 4$.

Then denoting by

$$f_k^{j,\pm} = \begin{pmatrix} f_{k,0}^{j,\pm} \\ \vdots \\ f_{k,j}^{j,\pm} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

the vector functions $(f_k^{j,\pm})_{0\leq j\leq \ell(k)}$ *constitute a basis of the homogeneous equation* (4.39), *on the intervals* $\big]0,\frac{1}{\sqrt{2}}\big[$ *and* $\big]\frac{1}{\sqrt{2}},\infty\big[\cdot$

Proof. Consider the linearization of (1.8) around ρ:

$$2\, v_{tt} - l_\rho v = 0\,, \tag{4.41}$$

where

$$l_\rho = \partial_\rho^2 + 6\Big(\frac{\partial_\rho}{\rho} + \frac{1}{\rho^2}\Big)\cdot \tag{4.42}$$

Writing

$$v(t,\rho) = tw(t,z) \quad \text{with} \quad z = \frac{\rho}{t}\,,$$

we clearly get under notation (4.11)

$$Lw = 0.$$

Observe also that (4.41) is equivalent to

$$2(\rho^3 v)_{tt} - (\rho^3 v)_{\rho\rho} = 0\,. \tag{4.43}$$

Set

$$G(t,z) = t^{\nu k+1}\,\big(\log t + \log\big|\frac{1}{\sqrt{2}} \pm z\big|\big)^j\, \frac{\big|\frac{1}{\sqrt{2}} \pm z\big|^{\alpha(\nu,k)}}{z^3}\cdot$$

Since

$$G(t,z) = \big(\log\big|\frac{t}{\sqrt{2}} \pm \rho\big|\big)^j\, \frac{\big|\frac{t}{\sqrt{2}} \pm \rho\big|^{\alpha(\nu,k)}}{\rho^3} = \frac{F\big(\big|\frac{t}{\sqrt{2}} \pm \rho\big|\big)}{\rho^3}\,,$$

for some function F, we infer that G satisfies

$$L(t^{-1}\, G) = 0\,.$$

This implies that

$$L\Big(t^{\nu k}\,\big(\log t + \log\big|\frac{1}{\sqrt{2}} \pm z\big|\big)^j\, \frac{\big|\frac{1}{\sqrt{2}} \pm z\big|^{\alpha(\nu,k)}}{z^3}\Big) = 0\,.$$

Since

$$t^{\nu k}\,\big(\log t + \log\big|\frac{1}{\sqrt{2}} \pm z\big|\big)^j\, \frac{\big|\frac{1}{\sqrt{2}} \pm z\big|^{\alpha(\nu,k)}}{z^3} = t^{\nu k} \sum_{\ell=0}^{j} \big(\log t\big)^\ell f_{k,\ell}^{j,\pm}(z)\,, \tag{4.44}$$

we obtain the result, recalling that

$$L\Big(t^{\nu k} \sum_{\ell=0}^{j} \big(\log t\big)^\ell f_{k,\ell}^{j,\pm}(z)\Big) = 0 \Leftrightarrow \mathcal{S}_k f_k^{j,\pm} = 0\,.$$

□

Remark 4.1. *Note that in view of the above lemma, the homogeneous equation*

$$\widetilde{\mathcal{L}}_k f = 0$$

admits the following basis:

$$\begin{cases} f^{0,+}_{k,0}(z) = \dfrac{\big(\frac{1}{\sqrt{2}} + z\big)^{\alpha(\nu,k)}}{z^3}\,, \\ f^{0,-}_{k,0}(z) = \dfrac{\big|\frac{1}{\sqrt{2}} - z\big|^{\alpha(\nu,k)}}{z^3}\,. \end{cases} \tag{4.45}$$

Before concluding this section, let us collect some useful properties about the elements of the basis $(f^{j,\pm}_k)_{0\leq j\leq \ell(k)}$ given above.

Lemma 4.2. *Under the above notations, the following asymptotic expansions hold*

$$[\nu k+\Lambda] f^{j,\pm}_{k,\ell}(z) + (\ell+1) f^{j,\pm}_{k,\ell+1}(z) = z^{\nu k} \sum_{0\leq\alpha\leq j-\ell}\ \sum_{p\in\mathbb{N}} \gamma^k_{p,\alpha}\big(\log z\big)^{\alpha} z^{-p}\,, \ \text{as } z\to\infty\,, \tag{4.46}$$

$$\begin{aligned} &[z^2\,\partial_z^2 + \nu k(\nu k+1-2z\partial_z)]\, f^{j,\pm}_{k,\ell}(z) + (\ell+1)\,[2\nu k+1-2z\partial_z]\, f^{j,\pm}_{k,\ell+1}(z) \\ &\quad + (\ell+1)(\ell+2)\, f^{j,\pm}_{k,\ell+2}(z) = z^{\nu k-1} \sum_{0\leq\alpha\leq j-\ell}\ \sum_{p\in\mathbb{N}} \hat{\gamma}^k_{p,\alpha}\big(\log z\big)^{\alpha} z^{-p}\,, \ \text{as } z\to\infty\,, \end{aligned} \tag{4.47}$$

for any integer $k\geq 3$ and all j,ℓ in $\big\{0,\cdots,\ell(k)\big\}$, respectively for some constants $\gamma^k_{p,\alpha}$ and $\hat{\gamma}^k_{p,\alpha}$.

Proof. In view of Formula (4.44), we have for large ρ

$$\big(\log\big(\rho\pm\frac{t}{\sqrt{2}}\big)\big)^j\, \frac{\big(\rho\pm\frac{t}{\sqrt{2}}\big)^{\alpha(\nu,k)}}{\rho^3} = t^{\nu k+1} \sum_{\ell=0}^{j} \big(\log t\big)^{\ell} f^{j,\pm}_{k,\ell}\big(\frac{\rho}{t}\big)\,. \tag{4.48}$$

Therefore taking the derivative of the above identity with respect to t, we deduce that

$$\begin{aligned} &\frac{1}{\sqrt{2}}\Big(j\big(\log\big(\rho\pm\frac{t}{\sqrt{2}}\big)\big)^{j-1} \frac{\big(\rho\pm\frac{t}{\sqrt{2}}\big)^{\alpha(\nu,k)-1}}{\rho^3} + \alpha(\nu,k)(\log\big(\rho\pm\frac{t}{\sqrt{2}}\big))^{j}\, \frac{\big(\rho\pm\frac{t}{\sqrt{2}}\big)^{\alpha(\nu,k)-1}}{\rho^3}\Big) \\ &= t^{\nu k} \sum_{\ell=0}^{j} \big(\log t\big)^{\ell}\Big((\nu k+\Lambda) f^{j,\pm}_{k,\ell} + (\ell+1) f^{j,\pm}_{k,\ell+1}\Big)\big(\frac{\rho}{t}\big)\,. \end{aligned} \tag{4.49}$$

Performing the change of variables $z=\dfrac{\rho}{t}$, we infer that

$$\begin{aligned} &\frac{t^{\nu k}}{\sqrt{2}} \frac{\big(z\pm\frac{1}{\sqrt{2}}\big)^{\alpha(\nu,k)-1}}{z^3} \Big(j\big(\log t + \log\big(z\pm\frac{1}{\sqrt{2}}\big)\big)^{j-1} + \alpha(\nu,k)\big(\log t+\log\big(z\pm\frac{1}{\sqrt{2}}\big)\big)^{j}\Big) \\ &= t^{\nu k} \sum_{\ell=0}^{j} \big(\log t\big)^{\ell}\big((\nu k+\Lambda) f^{j,\pm}_{k,\ell} + (\ell+1) f^{j,\pm}_{k,\ell+1}\big)(z)\,, \end{aligned} \tag{4.50}$$

which concludes the proof of (4.46).

Along the same lines taking the derivative with respect to t of (4.49) ensures Identity (4.47), which ends the proof of the lemma.

∎

4.2.2. *Study of the functions $w_{k,\ell}$.* The goal of this paragraph is to prove by induction that the system (4.28) admits a solution satisfying the matching conditions (4.29) coming out from the inner region.

For that purpose, let us start by the following usefull lemma which stems from standard techniques of ordinary differential equations. For the sake of completeness and the convenience of the reader, we outline its proof in Appendix D.

Lemma 4.3. *Under the above notations* [11]*, the following properties hold:*

- *For any function g in $\mathcal{C}^\infty(\mathbb{R}_+^*)$, the equation*
$$\widetilde{\mathcal{L}}_k f = g$$
admits a unique solution f in $\mathcal{C}^\infty(\mathbb{R}_+^*)$ satisfying $f\Big(\dfrac{1}{\sqrt{2}}\Big) = 0\cdot$
- *For any function h in $\mathcal{C}^\infty\big(]0, \dfrac{1}{\sqrt{2}}]\big)$, any $\gamma > 0$, and any integer q, the equation*
$$\widetilde{\mathcal{L}}_k f(z) = \big(\frac{1}{\sqrt{2}} - z\big)^\gamma \big(\log\big(\frac{1}{\sqrt{2}} - z\big)\big)^q h(z) \tag{4.51}$$
admits a unique solution f of the form:
$$f(z) = \big(\frac{1}{\sqrt{2}} - z\big)^{\gamma+1} \sum_{0\leq \ell\leq q} \big(\log\big(\frac{1}{\sqrt{2}} - z\big)\big)^\ell h_\ell(z)\,,$$
where for all $0 \leq \ell \leq q$, h_ℓ is a function in $\mathcal{C}^\infty\big(]0, \dfrac{1}{\sqrt{2}}]\big)$, provided that the exponent γ satisfies
$$\nu k + 4 - \gamma \notin \mathbb{N}^*\ . \tag{4.52}$$
- *Let g be a function in $\mathcal{C}^\infty\big(]0, \dfrac{1}{\sqrt{2}}[\big)$ with an asymptotic expansion at 0 of the form:*
$$g(z) = (\log z)^{\alpha_0} \sum_{\beta\geq\beta_0} g_\beta\, z^{\beta-2}\,,$$
for some integers α_0, β_0, then any solution f of the equation
$$\widetilde{\mathcal{L}}_k f = g \tag{4.53}$$
belongs to $\mathcal{C}^\infty\big(]0, \dfrac{1}{\sqrt{2}}[\big)$ and admits for z close to 0 an asymptotic expansion of the type:
$$f(z) = \sum_{\beta\geq -3} f_{0,\beta}\, z^\beta + \sum_{1\leq\alpha\leq\alpha_0}\sum_{\beta\geq\beta_0} f_{\alpha,\beta}(\log z)^\alpha z^\beta\,,$$
in the case when $\beta_0 \geq -1$, and of the type
$$f(z) = \sum_{\beta\geq\min(\beta_0,-3)} f_{0,\beta} z^\beta + \sum_{1\leq\alpha\leq\alpha_0}\sum_{\beta\geq\beta_0} f_{\alpha,\beta}(\log z)^\alpha z^\beta + \sum_{\beta\geq\max(\beta_0,-3)} f_{\alpha_0+1,\beta}(\log z)^{\alpha_0+1} z^\beta\,,$$
in the case when $\beta_0 \leq -2$.
- *If g denotes a function belonging to $\mathcal{C}^\infty\big(]\dfrac{1}{\sqrt{2}},\infty[\big)$ and admitting at infinity an asymptotic expansion of the form:*
$$g(z) = \sum_{0\leq\alpha\leq\alpha_0}\sum_{p\in\mathbb{N}} \hat{g}_{\alpha,p}\,(\log z)^\alpha\, z^{A-p}\,,$$

[11] and again with the convention that the sum is null if it is over an empty set.

for some real $A < \nu k$ and some integer α_0, then the equation

$$\widetilde{\mathcal{L}}_k f = g \tag{4.54}$$

admits a unique solution f in $\mathcal{C}^\infty\big(]\frac{1}{\sqrt{2}},\infty[\big)$ such that

$$f(z) = \sum_{0\leq\alpha\leq\alpha_0}\sum_{p\in\mathbb{N}} \hat{f}^k_{\alpha,p}(\log z)^\alpha \, z^{A-p}\,, z\to\infty\,.$$

The key result of this paragraph is the following proposition:

Proposition 4.1. *Under the above notations, the following properties hold:*

(1) <u>*Existence*</u>
The system (4.24) admits a solution $(w_{k,\ell},\lambda_{k,\ell})_{k\geq 3, 0\leq\ell\leq\ell(k)}$ such that for any integer $k\geq 3$ and any $\ell\in\{0,\cdots,\ell(k)\}$, the function $w_{k,\ell}$ belongs to $\mathcal{C}^{[\alpha(\nu,k)]}(\mathbb{R}^*_+)\cap\mathcal{C}^\infty\big(\mathbb{R}^*_+\setminus\{\frac{1}{\sqrt{2}}\}\big)$, and has the form [12]*:*

$$\begin{aligned} w_{k,\ell}(z) &= a^{\rm reg}_{k,\ell}(z) + \big(\frac{1}{\sqrt{2}}-z\big)^{k\nu+4}\sum_{0\leq\alpha\leq\frac{k-3}{2}-\ell} b^{\rm reg}_{k,\ell,\alpha}(z)\,\big(\log\big(\frac{1}{\sqrt{2}}-z\big)\big)^\alpha\,\chi_{]0,\frac{1}{\sqrt{2}}]}(z) \\ &+ \sum_{\substack{3\leq\beta\leq k-3\\ 0\leq\alpha\leq\frac{k-6}{2}-\ell}} b^{\rm reg}_{k,\ell,\alpha,\beta}(z)\big(\frac{1}{\sqrt{2}}-z\big)^{\beta\nu+4}\,\big(\log\big(\frac{1}{\sqrt{2}}-z\big)\big)^\alpha\,\chi_{]0,\frac{1}{\sqrt{2}}]}(z)\,, \end{aligned} \tag{4.55}$$

where the function χ denotes the characteristic function, namely

$$\begin{cases} \chi_{]0,\frac{1}{\sqrt{2}}]}(z) = 1 \ \text{ for } z\leq \dfrac{1}{\sqrt{2}} \quad \text{and} \\ \chi_{]0,\frac{1}{\sqrt{2}}]}(z) = 0 \ \text{ for } z > \dfrac{1}{\sqrt{2}}\,. \end{cases}$$

In addition, the following asymptotics hold:

$$w_{k,\ell}(z) = \sum_{\substack{0\leq\alpha\leq\frac{k-3}{2}-\ell\\ \beta\geq 1-k+2(\alpha+\ell)}} d^{k,\ell}_{\alpha,\beta}\,(\log z)^\alpha\, z^\beta\,,\ \text{ as } z\to 0\,, \tag{4.56}$$

with

$$d^{k,\ell}_{0,-2} = c^{k,\ell}_{0,-2}(\lambda)\,,\ d^{k,\ell}_{0,-3} = c^{k,\ell}_{0,-3}(\lambda)\,, \tag{4.57}$$

where $c^{k,\ell}_{0,\beta}(\lambda)$ are the coefficients related to the matching conditions coming out from the inner region involved in Formula (4.15).

Moreover for $z > \frac{1}{\sqrt{2}}$, $w_{k,\ell}$ can be splitted into two parts as follows:

$$w_{k,\ell}(z) = w^{\rm nl}_{k,\ell} + w^{\rm lin}_{k,\ell}\,, \tag{4.58}$$

where the nonlinear part $w^{\rm nl}_{k,\ell}$ is null if $k<6$ or $\ell > \frac{k-6}{2}$, and has in all other cases, as z tends to infinity, an asymptotic expansion of the form

$$w^{\rm nl}_{k,\ell}(z) = \sum_{\substack{3\leq\beta\leq k-3\\ 0\leq\alpha\leq\frac{k-6}{2}-\ell,\, p\in\mathbb{N}}} \hat{d}^{k,\ell}_{\alpha,\beta,p}\,(\log z)^\alpha\, z^{\beta\,\nu+1-p} + z^{\nu k+1}\sum_{0\leq\alpha\leq\frac{k-3}{2}-\ell,\, p\geq 2} \hat{d}^{k,\ell}_{\alpha,k,p}\,(\log z)^\alpha\, z^{-p}\,, \tag{4.59}$$

---

[12] Here and below, the notation $^{\rm reg}$ means that the corresponding function belongs to $\mathcal{C}^\infty(\mathbb{R}^*_+)$.

for some constants $\hat d^{k,\ell}_{\alpha,\beta,p}$, *and where the linear part* $w^{\rm lin}_{k,\ell}$ *is given by*

$$w^{\rm lin}_{k,\ell}(z) = \sum_{0\le j\le \ell(k)} \alpha_k^{j,+}\, f^{j,+}_{k,\ell} + \alpha_k^{j,-}\, f^{j,-}_{k,\ell}\,, \tag{4.60}$$

for some constants $\alpha_k^{j,\pm}$, *where* $f_k^{j,\pm} = (f^{j,\pm}_{k,\ell})_{0\le j\le \ell(k)}$ *are the solutions of the homogeneous equation* (4.39) *introduced in Lemma 4.1.*

(2) Uniqueness

Let $(\lambda_{k,\ell})_{k\ge 3, 0\le \ell\le \ell(k)}$ *be fixed, and let* $(w^0_{k,\ell})_{3\le k\le M, 0\le\ell\le\ell(k)}$ *and* $(w^1_{k,\ell})_{3\le k\le M, 0\le\ell\le\ell(k)}$ *be two solutions of*

$$\widetilde{\mathcal L}_k w_{k,\ell} = F_{k,\ell}(\lambda; w)\,, 3\le k\le M\,, \tag{4.61}$$

defined and $\mathcal C^\infty$ *in a neighborhood of* 0, *with* $w^i_{5,1}\equiv 0$ *for* $i\in\{0,1\}$, *and which have an asymptotic expansion of the form* (4.56), *as* z *tends to* 0*:*

$$w^i_{k,\ell}(z) = \sum_{\substack{0\le\alpha\le \frac{k-3}{2}-\ell\\ \beta\ge 1-k+2(\alpha+\ell)}} d^{k,\ell,i}_{\alpha,\beta}\,(\log z)^\alpha\, z^\beta\,.$$

If

$$d^{k,\ell,0}_{0,-2} = d^{k,\ell,1}_{0,-2}\,,\ d^{k,\ell,0}_{0,-3} = d^{k,\ell,1}_{0,-3}\,, \tag{4.62}$$

then $w^0_{k,\ell} = w^1_{k,\ell}$, *for all* $3\le k\le M$ *and all* $0\le\ell\le\ell(k)$.

Similarly, if $(w^0_{k,\ell})_{3\le k\le M, 0\le\ell\le\ell(k)}$ *and* $(w^1_{k,\ell})_{3\le k\le M, 0\le\ell\le\ell(k)}$ *are two solutions of the equation* (4.61) *defined and* $\mathcal C^\infty$ *around* $+\infty$, *with* $w^i_{5,1}\equiv 0$ *for* $i\in\{0,1\}$, *and which satisfy as* z *tends to infinity:*

$$\begin{aligned} w^i_{k,\ell} = &\sum_{0\le j\le\ell(k)} \alpha_k^{j,+,i}\, f^{j,+}_{k,\ell} + \alpha_k^{j,-,i}\, f^{j,-}_{k,\ell}\\ &+ \sum_{\substack{3\le\beta\le k-3\\ 0\le\alpha\le\frac{k-6}{2}-\ell,\, p\in\mathbb N}} \hat d^{k,\ell,i}_{\alpha,\beta,p}\,(\log z)^\alpha\, z^{\beta\,\nu+1-p} + z^{\nu k+1} \sum_{0\le\alpha\le\frac{k-6}{2}-\ell,\, p\ge 2} \hat d^{k,\ell,i}_{\alpha,k,p}\,(\log z)^\alpha\, z^{-p}\,, \end{aligned} \tag{4.63}$$

then

$$\alpha_k^{j,\pm,0} = \alpha_k^{j,\pm,1}\,, \forall\, 3\le k\le M \quad \text{and} \quad 0\le j\le \ell(k)\,, \tag{4.64}$$

implies that $w^0_{k,\ell} = w^1_{k,\ell}$, *for all* $3\le k\le M$ *and all* $0\le\ell\le\ell(k)$.

Remark 4.2. *By virtue of Lemma 4.1 and formulae* (4.59) *and* (4.60), *the functions* $w_{k,\ell}$ *admit an asymptotic expansion of the form:*

$$\begin{aligned} w_{k,\ell}(z) = &\sum_{\substack{3\le\beta\le k-3\\ 0\le\alpha\le\frac{k-6}{2}-\ell,\, p\in\mathbb N}} w_{k,\ell,\alpha,\beta,p}\,(\log z)^\alpha\, z^{\beta\,\nu+1-p}\\ &+ z^{k\,\nu+1} \sum_{\substack{0\le\alpha\le\frac{k-3}{2}-\ell\\ p\in\mathbb N}} w_{k,\ell,\alpha,p}\,(\log z)^\alpha\, z^{-p}\,, \text{ as } z\to\infty\,, \end{aligned} \tag{4.65}$$

for some constants $w_{k,\ell,\alpha,\beta,p}$ *and* $w_{k,\ell,\alpha,p}$, *as* z *tends to infinity.*

Proof of Proposition 4.1. Let us start with the existence part of the proposition, and first consider the indexes $k = 3, 4$ and 5. In view of the computations carried out in Section 4.1 (see Property (4.16)), we have in that case $w_{5,1} = 0$ and

$$\widetilde{\mathcal{L}}_k w_{k,0} = 0\,,\ \ k = 3, 4, 5\,. \tag{4.66}$$

In view of Remark 4.1, this implies that

$$\begin{cases} w_{k,0} = a^k_{0,+}\, f^{0,+}_{k,0}(z) + a^k_{0,-}\, f^{0,-}_{k,0}(z) \ \text{ for } z \leq \dfrac{1}{\sqrt{2}}\,, \\ w_{k,0} = a^k_{0,+}\, f^{0,+}_{k,0}(z) \ \text{ for } z > \dfrac{1}{\sqrt{2}}\,,\ k = 3, 4, 5\,, \end{cases} \tag{4.67}$$

where $a^3_{0,+} = -a^3_{0,-}$ and where $\big\{f^{0,+}_{k,0}, f^{0,-}_{k,0}\big\}$ denotes the basis of solutions associated to the operator $\widetilde{\mathcal{L}}_k$ given by (4.45). The coefficients $a^k_{0,\pm}$ are determined by (4.57):

$$\begin{cases} 2(3\nu+4)\Big(\dfrac{1}{\sqrt{2}}\Big)^{3\nu+3} a^3_{0,+} = c^{3,0}_{0,-2}\,, \\ \Big(\dfrac{1}{\sqrt{2}}\Big)^{\nu k+4}\big(a^k_{0,+} + a^k_{0,-}\big) = c^{k,0}_{0,-3}\,, \\ (\nu k+4)\Big(\dfrac{1}{\sqrt{2}}\Big)^{\nu k+3}\big(a^k_{0,+} - a^k_{0,-}\big) = c^{k,0}_{0,-2}\,,\ k = 4, 5\,. \end{cases} \tag{4.68}$$

Clearly the functions $w_{k,0}$, $k = 3, 4, 5$, satisfy properties (4.55)-(4.60).

Let us now consider the general case of any index $k \geq 6$. To this end, we shall proceed by induction assuming that, for any integer $3 \leq j \leq k-1$ and all $0 \leq \ell \leq \ell(j)$, $(w_{j,\ell}, \lambda_{j,\ell})$ satisfies the conclusion of part (1) of Proposition 4.1.

The first step consists to establish the following lemma:

Lemma 4.4. *Assume that $(w_{j,\ell}, \lambda_{j,\ell})_{0\leq \ell\leq \ell(j)}$ is a solution of the system (4.24) with $3 \leq j \leq k-1$, which satisfies (4.55), (4.56), (4.58), (4.59) and (4.60). Then $F^{\rm nl}_{k,\ell}$ has the following form:*

$$\begin{aligned} F^{\rm nl}_{k,\ell}(z) &= f^{\rm reg}_{k,\ell}(z) + \big(\frac{1}{\sqrt{2}} - z\big)^{k\nu+6} \sum_{0\leq\alpha\leq\frac{k-6}{2}-\ell} f^{\rm reg}_{k,\ell,\alpha}(z)\big(\log\big(\frac{1}{\sqrt{2}} - z\big)\big)^{\alpha}\, \chi_{]0,\frac{1}{\sqrt{2}}]}(z) \\ &\quad + \sum_{\substack{3\leq\beta\leq k-3\\ 0\leq\alpha\leq\frac{k-6}{2}-\ell}} f^{\rm reg}_{k,\ell,\beta,\alpha}(z)\,\big(\frac{1}{\sqrt{2}} - z\big)^{\beta\nu+3}\,\big(\log\big(\frac{1}{\sqrt{2}} - z\big)\big)^{\alpha}\, \chi_{]0,\frac{1}{\sqrt{2}}]}(z)\,, \end{aligned} \tag{4.69}$$

and has the following asymptotic expansions respectively close to 0 and at infinity:

$$F^{\rm nl}_{k,\ell}(z) = \sum_{\substack{0\leq\alpha\leq\frac{k-6}{2}-\ell\\ \beta\geq 1-k+2(\alpha+\ell)}} \widetilde{f}_{k,\ell,\alpha,\beta}\,(\log z)^{\alpha}\, z^{\beta-2}\,, \tag{4.70}$$

$$\begin{aligned} F^{\rm nl}_{k,\ell}(z) &= z^{k\nu-1} \sum_{\substack{0\leq\alpha\leq\frac{k-6}{2}-\ell\\ p\in\mathbb{N}}} \hat{f}_{k,\ell,\alpha,p}\big(\log z\big)^{\alpha} z^{-p} \\ &\quad + \sum_{\substack{3\leq\beta\leq k-3\\ 0\leq\alpha\leq\frac{k-6}{2}-\ell,\, p\in\mathbb{N}}} \hat{f}_{k,\ell,\alpha,\beta,p}\big(\log z\big)^{\alpha} z^{\nu\beta+1-p}\,, \end{aligned} \tag{4.71}$$

where the coefficients $\widetilde{f}_{k,\ell,\alpha,\beta}$ (resp. $\hat{f}_{k,\ell,\alpha,p}$ and $\hat{f}_{k,\ell,\alpha,\beta,p}$) are uniquely determined in terms of the coefficients $d^{j,\ell}_{\alpha,\beta}$ (resp. $w^{k,\ell}_{\alpha,\beta,p}$) involved in (4.56) (resp. (4.65)).

Proof. Let us first address the behavior of $F^{\mathrm{nl}}_{k,\ell}$ near $z=0$ and at infinity. To establish (4.70) and (4.71), we will use formulae (4.35)-(4.38) combining them with the corresponding asymptotic of $w^1_{j,\ell}$, $\widetilde{w}^1_{j,\ell}$, $w^2_{k,\ell}$, $\widetilde{w}^2_{j,\ell}$ and $\check{w}_{j,\ell}$ that we start to describe now.

Consider $\widetilde{w}^1_{j,\ell}$. It follows from (4.20) that if $w_{j,\ell}, 3\le j\le k-1$, verify (4.56) and (4.65), then for any $6\le j\le k+2$, $\widetilde{w}^1_{j,\ell}$ admits the following asymptotic expansions as $z\to 0$ and $z\to\infty$:

$$\widetilde{w}^1_{j,\ell}(z)=\sum_{\substack{0\le\alpha\le\frac{j-6}{2}-\ell\\ \beta\ge 4-j+2(\alpha+\ell)}}\widetilde{w}^{1,0}_{j,\ell,\alpha,\beta}\big(\log z\big)^\alpha z^\beta\,,\ \text{as } z\to 0\,, \tag{4.72}$$

$$\widetilde{w}^1_{j,\ell}(z)=\sum_{\substack{0\le\alpha\le\frac{j-6}{2}-\ell\\ 3\le\beta\le j-3,\, p\ge 0}}\widetilde{w}^{1,\infty}_{j,\ell,\alpha,\beta,p}\big(\log z\big)^\alpha z^{\beta\nu+1-p}\,,\ \text{as } z\to\infty\,. \tag{4.73}$$

Combining (4.20) together with (4.56) and (4.72), one sees that the function $w^1_{j,\ell}$ has asymptotic of the same form as $w_{j,\ell}$, as z tends to 0:

$$w^1_{j,\ell}(z)=\sum_{\substack{0\le\alpha\le\frac{j-3}{2}-\ell\\ \beta\ge 1-j+2(\alpha+\ell)}}w^{1,0}_{j,\ell,\alpha,\beta}\big(\log z\big)^\alpha z^\beta\,. \tag{4.74}$$

Furthermore, invoking (4.20), (4.58), (4.59), (4.60), (4.73) and taking into account (4.46), one obtains as $z\to\infty$

$$w^1_{j,\ell}(z)=z^{j\nu}\sum_{\substack{0\le\alpha\le\frac{j-3}{2}-\ell\\ p\in\mathbb{N}}}w^{1,\infty}_{j,\ell,\alpha,j,p}\big(\log z\big)^\alpha z^{-p}+\sum_{\substack{0\le\alpha\le\frac{j-6}{2}-\ell\\ 3\le\beta\le j-3,\, p\in\mathbb{N}}}w^{1,\infty}_{j,\ell,\alpha,\beta,p}\big(\log z\big)^\alpha z^{\nu\beta+1-p}\,, \tag{4.75}$$

for any integer $3\le j\le k-1$.

The function $\widetilde{w}^2_{j,\ell}$ can be analyzed along the same lines as $\widetilde{w}^1_{j,\ell}$. In particular, using Definition (4.22), one can show that under the assumptions of Lemma 4.4, for any $6\le j\le k+2$, $\widetilde{w}^2_{j,\ell}$ behaves in the same way as $\widetilde{w}^1_{j,\ell}$ when $z\to 0$ and $z\to\infty$, namely

$$\widetilde{w}^2_{j,\ell}(z)=\sum_{\substack{0\le\alpha\le\frac{j-6}{2}-\ell\\ \beta\ge 4-j+2(\alpha+\ell)}}\widetilde{w}^{2,0}_{j,\ell,\alpha,\beta}\big(\log z\big)^\alpha z^\beta\,,\ \text{as } z\to 0\,, \tag{4.76}$$

$$\widetilde{w}^2_{j,\ell}(z)=\sum_{\substack{0\le\alpha\le\frac{j-6}{2}-\ell\\ 3\le\beta\le j-3,\, p\ge 0}}\widetilde{w}^{2,\infty}_{j,\ell,\alpha,\beta,p}\big(\log z\big)^\alpha z^{\beta\nu+1-p}\,,\ \text{as } z\to\infty\,. \tag{4.77}$$

Combining (4.22), with (4.56), (4.58)-(4.60), (4.76), (4.77) and taking into account (4.47), we deduce, as we have done for $w^1_{j,\ell}$, that $w^2_{j,\ell}$ has the same form as $w_{j,\ell}$, $w^1_{j,\ell}$, as $z\to 0$

$$w^2_{j,\ell}(z)=\sum_{\substack{0\le\alpha\le\frac{j-3}{2}-\ell\\ \beta\ge 1-j+2(\alpha+\ell)}}w^{2,0}_{j,\ell,\alpha,\beta}\big(\log z\big)^\alpha z^\beta\,, \tag{4.78}$$

and as $z\to\infty$, one has:

$$w^2_{j,\ell}(z)=z^{j\nu-1}\sum_{\substack{0\le\alpha\le\frac{j-3}{2}-\ell\\ p\in\mathbb{N}}}w^{2,\infty}_{j,\ell,\alpha,j,p}\big(\log z\big)^\alpha z^{-p}+\sum_{\substack{0\le\alpha\le\frac{j-6}{2}-\ell\\ 3\le\beta\le j-3,\, p\in\mathbb{N}}}w^{2,\infty}_{j,\ell,\alpha,\beta,p}\big(\log z\big)^\alpha z^{\nu\beta+1-p}\,, \tag{4.79}$$

for all $3\le j\le k-1$.

Next we address $\breve{w}_{j,\ell}$. Writing

$$\breve{w}_{j,\ell} = \sum_{p\geq 1} \sum_{\substack{j_1+\cdots+j_p=j \\ \ell_1+\cdots+\ell_p=\ell}} (-1)^{p-1} z^{1-p} w_{j_1,\ell_1} \cdots w_{j_p,\ell_p}\,, \tag{4.80}$$

it is easy to check that if $w_{j,\ell}$, $3 \leq j \leq k-1$, verify (4.56), (4.65) then the same is true for $\breve{w}_{j,\ell}$, $3 \leq j \leq k-1$, namely the functions $\breve{w}_{j,\ell}$ admit asymptotic expansions of the form:

$$\breve{w}_{j,\ell}(z) = \sum_{\substack{0\leq\alpha\leq\frac{j-3}{2}-\ell \\ \beta\geq 1-j+2(\alpha+\ell)}} \breve{w}^0_{j,\ell,\alpha,\beta}\big(\log z\big)^\alpha z^\beta\,, \text{ as } z\to 0\,, \tag{4.81}$$

$$\breve{w}_{j,\ell}(z) = z^{j\nu+1} \sum_{\substack{0\leq\alpha\leq\frac{j-3}{2}-\ell \\ p\in\mathbb{N}}} \breve{w}^\infty_{j,\ell,\alpha,j,p}\big(\log z\big)^\alpha z^{-p} + \sum_{\substack{0\leq\alpha\leq\frac{j-6}{2}-\ell \\ 3\leq\beta\leq j-3,\, p\in\mathbb{N}}} \breve{w}^\infty_{j,\ell,\alpha,\beta,p}\big(\log z\big)^\alpha z^{\nu\beta+1-p}\,, \tag{4.82}$$

as $z\to\infty$.

Combining (4.30)-(4.34) with (4.74), (4.75), (4.76), (4.77), (4.78), (4.79), (4.81) and (4.82), we obtain (4.70) and (4.71).

To end the proof of the lemma, it remains to establish (4.69). To this end, we will use the representations (4.30)-(4.34).

Start with $F^{\mathrm{nl},1}_{k,\ell}$ defined by (4.31). It stems from the definition of $\widetilde{w}^{(2,')}_{j,\ell}$ given by (4.23) that for any $6 \leq j \leq k+2$, $\widetilde{w}^{(2,')}_{j,\ell}$ assumes the form:

$$f^{\mathrm{reg}}_{j,\ell}(z) + \sum_{\substack{3\leq\beta\leq j-3 \\ 0\leq\alpha\leq\frac{j-6}{2}-\ell}} \big(\frac{1}{\sqrt{2}}-z\big)^{\beta\nu+3}\big(\log\big(\frac{1}{\sqrt{2}}-z\big)\big)^\alpha h^{\mathrm{reg}}_{\alpha,\beta}(z)\,\chi_{]0,\frac{1}{\sqrt{2}}]}(z)\,, \tag{4.83}$$

which means that (4.69) holds for $F^{\mathrm{nl},1}_{k,\ell}$.

Next consider $F^{\mathrm{nl},i}_{k,\ell}$, $i=2,3$, respectively defined by (4.32) and (4.33). In view of (4.20) and (4.21), we deduce that $\widetilde{w}^1_{j,\ell}$ and $\widetilde{w}^3_{j,\ell}$ have the form (4.83) for any $6\leq j\leq k+2$, and therefore the functions $w^1_{j,\ell}$ and $w^3_{j,\ell}$ can be written in the following way

$$\begin{aligned} &f^{\mathrm{reg}}_{j,\ell}(z) + \big(\frac{1}{\sqrt{2}}-z\big)^{j\nu+3} \sum_{0\leq\alpha\leq\frac{j-3}{2}-\ell} \big(\log\big(\frac{1}{\sqrt{2}}-z\big)\big)^\alpha h^{\mathrm{reg}}_{j,\ell,\alpha}(z)\,\chi_{]0,\frac{1}{\sqrt{2}}]}(z) \\ &\qquad + \sum_{\substack{3\leq\beta\leq j-3 \\ 0\leq\alpha\leq\frac{j-6}{2}-\ell}} \big(\frac{1}{\sqrt{2}}-z\big)^{\beta\nu+3}\big(\log\big(\frac{1}{\sqrt{2}}-z\big)\big)^\alpha h^{\mathrm{reg}}_{j,\ell,\alpha,\beta}(z)\,\chi_{]0,\frac{1}{\sqrt{2}}]}(z)\,. \end{aligned} \tag{4.84}$$

Similarly, by (4.23) the same is true for $w^{(2,')}_{j,\ell}$. Finally using (4.80), one can easily check that the functions $\breve{w}_{j,\ell}$, $3\leq j\leq k-1$, have the form (4.55), which can be viewed as a particular case of (4.84).

Since all the functions involved in (4.32) and (4.33) have the form (4.84), one easily deduces that $F^{\mathrm{nl},i}_{k,\ell}$, $i=2,3$, verify (4.69).

Now consider $F^{\mathrm{nl},4}_{k,\ell}$ given by (4.34). It follows from the definition of A_0 (see (4.10)) that, for all $3 \le j \le k-1$, the function $A^0_{j,\ell}$ admits the same form as $w_{j,\ell}$:

$$\begin{aligned} A^0_{j,\ell}(z) = A^{\mathrm{reg}}_{j,\ell}(z) + \big(\frac{1}{\sqrt{2}} - z\big)^{j\nu+4} \sum_{0\le\alpha\le\frac{j-3}{2}-\ell} \big(\log\big(\frac{1}{\sqrt{2}} - z\big)\big)^{\alpha} A^{\mathrm{reg}}_{j,\ell,\alpha}(z)\, \chi_{]0,\frac{1}{\sqrt{2}}]}(z) \\ + \sum_{\substack{3\le\beta\le j-3\\ 0\le\alpha\le\frac{j-6}{2}-\ell}} \big(\frac{1}{\sqrt{2}} - z\big)^{\beta\nu+4} \big(\log\big(\frac{1}{\sqrt{2}} - z\big)\big)^{\alpha} A^{\mathrm{reg}}_{j,\ell,\alpha,\beta}(z)\, \chi_{]0,\frac{1}{\sqrt{2}}]}(z)\,. \end{aligned} \tag{4.85}$$

Furthermore by virtue of the required condition (4.19), the functions $A^0_{j,\ell}$, $3 \le j \le k-1$, vanish on $z = \dfrac{1}{\sqrt{2}}$, namely:

$$(A^0_{j,\ell})_{z=\frac{1}{\sqrt{2}}} = 0\,, \tag{4.86}$$

which together with (4.34), (4.55) and (4.85) give (4.69) for $F^{\mathrm{nl},4}_{k,\ell}$.

□

The second step in the proof of Proposition 4.1 relies on the following lemma:

Lemma 4.5. *Consider for $k \ge 6$ the non homogeneous equation*

$$\mathcal{S}_k\, X = \mathcal{F}^{\mathrm{nl}}_k\,, \tag{4.87}$$

where $\mathcal{S}_k$ is defined by (4.28) and $\mathcal{F}^{\mathrm{nl}}_k = \big(F^{\mathrm{nl}}_{k,\ell}\big)_{0\le\ell\le\ell(k)}$. Then the following properties hold:

(1) *The system (4.87) has a unique solution $X^0 = \big(X_{0,\ell}\big)_{0\le\ell\le\ell(k)}$ such that $X_{0,\ell} \equiv 0$ for any integer $\ell_1(k) < \ell \le \ell(k)$, where $\ell_1(k) = [\dfrac{k-6}{2}]$, and such that if $\ell \le \ell_1(k)$, then $X_{0,\ell}$ belongs to the functional space $\mathcal{C}^{[k\nu+4]}\big(\mathbb{R}^*_+\big) \cap \mathcal{C}^\infty\big(\mathbb{R}^*_+ \setminus \{\frac{1}{\sqrt{2}}\}\big)$ and has the following form:*

$$\begin{aligned} X_{0,\ell}(z) = X^{\mathrm{reg}}_{0,\ell}(z) + \big(\frac{1}{\sqrt{2}} - z\big)^{k\nu+7} \sum_{0\le\alpha\le\frac{k-6}{2}-\ell} \big(\log\big(\frac{1}{\sqrt{2}} - z\big)\big)^{\alpha} X^{\mathrm{reg}}_{0,\ell,\alpha}(z)\, \chi_{]0,\frac{1}{\sqrt{2}}]}(z) \\ + \sum_{\substack{3\le\beta\le k-3\\ 0\le\alpha\le\frac{k-6}{2}-\ell}} \big(\frac{1}{\sqrt{2}} - z\big)^{\beta\nu+4} \big(\log\big(\frac{1}{\sqrt{2}} - z\big)\big)^{\alpha} X^{\mathrm{reg}}_{0,\ell,\beta,\alpha}(z)\, \chi_{]0,\frac{1}{\sqrt{2}}]}(z)\,, \\ X_{0,\ell}\Big(\frac{1}{\sqrt{2}}\Big) = 0\,\cdot \end{aligned} \tag{4.88}$$

Moreover, it admits an expansion of the form (4.56) as $z \to 0$:

$$X_{0,\ell}(z) = \sum_{\substack{0\le\alpha\le\frac{k-4}{2}-\ell\\ \beta\ge 1-k+2(\alpha+\ell)}} X_{0,\ell,\alpha,\beta}\, (\log z)^{\alpha}\, z^{\beta}\,, \tag{4.89}$$

for some constants $X_{0,\ell,\alpha,\beta}$.

(2) *The system (4.87) has a unique solution $X^1 = \big(X_{1,\ell}\big)_{0\le\ell\le\ell(k)}$ such that $X_{1,\ell} \equiv 0$ for any integer $\ell_1(k) < \ell \le \ell(k)$, and such that if $\ell \le \ell_1(k)$, then $X_{1,\ell} \in \mathcal{C}^\infty\big(]\frac{1}{\sqrt{2}},\infty[\big)$ with the*

following asymptotic as $z \to \infty$:

$$X_{1,\ell}(z) = z^{k\nu-1} \sum_{\substack{0\leq\alpha\leq\frac{k-6}{2}-\ell \\ p\in\mathbb{N}}} X_{1,\ell,\alpha,p}\big(\log z\big)^{\alpha} z^{-p} + \sum_{\substack{3\leq\beta\leq k-3 \\ 0\leq\alpha\leq\frac{k-6}{2}-\ell,\, p\in\mathbb{N}}} X_{1,\ell,\alpha,\beta,p}\big(\log z\big)^{\alpha} z^{\nu\beta+1-p}\,, \tag{4.90}$$

where $X_{1,\ell,\alpha,p}$ and $X_{1,\ell,\alpha,\beta,p}$ denote some constants.

Proof. In order to establish this lemma, we shall proceed by induction on the index ℓ. Since for any integer $k \geq 6$, we have

$$F^{\rm nl}_{k,\ell} \equiv 0\,,\ \ \ell_1(k) < \ell \leq \ell(k)\,,$$

we get

$$X_{0,\ell} \equiv 0\,,\ \forall\, \ell_1(k) < \ell \leq \ell(k)\,.$$

Consider now

$$\widetilde{\mathcal{L}}_k X_{0,\ell_1(k)} = F^{\rm nl}_{k,\ell_1(k)}\,. \tag{4.91}$$

Invoking formulae (4.69), (4.70) together with Lemma 4.3, we easily check that the above equation has a unique solution $X_{0,\ell_1(k)}$ in $\mathcal{C}^{[k\nu+4]}\big(\mathbb{R}^*_+\big) \cap \mathcal{C}^{\infty}\big(\mathbb{R}^*_+ \setminus \{\frac{1}{\sqrt{2}}\}\big)$ which assumes the form (4.88) and admits asymptotic expansion of type (4.89), for z close to 0.

Let us assume now that for any integer $\ell < q \leq \ell_1(k)$, the equation

$$\widetilde{\mathcal{L}}_k X_{0,q} = F_{k,q}$$

admits a unique solution $X_{0,q}$ which satisfies (4.88) and (4.89). Then by virtue of Formula (4.27), we find that $F^{\rm lin}_{k,\ell}$ undertakes the following form:

$$F^{\rm lin}_{k,\ell}(z) = F^{\rm reg}_{k,\ell}(z) + \big(\frac{1}{\sqrt{2}} - z\big)^{k\nu+6} \sum_{0\leq\alpha\leq\frac{k-6}{2}-\ell-1} \big(\log\big(\frac{1}{\sqrt{2}} - z\big)\big)^{\alpha} F^{\rm reg}_{k,\ell,\alpha}(z)\,\chi_{]0,\frac{1}{\sqrt{2}}]}(z) + \sum_{\substack{3\leq\beta\leq k-3 \\ 0\leq\alpha\leq\frac{k-6}{2}-\ell-1}} \big(\frac{1}{\sqrt{2}} - z\big)^{\beta\nu+3} \big(\log\big(\frac{1}{\sqrt{2}} - z\big)\big)^{\alpha} F^{\rm reg}_{k,\ell,\alpha,\beta}(z)\,\chi_{]0,\frac{1}{\sqrt{2}}]}(z)\,, \tag{4.92}$$

and behaves as follows close to 0:

$$F^{\rm lin}_{k,\ell}(z) = \sum_{\substack{0\leq\alpha\leq\frac{k-6}{2}-\ell \\ \beta\geq 5-k+2(\alpha+\ell)}} \widetilde{F}_{k,\ell,\alpha,\beta}\,(\log z)^{\alpha}\, z^{\beta-2}\,, \tag{4.93}$$

for some constants $\widetilde{F}_{k,\ell,\alpha,\beta}$, which implies that $F_{k,\ell} = F^{\rm lin}_{k,\ell} + F^{\rm nl}_{k,\ell}$ verifies (4.69) and (4.70).

Therefore taking into account Lemma 4.3, we infer that the equation $\widetilde{\mathcal{L}}_k X_{0,\ell} = F_{k,\ell}$ has a unique solution $X_{0,\ell}$ in $\mathcal{C}^{[k\nu+4]}\big(\mathbb{R}^*_+\big) \cap \mathcal{C}^{\infty}\big(\mathbb{R}^*_+ \setminus \{\frac{1}{\sqrt{2}}\}\big)$ which satisfies (4.88) and (4.89).

The proof of the second part of the lemma is also by induction on ℓ. First taking into account Lemma 4.3 together with Formula (4.71), we infer that the equation

$$\widetilde{\mathcal{L}}_k X_{1,\ell_1(k)} = F^{\rm nl}_{k,\ell_1(k)}$$

admits a unique solution $X_{1,\ell_1(k)}$ which belongs to $\mathcal{C}^{\infty}\big(]\frac{1}{\sqrt{2}},\infty[\big)$ and verifies (4.90). Then assuming that for any integer $\ell < q \leq \ell_1(k)$, the equation

$$\widetilde{\mathcal{L}}_k X_{1,q} = F_{k,q}$$

admits a unique solution $X_{1,q}$ which satisfies (4.90), we deduce that $F^{\rm lin}_{k,\ell}$ defined by (4.27) has an expansion of the following form at infinity:

$$\begin{aligned} F^{\rm lin}_{k,\ell}(z) = \sum_{\substack{3\leq\beta\leq k-3\\ 0\leq\alpha\leq\frac{k-6}{2}-\ell-1,\, p\in\mathbb{N}}} \hat{F}_{k,\ell,\alpha,\beta,p}\,(\log z)^{\alpha}\, z^{\beta\,\nu+1-p} \\ + z^{k\,\nu-1} \sum_{\substack{0\leq\alpha\leq\frac{k-6}{2}-\ell-1\\ p\in\mathbb{N}}} \hat{F}^{k,\ell}_{k,\ell,\alpha,p}\,(\log z)^{\alpha}\, z^{-p}\,, \end{aligned} \tag{4.94}$$

with some constants $\hat{F}_{k,\ell,\alpha,\beta,p}$ and $\hat{F}^{k,\ell}_{k,\ell,\alpha,p}$.

Since

$$F_{k,\ell} = F^{\rm lin}_{k,\ell} + F^{\rm nl}_{k,\ell}\,,$$

it follows from (4.71), (4.94) and Lemma 4.3 that the equation $\widetilde{\mathcal{L}}_k X_{1,\ell} = F_{k,\ell}$ has a unique solution $X_{1,\ell}$ in $\mathcal{C}^\infty\big(]\frac{1}{\sqrt{2}},\infty[\big)$ admitting an asymptotic of the form (4.90) as $z\to\infty$. This achieves the proof of the lemma.

□

We now return to the proof of Proposition 4.1. Taking advantage of Lemma 4.5 (1), we get $W_k := \big(w_{k,\ell}\big)_{0\leq\ell\leq\frac{k-3}{2}}$ by setting

$$\begin{cases} W_k = X^0 + \displaystyle\sum_{0\leq j\leq\ell(k)} \big(a^k_{j,+}\, f^{j,+}_k + a^k_{j,-}\, f^{j,-}_k\big) \ \text{ for } z\leq\frac{1}{\sqrt{2}}, \\ W_k = X^0 + \displaystyle\sum_{0\leq j\leq\ell(k)} a^k_{j,+}\, f^{j,+}_k \ \text{ for } z>\frac{1}{\sqrt{2}}\,, \end{cases} \tag{4.95}$$

where $\big(f^{j,\pm}_k\big)_{0\leq j\leq\ell(k)}$ denotes the basis of solutions of $\mathcal{S}_k\, X=0$ given by Lemma 4.1, where X^0 is given by Lemma 4.5 (1), and where in view of Formula (4.15) the coefficients $a^k_{j,\pm}$ are determined by

$$\begin{cases} X_{0,\ell,0,-3} + \displaystyle\sum_{\ell\leq j\leq\ell(k)} \mu^{j,\ell}_{k,0}\big(a^k_{j,+} + a^k_{j,-}\big) = c^{k,\ell}_{0,-3}\,, \\ X_{0,\ell,0,-2} + \displaystyle\sum_{\ell\leq j\leq\ell(k)} \mu^{j,\ell}_{k,1}\big(a^k_{j,+} - a^k_{j,-}\big) = c^{k,\ell}_{0,-2}\,, \end{cases} \tag{4.96}$$

where respectively $X_{0,\ell,0,-3}$ and $X_{0,\ell,0,-2}$, $c^{k,\ell}_{0,-3}$ and $c^{k,\ell}_{0,-2}$, denote the coefficients involved in (4.89) and (4.15), [13] and where the coefficients $\mu^{j,\ell}_{k,0}$ and $\mu^{j,\ell}_{k,1}$ are defined so that

$$f^{j,+}_{k,\ell}(z) = \frac{\mu^{j,\ell}_{k,0}}{z^3} + \frac{\mu^{j,\ell}_{k,1}}{z^2} + \mathcal{O}\Big(\frac{1}{z}\Big),\ \text{as}\ \ z\to 0\,\cdot \tag{4.97}$$

By virtue of (4.40), we easily deduce that

$$\begin{cases} \mu^{j,\ell}_{k,0} = \Big(\dfrac{1}{\sqrt{2}}\Big)^{\alpha(\nu,k)} \dbinom{j}{\ell} \Big(\log\big(\dfrac{1}{\sqrt{2}}\big)\Big)^{j-\ell} \\ \mu^{j,\ell}_{k,1} = \sqrt{2}\,\Big(\alpha(\nu,k) - \dfrac{j-\ell}{\log(\sqrt{2})}\Big)\,\mu^{j,\ell}_{k,0}. \end{cases} \tag{4.98}$$

By Lemma 4.5 (2),

$$W_k = X^1 + \sum_{0\leq j\leq\ell(k)} \alpha^{j,+}_k\, f^{j,+}_{k,\ell} + \alpha^{j,-}_k\, f^{j,-}_{k,\ell}\,,$$

[13] with the convention $X_{0,\ell,0,\beta}=0$ if $\ell>\ell_1(k)$ and $c^{k,\ell}_{0,-3}=0$ if $\ell=\frac{k-3}{2}$.

with some coefficients $\alpha_k^{j,\pm}$, which concludes the proof of the first part of Proposition 4.1.

In order to establish the part (2) of the proposition, we shall again proceed by induction. Firstly, let us investigate the uniqueness for the solutions to (4.61) near 0, and consider the indexes $k = 3, 4$ and 5. By the computations carried out in Section 4.1 (see (4.16)), we have in that case $w^i_{k,1} = 0$ and

$$\widetilde{\mathcal{L}}_k w^i_{k,0} = 0\,,$$

which implies that

$$\widetilde{\mathcal{L}}_k (w^0_{k,0} - w^1_{k,0}) = 0\,.$$

Invoking Remark 4.1 together with Hypothesis (4.62), we easily gather that $w^0_{k,0} = w^1_{k,0}$ on a neighborhood of 0, for $k = 3, 4$ and 5.

Let us assume now that for any index $k \leq k_0 - 1 \leq M - 1$, the uniqueness for solutions to (4.61) near 0 holds under Hypothesis (4.62). Since $F^{\rm nl}_{k_0,\ell}(\lambda; w)$ only depends on $w_{j,\ell}$, $j \leq k_0 - 3$, this ensures that

$$\mathcal{S}_{k_0}(\mathcal{W}^0_{k_0} - \mathcal{W}^1_{k_0}) = 0\,, \tag{4.99}$$

where

$$\mathcal{W}^i_{k_0} = \begin{pmatrix} w^i_{k_0,0} \\ \vdots \\ w^i_{k_0,\ell} \\ \vdots \\ w^i_{k_0,\ell(k_0)} \end{pmatrix}. \tag{4.100}$$

In order to prove that $\mathcal{W}^0_{k_0} = \mathcal{W}^1_{k_0}$, we shall proceed by induction on the index ℓ starting by $\ell(k_0)$. Taking into account (4.28) together with (4.101), we infer that

$$\widetilde{\mathcal{L}}_{k_0} (w^0_{k_0,\ell(k_0)} - w^1_{k_0,\ell(k_0)}) = 0\,.$$

Thanks to Lemma 4.1 and Condition (4.62), this implies that

$$w^0_{k_0,\ell(k_0)} = w^1_{k_0,\ell(k_0)}\,.$$

Assume now that for any integer $\ell < q \leq \ell(k_0)$, we have on a neighborhood of 0

$$w^0_{k_0,q} = w^1_{k_0,q}\,.$$

Therefore in view of the definition of $\mathcal{S}_{k_0}$ page 20, we find that

$$\widetilde{\mathcal{L}}_{k_0} (w^0_{k_0,\ell} - w^1_{k_0,\ell}) = 0\,,$$

which, due to Lemma 4.1 and Condition (4.62), easily ensures that

$$w^0_{k_0,\ell} = w^1_{k_0,\ell}\,.$$

This achieves the proof of the uniqueness for solutions to (4.61) near 0.

Secondly, let us investigate the uniqueness for solutions to (4.61) around $+\infty$. Again, we shall proceed by induction starting with the indexes $k = 3, 4$ and 5. In that case, we have

$$\widetilde{\mathcal{L}}_k w^i_{k,0} = 0\,,$$

and the conclusion follows easily from (4.64). Now, assuming that the uniqueness holds under Hypothesis (4.64), for any index $k \leq k_0 - 1 \leq M - 1$, let us consider the index k_0. Again, by the induction hypothesis, we have

$$\mathcal{S}_{k_0}(\mathcal{W}^0_{k_0} - \mathcal{W}^1_{k_0}) = 0\,.$$

This gives the result thanks to Lemma 4.1 and Condition (4.64), which ends the proof of the proposition.

Remark 4.3. *By virtue of the uniqueness of the solutions to the system* (4.24) *near* 0 *established above, we readily gather from the matching conditions* (4.29) *coming out from the inner region that*

$$d^{k,\ell}_{\alpha,\beta} = c^{k,\ell}_{\alpha,\beta}(\lambda)\,,\ \forall\, k,\ell,\alpha,\beta. \tag{4.101}$$

4.3. **Estimate of the approximate solution in the self-similar region.** Under the above notations, set for any integer $N \geq 3$

$$V^{(N)}_{\rm ss}(t,y) = y + \lambda^{(N)}(t)t^{-\nu-1}W^{(N)}_{\rm ss}\Big(t, y\frac{t^{\nu+1}}{\lambda^{(N)}(t)}\Big)\,,\ u^{(N)}_{\rm ss}(t,\rho) = t^{\nu+1}V^{(N)}_{\rm ss}\Big(t, \frac{\rho}{t^{\nu+1}}\Big)\,, \tag{4.102}$$

with

$$W^{(N)}_{\rm ss}(t,z) = \sum_{k=3}^{N} t^{\nu k}\sum_{\ell=0}^{\ell(k)}(\log t)^{\ell}\, w_{k,\ell}\big(z\big) \quad \text{and} \quad \lambda^{(N)}(t) = t\Big(1+\sum_{k=3}^{N}\sum_{\ell=0}^{\ell(k)}\lambda_{k,\ell}\, t^{\nu k}\,(\log t)^{\ell}\Big)\,. \tag{4.103}$$

The purpose of this paragraph is firstly to estimate the radial function $V^{(N)}_{\rm ss}$ defined by (4.102), in the self-similar region:

$$\Omega_{\rm ss} := \Big\{Y \in \mathbb{R}^4,\ \frac{t^{\epsilon_1-\nu}}{10} \leq |Y| \leq 10\, t^{-\epsilon_2-\nu}\Big\}\,, \tag{4.104}$$

and secondly to study, for N sufficiently large, the remainder term.

Combining Identity (4.15) together with Lemma 4.1, we firstly get the following lemma:

Lemma 4.6. *There exist a positive constant C and a small positive time $T = T(N)$ such that the following L^∞ estimates hold, for any time $0 < t \leq T$:*

$$\|\langle\cdot\rangle^{|\alpha|-1}\nabla^{\alpha}(V^{(N)}_{\rm ss}(t,\cdot) - Q)\|_{L^\infty(\Omega_{\rm ss})} \leq C\,[t^{3(\nu-\epsilon_1)} + t^{3\nu(1-\epsilon_2)}],\ \forall\,|\alpha| < 3\nu+4\,, \tag{4.105}$$

$$\begin{aligned}\|\langle\cdot\rangle^{\beta}\nabla^{\alpha}(V^{(N)}_{\rm ss}(t,\cdot) - Q)\|_{L^\infty(\Omega_{\rm ss})} \leq C\,[&t^{2\nu+2(\nu-\epsilon_1)} + t^{(\nu-\epsilon_1)(N+1)}\\ &+ t^{\nu+1+(3\nu-1)(1-\epsilon_2)}],\ \forall\,\beta \leq |\alpha|-2 \quad \text{and} \quad 1 \leq |\alpha| < 3\nu+4\,.\end{aligned} \tag{4.106}$$

In addition $\partial_t V^{(N)}_{\rm ss}$ satisfies

$$\|\partial_t V^{(N)}_{\rm ss}(t,\cdot)\|_{L^\infty(\Omega_{\rm ss})} \leq C\, t^{-2-\nu}\,[t^{1+\nu+2(\nu-\epsilon_1)} + t^{(1+3\nu)(1-\epsilon_2)}]\,, \tag{4.107}$$

$$\|\nabla^{\alpha}\partial_t V^{(N)}_{\rm ss}(t,\cdot)\|_{L^\infty(\Omega_{\rm ss})} \leq C\, t^{-1}\,[t^{3(\nu-\epsilon_1)} + t^{3\nu(1-\epsilon_2)}],\ \forall\, 1 \leq |\alpha| < 3\nu+3\,. \tag{4.108}$$

Besides for any multi-index α of length $|\alpha| < 3\nu+3$, the function[14] $V^{(N)}_{{\rm ss},1}(t,y) := (\partial_t u^{(N)}_{\rm ss})(t,\rho)$ *satisfies*

$$\begin{aligned}&\|\langle\cdot\rangle^{\beta}\nabla^{\alpha}V^{(N)}_{{\rm ss},1}(t,\cdot)\|_{L^\infty(\Omega_{\rm ss})} \leq C\, t^{\nu}\,[t^{3(\nu-\epsilon_1)} + t^{3\nu(1-\epsilon_2)}],\ \forall\,\beta \leq |\alpha|-1\,,\\ &\|\langle\cdot\rangle^{\alpha}\nabla^{\alpha}V^{(N)}_{{\rm ss},1}(t,\cdot)\|_{L^\infty(\Omega_{\rm ss})} \leq C\,[t^{3\nu-2\epsilon_1} + t^{3\nu(1-\epsilon_2)}]\end{aligned} \tag{4.109}$$

$$\|\partial_t V^{(N)}_{{\rm ss},1}(t,\cdot)\|_{L^\infty(\Omega_{\rm ss})} \leq C\, t^{-1}\,[t^{3\nu-2\epsilon_1} + t^{3\nu(1-\epsilon_2)}]\,. \tag{4.110}$$

Finally for any multi-index α of length $|\alpha| < 3\nu+2$ and any integer $\beta \leq |\alpha|$, we have

$$\|\langle\cdot\rangle^{\beta}\nabla^{\alpha}V^{(N)}_{{\rm ss},2}(t,\cdot)\|_{L^\infty(\Omega_{\rm ss})} \leq C\,[t^{2\nu+2(\nu-\epsilon_1)} + t^{\nu+1+(3\nu-1)(1-\epsilon_2)}]\,, \tag{4.111}$$

where $V^{(N)}_{{\rm ss},2}(t,y) = t^{\nu+1}\,(\partial_t^2 u^{(N)}_{\rm ss})(t,\rho)$.

[14] We recall that $\rho = y\, t^{\nu+1}$.

In the same spirit as Lemma 3.3, we have the following result.

Lemma 4.7. *The following L^2 estimates hold for all $0 < t \leq T$:*

$$\begin{aligned}\|\nabla^\alpha(V_{\rm ss}^{(N)}(t,\cdot) - Q)\|_{L^2(\Omega_{\rm ss})} \leq C\,[t^{\nu|\alpha|-\epsilon_1(|\alpha|-2)} + t^{(\nu-\epsilon_1)(N+|\alpha|-3)}\,, \\ + t^{\nu|\alpha|-\epsilon_2(3\nu+3-|\alpha|)}]\,,\ \forall\, 1 \leq |\alpha| < 3\nu+4+\frac{1}{2}\,,\end{aligned} \tag{4.112}$$

$$\|\nabla^\alpha(V_{\rm ss,1}^{(N)})(t,\cdot)\|_{L^2(\Omega_{\rm ss})} \leq C\,t^{\nu(|\alpha|+1)}\,[t^{-\epsilon_1|\alpha|} + t^{-\epsilon_2(3\nu+2-|\alpha|)}]\,,\ \forall\, 0 \leq |\alpha| < 3\nu+3+\frac{1}{2}\,, \tag{4.113}$$

$$\begin{aligned}\|\nabla^\alpha(V_{\rm ss,2}^{(N)})(t,\cdot)\|_{L^2(\Omega_{\rm ss})} \leq C\,t^{\nu(|\alpha|+2)}\,[t^{-\epsilon_1|\alpha|} + t^{-\epsilon_2(3\nu+1-|\alpha|)}(1+t^{3\nu-2\epsilon_2})]\,, \\ \forall\, 0 \leq |\alpha| < 3\nu+2+\frac{1}{2}\,\cdot\end{aligned} \tag{4.114}$$

Let us now consider the remainder

$$\mathcal{R}_{\rm ss}^{(N)}(t,y) := \big[(3.3)V_{\rm ss}^{(N)}\big](t,y)\,.$$

Clearly,

$$\mathcal{R}_{\rm ss}^{(N)}(t,y) = \frac{t^{\nu+1}}{\lambda^{(N)}(t)}\widetilde{\mathcal{R}}_{\rm ss}^{(N)}\Big(t, y\frac{t^{\nu+1}}{\lambda^{(N)}(t)}\Big),$$

where

$$\widetilde{\mathcal{R}}_{\rm ss}^{(N)}(t,z) = \big[(4.7)W_{\rm ss}^{(N)}\big](t,z)\,.$$

By construction,

$$\widetilde{\mathcal{R}}_{\rm ss}^{(N)}(t,z) = \sum_{\substack{k\geq N+1\\ \ell\leq\frac{k-6}{2}}} t^{\nu k}(\log t)^\ell r_{k,\ell}(z) \quad \text{with} \quad r_{k,\ell}(z) = F_{k,\ell}^{\rm nl}(W_{\rm ss}^{(N)},\lambda^{(N)})\,.$$

In view of computations carried out in Section 4.1, we have

$$\begin{aligned}r_{k,\ell}(z) = r_{k,\ell}^{\rm reg}(z) + \big(\frac{1}{\sqrt{2}} - z\big)^{k\nu+6} \sum_{0\leq\alpha\leq\frac{k-6}{2}-\ell} r_{k,\ell,\alpha}^{\rm reg}(z)\,\big(\log\big(\frac{1}{\sqrt{2}} - z\big)\big)^\alpha\,\chi_{]0,\frac{1}{\sqrt{2}}]}(z) \\ + \sum_{\substack{3\leq\beta\leq k-3\\ 0\leq\alpha\leq\frac{k-6}{2}-\ell}} r_{k,\ell,\alpha,\beta}^{\rm reg}(z)\big(\frac{1}{\sqrt{2}} - z\big)^{\beta\nu+2}\,\big(\log\big(\frac{1}{\sqrt{2}} - z\big)\big)^\alpha\,\chi_{]0,\frac{1}{\sqrt{2}}]}(z)\,\cdot\end{aligned} \tag{4.115}$$

Furthermore, as $z \to 0$ and as $z \to \infty$, $r_{k,\ell}$ satisfies (4.70), (4.71) respectively.

As a direct consequence of these properties, we obtain the following lemma:

Lemma 4.8. *There exist a small positive time $T = T(N)$ and a positive constant C_N such that for all $0 < t \leq T$, the remainder term $\mathcal{R}_{\rm ss}^{(N)}$ satisfies the following estimate*

$$\|\langle\cdot\rangle^{\frac{3}{2}}\,\mathcal{R}_{\rm ss}^{(N)}(t,\cdot)\|_{H^{K_0}(\Omega_{\rm ss})} \leq C_N\,[t^{(\nu-\epsilon_1)(N-\frac{3}{2})} + t^{\nu(1-\epsilon_2)(N+1)-\frac{5}{2}(\nu+1)}]\,, \tag{4.116}$$

where $K_0 = [3\nu + \dfrac{5}{2}]\cdot$

Let us end this section by investigating $V_{\rm in}^{(N)} - V_{\rm ss}^{(N)}$ in the intersection of the inner and self-similar regions, namely in

$$\Omega_{\rm in} \cap \Omega_{\rm ss} = \Big\{Y \in \mathbb{R}^4,\, \frac{t^{\epsilon_1-\nu}}{10} \leq |Y| \leq t^{\epsilon_1-\nu}\Big\}\,\cdot$$

In view of (4.15), (4.55) and Remark 4.3, we have for any multi-index α and any integer m

$$\big|\partial_y^\alpha\partial_t^m(V_{\rm in}^{(N)} - V_{\rm ss}^{(N)})(t,y)\big| \lesssim t^{2\nu(N+1)-m}\,y^{2N-|\alpha|} + t^{-m}y^{-N-|\alpha|}\,,$$

provided that y belongs to $\Omega_{\rm in}\cap\Omega_{\rm ss}$, and t is sufficiently small, which leads to the following result:

Lemma 4.9. *For any integer m and any multi-index α, the following estimate holds*

$$\|\nabla^\alpha\partial_t^m(V_{\rm in}^{(N)}-V_{\rm ss}^{(N)})(t,\cdot)\|_{L^\infty(\Omega_{\rm in}\cap\Omega_{\rm ss})}\leq C_{N,\alpha,m}t^{-m+|\alpha|(\nu-\epsilon_1)}\left(t^{2\nu+2N\epsilon_1}+t^{N(\nu-\epsilon_1)}\right), \tag{4.117}$$

for all $0<t\leq T=T(\alpha,m,N)$.

5. Approximate solution in the remote region

5.1. **General scheme of the construction of the approximate solution in the remote region.** In the previous section, we built in the self-similar region an approximate solution $u_{\rm ss}^{(N)}$ which extends the approximation solution $u_{\rm in}^{(N)}$ constructed in Section 3 in the inner region. Our goal here is to extend $u_{\rm ss}^{(N)}$ to the whole space.

Recall that the approximate solution $u_{\rm ss}^{(N)}$ built in in Section 4 assumes the following form:

$$u_{\rm ss}^{(N)}(t,\rho)=\rho+\lambda^{(N)}(t)\sum_{k=3}^{N}t^{\nu k}\sum_{\ell=0}^{\ell(k)}\big(\log t\big)^\ell\,w_{k,\ell}\Big(\frac{\rho}{\lambda^{(N)}(t)}\Big)\,,$$

where $\ell(k)=[\dfrac{k-3}{2}]$, and where $\lambda^{(N)}(t)$ is the perturbation of t defined by (4.103).

To achieve our goal, let us start by introducing the function $u^{{\rm lin},(N)}$ defined by

$$u^{{\rm lin},(N)}(t,\rho):=t\sum_{k=3}^{N}t^{\nu k}\sum_{\ell=0}^{\ell(k)}\big(\log t\big)^\ell\,w_{k,\ell}^{\rm lin}\Big(\frac{\rho}{t}\Big)\,, \tag{5.1}$$

where $w_{k,\ell}^{\rm lin}$ denotes the linear part of the function $w_{k,\ell}$ involved in the asymptotic expansion (4.18) and given by (4.60).

The function $u^{{\rm lin},(N)}$ solves the Cauchy problem:

$$\begin{cases}(2\partial_t^2-l_\rho)\,u^{{\rm lin},(N)}=0\\ u^{{\rm lin},(N)}{}_{|t=0}=u_0^{{\rm lin},(N)}\\ (\partial_t u^{{\rm lin},(N)})_{|t=0}=u_1^{{\rm lin},(N)}\,,\end{cases} \tag{5.2}$$

where l_ρ is defined by (4.42), and where

$$\begin{cases}u_0^{{\rm lin},(N)}(\rho)=\displaystyle\sum_{k=3}^{N}\sum_{\ell=0}^{\ell(k)}\mu_{k,\ell}^0\,\rho^{k\,\nu+1}\big(\log\rho\big)^\ell\,,\\ u_1^{{\rm lin},(N)}(\rho)=\displaystyle\sum_{k=3}^{N}\sum_{\ell=0}^{\ell(k)}\mu_{k,\ell}^1\,\rho^{k\,\nu}\big(\log\rho\big)^\ell\,,\end{cases} \tag{5.3}$$

with under notations (4.60)

$$\begin{cases}\mu_{k,\ell}^0=\alpha_k^{\ell,+}+\alpha_k^{\ell,-}\,,\\ \mu_{k,\ell}^1=\dfrac{1}{\sqrt{2}}\Big((\nu k+4)\big(\alpha_k^{\ell,+}-\alpha_k^{\ell,-}\big)+(\ell+1)\big(\alpha_k^{\ell+1,+}-\alpha_k^{\ell+1,-}\big)\Big)\,,\end{cases} \tag{5.4}$$

using again the convention that $\alpha_k^{\ell+1,\pm}=0$ if $\ell+1>\dfrac{k-3}{2}\cdot$

Indeed, combining (4.60) together with (5.1), we infer that

$$u^{\mathrm{lin},(N)}(t,\rho)=\sum_{k=3}^{N} t^{\nu k+1}\sum_{\ell=0}^{\ell(k)}\big(\log t\big)^{\ell}\sum_{0\leq j\leq \frac{k-3}{2}}\Big(\alpha_k^{j,+}\, f_{k,\ell}^{j,+}\Big(\frac{\rho}{t}\Big)+\alpha_k^{j,-}\, f_{k,\ell}^{j,-}\Big(\frac{\rho}{t}\Big)\Big)\cdot$$

Taking advantage of (4.44), this gives rise to

$$\begin{aligned}u^{\mathrm{lin},(N)}(t,\rho)=\sum_{k=3}^{N} t^{\nu k+1}\sum_{0\leq j\leq \frac{k-3}{2}}\Big(\alpha_k^{j,+}\big(\log t+\log\big((\frac{\rho}{t})+\frac{1}{\sqrt{2}}\big)\big)^{j}\,\frac{\big((\frac{\rho}{t})+\frac{1}{\sqrt{2}}\big)^{\nu k+4}}{\big(\frac{\rho}{t}\big)^{3}}\\ +\alpha_k^{j,-}\big(\log t+\log\big((\frac{\rho}{t})-\frac{1}{\sqrt{2}}\big)\big)^{j}\,\frac{\big((\frac{\rho}{t})-\frac{1}{\sqrt{2}}\big)^{\nu k+4}}{\big(\frac{\rho}{t}\big)^{3}}\Big)\\ =\sum_{k=3}^{N}\sum_{0\leq j\leq \frac{k-3}{2}}\Big(\alpha_k^{j,+}\big(\log\big(\rho+\frac{t}{\sqrt{2}}\big)\big)^{j}\,\frac{\big(\rho+\frac{t}{\sqrt{2}}\big)^{\nu k+4}}{\rho^{3}}+\alpha_k^{j,-}\big(\log\big(\rho-\frac{t}{\sqrt{2}}\big)\big)^{j}\,\frac{\big(\rho-\frac{t}{\sqrt{2}}\big)^{\nu k+4}}{\rho^{3}}\Big),\end{aligned}$$

which ensures the result.

Let now χ_0 be a radial smooth cutoff function on $\mathbb{R}^4$ equal to 1 on the unit ball centered at the origin and vanishing outside the ball of radius 2 centered at the origin, and consider for a small positive real number δ, the compact support functions:

$$\left\{\begin{array}{l}\mathfrak{g}_0(\rho)=\chi_\delta(\rho)\,u_0^{\mathrm{lin},(N)}(\rho)\,,\\ \mathfrak{g}_1(\rho)=\chi_\delta(\rho)\,u_1^{\mathrm{lin},(N)}(\rho)\,,\end{array}\right.\tag{5.5}$$

where $u_0^{\mathrm{lin},(N)}$ and $u_1^{\mathrm{lin},(N)}$ are the functions defined above by (5.3), and where $\chi_\delta(\rho)=\chi_0\Big(\dfrac{\rho}{\delta}\Big)\cdot$

Remark 5.1. *Invoking* (5.3) *together with* (5.5), *we infer that there* [15] *is* $\delta_0(N)>0$ *such that for all positive real* $\delta\leq\delta_0(N)$ *and any integer* $m<3\nu+2$, *the above functions* $\mathfrak{g}_0$ *and* $\mathfrak{g}_1$ *belong respectively to the Sobolev spaces* $\dot H^{m+1}(\mathbb{R}^4)$ *and* $\dot H^{m}(\mathbb{R}^4)$, *and satisfy*

$$\|\mathfrak{g}_0\|_{\dot H^{m+1}(\mathbb{R}^4)}\leq C\,\delta^{3\nu-m+2}\quad \textit{and}\quad \|\mathfrak{g}_1\|_{\dot H^{m}(\mathbb{R}^4)}\leq C\,\delta^{3\nu-m+2}\,.$$

We shall look for the solution in the remote region under the form:

$$u_{\mathrm{out}}(t,\rho)=\rho+\mathfrak{g}_0(\rho)+t\mathfrak{g}_1(\rho)+\sum_{k\geq 2}t^{k}\mathfrak{g}_k(\rho)\,.\tag{5.6}$$

To this end, we shall apply the lines of reasoning of Sections 3 and 4 and determine by induction the functions $\mathfrak{g}_k$, for $k\geq 2$, making use of the fact that the function u_{out} is a formal solution to the Cauchy problem:

$$\left\{\begin{array}{l}(1.8)\,u_{\mathrm{out}}=0\\ u_{\mathrm{out}|t=0}=\rho+\mathfrak{g}_0\\ (\partial_t u_{\mathrm{out}})_{|t=0}=\mathfrak{g}_1\,.\end{array}\right.\tag{5.7}$$

For that purpose, we substitute (5.6) into (1.8), which by straightforward computations leads to the following recurrent relation for $k\geq 2$

$$\mathfrak{g}_k=\frac{1}{k(k-1)(2+2(\mathfrak{g}_0)_\rho+(\mathfrak{g}_0)_\rho^2)}\,\mathcal{H}_k\big(\mathfrak{g}_j,j\leq k-1\big)\cdot\tag{5.8}$$

[15] In what follows, the parameter $\delta_0(N)$ may vary from line to line.

The source term $\mathcal{H}_k$ involved in the above identity can be splitted into three parts as follows:

$$\mathcal{H}_k = \mathcal{H}_k^{(1)} + \mathcal{H}_k^{(2)} + \mathcal{H}_k^{(3)}\,, \tag{5.9}$$

with

$$\mathcal{H}_k^{(1)} = l_\rho\, \mathfrak{g}_{k-2}\,, \tag{5.10}$$

$$\begin{aligned} \mathcal{H}_k^{(2)} &= -2 \sum_{\substack{k_1+k_2=k \\ k_2>0}} k_1(k_1-1-k_2)\mathfrak{g}_{k_1}(\mathfrak{g}_{k_2})_\rho \\ &\quad + 6 \sum_{k_1+k_2=k-2} \Big(- \mathfrak{g}_{k_1} \frac{\check{u}_{k_2}}{\rho^3} + (\mathfrak{g}_{k_1})_\rho \Big(\frac{\check{u}_{k_2}}{\rho^2} + \frac{(\mathfrak{g}_{k_2})_\rho}{\rho} \Big) \Big)\,, \end{aligned} \tag{5.11}$$

$$\begin{aligned} \mathcal{H}_k^{(3)} &= \sum_{k_1+k_2+k_3=k} k_1 k_2 \Big(- \mathfrak{g}_{k_1}\mathfrak{g}_{k_2}(\mathfrak{g}_{k_3})_{\rho\rho} + 2\mathfrak{g}_{k_1}(\mathfrak{g}_{k_2})_\rho(\mathfrak{g}_{k_3})_\rho - 3\mathfrak{g}_{k_1}\mathfrak{g}_{k_2}\Big(\frac{\check{u}_{k_3}}{\rho^2} + \frac{(\mathfrak{g}_{k_3})_\rho}{\rho}\Big)\Big) \\ &\quad - \sum_{\substack{k_1+k_2+k_3=k \\ 2\le k_1<k}} k_1(k_1-1)\mathfrak{g}_{k_1}(\mathfrak{g}_{k_2})_\rho(\mathfrak{g}_{k_3})_\rho + 3 \sum_{k_1+k_2+k_3=k-2} (\mathfrak{g}_{k_1})_\rho(\mathfrak{g}_{k_2})_\rho \Big(\frac{\check{u}_{k_3}}{\rho^2} + \frac{(\mathfrak{g}_{k_3})_\rho}{\rho}\Big)\,, \end{aligned} \tag{5.12}$$

where $\check{u}_k$ is given by

$$\check{u} = \frac{u-\rho}{1+\frac{u-\rho}{\rho}} = \sum_{k\ge 0} t^k \check{u}_k\,\cdot \tag{5.13}$$

Note that $\check{u}_k$ only depends on $\mathfrak{g}_{k_i}$, with $k_i \le k$.

5.2. **Analysis of the fuctions $\mathfrak{g}_k$.** The aim of the present paragraph is to investigate the fuctions $\mathfrak{g}_k$ defined above by (5.8)-(5.12). To this end, let us start by introducing the following definition.

Definition 5.1. *We denote by $\mathcal{A}$ the set of functions a in $\mathcal{C}^\infty(\mathbb{R}_+^*)$ supported in $\{0<\rho\le 2\delta\}$, where δ is the positive parameter introduced in* (5.5)*, and admitting for $\rho<\delta$ an absolutely convergent expansion of the form:*

$$a(\rho) = \sum_{j\ge 3} \sum_{0\le \ell \le \frac{j-3}{2}} a_{j,\ell}\, \rho^{\nu j} \big(\log \rho\big)^\ell\,. \tag{5.14}$$

Remark 5.2. *The functional space $\mathcal{A}$ given by Definition 5.1 is an algebra, and we have for any function a in $\mathcal{A}$ and any integer m,*

$$\partial^m a \in \rho^{-m}\mathcal{A}\,. \tag{5.15}$$

Our aim now is to establish the following key result which describes the behavior of the functions $\mathfrak{g}_k$.

Lemma 5.1. *There exists $\delta_0(N)>0$ such that for all positive real $\delta \le \delta_0(N)$, we have, under the above notations, for any integer k*

$$\mathfrak{g}_k \in \rho^{1-k}\mathcal{A}\,.$$

Proof. Firstly note that in view of (5.3) and (5.5), $\mathfrak{g}_0 \in \rho\,\mathcal{A}$ and $\mathfrak{g}_1 \in \mathcal{A}$ for any $\delta>0$, and there exists $\delta_0(N)>0$ such that

$$\frac{1}{1+(1+(\mathfrak{g}_0)_\rho)^2}\,\mathcal{A} \subset \mathcal{A}\,, \quad \frac{1}{1+\dfrac{\mathfrak{g}_0}{\rho}}\,\mathcal{A} \subset \mathcal{A}\,, \tag{5.16}$$

for any $\delta \le \delta_0(N)$.

Let us now show that for any $\delta \leq \delta_0(N)$, $\mathfrak{g}_k \in \rho^{1-k}\mathcal{A}$, for all $k \geq 2$. To this end, we shall proceed by induction assuming that, for any integer $j \leq k-1$, the function $\mathfrak{g}_j$ belongs to $\rho^{1-j}\mathcal{A}$.

Recalling that

$$l_\rho v = v_{\rho\rho} + 6\Big(\frac{v}{\rho^2} + \frac{v_\rho}{\rho}\Big),$$

we infer taking into account (5.15) that the function $\mathcal{H}_k^{(1)}$ given by (5.10) belongs to $\rho^{1-k}\mathcal{A}$.

Since $\check{u}_k$ is defined by

$$\check{u} = \frac{u-\rho}{1+\frac{u-\rho}{\rho}} = \sum_{k\geq 0} t^k \check{u}_k\,,$$

it readily follows from the induction hypothesis that for any integer $j \leq k-1$, $\check{u}_j$ belongs to the functional space $\rho^{1-j}\mathcal{A}$.

Combining the fact that $\mathcal{A}$ is an algebra together with (5.15) and (5.16), we deduce that the function $\mathcal{H}_k^{(2)}$ defined by (5.11) belongs to $\rho^{1-k}\mathcal{A}$.

Along the same lines, taking into account (5.11), we readily gather that $\mathcal{H}_k^{(3)} \in \rho^{1-k}\mathcal{A}$. This concludes the proof of the result thanks to (5.8), (5.9) and (5.16).

□

Remark 5.3. *Combining Definition 5.1 together with Lemma 5.1, we infer that for any integer k, the function $\mathfrak{g}_k$ involved in the asymptotic formula* (5.6) *admits an absolutely convergent expansion of the form:*

$$\mathfrak{g}_k(\rho) = \rho^{1-k}\sum_{j\geq 3}\sum_{0\leq \ell\leq \frac{j-3}{2}} a^k_{j,\ell}\,\rho^{\nu j}\,\big(\log\rho\big)^\ell\,, \tag{5.17}$$

for $\rho < \delta$, with some coefficients $a^k_{j,\ell}$ satisfying

$$a^0_{j,\ell} = \mu^0_{j,\ell}\,,\; a^1_{j,\ell} = \mu^1_{j,\ell} \text{ if } 3\leq j\leq N \quad\text{and}\quad a^0_{j,\ell} = a^1_{j,\ell} = 0 \text{ if } j \geq N+1\,. \tag{5.18}$$

5.3. **Estimate of the approximate solution in the remote region.** Under the above notations, set for any integer $N \geq 3$

$$u^{(N)}_{\text{out}}(t,\rho) = \rho + \sum_{k=0}^{N} t^k \mathfrak{g}_k(\rho)\,,\quad V^{(N)}_{\text{out}}(t,y) = t^{-(\nu+1)}\,u^{(N)}_{\text{out}}(t, t^{\nu+1}\,y)\,. \tag{5.19}$$

Invoking Lemma 5.1, and recalling that for any integer k the function $\mathfrak{g}_k$ is compactly supported in $\{0 \leq \rho \leq 2\delta\}$, we infer that the function $V^{(N)}_{\text{out}}$ defined by (5.19) satisfies the following L^∞ estimates in the remote region

$$\Omega_{\text{out}} := \big\{Y \in \mathbb{R}^4,\, y = |Y| \geq t^{-\epsilon_2-\nu}\big\}\,.$$

Lemma 5.2. *For any multi-index α, there exists $\delta_0(\alpha, N) > 0$ such that for all positive real number $\delta \leq \delta_0(\alpha, N)$, we have*

$$\begin{aligned}
&(5.20) & \|\langle\cdot\rangle^{|\alpha|}\nabla^\alpha(V_{\rm out}^{(N)}(t,\cdot)-Q)\|_{L^\infty(\Omega_{\rm out})} &\leq C_\alpha\, t^{-(\nu+1)}\,\delta^{3\nu+1}\,,\\
&(5.21) & \|\langle\cdot\rangle^{|\alpha|-1}\nabla^\alpha(V_{\rm out}^{(N)}(t,\cdot)-Q)\|_{L^\infty(\Omega_{\rm out})} &\leq C_\alpha\,\delta^{3\nu}\,,\\
&(5.22) & \|\langle\cdot\rangle^{\beta}\nabla^\alpha(V_{\rm out}^{(N)}(t,\cdot)-Q)\|_{L^\infty(\Omega_{\rm out})} &\leq C_{\alpha,\beta}\left(t^{3\nu(\nu+1)}+t^{\nu+1}\right),\ \forall\,\beta\leq|\alpha|-2\,,\\
&(5.23) & \|\partial_t V_{\rm out}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\rm out})} &\leq C\, t^{-(\nu+2)}\,\delta^{3\nu+1}\,,\\
&(5.24) & \|\langle\cdot\rangle^{|\alpha|}\nabla^\alpha V_{\rm out,1}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\rm out})} &\leq C_\alpha\,\delta^{3\nu}\,,\\
&(5.25) & \|\langle\cdot\rangle^{\beta}\nabla^\alpha V_{\rm out,1}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\rm out})} &\leq C_{\alpha,\beta}\left(t^{3\nu(\nu+1)}+t^{\nu+1}\right),\ \forall\,\beta\leq|\alpha|-1\,,\\
&(5.26) & \|\partial_t V_{\rm out,1}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\rm out})} &\leq C\, t^{-1}\,\delta^{3\nu}\,,\\
&(5.27) & \|\langle\cdot\rangle^{\beta}\nabla^\alpha V_{\rm out,2}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\rm out})} &\leq C_{\alpha,\beta}\left(t^{3\nu(\nu+1)}+\delta^{3\nu-1}\,t^{\nu+1}\right),\ \forall\,\beta\leq|\alpha|\,,
\end{aligned}$$

for all $0 < t \leq T$ with $T = T(\alpha, \delta, N)$, and where

$$V_{\rm out,1}^{(N)}(t,y) := (\partial_t u_{\rm out}^{(N)})(t,\rho)\,,\quad V_{\rm out,2}^{(N)}(t,y) := t^{\nu+1}\,(\partial_t^2 u_{\rm out}^{(N)})(t,\rho)\,.$$

Besides, for any multi-index $|\alpha| \geq 1$

$$\|\nabla^\alpha\partial_t V_{\rm out}^{(N)}(t,\cdot)\|_{L^\infty(\Omega_{\rm out})} \leq C_\alpha\, t^{-1}\,\delta^{3\nu}\,, \tag{5.28}$$

for all $0 < t \leq T$.

Denote

$$\Omega_{\rm out}^x := \left\{x\in\mathbb{R}^4,\ |x|\geq t^{1-\epsilon_2}\right\}. \tag{5.29}$$

One has the following estimates in L^2 framework:

Lemma 5.3. *Under the above notations, the following estimates occur for any $0 < \delta \leq \delta_0(\alpha, N)$ and all $0 < t \leq T = T(\alpha, \delta, N)$:*

$$\begin{aligned}
\left\|\nabla_x^\alpha\left(u_{\rm out}^{(N)}(t,\cdot)-t^{\nu+1}Q\Big(\frac{\cdot}{t^{\nu+1}}\Big)-\mathfrak{g}_0\right)\right\|_{L^2(\Omega_{\rm out}^x)} &\leq C_\alpha t(1+t^{(1-\epsilon_2)(3\nu+2-|\alpha|)}),\ \forall\,|\alpha|\geq 1,\\
\left\|\nabla_x^\alpha\left(\partial_t^\ell u_{\rm out}^{(N)}(t,\cdot)-\mathfrak{g}_\ell\right)\right\|_{L^2(\Omega_{\rm out}^x)} &\leq C_\alpha t(1+t^{(1-\epsilon_2)(3\nu+2-\ell-|\alpha|)}),\ \forall\,|\alpha|\geq 0,
\end{aligned}$$

for all $\ell = 1, 2$.

Remark 5.4. *Combining Formula (5.19) with Lemma 5.3, we infer that $V_{\rm out}^{(N)}(t,\cdot)$ satisfies, for all $0 < t \leq T$*

$$\|\nabla^\alpha(V_{\rm out}^{(N)}(t,\cdot)-Q)\|_{L^2(\Omega_{\rm out})} \leq C_\alpha t^{(|\alpha|-3)\,(\nu+1)}[\delta^{3\nu+3-|\alpha|}+t^{(1-\epsilon_2)(3\nu+3-|\alpha|)}],\ \forall\alpha\geq 1\,, \tag{5.30}$$

$$\|\nabla^\alpha V_{\rm out,\ell}^{(N)}(t,\cdot)\|_{L^2(\Omega_{\rm out})} \leq C_\alpha t^{(|\alpha|-3+\ell)\,(\nu+1)}[\delta^{3\nu+3-\ell-|\alpha|}+t^{(1-\epsilon_2)(3\nu+3-\ell-|\alpha|)}],\forall\alpha\geq 0, \tag{5.31}$$

for all $\ell = 1, 2$.

Let us now consider the remainder

$$\mathcal{R}_{\rm out}^{(N)} := (3.3)V_{\rm out}^{(N)}\,. \tag{5.32}$$

We have

$$\mathcal{R}_{\rm out}^{(N)}(t,y) = t^{\nu+1}\widetilde{\mathcal{R}}_{\rm out}^{(N)}(t,t^{\nu+1}y)$$

where

$$\widetilde{\mathcal{R}}_{\rm out}^{(N)}(t,y) = [(1.8)u_{\rm out}^{(N)}](t,t^{\nu+1}y)\,.$$

It follows readily from the proof of Lemma 5.1 that

$$\| |\cdot|^{\frac{3}{2}}\, \nabla_x^\alpha \widetilde{\mathcal{R}}_{\rm out}^{(N)}(t,\cdot)\|_{L^2(\Omega_{\rm out}^x)} \leq C_{\alpha,N}\, t^{N-1-(1-\epsilon_2)(|\alpha|+N-3\nu-\frac{7}{2})}\,, \tag{5.33}$$

for any $|\alpha| \geq 0$, provided that $N \geq 3\nu + \dfrac{7}{2}$ which leads to the following lemma:

Lemma 5.4. *For any multi-index α, the following estimate holds :*

$$\|\langle\cdot\rangle^{\frac{3}{2}}\, \nabla^\alpha \mathcal{R}_{\rm out}^{(N)}(t,\cdot)\|_{L^2(\Omega_{\rm out})} \leq t^{\epsilon_2 N - \frac{5}{2}(\nu+1)}\,, \tag{5.34}$$

for all $0 < t \leq T = T(\alpha,\delta,N)$, provided that $N \geq 3\nu + \dfrac{7}{2}\cdot$

We next investigate $V_{\rm out}^{(N)} - V_{\rm ss}^{(N)}$ in $\Omega_{\rm out} \cap \Omega_{\rm ss}$. Assuming $\rho < \delta$, and rewriting $u_{\rm out}^{(N)}$ in terms of the variable $z = \dfrac{\rho}{\lambda^{(N)}(t)}$, we get:

$$\begin{aligned} u_{\rm out}^{(N)}(t,\rho) = \lambda^{(N)}(t)\Bigg[z + \sum_{k=3}^{N} \sum_{0\leq \ell\leq \frac{k-3}{2}} t^{\nu k}(\log t)^\ell \Bigg(&\sum_{\substack{3\leq\beta\leq k-3\\ 0\leq\alpha\leq\frac{k-6}{2}-\ell,\, p\geq 0}} w_{k,\ell,\alpha,\beta,p}^{\rm out}(\log z)^\alpha z^{\nu\beta+1-p} \\ &+ z^{\nu k+1} \sum_{0\leq\alpha\leq\frac{k-3}{2}-\ell,\, p\geq 0} w_{k,\ell,\alpha,p}^{\rm out}(\log z)^\alpha z^{-p}\Bigg)\Bigg], \end{aligned}$$

with some coefficients $w_{k,\ell,\alpha,\beta,p}^{\rm out}$, $w_{k,\ell,\alpha,p}^{\rm out}$ that can be expressed explicitly in terms of the coefficients $(\lambda_{j,\ell})$, for $3 \leq j \leq N, 0 \leq \ell \leq \ell(j)$ and of the constants $(a_{j,\ell}^k)$, $k \geq 0, j \geq 3, 0 \leq \ell \leq \ell(j)$ introduced in Remark 5.3.

In particular

$$w_{k,\ell,\alpha,p}^{\rm out} = \binom{\alpha+\ell}{\alpha} a_{k,\alpha+\ell}^p\,, \tag{5.35}$$

for all $k \geq 3$, $\ell \leq \dfrac{k-3}{2}$, $\alpha \leq \dfrac{k-3}{2} - \ell$, $p \geq 0\cdot$

Combining (5.4), (5.18) with (5.35), we infer that

$$\sum_{0\leq j\leq \ell(k)} \big(\alpha_k^{j,+} f_{k,\ell}^{j,+} + \alpha_k^{j,-} f_{k,\ell}^{j,-}\big) = \sum_{\substack{0\leq\alpha\leq\frac{j-3}{2}-\ell\\ p=0,1}} z^{\nu k+1-p}(\log z)^\alpha w_{k,\ell,\alpha,p}^{\rm out} + \mathcal{O}(z^{\nu k-1}(\log z)^{\ell(j)-\ell})\,,$$

as $z \to 0$, which by Proposition 4.1 (2) (uniqueness around infinity) implies that

$$w_{k,\ell,\alpha,\beta,p}^{\rm out} = w_{k,\ell,\alpha,\beta,p}\,,\ w_{k,\ell,q,p}^{\rm out} = w_{k,\ell,q,p}\,, \tag{5.36}$$

for any $3 \leq k \leq N$, $0 \leq \ell \leq \ell(k)$, $0 \leq \alpha \leq \frac{k-6}{2} - \ell$, $0 \leq q \leq \frac{k-3}{2} - \ell$, $3 \leq \beta \leq k-3$, $p \geq 0$, where $w_{k,\ell,\alpha,\beta,p}$, $w_{k,\ell,q,p}$ are the coefficients involved in (4.65).

As a direct consequence of (5.36), we obtain

Lemma 5.5. *For any multi-index $\alpha \in \mathbb{N}^4$ and any integer m, we have*

$$\|\partial_t^m \nabla^\alpha (V_{\rm out}^{(N)} - V_{\rm ss}^{(N)})(t,\cdot)\|_{L^\infty(\Omega_{\rm out}\cap\Omega_{\rm ss})} \leq t^{-m-\nu+|\alpha|(\nu+\epsilon_2)}\big(t^{\epsilon_2 N} + t^{-\epsilon_2+(1-\epsilon_2)\nu N}\big)\,, \tag{5.37}$$

for all $0 < t < T = T(\alpha,m,N)$.

6. Approximate solution in the whole space

Let Θ be a radial function in $\mathcal{D}(\mathbb{R})$ satisfying

$$\Theta(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \leq \frac{1}{4} \\ 0 & \text{if} \quad |\xi| \geq \frac{1}{2} \end{cases}.$$

Set

$$\begin{aligned} V^{(N)}(t,y) &:= \Theta\big(y\, t^{\nu-\epsilon_1}\big) V_{\rm in}^{(N)})(t,y) \\ &\quad + \big(\Theta\big(y\, t^{\nu+\epsilon_2}\big) - \Theta\big(y\, t^{\nu-\epsilon_1}\big)\big) V_{\rm ss}^{(N)})(t,y) + \big(1-\Theta\big(y\, t^{\nu+\epsilon_2}\big)\big) V_{\rm out}^{(N)})(t,y)\,, \\ u^{(N)}(t,\rho) &:= t^{\nu+1} V^{(N)}\big(t, \frac{\rho}{t^{\nu+1}}\big)\,. \end{aligned} \tag{6.1}$$

Combining Lemmas 3.2, 4.6 and 5.2 together with Lemmas 4.9 and 5.5, we infer that for N sufficiently large there exists a positive parameter $\delta_0(N)$ such that for any $\delta \leq \delta_0(N)$ there exists a positive time $T = T(\delta, N)$ so that the above approximate solution $V^{(N)}$ defined by (6.1) satisfies the following L^∞ estimates:

Lemma 6.1. *The following estimates hold for $V^{(N)}$, for all $0 < t \leq T$*

$$\|\langle\cdot\rangle^{|\alpha|-1}\, \nabla^\alpha\, (V^{(N)} - Q)(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \leq C\, \delta^{3\nu}\,,\ \forall\, 0 \leq |\alpha| < 3\nu+4\,. \tag{6.2}$$

$$\|\langle\cdot\rangle^{\beta} \nabla^\alpha (V^{(N)} - Q)(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \leq C\, t^{\nu}\,,\ \forall\, 1 \leq |\alpha| < 3\nu+4 \text{ and } \beta \leq |\alpha| - 2. \tag{6.3}$$

$$\|\nabla^\alpha \frac{y}{|y|^2} \cdot \nabla (V^{(N)} - Q)(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \leq C\, t^{\nu}\,,\ \forall\, 0 \leq |\alpha| < 3\nu+3\,. \tag{6.4}$$

Besides the time derivative of $V^{(N)}$ satisfies

$$\|\partial_t V^{(N)}(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \leq C\, t^{-2-\nu}\, \delta^{3\nu+1} \tag{6.5}$$

$$\|\nabla^\alpha \partial_t V^{(N)}(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \leq C\, t^{-1}\, \delta^{3\nu}\,,\ \forall\ 1 \leq |\alpha| < 3\nu+3\,. \tag{6.6}$$

In addition for any multi-index α of length $|\alpha| < 3\nu+3$, the function $V_1^{(N)}(t,y) := (\partial_t u^{(N)})(t,\rho)$ and its time derivative verify

$$\|\langle\cdot\rangle^{|\alpha|} \nabla^\alpha V_1^{(N)}\|_{L^\infty(\mathbb{R}^4)} \leq C\, \delta^{3\nu}\,, \tag{6.7}$$

$$\|\langle\cdot\rangle^{\beta} \nabla^\alpha V_1^{(N)}(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \leq C\, t^{\nu}\,,\ \forall\, \beta \leq |\alpha| - 1\,, \tag{6.8}$$

$$\|\partial_t V_1^{(N)}(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \leq C\, t^{-1} \delta^{3\nu}\,. \tag{6.9}$$

Finally for any multi-index α of length $|\alpha| < 3\nu+2$ and any integer $\beta \leq |\alpha|$, we have

$$\|\langle\cdot\rangle^{\beta} \nabla^\alpha V_2^{(N)}(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \leq C t^{\nu}\,, \tag{6.10}$$

where $V_2^{(N)}(t,y) := t^{\nu+1} (\partial_t^2 u^{(N)})(t,\rho)$.

Along the same lines, taking advantage of Lemmas 3.3, 4.7 and 5.3, we get the following L^2 estimates, as before for N sufficiently large, $\delta \leq \delta_0(N)$ and $0 < t \leq T(\delta, N)$:

Lemma 6.2. *For any $1 \leq |\alpha| < 3\nu+3$, we have*

$$\|\nabla^\alpha (u^{(N)}(t,\cdot) - t^{\nu+1} Q\big(\frac{\cdot}{t^{\nu+1}}\big) - \mathfrak{g}_0)\|_{L^2((\mathbb{R}^4)} \leq C\big(t + t^{(1-\epsilon_2)(3\nu+3-|\alpha|)} + t^{3+5\nu-|\alpha|(1+\nu)}\big), \tag{6.11}$$

and for any $0 \leq |\alpha| < 3\nu+2$

$$\|\nabla^\alpha (u_t^{(N)}(t,\cdot) - \mathfrak{g}_1)\|_{L^2(\mathbb{R}^4)} \leq C\big(t + t^{(1-\epsilon_2)(3\nu+2-|\alpha|)} + t^{2+3\nu-|\alpha|(1+\nu)}\big)\,. \tag{6.12}$$

Besides, we have

$$\|\nabla^\alpha(V^{(N)}(t,\cdot)-Q)\|_{L^2(\mathbb{R}^4)} \leq Ct^{2\nu},\ \forall\, 3\nu+3<|\alpha|<3\nu+4+\frac{1}{2}, \tag{6.13}$$

$$\|\nabla^\alpha V_1^{(N)}(t,\cdot)\|_{L^2(\mathbb{R}^4)} \leq C\,t^{\nu},\ \forall\, 3\nu+2<|\alpha|<3\nu+3+\frac{1}{2}, \tag{6.14}$$

$$\|\nabla^\alpha V_2^{(N)}(t,\cdot)\|_{L^2(\mathbb{R}^4)} \leq C\,t^{2\nu},\ \forall\, 3\nu+1<|\alpha|<3\nu+2+\frac{1}{2}. \tag{6.15}$$

Remark 6.1. *Lemma 6.2 implies that*

$$\|\nabla^\alpha(V^{(N)}(t,\cdot)-Q)\|_{L^2(\mathbb{R}^4)} \leq C\left(t^{2\nu}+t^{(1+\nu)(|\alpha|-3)}\delta^{3\nu+3-|\alpha|}\right),\ \forall 1\leq|\alpha|<3\nu+4+\frac{1}{2}, \tag{6.16}$$

$$\|\nabla^\alpha V_1^{(N)}(t,\cdot)\|_{L^2(\mathbb{R}^4)} \leq C\left(t^{\nu}+t^{(1+\nu)(|\alpha|-2)}\delta^{3\nu+2-|\alpha|}\right),\ \forall 0\leq|\alpha|<3\nu+3+\frac{1}{2}, \tag{6.17}$$

and

$$\begin{aligned}\|\nabla^\alpha(u^{(N)}(t,\cdot)-t^{\nu+1}Q(\tfrac{\cdot}{t^{\nu+1}})-\mathfrak{g}_0)\|_{L^2(\mathbb{R}^4)} &\overset{t\to 0}{\longrightarrow} 0 \quad,\ \forall\ 1\leq|\alpha|<3+\frac{2\nu}{\nu+1}\,,\\ \|\nabla^\alpha(u_t^{(N)}(t,\cdot)-\mathfrak{g}_1)\|_{L^2(\mathbb{R}^4)} &\overset{t\to 0}{\longrightarrow} 0 \quad,\ \forall\ 0\leq|\alpha|<2+\frac{\nu}{\nu+1}\,.\end{aligned}$$

Finally, if we denote by

$$\mathcal{R}^{(N)} := (3.3)\, V^{(N)}\,,$$

then invoking Lemmas 3.4, 4.8, 4.9, 5.4 and 5.5, we infer that the following result holds:

Lemma 6.3. *There exist $N_0\in\mathbb{N}$ and $\kappa>0$ such that*

$$\|\langle\cdot\rangle^{\frac{3}{2}}\,\mathcal{R}^{(N)}(t,\cdot)\|_{H^{K_0}(\mathbb{R}^4)} \leq t^{\kappa N+\nu}\,, \tag{6.18}$$

for all $0<t\leq T(\delta,N)$, where $K_0=[3\nu+\frac{5}{2}]$ denotes the integer introduced in Lemma 4.8.

Re-denoting N, one can always assume that the approximate solutions $u^{(N)}$ are defined and satisfy Lemmas 6.1-6.2 for any integer $N\geq 1$, and that (6.18) holds with $\kappa=1$ for all $N\geq 1$.

7. Proof of the blow up result

7.1. **Key estimates.** The approximated solutions $u^{(N)}$ constructed in the previous sections verify, for any integer $N\geq 1$

$$(\nabla(u^{(N)}-Q),\partial_t u^{(N)})\in C(]0,T],H^{K_0+1}(\mathbb{R}^4))\,,$$

for [16] some $T=T(\delta,N)>0$. Furthermore, by (6.2), (6.8), there are positive constants c_0 and c_1 such that

$$u^{(N)}(t,\cdot)\geq c_0\,t^{\nu+1} \quad \text{and} \tag{7.1}$$

$$(1+|\nabla u^{(N)}|^2-(\partial_t u^{(N)})^2)(t,\cdot)\geq c_1\,, \tag{7.2}$$

for any $N\geq 1$, and all t in $]0,T]$. This ensures that

$$(u^{(N)}(t,\cdot),\partial_t u^{(N)}(t,\cdot))\in X_{K_0+2}\,,\forall t\in]0,T]\,.$$

The goal of this paragraph is to achieve the proof of Theorem 1.2 by complementing these approximate solutions $u^{(N)}$ to an actual solution u to the quasilinear wave equation (1.8) which

[16] In what follows, δ is assumed to be less than $\delta_0(N)$, which may vary from line to line.

blows up at $t=0$, and which for N fixed large enough is close to $u^{(N)}$, in the sense that there is a positive time $T=T(\delta,N)$ such that the following estimate holds

$$\|\langle\cdot\rangle^{\frac{3}{2}}\,\partial_t(u-u^{(N)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)}+\|\langle\cdot\rangle^{\frac{3}{2}}\,\nabla(u-u^{(N)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)}\leq t^{\frac{N}{2}}\,,\tag{7.3}$$

for all time t in $]0,T]$, where the regularity index[17] $L_0=2M+1$, with $M=[\frac{K_0}{2}]\cdot$

Note that since $\nu>\frac{1}{2}$, we have $M\geq 2$, and thus $L_0\geq 5$.

The mechanism for achieving this will rely on the following crucial result:

Proposition 7.1. *There is N_0 in $\mathbb{N}$ such that for any integer $N\geq N_0$, there exists a small positive time $T=T(\delta,N)$ such that, for any time $0<t_1\leq T$, the Cauchy problem:*

$$(\mathrm{NW})^{(N)}\begin{cases}(1.6)\,u=0\,,\\ u_{|t=t_1}=u^{(N)}(t_1,\cdot)\\ (\partial_t u)_{|t=t_1}=\partial_t u^{(N)}(t_1,\cdot)\end{cases}\tag{7.4}$$

admits a unique solution u on the interval $[t_1,T]$ which satisfies

$$\|\langle\cdot\rangle^{\frac{3}{2}}\,\partial_t(u-u^{(N)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)}+\|\langle\cdot\rangle^{\frac{3}{2}}\,\nabla(u-u^{(N)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)}\leq t^{\frac{N}{2}}\,,\tag{7.5}$$

for all $t_1\leq t\leq T$.

Proof. As mentioned above, for any t_1 sufficiently small, the initial data $(u^{(N)}(t_1,\cdot),\partial_t u^{(N)}(t_1,\cdot))$ belongs to X_{K_0+2}, and thus satisfy the hypothesis of Theorem 1.1. By construction $u^{(N)}(t,\rho)-\rho$ is compactly supported. Thus, to prove Proposition 7.1, it is enough to show that there exists a time $T=T(\delta,N)>0$ such that the solution to the Cauchy problem (7.4) satisfies the energy estimate (7.5), for any time $t_1\leq t<\min\big\{T(\delta,N),T^*\big\}$, where T^* is the maximal time of existence. This will be achieved in two steps:

(1) First writing $u(t,x)=t^{\nu+1}V(t,y)$, $V(t,y)=V^{(N)}(t,y)+\varepsilon^{(N)}(t,y)$, with $y=\frac{x}{t^{\nu+1}}$, $x\in\mathbb{R}^4$, we derive the equation satisfied by the remainder term $\varepsilon^{(N)}$. We next set
$$\varepsilon^{(N)}(t,y)=H(y)\,r^{(N)}(t,y)\,,$$
where H is the function defined by (2.8), and rewrite the obtained equation in terms of $r^{(N)}$. As we will see later, the equation for $r^{(N)}$ involves the operator $\mathfrak{L}$ introduced in (2.10).

(2) We deduce the desired result (inequalities (7.5)) by suitable energy estimates by making use of the behavior of the approximate solution $u^{(N)}$ described by Lemmas 6.1, 6.2, and the spectral properties of the operator $\mathfrak{L}$ which turns out to be close to the Laplace operator.

In order to make notations as light as possible, we shall omit in the sequel the dependence of the functions $\varepsilon^{(N)}$ and $r^{(N)}$ on N.

Denote by

$$V_1(t,y):\;=\;a(t)\,V_t(t,y)+a'(t)\,\Lambda V(t,y)=u_t(t,x)\tag{7.6}$$
$$V_2(t,y):\;=\;a(t)\,(V_1)_t(t,y)-a'(t)\,(y\cdot\nabla V_1)(t,y)=t^{\nu+1}\,u_{tt}(t,x)\,,\tag{7.7}$$

with $a(t)=t^{\nu+1}$ and $\Lambda V=V-y\cdot\nabla V$.

[17] we take L to be odd just to make the estimates we are dealing more easier, but it is not important.

By straightforward computations, we readily gather that the quasilinear wave equation (1.6) multiplied by $a(t)$ undertakes the following form in terms of the function V with respect to the variables $(t, y) = \Big(t, \dfrac{x}{t^{1+\nu}}\Big)$

$$\begin{aligned}(1+|\nabla V|^2)V_2 - 2(\nabla V\cdot\nabla V_1)V_1 - (1-V_1^2+|\nabla V|^2)\,\Delta V \\ + \sum_{j,k=1}^{4} V_{y_j}V_{y_k}\partial^2_{y_jy_k}V + \frac{3}{V}(1-V_1^2+|\nabla V|^2) = 0\,\cdot\end{aligned} \tag{7.8}$$

Thus recalling that the approximate solution $V^{(N)}$ satisfies (7.8) up to a remainder term $\mathcal{R}^{(N)}$, we infer that saying that the function u solves the equation (1.6) $u = 0$ is equivalent to say that the remainder term ε satisfies the following equation:

$$\begin{aligned}(1+|\nabla V|^2)\,\varepsilon_2 - \mathcal{L}\varepsilon - 2V_1\nabla V\cdot\nabla\varepsilon_1 + (V_1^2-|\nabla\varepsilon|^2)\,\Delta\varepsilon \\ + \sum_{j,k=1}^{4}\varepsilon_{y_j}\varepsilon_{y_k}\partial^2_{y_jy_k}\varepsilon + \mathcal{F} + \mathcal{R}^{(N)} = 0\,,\end{aligned} \tag{7.9}$$

where

$$\varepsilon_2 = a(t)\,(\varepsilon_1)_t - a'(t)\,(y\cdot\nabla\varepsilon_1),\quad \varepsilon_1 = a(t)\,\varepsilon_t + a'(t)\,\Lambda\varepsilon\,, \tag{7.10}$$

with $\mathcal{L}$ the linearized operator introduced in (1.16):

$$\mathcal{L}\varepsilon = \Delta\varepsilon + 3\Big(\frac{(3\,y\cdot\nabla Q)\,\nabla Q}{|y|^2} - \frac{2\,\nabla Q}{Q}\Big)\cdot\nabla\varepsilon + 3\frac{1+|\nabla Q|^2}{Q^2}\,\varepsilon\,,$$

and where the term $\mathcal{F}$ is given by:

$$\begin{aligned}\mathcal{F} &= (|\nabla V|^2 - |\nabla V^{(N)}|^2)\,V_2^{(N)} - 2\,(V_1\nabla V - V_1^{(N)}\nabla V^{(N)})\cdot\nabla V_1^{(N)} + (V_1^2 - (V_1^{(N)})^2)\Delta V^{(N)} \\ &\quad - \frac{3}{V\,V^{(N)}}\big(V_1^2\,V^{(N)} - (V_1^{(N)})^2\,V\big) + 3\Big[\frac{1}{V^{(N)}}\big(|\nabla V|^2 - |\nabla V^{(N)}|^2\big) - \frac{2}{Q}\,\nabla Q\cdot\nabla\varepsilon\Big] \\ &\quad - 3\,\varepsilon\Big[\frac{(1+|\nabla V|^2)}{VV^{(N)}} - \frac{(1+|\nabla Q|^2)}{Q^2}\Big] - 9\,(|\nabla V^{(N)}|^2 - |\nabla Q|^2)\frac{y\cdot\nabla\varepsilon}{|y|^2} - 9\frac{y\cdot\nabla V^{(N)}}{|y|^2}|\nabla\varepsilon|^2\,.\end{aligned}$$

Next, set

$$\varepsilon(t,y) = H(y)\,r(t,y)\,, \tag{7.11}$$

with

$$H = \frac{(1+|\nabla Q|^2)^{\frac14}}{Q^{\frac32}}\,\cdot \tag{7.12}$$

Let us emphasize that in view of Lemma 2.1, the above function H enjoys the following property: for any multi-index α in $\mathbb{N}^4$, there exists a positive constant C_α such that, for any y in $\mathbb{R}^4$ the following estimate holds

$$\frac{1}{C_\alpha\,\langle y\rangle^{\frac32+|\alpha|}} \leq |\,\nabla^\alpha H(y)| \leq \frac{C_\alpha}{\langle y\rangle^{\frac32+|\alpha|}}\,\cdot \tag{7.13}$$

Now in light of the definitions introduced in (7.10), we have

$$\varepsilon_1(t,y) = H(y)\,r_1(t,y),\quad \varepsilon_2(t,y) = H(y)\,r_2(t,y)\,,$$

where

$$r_1 = a\,r_t + a'\,\Lambda r - a'\,\frac{y\cdot\nabla H}{H}\,r,\quad r_2 = a\,(r_1)_t - a'\,y\cdot\nabla r_1 - a'\,\frac{y\cdot\nabla H}{H}\,r_1\,\cdot \tag{7.14}$$

Thus taking advantage of (7.9), we readily gather that the remainder term r given by (7.11) satisfies

$$\begin{aligned}(1+|\nabla V|^2)\, r_2 + (1+|\nabla Q|^2)\, \mathfrak{L} r - \frac{2V_1}{H}\nabla V\cdot\nabla(Hr_1) + (V_1^2 - |\nabla(Hr)|^2)\,\Delta r\\ - \frac{2V_1}{H}\nabla V\cdot\nabla(Hr_1) + \frac{V_1^2-|\nabla(Hr)|^2}{H}[\Delta, H]\, r + \sum_{j,k=1}^{4}(Hr)_{y_j}(Hr)_{y_k}\partial^2_{y_jy_k} r\\ + \sum_{j,k=1}^{4}\frac{(Hr)_{y_j}(Hr)_{y_k}}{H}[\partial^2_{y_jy_k}, H]r + \frac{\mathcal{F}}{H} + \frac{\mathcal{R}^{(N)}}{H} = 0\,,\end{aligned}\tag{7.15}$$

where $[A,B]=AB-BA$ denotes the commutator of the operators A and B, and where

$$\mathfrak{L} = -\frac{1}{H(1+|\nabla Q|^2)}\,\mathcal{L}H\,\cdot\tag{7.16}$$

Let us recall that in view of (2.10)

$$\mathfrak{L} = -q\,\Delta\, q + \mathcal{P}\,,$$

with $q = \dfrac{1}{(1+|\nabla Q|^2)^{\frac12}}$, and $\mathcal{P}$ a radial $\mathcal{C}^\infty$ function which satisfies

$$\mathcal{P} = -\frac{3}{8\rho^2}\big(1+\circ(1)\big)\,,$$

as ρ tends to infinity.

Now dividing the equation at hand by $(1+|\nabla V|^2)$, we infer that the function r solves the following equation:

$$\begin{aligned}r_2 + \frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\,\mathfrak{L} r - \frac{2\,V_1}{1+|\nabla V|^2}\nabla V\cdot\nabla r_1 + \frac{V_1^2-|\nabla(Hr)|^2}{1+|\nabla V|^2}\,\Delta r\\ + \frac{1}{1+|\nabla V|^2}\sum_{j,k=1}^{4}(Hr)_{y_j}(Hr)_{y_k}\partial^2_{y_jy_k} r + \widetilde{\mathcal{F}} + \widetilde{\mathcal{R}}^{(N)} = 0\,,\end{aligned}\tag{7.17}$$

where

$$\widetilde{\mathcal{R}}^{(N)} := \frac{\mathcal{R}^{(N)}}{(1+|\nabla V|^2)\,H}\,,\tag{7.18}$$

and

$$\begin{aligned}\widetilde{\mathcal{F}} :=& \frac{\mathcal{F}}{(1+|\nabla V|^2)\,H} - \frac{2V_1}{(1+|\nabla V|^2)\,H}\nabla V\cdot(\nabla H)\, r_1\\ &+ \frac{V_1^2-|\nabla(Hr)|^2}{(1+|\nabla V|^2)\,H}[\Delta, H]\, r + \sum_{j,k=1}^{4}\frac{(Hr)_{y_j}(Hr)_{y_k}}{(1+|\nabla V|^2)\,H}[\partial^2_{y_jy_k}, H]r\,\cdot\end{aligned}\tag{7.19}$$

Note that we split Equation (7.15) into a first part which behaves as a quasilinear wave equation and a second part depending only on the remainder term r and its first derivatives. This achieves the goal of the first step.

The proof of energy inequalities (7.5) is based on suitable priori estimates. These priori estimates are established by combining the key properties of the operator $\mathfrak{L}$ stated page 9 (and established in Appendix B) together with the asymptotic formula (2.4) as well as some properties of the approximate solution $V^{(N)}$. We shall argue by bootstrap argument by proving the following key result:

Lemma 7.1. *There is N_0 in $\mathbb{N}$ such that for any integer $N \geq N_0$, there exists $T = T(N, \delta) > 0$ such that for any $t_1 \in]0, T]$, and any $t_2 \in [t_1, T]$ the following property holds.*

If we have for all time t in $[t_1, t_2]$,

$$\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} \leq t^{2N}\,, \tag{7.20}$$

then

$$\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} \leq \frac{C}{N}\, t^{2N}\,, \tag{7.21}$$

for any time t in $[t_1, t_2]$, where C is an absolute constant.

Proof of Lemma 7.1. In order to establish Inequality (7.21), let us start by applying the operator $\mathfrak{L}^M$ to Equation (7.17). This gives rise to

$$\begin{aligned}
&\mathfrak{L}^M\, r_2 + \frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\,\mathfrak{L}^{M+1} r - \frac{2\,V_1}{1+|\nabla V|^2}\nabla V\cdot\nabla\mathfrak{L}^M r_1 + \frac{V_1^2 - |\nabla\varepsilon|^2}{1+|\nabla V|^2}\,\Delta\mathfrak{L}^M r \\
&+ \frac{1}{1+|\nabla V|^2}\sum_{j,k=1}^4 (Hr)_{y_j}(Hr)_{y_k}\partial^2_{y_j y_k}(\mathfrak{L}^M r) + \widetilde{\mathcal{F}}_M + \mathfrak{L}^M\widetilde{\mathcal{R}}^{(N)} = 0\,,
\end{aligned} \tag{7.22}$$

with $\widetilde{\mathcal{F}}_M = \mathfrak{L}^M\,\widetilde{\mathcal{F}} + \mathcal{G}_M$, where

$$\begin{aligned}
\mathcal{G}_M :=& \Big[\mathfrak{L}^M, \frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\Big]\,\mathfrak{L} r - 2\,\Big[\mathfrak{L}^M, \frac{V_1}{1+|\nabla V|^2}\nabla V\cdot\nabla\Big]\, r_1 \\
&+ \Big[\mathfrak{L}^M, \frac{V_1^2 - |\nabla\varepsilon|^2}{1+|\nabla V|^2}\,\Delta\Big]\, r + \sum_{j,k=1}^4 \Big[\mathfrak{L}^M, \frac{(Hr)_{y_j}(Hr)_{y_k}}{1+|\nabla V|^2}\partial^2_{y_j y_k}\Big]\, r\,.
\end{aligned} \tag{7.23}$$

Now let us respectively multiply Equation (7.17) by $a^{-1}\, r_1$ and Equation (7.22) by $a^{-1}\,\mathfrak{L}^M r_1$, and then integrate over $\mathbb{R}^4$. This easily gives rise to the following identity

$$a^{-1}(t)\int_{\mathbb{R}^4}\Big[r_1(7.17) + (\mathfrak{L}^M r_1)(7.22)\Big](t,y)\, dy = 0\,. \tag{7.24}$$

Making use of formulae (7.17) and (7.22), we deduce that (7.24) can be splited in several parts as follows:

$$\begin{aligned}
&(I) + (II) + (III) + (IV) = \\
&-a^{-1}(t)\int_{\mathbb{R}^4}\Big[r_1\big(\widetilde{\mathcal{F}} + \widetilde{\mathcal{R}}^{(N)}\big) + (\mathfrak{L}^M r_1)\big(\widetilde{\mathcal{F}}_M + \mathfrak{L}^M\widetilde{\mathcal{R}}^{(N)}\big)\Big](t,y)\, dy\,,
\end{aligned}$$

with

$$\begin{aligned}
(I) &= a^{-1}(t)\int_{\mathbb{R}^4}\big(r_2\, r_1 + \mathfrak{L}^M r_2\,\mathfrak{L}^M r_1\big)(t,y)\, dy\,, \\
(II) &= a^{-1}(t)\int_{\mathbb{R}^4}\frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\Big[(\mathfrak{L} r)\, r_1 + (\mathfrak{L}^{M+1} r)\,(\mathfrak{L}^M r_1)\Big](t,y)\, dy\,, \\
(III) &= -2a^{-1}(t)\int_{\mathbb{R}^4}\frac{V_1}{1+|\nabla V|^2}\,\nabla V\cdot\Big[(\nabla r_1)\, r_1 + (\nabla\mathfrak{L}^M r_1)\,(\mathfrak{L}^M r_1)\Big](t,y)\, dy \quad \text{and} \\
(IV) &= a^{-1}(t)\sum_{i,j=1}^4\int_{\mathbb{R}^4} g_{i,j}\Big[\partial^2_{y_i y_j} r\, r_1 + (\partial^2_{y_i y_j}\mathfrak{L}^M r)\,(\mathfrak{L}^M r_1)\Big](t,y)\, dy\,,
\end{aligned}$$

where for all $1 \leq i, j \leq 4$ the coefficients $g_{i,j}$ in the latter integral are defined by

$$g_{i,j} = \frac{V_1^2 - |\nabla\varepsilon|^2}{1+|\nabla V|^2}\,\delta_{i,j} + \frac{(Hr)_{y_i}(Hr)_{y_j}}{1+|\nabla V|^2}\,, \tag{7.25}$$

and obviously satisfy the symmetry relations $g_{i,j} = g_{j,i}$.

Firstly, let us investigate the term (I). By definition

$$r_2 = a\,(r_1)_t - a'\,y\cdot\nabla r_1 - a'\,\frac{y\cdot\nabla H}{H}\,r_1\,,$$

and thus

$$\mathfrak{L}^M r_2 = a\,(\mathfrak{L}^M r_1)_t - a'\,y\cdot\nabla(\mathfrak{L}^M r_1) - a'\,X\,,$$

with

$$X = [\mathfrak{L}^M, y\cdot\nabla]\,r_1 + \mathfrak{L}^M\,\frac{y\cdot\nabla H}{H}\,r_1\,\cdot$$

We deduce that

$$(I) = \frac{1}{2}\frac{d}{dt}\Big[\|r_1(t)\|^2_{L^2(\mathbb{R}^4)} + \|\mathfrak{L}^M r_1(t)\|^2_{L^2(\mathbb{R}^4)}\Big] - \frac{1+\nu}{t}\int_{\mathbb{R}^4}\Big[r_1\,y\cdot\nabla r_1 + (\mathfrak{L}^M r_1)\,y\cdot\nabla(\mathfrak{L}^M r_1)$$
$$+ r_1\,\frac{y\cdot\nabla H}{H}\,r_1 + \,(\mathfrak{L}^M r_1)\,X\Big](t,y)\,dy\,\cdot$$

Integrating by parts and taking into account that

$$\|X\|_{L^2(\mathbb{R}^4)} \lesssim \|r_1\|_{H^{L_0-1}(\mathbb{R}^4)},$$

we find

$$(I) = \frac{1}{2}\frac{d}{dt}\Big[\|r_1(t)\|^2_{L^2(\mathbb{R}^4)} + \|\mathfrak{L}^M r_1(t)\|^2_{L^2(\mathbb{R}^4)}\Big] + \frac{1}{t}\,\mathcal{O}\big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\big)\,, \tag{7.26}$$

in the sense that (and all along this proof)

$$\Big|\mathcal{O}\big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\big)\Big| \lesssim \|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\,\cdot$$

Let us now estimate the part (II). Firstly, let us point out that it stems from Hardy inequality and the asymptotic expansion (2.4) that for any function f in $\dot H^1(\mathbb{R}^4)$ the following inequality holds

$$\|\nabla(q\,f)\|_{L^2(\mathbb{R}^4)} \le C\|\nabla f\|_{L^2(\mathbb{R}^4)}\,. \tag{7.27}$$

Therefore performing an integration by parts, we get

$$(II) = \int_{\mathbb{R}^4}\nabla\Big(\frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\Big)\cdot\big[qr_1\nabla(qr) + q\mathfrak{L}^M r_1\nabla(q\mathfrak{L}^M r)\big](t,y)dy$$
$$+\int_{\mathbb{R}^4}\frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\big[\nabla(qr_1)\cdot\nabla(qr) + \nabla(q\mathfrak{L}^M r_1)\cdot\nabla(q\mathfrak{L}^M r) + \mathcal{P}(rr_1 + \mathfrak{L}^M r\mathfrak{L}^M r_1)\big](t,y)dy\,\cdot$$

A straightforward computation gives

$$\nabla\Big(\frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\Big) = \nabla\Big(1+\frac{|\nabla Q|^2-|\nabla V|^2}{1+|\nabla V|^2}\Big) = \nabla\Big(\frac{(\nabla Q-\nabla V)(\nabla Q+\nabla V)}{1+|\nabla V|^2}\Big)\cdot$$

We claim that there is a positive constant C such that the following estimate holds for any time t in $[t_1, t_2]$, with $0 < t_1 \le t_2 \le T$:

$$\Big\|\nabla\Big(\frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\Big)(t,\cdot)\Big\|_{L^\infty(\mathbb{R}^4)} \le Ct^\nu\,. \tag{7.28}$$

In order to establish the above estimate, let us start by observing that for any time t in $[t_1, t_2]$ we have [18]

$$\|\nabla^2(V-Q)(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \le C\,t^\nu\,.$$

[18] Here and bellow, we assume that $N > \nu$.

Indeed by definition

$$V = V^{(N)} + \varepsilon\,, \quad \text{with} \quad \varepsilon = H\,r\,,$$

which gives the result by applying the triangle inequality and making use of Lemma 6.1, Hardy inequality, the estimates (6.3), (7.13) and the bootstrap assumption (7.20).

Along the same lines, we find that

$$\|\langle\cdot\rangle^{-1}\nabla(V-Q)(t,\cdot)\|_{L^\infty} \leq Ct^{\nu},\ \|\langle\cdot\rangle\nabla^2 V(t,\cdot)\|_{L^\infty} \leq C \quad \text{and} \quad \|\nabla V(t,\cdot)\|_{L^\infty} \leq C\,,$$

which achieves the proof of the claim (7.28).

We deduce that

$$(II) = \frac{1}{t}\mathcal{O}\Big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\Big)$$
$$+ \int_{\mathbb{R}^4} \frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\big[\nabla(qr_1)\cdot\nabla(qr) + \nabla(q\mathfrak{L}^M r_1)\cdot\nabla(q\mathfrak{L}^M r) + \mathcal{P}(rr_1 + \mathfrak{L}^M r\mathfrak{L}^M r_1)\big](t,y)dy\cdot$$

Besides, remembering that

$$r_1 = a\,r_t + a'\,\Lambda r - a'\,\frac{y\cdot\nabla H}{H}\,r\,,$$

we obtain

$$\nabla(qr_1) = a\,\partial_t(\nabla qr) + a'\,\Lambda\nabla qr - a'\,\mathcal{Y}_0\,,$$

with

$$\mathcal{Y}_0 = \nabla\left(q\frac{y\cdot\nabla H}{H}\,r\right) - [\nabla q,\Lambda]r\cdot$$

Invoking (2.4) together with (7.13) and Hardy inequality, we infer that

$$\|\mathcal{Y}_0\|_{L^2(\mathbb{R}^4)} \lesssim \|\nabla r\|_{L^2(\mathbb{R}^4)}\,. \tag{7.29}$$

Along the same lines, we readily gather that

$$\mathfrak{L}^M r_1 = a\,\partial_t(\mathfrak{L}^M r) + a'\,\Lambda\mathfrak{L}^M r - a'\,\mathcal{Y}_1\,,\quad \nabla q\mathfrak{L}^M r_1 = a\,\partial_t(\nabla q\mathfrak{L}^M r) + a'\,\Lambda\nabla q\mathfrak{L}^M r - a'\,\mathcal{Y}_2\,,$$

with

$$\mathcal{Y}_1 = \mathfrak{L}^M\frac{y\cdot\nabla H}{H}\,r - [\mathfrak{L}^M,\Lambda]r\,,\quad \mathcal{Y}_2 = -[\nabla q,\Lambda]\mathfrak{L}^M r + \nabla(q\mathcal{Y}_1)\,,$$

that clearly satisfy:

$$\|\mathcal{Y}_1\|_{H^1(\mathbb{R}^4)} + \|\mathcal{Y}_2\|_{L^2(\mathbb{R}^4)} \lesssim \|\nabla r\|_{H^{L_0-1}(\mathbb{R}^4)}\,. \tag{7.30}$$

Taking advantage of (7.29) and (7.30), we infer that

$$(II) = \frac{d}{dt}\mathcal{E}_1(t) + (II)_1 + (II)_2 + \frac{1}{t}\mathcal{O}\Big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\Big)\,,$$

with

$$\mathcal{E}_1(t) := \frac{1}{2}\int_{\mathbb{R}^4}\frac{1+|\nabla Q(y)|^2}{1+|\nabla V(t,y)|^2}\Big[|\nabla(qr)|^2 + |\nabla(q\mathfrak{L}^M r)|^2 + \mathcal{P}(r^2 + (\mathfrak{L}^M r)^2)\Big](t,y)dy\,, \tag{7.31}$$

$$(II)_1 := -\frac{1}{2}\int_{\mathbb{R}^4}\partial_t\Big(\frac{1+|\nabla Q(y)|^2}{1+|\nabla V(t,y)|^2}\Big)\Big[|\nabla(qr)|^2 + |\nabla(q\mathfrak{L}^M r)|^2 + \mathcal{P}(r^2 + (\mathfrak{L}^M r)^2)\Big](t,y)dy\,,$$

and

$$(II)_2 := \frac{1+\nu}{t}\int_{\mathbb{R}^4}\frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\big[\nabla(qr)\cdot\Lambda\nabla(qr) + \nabla(q\mathfrak{L}^M r)\cdot\Lambda\nabla(q\mathfrak{L}^M r)\big](t,y)dy\cdot$$

Again combining the bootstrap assumption (7.20) with Estimate (6.6), we claim that for any time t in $[t_1,t_2]$, with $0 < t_1 \leq t_2 \leq T$

$$\Big\|\partial_t\Big(\frac{1+|\nabla Q(y)|^2}{1+|\nabla V(t,y)|^2}\Big)(s,\cdot)\Big\|_{L^\infty(\mathbb{R}^4)} \leq Ct^{-1}\,. \tag{7.32}$$

It is obvious that (7.32) reduces to the following inequality

$$\|\partial_t \nabla V(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} \leq Ct^{-1} \cdot \tag{7.33}$$

Now to establish (7.33), let us first recall that

$$V = V^{(N)} + \varepsilon \quad \text{with} \quad \varepsilon = H\, r\,.$$

Applying the triangle inequality and invoking Estimate (6.6), we deduce that

$$\begin{aligned}\|\partial_t \nabla V(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} &\leq \|\partial_t \nabla V^{(N)}(t,\cdot)\|_{L^\infty(\mathbb{R}^4)} + \|\partial_t \nabla (H\, r)(t,\cdot)\|_{L^\infty(\mathbb{R}^4)}\\ &\leq Ct^{-1} + \|\nabla (H\, \partial_t r)(t,\cdot)\|_{L^\infty(\mathbb{R}^4)}\cdot\end{aligned}$$

But in view of (7.14), we have

$$a\, r_t = a\, r_1 - a'\, \Lambda r + a'\, \frac{y\cdot \nabla H}{H}\, r\,,$$

which ends the proof of the result thanks to the bootstrap hypothesis (7.20).

Consequently, we get

$$(II)_1 = \frac{1}{t}\mathcal{O}\Big(\|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\Big)\cdot \tag{7.34}$$

To end the estimate of the second part, it remains to investigate the term $(II)_2$. For that purpose we perform an integration by parts, which implies that

$$(II)_2 = \frac{(1+\nu)}{2}\int_{\mathbb{R}^4} \Big((\nabla\cdot y)\frac{1+|\nabla Q|^2}{1+|\nabla V|^2}\Big)\Big[|\nabla (qr)|^2 + |\nabla (q\mathfrak{L}^M r)|^2\Big](t,y)dy\cdot$$

Taking into account Lemma 6.1 and the bootstrap assumption (7.20), this gives rise to

$$(II)_2 = \frac{1}{t}\mathcal{O}\Big(\|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\Big)\cdot \tag{7.35}$$

In summary, we have

$$(II) = \frac{d}{dt}\mathcal{E}_1(t) + \frac{1}{t}\mathcal{O}\Big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\Big)\cdot \tag{7.36}$$

Besides, it stems from the definition of the operator $\mathfrak{L}$ and the estimates (6.2), (7.20) that there is a positive constant C such that

$$\Big|\mathcal{E}_1(t) - \frac{1}{2}\big(\mathfrak{L}r(t,\cdot)|r(t,\cdot)\big)_{L^2} - \frac{1}{2}\big(\mathfrak{L}^{M+1}r(t,\cdot)|\mathfrak{L}^M r(t,\cdot)\big)_{L^2}\Big| \leq C\delta^{3\nu}\|\nabla r(t,\cdot)\|^2_{H^{L_0-1}}\cdot \tag{7.37}$$

Let us now estimate the third term (III). Integrating again by parts, we easily get

$$\begin{aligned}(III) &= -2a^{-1}(t)\int_{\mathbb{R}^4}\frac{V_1}{1+|\nabla V|^2}\,\nabla V\cdot\Big[(\nabla r_1)\, r_1 + (\nabla \mathfrak{L}^M r_1)\,(\mathfrak{L}^M r_1)\Big](t,y)\, dy\\ &= a^{-1}(t)\int_{\mathbb{R}^4}\partial_{y_j}\Big(\frac{V_1}{1+|\nabla V|^2}\,\partial_{y_j}V\Big)\Big[(r_1)^2 + (\mathfrak{L}^M r_1)^2\Big](t,y)\, dy\cdot\end{aligned}$$

Arguing as above, we infer that for any time t in $[t_1,t_2]$, with $0<t_1\leq t_2\leq T$, we have

$$\Big\|\nabla\Big(\frac{V_1}{1+|\nabla V|^2}\,\nabla V\Big)(t,\cdot)\Big\|_{L^\infty} \leq Ct^\nu\cdot \tag{7.38}$$

The latter estimate is a direct consequence of the following inequalities

$$\|\nabla V_1(t,\cdot)\Big\|_{L^\infty} \leq Ct^\nu, \qquad \|\langle\cdot\rangle^{-1}V_1(t,\cdot)\|_{L^\infty}\leq Ct^\nu \quad \text{and} \quad \|\langle\cdot\rangle\nabla^2 V(t,\cdot)\|_{L^\infty}\leq C\,,$$

which readily stem from the bootstrap assumption (7.20) and Lemma 6.1.

It proceeds to say that

$$(III) = \frac{1}{t}\mathcal{O}\Big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\Big)\cdot \tag{7.39}$$

Finally the last term (IV) can be dealt with along the same lines as the second term (II). Firstly performing an integration by parts, we get

$$\begin{aligned}(IV) = &- \sum_{i,j=1}^{4} a^{-1}(t)\int_{\mathbb{R}^4} g_{i,j}\Big[(\partial_{y_i} r)\,(\partial_{y_j} r_1) + (\partial_{y_i}\mathfrak{L}^M r)\,(\partial_{y_j}\mathfrak{L}^M r_1)\Big](t,y)\,dy \\ &- \sum_{i,j=1}^{4} a^{-1}(t)\int_{\mathbb{R}^4} (\partial_{y_j} g_{i,j})\Big[(\partial_{y_i} r)\, r_1 + (\partial_{y_i}\mathfrak{L}^M r)\,(\mathfrak{L}^M r_1)\Big](t,y)\,dy\,,\end{aligned}$$

where the coefficients $g_{i,j}$ are defined by (7.25).

For any time t in $[t_1,t_2]$, with $0 < t_1 \leq t_2 \leq T$, the functions $g_{i,j}$ for $1 \leq i,j \leq 4$ enjoy the following properties

$$\|g_{i,j}(t)\|_{L^\infty(\mathbb{R}^4)} \leq C\delta^{6\nu} \quad \text{and} \quad \|\nabla g_{i,j}(t)\|_{L^\infty(\mathbb{R}^4)} \leq Ct^{\nu}\,. \tag{7.40}$$

Indeed by definition

$$g_{i,j} = \frac{V_1^2 - |\nabla\varepsilon|^2}{1+|\nabla V|^2}\,\delta_{i,j} + \frac{(Hr)_{y_i}(Hr)_{y_j}}{1+|\nabla V|^2}\,,$$

which leads to the result thanks to the following estimates

$$\Big\|\Big(\frac{V_1^2 - |\nabla\varepsilon|^2}{1+|\nabla V|^2}\Big)(t,\cdot)\Big\|_{L^\infty(\mathbb{R}^4)} \leq C\delta^{6\nu}\,, \quad \Big\|\nabla\Big(\frac{V_1^2 - |\nabla\varepsilon|^2}{1+|\nabla V|^2}\Big)(t,\cdot)\Big\|_{L^\infty(\mathbb{R}^4)} \leq Ct^{\nu}\,,$$
$$\Big\|\nabla^{\ell}\Big(\frac{(Hr)_{y_j}(Hr)_{y_k}}{1+|\nabla V|^2}\Big)(t,\cdot)\Big\|_{L^\infty(\mathbb{R}^4)} \leq Ct^{2N}\,,\ \ell = 0,1\,,$$

that can be proved by the same way as (7.28), making use of the bootstrap assumption (7.20) and Lemma 6.1.

Now remembering that

$$r_1 = a\,r_t + a'\,\Lambda r - a'\,\frac{y\cdot\nabla H}{H}\,r\,,$$

we find that

$$\nabla(r_1) = a\,\partial_t(\nabla r) + a'\,\Lambda\nabla r - a'\,\widetilde{\mathcal{Y}}_0\,,$$

with

$$\widetilde{\mathcal{Y}}_0 = \nabla\Big(\frac{y\cdot\nabla H}{H}\,r\Big) - [\nabla,\Lambda]r\cdot$$

Along the same lines as for $\mathcal{Y}_0$, we have

$$\|\widetilde{\mathcal{Y}}_0\|_{L^2(\mathbb{R}^4)} \lesssim \|\nabla r\|_{L^2(\mathbb{R}^4)}\,. \tag{7.41}$$

Similarly, we easily check that

$$\nabla\mathfrak{L}^M r_1 = a\,\partial_t(\nabla\mathfrak{L}^M r) + a'\,\Lambda\nabla\mathfrak{L}^M r - a'\,\widetilde{\mathcal{Y}}_2\,,$$

with

$$\widetilde{\mathcal{Y}}_2 = -[\nabla,\Lambda]\mathfrak{L}^M r + \nabla(\mathcal{Y}_1)\,,$$

that clearly satisfies

$$\|\widetilde{\mathcal{Y}}_2\|_{L^2(\mathbb{R}^4)} \lesssim \|\nabla r\|_{H^{L_0-1}(\mathbb{R}^4)}\,. \tag{7.42}$$

Therefore

$$(IV) = \frac{d}{dt}\mathcal{E}_2(t) + \frac{1}{2}\sum_{i,j=1}^{4}\int_{\mathbb{R}^4}(\partial_t g_{i,j})\Big[(\partial_{y_i} r)\,(\partial_{y_j} r) + (\partial_{y_i}\mathfrak{L}^M r)\,(\partial_{y_j}\mathfrak{L}^M r)\Big](t,y)\,dy$$
$$- \frac{(\nu+1)}{t}\sum_{i,j=1}^{4}\int_{\mathbb{R}^4} g_{i,j}\Big[\partial_{y_i} r\,\Lambda\partial_{y_j} r + (\partial_{y_i}\mathfrak{L}^M r)\,(\Lambda\partial_{y_j}\mathfrak{L}^M r)\Big](t,y)\,dy$$
$$+ \frac{1}{t}\mathcal{O}\Big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\Big)\,,$$

with

$$\mathcal{E}_2(t) = -\frac{1}{2}\sum_{i,j=1}^{4}\int_{\mathbb{R}^4} g_{i,j}\Big[(\partial_{y_i} r)\,(\partial_{y_j} r) + (\partial_{y_i}\mathfrak{L}^M r)\,(\partial_{y_j}\mathfrak{L}^M r)\Big](t,y)\,dy\,\cdot \tag{7.43}$$

In view of the bootstrap assumption (7.20) and Lemma 6.1, we have the following estimates

$$\|\langle\cdot\rangle\nabla g_{i,j}\|_{L^\infty(\mathbb{R}^4)} \leq C\,, \tag{7.44}$$

which follow easily from the fact that we have for any time t in $[t_1,t_2]$

$$\Big\|\langle\cdot\rangle\nabla\Big(\frac{V_1^2 - |\nabla\varepsilon|^2}{1+|\nabla V|^2}\Big)(t,\cdot)\Big\|_{L^\infty(\mathbb{R}^4)} \leq C \quad\text{and}\quad \Big\|\langle\cdot\rangle\nabla\Big(\frac{(Hr)_{y_j}(Hr)_{y_k}}{1+|\nabla V|^2}\Big)(t,\cdot)\Big\|_{L^\infty(\mathbb{R}^4)} \leq Ct^{2N}\,.$$

An integration by parts thus gives rise to

$$(IV) = \frac{d}{dt}\mathcal{E}_2(t) + \frac{1}{2}\sum_{i,j=1}^{4}\int_{\mathbb{R}^4}(\partial_t g_{i,j})\Big[(\partial_{y_i} r)\,(\partial_{y_j} r) + (\partial_{y_i}\mathfrak{L}^M r)\,(\partial_{y_j}\mathfrak{L}^M r)\Big](t,y)\,dy$$
$$+ \frac{1}{t}\mathcal{O}\Big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\Big)\,\cdot$$

Now we claim that

$$\|\partial_t g_{i,j}\|_{L^\infty(\mathbb{R}^4)} \leq C\,t^{-1}\,. \tag{7.45}$$

The latter estimate is shown making use again of the bootstrap assumption (7.20) and Lemma 6.1 which assert that there is a positive constant C such that for any time t in $[t_1,t_2]$, we have

$$\Big\|\partial_t\Big(\frac{V_1^2 - |\nabla\varepsilon|^2}{1+|\nabla V|^2}\Big)(t,\cdot)\Big\|_{L^\infty(\mathbb{R}^4)} \leq C\,t^{-1} \quad\text{and}\quad \Big\|\partial_t\Big(\frac{(Hr)_{y_j}(Hr)_{y_k}}{1+|\nabla V|^2}\Big)(t,\cdot)\Big\|_{L^\infty(\mathbb{R}^4)} \leq Ct^{N}\,\cdot$$

Therefore, we obtain

$$(IV) = \frac{d}{dt}\mathcal{E}_2(t) + \frac{1}{t}\mathcal{O}\Big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\Big)\,\cdot \tag{7.46}$$

Observe also that by (7.40), we have

$$\big|\mathcal{E}_2(t)\big| \leq C\delta^{6\nu}\Big(\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)}\Big)\,\cdot \tag{7.47}$$

We finally address the terms $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}_M$ and $\widetilde{\mathcal{R}}^{(N)}$. Using the bootstrap assumption (7.20) and Lemmas 6.1, 6.2 it is not difficult to show that they admit the following estimates.

Lemma 7.2. *There is a positive constant C such that under Assumption* (7.20), *the following estimates occur for any time t in $[t_1,t_2]$, where $0 < t_1 \leq t_2 \leq T$*

$$\|\widetilde{\mathcal{F}}(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} \leq Ct^{\nu}\Big(\|\nabla r(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} + \|r_1(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)}\Big)\,, \tag{7.48}$$

$$\|\widetilde{\mathcal{R}}^{(N)}(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} \leq C\,t^{N+\nu}\,, \tag{7.49}$$

$$\|\widetilde{\mathcal{F}}_M(t,\cdot)\|_{L^2(\mathbb{R}^4)} \leq C\, t^{\nu} \Big(\|\nabla r(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} + \|r_1(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} \Big) \cdot \tag{7.50}$$

We now gather the latter lemma with the bootstrap hypothesis (7.20) and the above estimates (7.26), (7.37), (7.39), (7.47), (7.49) and (7.50). This yields the following estimate

$$\frac{d}{dt}\mathcal{E}(t) \leq C t^{2N-1}\,, \tag{7.51}$$

with

$$\mathcal{E}(t) = \frac{1}{2}\Big[\|r_1(t)\|^2_{L^2(\mathbb{R}^4)} + \|\mathfrak{L}^M r_1(t)\|^2_{L^2(\mathbb{R}^4)} \Big] + \mathcal{E}_1(t) + \mathcal{E}_2(t),$$

and where $\mathcal{E}_1$ and $\mathcal{E}_2$ are respectively given by (7.31) and (7.43).

It follows from (2.4), (2.12), (7.37) and (7.47) that

$$\mathcal{E}(t) \geq C(\|r_1(t,\cdot)\|^2_{H^{L_0-1}} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}},$$

for some positive constant C, provided that δ is taken sufficiently small. Therefore, integrating inequality (7.51) and taking into account that $r(t_1) = r_1(t_1) = 0$, we get

$$\|r_1(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla r(t,\cdot)\|^2_{H^{L_0-1}(\mathbb{R}^4)} \leq \frac{C}{N} t^{2N}\,,$$

which achieves the proof of Lemma 7.1. □

Since by construction, we have

$$\begin{aligned} (u - u^{(N)})(t,x) &= t^{\nu+1}(V - V^{(N)})\Big(t, \frac{x}{t^{\nu+1}}\Big) = t^{\nu+1}(H\, r)\Big(t, \frac{x}{t^{\nu+1}}\Big)\,, \\ \partial_t(u - u^{(N)})(t,x) &= (V_1 - V_1^{(N)})\Big(t, \frac{x}{t^{\nu+1}}\Big) = (H\, r_1)\Big(t, \frac{x}{t^{\nu+1}}\Big)\,, \end{aligned}$$

Proposition 7.1 follows readily from (7.13) and Lemma 7.1 by standard continuity arguments. □

Remark 7.1. *Combining Proposition 7.1 with the bounds* (7.1) *and* (7.2)*, we get that for any time* $t \in [t_1, T]$*,*

$$u(t,\cdot) \geq \tilde{c}_0\, t^{\nu+1} \quad \textit{and} \quad (1 + |\nabla u|^2 - (\partial_t u)^2(t,\cdot) \geq \tilde{c}_1\,, \tag{7.52}$$

with some positive constants $\tilde{c}_0$ *and* $\tilde{c}_1$*, provided that* N_0 *is sufficiently large.*

Furthermore, injecting the bounds (7.20) *into* (7.17) *and taking into account Lemma 7.2, one easily deduces that the solution to the Cauchy problem* (7.4) *satisfies*

$$\|\langle\cdot\rangle^{\frac{3}{2}}\, \partial_t^2 (u - u^{(N)})(t,\cdot)\|_{H^{L_0-2}(\mathbb{R}^4)} \leq t^{\frac{N}{2}}\,, \tag{7.53}$$

for all $t \in [t_1, T]$*.*

7.2. **End of the proof.** We are now in position to finish the proof of Theorem 1.2. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers in $]0, T]$ converging to 0, and consider the Cauchy problem $(\mathrm{NW})_{n,N}$ defined by:

$$(\mathrm{NW})_{n,N} \begin{cases} (1.6)\, u_n = 0 \\ u_{n|t=t_n} = u^{(N)}(t_n,\cdot) \\ (\partial_t u_n)_{|t=t_n} = (\partial_t u^{(N)})(t_n,\cdot)\,\cdot \end{cases}$$

In view of Proposition 7.1 and Remark 7.1, we straightforward have the following uniform result of local well-posedness:

Corollary 7.1. *There exists an integer N_0 such that the Cauchy problem* $(\mathrm{NW})_{n,N_0}$ *admits a unique solution u_n on $[t_n, T]$ which satisfies the following energy estimates*

$$\|\langle\cdot\rangle^{\frac{3}{2}}\,\partial_t(u_n - u^{(N_0)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} + \|\langle\cdot\rangle^{\frac{3}{2}}\,\nabla(u_n - u^{(N_0)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} \leq t^{\frac{N_0}{2}}\,, \tag{7.54}$$

for any time $t_n \leq t \leq T$.

Furthermore,

$$u_n(t,x) \geq \tilde{c}_0\, t^{\nu+1},\;\; 1 + |\nabla u_n(t,x)|^2 - (\partial_t u_n(t,x))^2 \geq \tilde{c}_1, \quad \forall\,(t,x) \in [t_n, T] \times \mathbb{R}^4\,. \tag{7.55}$$

By Ascoli theorem, the bounds (7.54), (7.55) imply that there exists a solution u to the Cauchy problem (1.11) on $]0,T]$ satisfying $(u, \partial_t u) \in C(]0,T], X_{L_0})$ and such that after passing to a subsequence, the sequence $((\nabla u_n, \partial_t u_n))_{n\in\mathbb{N}}$ converges to $(\nabla u, \partial_t u)$ in $C([T_1,T], H^{s-1}(\mathbb{R}^4))$ for any $T_1 \in]0,T]$ and any $s < L_0$. Clearly the solution u satisfies:

$$\|\partial_t(u - u^{(N_0)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} + \|\nabla(u - u^{(N_0)})(t,\cdot)\|_{H^{L_0-1}(\mathbb{R}^4)} \leq t^{\frac{N_0}{2}}, \quad \forall t \in]0,T]\,,$$

$$u(t,x) \geq \tilde{c}_0\, t^{\nu+1},\;\; 1 + |\nabla u(t,x)|^2 - (\partial_t u(t,x))^2 \geq \tilde{c}_1 \quad \forall (t,x) \in]0,T] \times \mathbb{R}^2.$$

Taking into account Lemma 6.2 and Remaks 5.1, 6.1, this concludes the proof of Theorem 1.2.

Appendix A. Derivation of the equation

It is well-known that if we consider in the Minkowski space $\mathbb{R}^{1,m}$ regular time-like hypersurfaces with vanishing mean curvature which for fixed t are graphs of functions $\varphi(t,x)$ over $\mathbb{R}^m$, then φ satisfies the following quasilinear wave equation:

$$\partial_t\Big(\frac{\varphi_t}{\sqrt{1-(\varphi_t)^2+|\nabla\varphi|^2}}\Big) - \sum_{j=1}^{m} \partial_{x_j}\Big(\frac{\varphi_{x_j}}{\sqrt{1-(\varphi_t)^2+|\nabla\varphi|^2}}\Big) = 0\,.$$

Our purpose in this appendix is to carry out the computations for the equation in the case of time-like surfaces with vanishing mean curvature that for fixed t are parametrized as follows

$$\mathbb{R}^n \times \mathbb{S}^{n-1} \ni (x,\omega) \to (x, u(t,x)\omega) \in \mathbb{R}^{2n}\,, \tag{A.1}$$

with some positive function u. An elementary computation shows that in that case, the pull-back metric is:

$$g = -dt^2 + dx^2 + u^2 d\omega^2 + du^2\,. \tag{A.2}$$

Recalling the obvious identities

$$dx^2 = \sum_{j=1}^{n} dx_j^2 \quad \text{and} \quad du = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} dx_j + \frac{\partial u}{\partial t} dt\,,$$

we infer that the associated Lagrangian density is given by

$$\mathcal{L}(u, u_t, \nabla u) = u^{n-1}\sqrt{1-(u_t)^2+|\nabla u|^2}\,. \tag{A.3}$$

Using that the mean curvature is the first variation of the volume form, we can determine the equation of motion by considering formally the Euler-Lagrange equation associated to the Lagrangian density $\mathcal{L}$, which gives rise to

$$\frac{\partial\mathcal{L}}{\partial u} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\frac{\partial\mathcal{L}}{\partial u_{x_j}} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial u_t} = 0\,.$$

According to (A.3), this leads to

$$(n-1)\,u^{n-2}\sqrt{1-(u_t)^2+|\nabla u|^2}-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\frac{u^{n-1}\,u_{x_j}}{\sqrt{1-(u_t)^2+|\nabla u|^2}} \\ +\frac{\partial}{\partial t}\frac{u^{n-1}\,u_t}{\sqrt{1-(u_t)^2+|\nabla u|^2}}\,\cdot$$

Therefore the quasilinear wave equation at hand undertakes the following form:

$$\partial_t\Big(\frac{u_t}{\sqrt{1-(u_t)^2+|\nabla u|^2}}\Big)-\sum_{j=1}^{n}\partial_{x_j}\Big(\frac{u_{x_j}}{\sqrt{1-(u_t)^2+|\nabla u|^2}}\Big)+\frac{n-1}{u\sqrt{1-(u_t)^2+|\nabla u|^2}}=0\,. \tag{A.4}$$

Straightforward computations show that the above equation (A.4) rewrites as follows:

$$\begin{aligned} &u_{tt}(1+|\nabla u|^2)-\Delta u\,(1-(u_t)^2+|\nabla u|^2)+\sum_{j,k=1}^{n}u_{x_j}u_{x_k}u_{x_jx_k}\\ &\qquad -2u_t(\nabla u\cdot\nabla u_t)+\frac{(n-1)}{u}\,(1-(u_t)^2+|\nabla u|^2)=0\,, \end{aligned} \tag{A.5}$$

which achieves the proof of (1.6).

Appendix B. Study of the linearized operator of the quasilinear wave equation around the ground state

The aim of this section is to investigate the linearized operator $\mathcal{L}$ introduced in (1.16). To this end, consider under notations (1.17) the change of function

$$w(\rho)=H(\rho)f(\rho)\quad\text{with}\quad H=\frac{(1+Q_\rho^2)^{\frac14}}{Q^{\frac32}}\,\cdot$$

By easy computations, we deduce that

$$\mathcal{L}w=-H(1+Q_\rho^2)\mathfrak{L}f\quad\text{with}\quad \mathfrak{L}=-q\,\Delta\,q+\mathcal{P}\,,$$

where $q=\dfrac{1}{(1+Q_\rho^2)^{\frac12}}$ and $\mathcal{P}=\dfrac{V^\flat}{(1+Q_\rho^2)}$, with

$$V^\flat=\frac{-3\,(1+Q_\rho^2)}{Q^2}+\frac12(B_1)\rho-\frac14B_1^2-\frac32B_1\Big(-\frac{(1+Q_\rho^2)}{\rho}+2Q_\rho\big(\frac1Q-\frac{Q_\rho}{\rho}\big)\Big)\,\cdot \tag{B.1}$$

In view of Lemma 2.1, the potential $\mathcal{P}$ belongs to $\mathcal{C}^\infty_{rad}(\mathbb{R}^4)$ and satisfies

$$\mathcal{P}=-\frac{3}{8\rho^2}\big(1+\circ(1)\big)\,,\ \text{as}\ \rho\to\infty\,\cdot \tag{B.2}$$

The operator $\mathfrak{L}$ is at the heart of the analysis carried out in this article. The following lemma summarizes some of its useful properties.

Lemma B.1. *Under the above notations, we have*

- *The operator $\mathfrak{L}$ with domain $H^2(\mathbb{R}^4)$ is self-adjoint on $L^2(\mathbb{R}^4)$.*
- *There is a positive constant c such that for any function f in $\dot H^1_{rad}(\mathbb{R}^4)$, the following inequality holds:*

$$\big(\mathfrak{L}f|f\big)_{L^2(\mathbb{R}^4)}\geq c\,\|\nabla f\|^2_{L^2(\mathbb{R}^4)}\,\cdot \tag{B.3}$$

Remark B.1. *Taking into account* (2.4), *one easily deduces from* (B.3) *that for any integer* m, *there exists a positive constant* c_m *such that*

$$\big(\mathfrak{L}^{m+1} f|f\big)_{L^2(\mathbb{R}^4)} + \big(\mathfrak{L} f|f\big)_{L^2(\mathbb{R}^4)} \geq c_m \, \|\nabla f\|^2_{H^m(\mathbb{R}^4)} \, , \; \forall f \in \dot{H}^1(\mathbb{R}^4) \cap \dot{H}^{m+1}(\mathbb{R}^4) \, ,$$

and

$$\big(\mathfrak{L}^{m+1} f|f\big)_{L^2(\mathbb{R}^4)} + \big(f|f\big)_{L^2(\mathbb{R}^4)} \geq c_m \, \|f\|^2_{H^m(\mathbb{R}^4)} \, , \; \forall f \in H^{m+1}(\mathbb{R}^4) \, .$$

Proof. The fact that $\mathfrak{L}$ is self-adjoint on $L^2(\mathbb{R}^4)$ stems easily from (2.10). Consequently the spectrum of $\mathfrak{L}$ which will be denoted in what follows by $\sigma(\mathfrak{L})$ is real. Since the potential $\mathcal{P}$ is a regular function which behaves as $-\dfrac{3}{8\rho^2}$ as ρ tends to infinity, we deduce that $\sigma(\mathfrak{L}) \cap \mathbb{R}^*_-$ is a discrete set. Besides if $\sigma(\mathfrak{L}) \cap \mathbb{R}^*_- \neq \emptyset$, then it admits a minimum $\lambda_0 < 0$ which is an eigenvalue of $\mathfrak{L}$ and an associated eigenfunction u_0 in $\mathcal{S}(\mathbb{R}^4)$ which is positive.

Recalling that the positive function $\Lambda Q = Q - \rho \, Q_\rho$ solves the homogeneous equation $\mathcal{L} w = 0$, we infer that the function

$$G := \frac{\Lambda Q}{H} \quad \text{with} \quad H = \frac{(1+Q_\rho^2)^{\frac14}}{Q^{\frac32}} \, , \tag{B.4}$$

defines a regular positive solution to the homogeneous equation $\mathfrak{L} f = 0$. We deduce that

$$0 = \big(\mathfrak{L} G|u_0\big)_{L^2(\mathbb{R}^4)} = \big(G|\mathfrak{L} u_0\big)_{L^2(\mathbb{R}^4)} = \lambda_0 \big(G|u_0\big)_{L^2(\mathbb{R}^4)} < 0 \, ,$$

which yields a contradiction. This implies that $\sigma(\mathfrak{L}) \cap \mathbb{R}^*_- = \emptyset$ and ends the proof of the fact the operator $\mathfrak{L}$ is positive in the sense that for any function f in $H^1_{rad}(\mathbb{R}^4)$, we have

$$\big(\mathfrak{L} f|f\big)_{L^2(\mathbb{R}^4)} \geq 0 \, . \tag{B.5}$$

In order to prove Inequality (B.3), we shall proceed by contradiction assuming that there is a sequence $(u_n)_{n\in\mathbb{N}}$ in $\dot{H}^1_{rad}(\mathbb{R}^4)$ satisfying $\|\nabla u_n\|_{L^2(\mathbb{R}^4)} = 1$ and

$$\big(\mathfrak{L} u_n|u_n\big)_{L^2(\mathbb{R}^4)} \overset{n\to\infty}{\longrightarrow} 0 \, . \tag{B.6}$$

Since the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $\dot{H}^1_{rad}(\mathbb{R}^4)$, there is a function u in $\dot{H}^1_{rad}(\mathbb{R}^4)$ such that, up to a subsequence extracting (still denoted by u_n for simplicity)

$$u_n \overset{n\to\infty}{\rightharpoonup} u \quad \text{in} \quad \dot{H}^1(\mathbb{R}^4) \, . \tag{B.7}$$

We claim that the function $u \neq 0$ and satisfies $\mathfrak{L} u = 0$. Indeed by definition, we have

$$\mathfrak{L} u_n = (-q \, \Delta \, q + \mathcal{P}) u_n \, ,$$

which, with Notation (B.1), gives by integration

$$\big(\mathfrak{L} u_n|u_n\big)_{L^2(\mathbb{R}^4)} = \int_{\mathbb{R}^4} |\nabla(q u_n)(x)|^2 \, dx + \int_{\mathbb{R}^4} |(q u_n)(x)|^2 \, V^\flat(x) \, dx \, .$$

Firstly, let us observe that there is a positive constant C such that, for any integer n we have

$$\|\nabla(q u_n) \, \|_{L^2(\mathbb{R}^4)} > C \cdot \tag{B.8}$$

Indeed, one has

$$\|\nabla u_n\|_{L^2(\mathbb{R}^4)} \leq \big\|\frac{1}{q} \nabla(q u_n)\big\|_{L^2(\mathbb{R}^4)} + \big\|\nabla(\frac{1}{q}) q u_n\big\|_{L^2(\mathbb{R}^4)} \, ,$$

which in view of Hardy inequality and Lemma 2.1 leads to (B.8).

Secondly, consider θ a smooth radial function valued in $[0,1]$ and satisfying

$$\begin{cases} \theta(x) = 0 \text{ for } |x| \leq 1 \\ \theta(x) = 1 \text{ for } |x| \geq 2 \,, \end{cases}$$

and write

$$\big(\mathfrak{L} u_n | u_n\big)_{L^2(\mathbb{R}^4)} = \int_{\mathbb{R}^4} |\nabla (q u_n)(x)|^2 dx - \frac{3}{4} \int_{\mathbb{R}^4} |(q u_n)(x)|^2 \, \frac{\theta(x)}{|x|^2} dx + \int_{\mathbb{R}^4} |(q u_n)(x)|^2 \tilde{V}(x) dx \,,$$

where of course

$$\tilde{V}(x) = V^\flat(x) + \frac{3}{4} \frac{\theta(x)}{|x|^2} \cdot$$

Invoking Formula (B.2), we infer that there is a positive constant δ such that $\tilde{V}$ satisfies at infinity

$$|\tilde{V}(x)| \lesssim \frac{1}{\langle x \rangle^{2+\delta}} \cdot$$

Invoking Rellich theorem and Hardy inequality, we deduce that

$$\int_{\mathbb{R}^4} |(q u_n)(x)|^2 \, \tilde{V}(x) \, dx \overset{n \to \infty}{\longrightarrow} \int_{\mathbb{R}^4} |(q u)(x)|^2 \, \tilde{V}(x) \, dx \cdot \tag{B.9}$$

Now for any functions f and g in $\dot{H}^1(\mathbb{R}^4)$, denote by

$$a(f,g) := \int_{\mathbb{R}^4} \nabla (qf)(x) \cdot \overline{\nabla (qg)(x)} dx - \frac{3}{4} \int_{\mathbb{R}^4} \frac{\theta(x)}{|x|^2} \, (qf)(x) \overline{(qg)(x)} dx \cdot$$

Combining Hardy inequality with Lemma 2.1, we easily gather that there exist two positive constants $\alpha_0 < \alpha_1$ such that for any function f in $\dot{H}^1(\mathbb{R}^4)$, we have

$$\alpha_0 \|\nabla f\|^2_{L^2(\mathbb{R}^4)} \leq a(f,f) \leq \alpha_1 \|\nabla f\|^2_{L^2(\mathbb{R}^4)} \,,$$

which ensures that $a(f,g)$ is a scalar product on $\dot{H}^1(\mathbb{R}^4)$ and that the norms $\sqrt{a(\cdot,\cdot)}$ and $\|\cdot\|_{\dot{H}^1(\mathbb{R}^4)}$ are equivalent.

Since $u_n \overset{n \to \infty}{\rightharpoonup} u$ in $\dot{H}^1(\mathbb{R}^4)$, we deduce that $a(u,u) \leq \liminf_{n \to \infty} a(u_n, u_n)$, and thus

$$\big(\mathfrak{L} u | u\big)_{L^2(\mathbb{R}^4)} \leq \liminf_{n \to \infty} \Big(\big(\mathfrak{L} u_n | u_n\big)_{L^2(\mathbb{R}^4)} \cdot$$

Taking into account (B.5), (B.6) and (B.9), we deduce that

$$\big(\mathfrak{L} u | u\big)_{L^2(\mathbb{R}^4)} = 0 \,, \tag{B.10}$$

which according to the fact that $\mathfrak{L}$ is positive implies that $\mathfrak{L} u = 0$.

To end the proof of the claim, it remains to establish that $u \neq 0$. For that purpose, let us start by observing that by virtue of (B.6), (B.9) and (B.10), we have

$$\begin{aligned} &\int_{\mathbb{R}^4} |\nabla (q u_n)(x)|^2 dx - \frac{3}{4} \int_{\mathbb{R}^4} |(q u_n)(x)|^2 \frac{\theta(x)}{|x|^2} dx \\ &\qquad \overset{n \to \infty}{\longrightarrow} \int_{\mathbb{R}^4} |\nabla (q u)(x)|^2 dx - \frac{3}{4} \int_{\mathbb{R}^4} |(q u_n)(x)|^2 \frac{\theta(x)}{|x|^2} dx \,. \end{aligned} \tag{B.11}$$

But in view of Hardy inequality and the bound (B.8), we have

$$\begin{aligned} &\int_{\mathbb{R}^4} |\nabla (q u_n)(x)|^2 \, dx - \frac{3}{4} \int_{\mathbb{R}^4} |(q u_n)(x)|^2 \, \frac{\theta(x)}{|x|^2} \, dx \\ &\qquad \geq \frac{1}{4} \int_{\mathbb{R}^4} |\nabla (q u_n)(x)|^2 \, dx \geq \frac{C}{4} \cdot \end{aligned} \tag{B.12}$$

By passing to the limit, we obtain

$$\int_{\mathbb{R}^4} |\nabla(qu)(x)|^2\, dx - \frac{3}{4}\int_{\mathbb{R}^4} |(qu)(x)|^2\, \frac{\theta(x)}{|x|^2}\, dx \geq \frac{C}{4}\,,$$

which achieves the proof of the fact that u is not null.

By construction the function u belongs to $\dot H^1_{rad}(\mathbb{R}^4)$ and satisfies

$$\mathfrak{L}u = -q\,\Delta\, qu + \mathcal{P}u = 0 \quad \text{with} \quad \mathcal{P} = -\frac{3}{8\rho^2}(1+\circ(1))\,,\ \text{as}\ \rho\to\infty\,\cdot$$

Therefore in view of Hardy inequality, $\mathcal{P}u \in L^2_{rad}(\mathbb{R}^4)$ and thus $q\,\Delta\, qu$ belongs to $L^2_{rad}(\mathbb{R}^4)$, which ensures that $u \in \dot H^2_{rad}(\mathbb{R}^4)$.

Now the homogeneous equation $\mathfrak{L}u = 0$ admits a basis of solutions $\{f_1, f_2\}$ given by [19]:

$$\begin{cases} f_1(\rho) = G(\rho) \quad \text{and} \\ f_2(\rho) = G(\rho) \displaystyle\int_1^\rho \frac{(1+(Q_r(r))^2)^{\frac{3}{2}}}{Q^3(r)\, r^3\,(\Lambda Q)^2(r)}\, dr\,, \end{cases}$$

where G denotes the function defined by (B.4). By Lemma 2.1, one then has

$$\begin{cases} f_1(\rho) \sim 1 \\ f_2(\rho) \sim \dfrac{1}{\rho^2}\,, \end{cases}$$

near $\rho = 0$. Since f_2 does not belong to $\dot H^1_{rad}(\mathbb{R}^4)$, we deduce that u is collinear to G. This yields a contradiction because in view of (2.4), the function G behaves as $\dfrac{1}{\sqrt{\rho}}$ when ρ tends to infinity and thus it does not belong to $\dot H^1_{rad}(\mathbb{R}^4)$. This finally completes the proof of the lemma. □

Appendix C. Proof of the local well-posedness result

The aim of this appendix is to give an outline of the proof of Theorem 1.1. Since the subject is so well known, we only indicate the main arguments. One can proceed on three steps:

(1) First, one proves that for some positive time sufficiently small

$$T = T\big(\|\nabla(u_0 - Q)\|_{H^{L-1}}, \|u_1\|_{H^{L-1}}, \inf u_0, \inf\,(1+|\nabla u_0|^2 - (u_1)^2)\big),$$

the Cauchy problem (1.11) admits a solution u such that (u, u_t) belongs to the functional space $C([0,T], X_L)$, $u_t \in C^1([0,T], H^{L-1})$, and which satisfies for all t in $[0,T]$

$$\|\nabla(u-Q)(t,\cdot)\|_{H^{L-1}} + \|u_t(t,\cdot)\|_{H^{L-1}} \leq C\big(\|\nabla(u_0-Q)\|_{H^{L-1}} + \|u_1\|_{H^{L-1}}\big)\,,$$

for some positive constant

$$C = C\big(\|\nabla(u_0 - Q)\|_{H^{L-1}}, \|u_1\|_{H^{L-1}}, \inf u_0, \inf\,(1+|\nabla u_0|^2 - (u_1)^2)\big).$$

(2) Second, one shows the uniqueness of solutions thanks to a continuity argument.

(3) Third, one establishs the blow up criterion (1.13).

Let us then consider the Cauchy problem (1.11) and assume that $\nabla(u_0 - Q)$ and u_1 belong to $H^{L-1}(\mathbb{R}^4)$, with L an integer strictly larger than 4, and that there is $\varepsilon > 0$ such that

$$u_0 \geq 2\varepsilon \quad \text{and} \quad \frac{1-(u_1)^2 + |\nabla u_0|^2}{1+|\nabla u_0|^2} \geq 2\varepsilon\,.$$

[19] see Appendix D for a proof of this fact.

Defining for $1 \leq i, j \leq 4$

$$
\left\{
\begin{aligned}
a_{ij}\big(\nabla u, u_t\big) &= \delta_{ij}\Big(1 - \frac{(u_t)^2}{(1+|\nabla u|^2)}\Big) - \frac{u_{x_j} u_{x_i}}{(1+|\nabla u|^2)}\,,\\
b_i\big(\nabla u, u_t\big) &= \frac{2\, u_{x_i} u_t}{(1+|\nabla u|^2)}\,,\\
c\big(u, \nabla u, u_t\big) &= -\frac{3\,(1-(u_t)^2+|\nabla u|^2)}{u(1+|\nabla u|^2)}\,,
\end{aligned}
\right.
\tag{C.1}
$$

we readily gather that (1.6) takes the form

$$
u_{tt} - \sum_{i,j=1}^{4} a_{i,j}\big(\nabla u, u_t\big) u_{x_i x_j} - \sum_{i=1}^{4} b_i\big(\nabla u, u_t\big) u_{t x_i} - c\big(u, \nabla u, u_t\big) = 0\,. \tag{C.2}
$$

To prove existence, we shall use an iterative scheme. To this end, under the above notations, introduce the sequence $(u^{(n)})_{n\in\mathbb{N}}$ defined by $u^{(0)} = Q$ which according to (2.1) satisfies

$$
c\big(Q, \nabla Q, 0\big) + \sum_{i,j=1}^{4} a_{i,j}\big(\nabla Q, 0\big) Q_{x_i x_j} = 0\,,
$$

and

$$
(\mathrm{W})_{n+1}
\left\{
\begin{aligned}
& u^{(n+1)}_{tt} - \sum_{i,j=1}^{4} a_{i,j}\big(\nabla u^{(n)}, u^{(n)}_t\big) u^{(n+1)}_{x_i x_j} - \sum_{i=1}^{4} b_i\big(\nabla u^{(n)}, u^{(n)}_t\big) u^{(n+1)}_{t x_i} - c\big(u^{(n)}, \nabla u^{(n)}, u^{(n)}_t\big) = 0\\
& {u^{(n+1)}}_{|t=0} = u_0\\
& (\partial_t u^{(n+1)})_{|t=0} = u_1\,.
\end{aligned}
\right.
$$

In order to investigate the sequence $(u^{(n)})_{n\in\mathbb{N}}$ defined above by induction, let us begin by proving that this sequence of functions is well defined for any time t in some fixed interval $[0, T]$ which depends only on $\|\nabla(u_0 - Q)\|_{H^{L-1}}$, $\|u_1\|_{H^{L-1}}$ and ε. This will be deduced from the following result.

Proposition C.1. *Let u be such that $(u, u_t) \in C([0,T], X_L)$, $u_{tt} \in C([0,T], H^{L-2})$ for some integer $L > 4$ and some $0 < T \leq 1$. Assume that*

$$
\|u_t\|_{L^\infty([0,T],H^{L-1})} + \|\nabla(u - Q)\|_{L^\infty([0,T],H^{L-1})} \leq A\,, \tag{C.3}
$$

$$
\|u_{tt}\|_{L^\infty([0,T],H^{L-2})} \leq A_1\,, \tag{C.4}
$$

$$
u(t,x) \geq \varepsilon \quad \text{and} \quad \frac{1 - (u_t(t,x))^2 + |\nabla u(t,x)|^2}{1 + |\nabla u(t,x)|^2} \geq \varepsilon\,,\ \forall (t,x) \in [0,T]\times\mathbb{R}^d\,\cdot \tag{C.5}
$$

Consider the Cauchy problem

$$
\left\{
\begin{aligned}
& \Phi_{tt} - \sum_{i,j=1}^{4} a_{i,j}\big(\nabla u, u_t\big)\Phi_{x_i x_j} - \sum_{i=1}^{4} b_i\big(\nabla u, u_t\big)\Phi_{t x_i} = c\big(u, \nabla u, u_t\big)\\
& \Phi_{|t=0} = \Phi_0\\
& (\partial_t \Phi)_{|t=0} = \Phi_1\,,
\end{aligned}
\right.
\tag{C.6}
$$

assuming that $\nabla(\Phi_0 - Q)$ and Φ_1 belong to $H^{L-1}(\mathbb{R}^4)$. Then the Cauchy problem (C.6) admits a unique solution Φ on $[0,T]$ and the following energy inequalities hold:

$$
\begin{aligned}
\|\Phi_t(t,\cdot)\|_{H^{L-1}} + \|\nabla(\Phi(t,\cdot) - Q)\|_{H^{L-1}} &\leq C_\varepsilon\, \mathrm{e}^{t\, C_{\varepsilon,A,A_1}} \big(\|\Phi_1\|_{H^{L-1}} + \|\nabla(\Phi_0 - Q)\|_{H^{L-1}}\big)\\
&\quad + C_{\varepsilon,A,A_1} \int_0^t \big(\|u_t(s,\cdot)\|_{H^{L-1}} + \|\nabla(u - Q)(s,\cdot)\|_{H^{L-1}}\big) ds\,,
\end{aligned}
\tag{C.7}
$$

and

$$\begin{aligned}\|\Phi_{tt}(t,\cdot)\|_{H^{L-2}} \leq C_A \big(\|\Phi_t(t,\cdot)\|_{H^{L-1}} + \|\nabla(\Phi(t,\cdot) - Q)\|_{H^{L-1}}\big) \\ + C_{\varepsilon,A}\big(\|u_t(t,\cdot)\|_{H^{L-1}} + \|\nabla(u-Q)(t,\cdot)\|_{H^{L-1}}\big)\,.\end{aligned} \tag{C.8}$$

Proof. Invoking Hypothesis (C.40), we easily check that for any $\xi \in \mathbb{R}^4 \setminus \{0\}$, the characteristic polynomial of the wave equation (C.6)

$$\tau^2 - \tau \sum_{i=1}^{4} b_i\big(\nabla u, u_t\big)\xi_i - \sum_{i,j=1}^{4} a_{i,j}\big(\nabla u, u_t\big)\xi_i\xi_j \tag{C.9}$$

has two distinct real roots τ_1 and τ_2. Indeed taking account of (C.1), we find that Δ the discriminant of (C.9) is given by

$$\begin{aligned}\Delta &= \frac{4(u_t)^2}{(1+|\nabla u|^2)^2}\big(\sum_{i=1}^{4} u_{x_i}\xi_i\big)^2 + \frac{4(1-(u_t)^2+|\nabla u|^2)}{(1+|\nabla u|^2)}\,|\xi|^2 - \frac{4}{(1+|\nabla u|^2)}\big(\sum_{i=1}^{4} u_{x_i}\xi_i\big)^2 \\ &= \frac{4(1-(u_t)^2+|\nabla u|^2)}{(1+|\nabla u|^2)^2}\,|\xi|^2 + \frac{4(1-(u_t)^2+|\nabla u|^2)}{(1+|\nabla u|^2)^2}\big(|\nabla u|^2\,|\xi|^2 - \big(\sum_{i=1}^{4} u_{x_i}\xi_i\big)^2\big)\,,\end{aligned}$$

which implies that

$$\Delta \geq \frac{4(1-(u_t)^2+|\nabla u|^2)}{(1+|\nabla u|^2)^2}\,|\xi|^2\,. \tag{C.10}$$

This ends the proof of the claim and ensures that (C.6) is strictly hyperbolic as long as

$$(1-(u_t)^2+|\nabla u|^2) > 0\,,$$

and thus, in view of (C.40), on $[0,T]\times\mathbb{R}^4$.

Let us emphasize that under the above notations, the function $\widetilde{\Phi} := \Phi - Q$ satisfies :

$$\begin{cases} \widetilde{\Phi}_{tt} - \displaystyle\sum_{i,j=1}^{4} a_{i,j}\big(\nabla u, u_t\big)\widetilde{\Phi}_{x_ix_j} - \sum_{i=1}^{4} b_i\big(\nabla u, u_t\big)\widetilde{\Phi}_{tx_i} = f\big(u, \nabla u, u_t\big) \\ \widetilde{\Phi}_{|t=0} = \Phi_0 - Q \\ (\partial_t\widetilde{\Phi})_{|t=0} = \Phi_1\,, \end{cases} \tag{C.11}$$

with

$$f\big(u,\nabla u,u_t\big) = c\big(u,\nabla u,u_t\big) + \sum_{i,j=1}^{4} a_{i,j}\big(\nabla u, u_t\big)Q_{x_ix_j}\,. \tag{C.12}$$

Firstly note that the source term f belongs to the functional space $L^\infty([0,T],H^{L-1}(\mathbb{R}^4))$ and thus to $L^1([0,T],H^{L-1}(\mathbb{R}^4))$. Let us start by establishing that $f \in L^\infty([0,T],L^2(\mathbb{R}^4))$. Recalling that by virtue of (2.1), we have

$$c\big(Q,\nabla Q,0\big) + \sum_{i,j=1}^{4} a_{i,j}\big(\nabla Q,0\big)Q_{x_ix_j} = 0\,,$$

we deduce that f rewrites on the following way:

$$f = c\big(u,\nabla u,u_t\big) - c\big(Q,\nabla Q,0\big) + \sum_{i,j=1}^{4}\big(a_{i,j}\big(\nabla u,u_t\big) - a_{i,j}\big(\nabla Q,0\big)\big)Q_{x_ix_j} = -3\Big(\frac{1}{u} - \frac{1}{Q}\Big) + \widetilde{f}\,,$$

where

$$\widetilde{f} = c\big(u, \nabla u, u_t\big) + \frac{3}{u} + \sum_{i,j=1}^{4} \big(a_{i,j}\big(\nabla u, u_t\big) - a_{i,j}\big(\nabla Q, 0\big)\big) Q_{x_i x_j}\,. \tag{C.13}$$

Combining Lemma 2.1 together with the hypotheses (C.3) and (C.40), we obtain making use of Taylor's formula:

$$|\widetilde{f}| \leq C_{\varepsilon,A}\,\big(|u_t| + |\nabla(u-Q)|\big)\,,$$

which easily ensures that for all t in $[0,T]$, we have

$$\|\widetilde{f}(t,\cdot)\|_{L^2} \leq C_{\varepsilon,A}\,\big(\|u_t(t,\cdot)\|_{L^2} + \|\nabla(u-Q)(t,\cdot)\|_{L^2}\big)\,. \tag{C.14}$$

Therefore, we are reduced to the study of the part

$$-3\Big(\frac{1}{u} - \frac{1}{Q}\Big) = 3\,\frac{u-Q}{u\,Q}\,\cdot$$

We claim that

$$\Big|\frac{1}{u} - \frac{1}{Q}\Big| \leq C_{\varepsilon,A}\,\frac{|u-Q|}{Q^2}\,\cdot \tag{C.15}$$

Indeed on the one hand according to Estimate (C.3), the function $u-Q$ is bounded on $[0,T]\times\mathbb{R}^4$. Then writing

$$u = Q + (u-Q)\,,$$

and recalling that the stationary solution Q behaves as ρ at infinity, we infer that there is a positive real number $R_0 = R_0(A)$ such that for any $|x| \geq R_0$ and any t in $[0,T]$, we have

$$u(t,x) \geq \frac{Q(x)}{2}\,\cdot$$

On the other hand, invoking (C.40) together with Lemma 2.1, we infer that there is a positive constant $C(\varepsilon, R_0)$ such that if $|x| \leq R_0$, then we have for all $0 \leq t \leq T$

$$\frac{1}{u(t,x)} \leq \frac{C(\varepsilon,R_0)}{Q(x)}\,\cdot$$

Now taking advantage of the Sobolev embedding $\dot{H}^1(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$, we deduce that

$$\begin{aligned}\Big\|\Big(\frac{1}{u} - \frac{1}{Q}\Big)(t,\cdot)\Big\|_{L^2} &\leq C_{\varepsilon,A}\,\|(u-Q)(t,\cdot)\|_{L^4}\Big\|\frac{1}{Q^2}\Big\|_{L^4}\\ &\leq C_{\varepsilon,A}\,\|\nabla(u-Q)(t,\cdot)\|_{L^2}\Big\|\frac{1}{Q^2}\Big\|_{L^4},\end{aligned}$$

which according to the fact that $\dfrac{1}{Q(\rho)} \lesssim \dfrac{1}{\langle\rho\rangle}$ ensures that

$$\Big\|\Big(\frac{1}{u} - \frac{1}{Q}\Big)(t,\cdot)\Big\|_{L^2} \leq C_{\varepsilon,A}\,\|\nabla(u-Q)(t,\cdot)\|_{L^2}\,\cdot \tag{C.16}$$

Together with (C.14), this implies that for all t in $[0,T]$

$$\|f(t,\cdot)\|_{L^2} \leq C_{\varepsilon,A}\big(\|u_t(t,\cdot)\|_{L^2} + \|\nabla(u-Q)(t,\cdot)\|_{L^2}\big)\,. \tag{C.17}$$

Thanks to the bound (C.3), this ends the proof of the fact that f belongs to $L^\infty([0,T], L^2(\mathbb{R}^4))$.

In order to establish that $f \in L^\infty([0,T], H^{L-1}(\mathbb{R}^4))$, let us firstly observe that by virtue of the assumption (C.3), the functions $\big(b_i\big(\nabla u, u_t\big)\big)_{1\leq i\leq 4}$, $\big(a_{i,j}\big(\nabla u, u_t\big) - a_{i,j}\big(\nabla Q, 0\big)\big)_{1\leq i,j\leq 4}$ as well as the function $c\big(u, \nabla u, u_t\big) + \dfrac{3}{u}$ belong to $L^\infty([0,T], H^{L-1}(\mathbb{R}^4))$.

Thus taking advantage of Lemma 2.1 and recalling that $L > 4$, we find that the function $\widetilde{f}$ belongs to $L^\infty([0,T], H^{L-1}(\mathbb{R}^4))$ and satisfies the following estimate uniformly on $[0,T]$:

$$\|\widetilde{f}(t,\cdot)\|_{H^{L-1}} \leq C_{\varepsilon,A}\left(\|u_t(t,\cdot)\|_{H^{L-1}} + \|\nabla(u-Q)(t,\cdot)\|_{H^{L-1}}\right).$$

Besides applying Leibniz's formula to the term $\dfrac{u-Q}{u\,Q}$ and taking account (C.16), we infer that there is a positive constant $C_{\varepsilon,A}$ such that for all t in $[0,T]$, we have

$$\Big\|\Big(\frac{1}{u}-\frac{1}{Q}\Big)(t,\cdot)\Big\|_{H^{L-1}} \leq C_{\varepsilon,A}\,\|\nabla(u-Q)(t,\cdot)\|_{H^{L-1}}.$$

Combining the two last inequalities, we get

$$\|f(t,\cdot)\|_{H^{L-1}} \leq C_{\varepsilon,A}\left(\|u_t(t,\cdot)\|_{H^{L-1}} + \|\nabla(u-Q)(t,\cdot)\|_{H^{L-1}}\right). \tag{C.18}$$

This concludes the proof of the desired result.

Finally, since the coefficients of Equation (C.11) as well as their time and spatial derivatives are bounded on $[0,T]\times\mathbb{R}^4$, applying classical arguments, we infer that the Cauchy problem (C.6) admits a unique solution on $[0,T]\times\mathbb{R}^4$.

To avoid heaviness, we shall notice all along this proof by $\mathcal{A}$ the matrix $(a_{i,j})_{1\leq i,j\leq 4}$ and by b the vector $(b_1,\cdots,b_4)$, and omit on what follows the dependence of all the functions $a_{i,j}$ and b_i on $(\nabla u, u_t)$ and the source term f on $(u,\nabla u, u_t)$.

Now to establish the energy inequality (C.7), we can proceed as follows. First we take the L^2-scalar product of (C.11) with $(\widetilde{\Phi}_t - \frac{b}{2}\cdot\nabla\widetilde{\Phi})$, which gives rise to

$$\Big([\partial_t(\widetilde{\Phi}_t - \frac{b}{2}\cdot\nabla\widetilde{\Phi}) - \sum_{i,j=1}^4 a_{i,j}\widetilde{\Phi}_{x_ix_j} - \frac{b}{2}\cdot\nabla\widetilde{\Phi}_t + \frac{b_t}{2}\cdot\nabla\widetilde{\Phi}](t,\cdot)\Big|(\widetilde{\Phi}_t - \frac{b}{2}\cdot\nabla\widetilde{\Phi})(t,\cdot)\Big)_{L^2}$$
$$= \Big(f(t,\cdot)\Big|(\widetilde{\Phi}_t - \frac{b}{2}\cdot\nabla\widetilde{\Phi})(t,\cdot)\Big)_{L^2}.$$

Performing integrations by parts, we deduce that

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}(\widetilde{\Phi})(t,\cdot) = I_0(t) + \Big(f(t,\cdot)\Big|(\widetilde{\Phi}_t - \frac{b}{2}\cdot\nabla\widetilde{\Phi})(t,\cdot)\Big)_{L^2}, \tag{C.19}$$

where

$$\mathcal{E}(\widetilde{\Phi})(t,\cdot) := \|(\widetilde{\Phi}_t - \frac{b}{2}\cdot\nabla\widetilde{\Phi})(t,\cdot)\|^2_{L^2} + \sum_{i,j=1}^4\Big(a_{i,j}\widetilde{\Phi}_{x_i}(t,\cdot)\Big|\widetilde{\Phi}_{x_j}(t,\cdot)\Big)_{L^2} + \|(\frac{b}{2}\cdot\nabla\widetilde{\Phi})(t,\cdot)\|^2_{L^2}, \tag{C.20}$$

and where I_0 admits the estimate

$$|I_0(t)| \leq a_0(t)\left(\|(\nabla\widetilde{\Phi})(t,\cdot)\|^2_{L^2} + \|\widetilde{\Phi}_t(t,\cdot)\|^2_{L^2}\right), \tag{C.21}$$

with

$$a_0(t,\cdot) = \mathcal{T}\big(\|\mathcal{A}(t,\cdot)\|_{L^\infty}, \|(\nabla_{t,x}\mathcal{A})(t,\cdot)\|_{L^\infty}, \|b(t,\cdot)\|_{L^\infty}, \|(\nabla_{t,x}b)(t,\cdot)\|_{L^\infty}\big), \tag{C.22}$$

$\mathcal{T}$ denoting a polynomial function of all its arguments.

By virtue of estimates (C.3) and (C.4), we have

$$\|a_0\|_{L^\infty([0,T]\times\mathbb{R}^4)} \leq C_{A,A_1}. \tag{C.23}$$

Observe also that thanks to (C.10), we have

$$\begin{aligned}\mathcal{E}(\widetilde{\Phi})(t,\cdot) &\leq 4\big(\|(\nabla\widetilde{\Phi})(t,\cdot)\|^2_{H^{L-1}} + \|\widetilde{\Phi}_t(t,\cdot)\|^2_{H^{L-1}}\big) \quad \text{and} \\ \mathcal{E}(\widetilde{\Phi})(t,\cdot) &\geq \|(\widetilde{\Phi}_t - \frac{b}{2}\cdot\nabla\widetilde{\Phi})(t,\cdot)\|^2_{L^2} + \varepsilon\,\|\nabla\widetilde{\Phi}(t,\cdot)\|^2_{L^2}\,.\end{aligned} \tag{C.24}$$

Now in order to investigate $\Phi_t(t,\cdot)$ and $\nabla(\Phi-Q)(t,\cdot)$ in the setting of H^{L-1}, we differentiate the nonlinear wave equation (C.11) with respect to the variable space up to the order $L-1$. By straightforward computations, we obtain formally for any multi-index α of length $|\alpha|\leq L-1$

$$\begin{cases}(\partial^\alpha\widetilde{\Phi})_{tt} - \displaystyle\sum_{i,j=1}^{4} a_{i,j}(\partial^\alpha\widetilde{\Phi})_{x_ix_j} - \sum_{i=1}^{4} b_i(\partial^\alpha\widetilde{\Phi})_{tx_i} = f_\alpha \\ (\partial^\alpha\widetilde{\Phi})_{|t=0} = \partial^\alpha(\Phi_0 - Q) \\ (\partial_t(\partial^\alpha\widetilde{\Phi}))_{|t=0} = \partial^\alpha\Phi_1\,,\end{cases} \tag{C.25}$$

with under the above notations

$$f_\alpha = \partial^\alpha f + \widetilde{f}_\alpha\,.$$

where

$$\widetilde{f}_\alpha := \sum_{i,j=1}^{4}\sum_{\beta<\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(\partial^{\alpha-\beta}a_{i,j})(\partial^\beta\widetilde{\Phi})_{x_ix_j} + \sum_{i=1}^{4}\sum_{\beta<\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(\partial^{\alpha-\beta}b_i)(\partial^\beta\widetilde{\Phi})_{tx_i}\,. \tag{C.26}$$

Then taking the L^2-scalar product of Equation (C.25) with $((\partial^\alpha\widetilde{\Phi})_t - \frac{b}{2}\cdot\nabla(\partial^\alpha\widetilde{\Phi}))$, we get applying the same lines of reasoning as above

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}(\partial^\alpha\widetilde{\Phi})(t,\cdot) = I_\alpha(t) + \big(f_\alpha(t,\cdot)\big|[(\partial^\alpha\widetilde{\Phi})_t - \frac{b}{2}\cdot\nabla(\partial^\alpha\widetilde{\Phi})](t,\cdot)\big)_{L^2}\,, \tag{C.27}$$

where

$$|I_\alpha(t)| \leq a_0(t)\left(\|\nabla(\partial^\alpha\widetilde{\Phi})(t,\cdot)\|^2_{L^2} + \|(\partial^\alpha\widetilde{\Phi})_t(t,\cdot)\|^2_{L^2}\right). \tag{C.28}$$

Now since the functions $\big(\nabla a_{i,j}\big)_{1\leq i,j\leq 4}$ and $(\nabla b_i)_{1\leq i\leq 4}$ belong to the Sobolev space $H^{L-2}(\mathbb{R}^4)$, the function $\widetilde{f}_\alpha$ belongs to $L^2(\mathbb{R}^4)$ and satisfies uniformly on $[0,T]$

$$\|\widetilde{f}_\alpha(t,\cdot)\|_{L^2} \leq C_A\big[\|\nabla\widetilde{\Phi}(t,\cdot)\|_{H^{|\alpha|}} + \|\widetilde{\Phi}_t(t,\cdot)\|_{H^{|\alpha|}}\big]. \tag{C.29}$$

Therefore taking into account (C.18), we get for any $|\alpha|\leq L-1$

$$\begin{aligned}\big|\big(f_\alpha(t,\cdot)\big|[(\partial^\alpha\widetilde{\Phi})_t - \frac{b}{2}\cdot\nabla(\partial^\alpha\widetilde{\Phi})](t,\cdot)\big)_{L^2}\big| &\leq C_{\varepsilon,A}\left[\|\nabla\widetilde{\Phi}(t,\cdot)\|^2_{H^{|\alpha|}} + \|\widetilde{\Phi}_t(t,\cdot)\|^2_{H^{|\alpha|}}\right. \\ &\quad + \big(\|\nabla\widetilde{\Phi}(t,\cdot)\|_{H^{|\alpha|}} + \|\widetilde{\Phi}_t(t,\cdot)\|_{H^{|\alpha|}}\big)\big(\|u_t(t,\cdot)\|_{H^{L-1}} + \|\nabla(u-Q)(t,\cdot)\|_{H^{L-1}}\big)\big]\,.\end{aligned} \tag{C.30}$$

Combining (C.19), (C.21), (C.23), (C.27), (C.28) and (C.30), we easily gather that

$$\begin{aligned}\Big|\sum_{|\alpha|\leq L-1}\frac{d}{dt}\mathcal{E}(\partial^\alpha\widetilde{\Phi})(t,\cdot)\Big| &\leq C_{\varepsilon,A,A_1}\left[\|\nabla\widetilde{\Phi}(t,\cdot)\|^2_{H^{L-1}} + \|\widetilde{\Phi}_t(t,\cdot)\|^2_{H^{L-1}}\right. \\ &\quad + \big(\|\nabla\widetilde{\Phi}(t,\cdot)\|_{H^{L-1}} + \|\widetilde{\Phi}_t(t,\cdot)\|_{H^{L-1}}\big)\big(\|u_t(t,\cdot)\|_{H^{L-1}} + \|\nabla(u-Q)(t,\cdot)\|_{H^{L-1}}\big)\big]\,.\end{aligned} \tag{C.31}$$

Applying Gronwall lemma and taking into account (C.24), we deduce for all t in $[0,T]$

$$\begin{aligned}\|\nabla\widetilde{\Phi}(t,\cdot)\|_{H^{L-1}} + \|\widetilde{\Phi}_t(t,\cdot)\|_{H^{L-1}} &\leq C_\varepsilon\,\mathrm{e}^{t\,C_{\varepsilon,A,A_1}}\big(\|\Phi_1\|_{H^{L-1}} + \|\nabla(\Phi_0-Q)\|_{H^{L-1}}\big) \\ &\quad + C_{\varepsilon,A,A_1}\int_0^t\big(\|u_s(s,\cdot)\|_{H^{L-1}} + \|\nabla(u-Q)(s,\cdot)\|_{H^{L-1}}\big)ds\,.\end{aligned} \tag{C.32}$$

To achieve the proof of the energy estimates, it remains to estimate $\|\Phi_{tt}(t,\cdot)\|_{H^{L-2}(\mathbb{R}^4)}$. To this end, we make use of Equation (C.11) which implies that

$$\Phi_{tt} = \sum_{i,j=1}^{4} a_{i,j}\big(\nabla u, u_t\big)\widetilde{\Phi}_{x_i x_j} + \sum_{i=1}^{4} b_i\big(\nabla u, u_t\big)\widetilde{\Phi}_{t x_i} + f\big(u, \nabla u, u_t\big)\,.$$

This ensures the result according to (C.3) and (C.18). □

Let us now return to the proof of Theorem 1.1. The first step can be deduced from Proposition C.1 by a standard argument that can be found for instance in the monographs [5, 13, 32]. The key point consists to prove that the sequence $(u^{(n)})_{n\in\mathbb{N}}$ of solutions to the initial value problem $(\mathrm{W})_n$ introduced page 60 is uniformly bounded, in the sense that there exist a small positive time

$$T = T\big(\|u_1\|_{H^{L-1}}, \|\nabla(u_0 - Q)\|_{H^{L-1}}, \varepsilon\big), \tag{C.33}$$

and a positive constant $C = \big(\|u_1\|_{H^{L-1}}, \|\nabla(u_0 - Q)\|_{H^{L-1}}, \varepsilon\big)$ such that for any integer n and any time t in $[0,T]$, we have

$$\|u_t^{(n)}(t,\cdot)\|_{H^{L-1}} + \|\nabla(u^{(n)} - Q)(t,\cdot)\|_{H^{L-1}} + \|u_{tt}^{(n)}(t,\cdot)\|_{H^{L-2}} \leq C\,, \tag{C.34}$$

and

$$u^{(n)}(t,\cdot) \geq \varepsilon \quad \text{and} \quad (1 - (u_t^{(n)})^2 + |\nabla u^{(n)}|^2)(t,\cdot) \geq \varepsilon\,. \tag{C.35}$$

In order to establish the uniform estimate (C.34), set

$$A = 2\, C_\varepsilon \left(\|u_1\|_{H^{L-1}} + \|\nabla(u_0 - Q)\|_{H^{L-1}}\right), \tag{C.36}$$

and

$$A_1 = (C_A + C_{\varepsilon,A})\, A\,, \tag{C.37}$$

where C_ε, C_A and $C_{\varepsilon,A}$ are the constants introduced in (C.7)- (C.8).

We claim that there exists a positive time $T \leq 1$ under the form (C.33) such that for any integer $n \geq 0$:
if for any time t in $[0,T]$, we have

$$\|u_t^{(n)}(t,\cdot)\|_{H^{L-1}} + \|\nabla(u^{(n)} - Q)(t,\cdot)\|_{H^{L-1}} \leq A\,, \tag{C.38}$$

$$\|u_{tt}^{(n)}(t,\cdot)\|_{H^{L-2}} \leq A_1\,, \tag{C.39}$$

and

$$u^{(n)}(t,x) \geq \varepsilon\,,\ \frac{1 - (u_t^{(n)}(t,x))^2 + |\nabla u^{(n)}(t,x)|^2}{1 + |\nabla u^{(n)}(t,x)|^2} \geq \varepsilon\,,\ \forall (t,x) \in [0,T]\times\mathbb{R}^d\,, \tag{C.40}$$

then the same bounds remain true for $u^{(n+1)}$.

Indeed, by virtue of the energy estimate (C.7) in Proposition C.1, $u^{(n+1)}$ verifies for all $t \in [0,T]$

$$\begin{aligned}\|u_t^{(n+1)}(t,\cdot)\|_{H^{L-1}} + \|\nabla(u^{(n+1)} - Q)(t,\cdot)\|_{H^{L-1}} \leq C_\varepsilon\, \mathrm{e}^{t\, C_{\varepsilon,A,A_1}}\big(\|u_1\|_{H^{L-1}} + \|\nabla(u_0 - Q)\|_{H^{L-1}}\big)\\ + A\, C_{\varepsilon,A,A_1}\, t\,,\end{aligned}$$

which implies that there exists $T(A,\varepsilon) > 0$ such that if $t \leq T(A,\varepsilon)$, then

$$\|u_t^{(n+1)}(t,\cdot)\|_{H^{L-1}} + \|\nabla(u^{(n+1)} - Q)(t,\cdot)\|_{H^{L-1}} \leq 2\, C_\varepsilon \left(\|u_1\|_{H^{L-1}} + \|\nabla(u_0 - Q)\|_{H^{L-1}}\right) = A\,.$$

Invoking then (C.8), we get for all $t \leq T(A,\varepsilon)$

$$\|u_{tt}^{(n+1)}(t,\cdot)\|_{H^{L-2}} \leq (C_A + C_{\varepsilon,A})\, A = A_1\,.$$

This ends the proof of the fact that $u^{(n+1)}$ satisfies the bounds (C.38) and (C.39).

Finally, Property (C.40) results directly from the following straightforward estimates:

$$\|u^{(n+1)}(t,\cdot)-u_0\|_{L^\infty(\mathbb{R}^4)}\leq\int_0^t\|\partial_s u^{(n+1)}(s,\cdot)\|_{L^\infty(\mathbb{R}^4)}\,ds\lesssim A\,t\,,$$
$$\|(\partial_t u^{(n+1)})(t,\cdot)-u_1\|_{L^\infty(\mathbb{R}^4)}\leq\int_0^t\|\partial_s^2 u^{(n+1)}(s,\cdot)\|_{L^\infty(\mathbb{R}^4)}\,ds\lesssim A_1\,t\,,$$
$$\|(\nabla u^{(n+1)})(t,\cdot)-\nabla u_0\|_{L^\infty(\mathbb{R}^4)}\leq\int_0^t\|(\partial_s\nabla(u^{(n+1)})(s,\cdot)\|_{L^\infty(\mathbb{R}^4)}\,ds\lesssim A\,t\,,$$

which implies (C.40) provided that $T=T\big(A,A_1,\varepsilon\big)$ is chosen sufficiently small. This achieves the proof of the claim.

To end the proof of the local well-posedness for the Cauchy problem (1.11), it suffices to establish that the sequences $(\partial_t u^{(n)})_{n\in\mathbb{N}}$ and $(\nabla(u^{(n)}-Q))_{n\in\mathbb{N}}$ are Cauchy sequences in the functional space $L^\infty([0,T],H^{L-2}(\mathbb{R}^4))$. By a standard argument, this fact follows easily from (C.34). Indeed setting $w^{(n+1)}:=u^{(n+1)}-u^{(n)}$, we readily gather that for all $n\geq 0$

$$\begin{cases} w_{tt}^{(n+1)}-\displaystyle\sum_{i,j=1}^4 a_{i,j}\big(\nabla u^{(n)},u_t^{(n)}\big)w_{x_ix_j}^{(n+1)}-\sum_{i=1}^4 b_i\big(\nabla u^{(n)},u_t^{(n)}\big)w_{tx_i}^{(n+1)}=g^{(n)}\\ w^{(n+1)}{}_{|t=0}=0\\ (\partial_t w^{(n+1)})_{|t=0}=0\,,\end{cases}$$

where

$$g^{(n)}=\sum_{i,j=1}^4\big(a_{i,j}\big(\nabla u^{(n)},u_t^{(n)}\big)-a_{i,j}\big(\nabla u^{(n-1)},u_t^{(n-1)}\big)\big)u_{x_ix_j}^{(n)}$$
$$+\sum_{i=1}^4\big(b_i\big(\nabla u^{(n)},u_t^{(n)}\big)-b_i\big(\nabla u^{(n-1)},u_t^{(n-1)}\big)\big)u_{tx_i}^{(n)}+c\big(u^{(n)},\nabla u^{(n)},u_t^{(n)}\big)-c\big(u^{(n-1)},\nabla u^{(n-1)},u_t^{(n-1)}\big)\,.$$

Since by construction, we have for any (t,x) in $[0,T]\times\mathbb{R}^d$

$$u^{(n)}(t,x)\geq\varepsilon\quad\text{and}\quad\frac{1-(u_t^{(n)}(t,x))^2+|\nabla u^{(n)}(t,x)|^2}{1+|\nabla u^{(n)}(t,x)|^2}\geq\varepsilon\,,$$

we obtain arguing as for the proof of Proposition C.1

$$\|w_t^{(n+1)}(t,\cdot)\|_{L^\infty([0,T],H^{L-2})}+\|\nabla w^{(n+1)}(t,\cdot)\|_{L^\infty([0,T],H^{L-2})}$$
$$\leq C\,T\Big(\|w_t^{(n)}(t,\cdot)\|_{L^\infty([0,T],H^{L-2})}+\|\nabla w^{(n)}(t,\cdot)\|_{L^\infty([0,T],H^{L-2})}\Big)\,.$$

This ensures the result provided that T is small enough and completes the proof of the first step.

Let us now address the second step, and establish the uniqueness of solutions to the Cauchy problem (1.11). For that purpose, we shall prove the following continuation criterion which easily gives the result:

Lemma C.1. *Consider u and v two solutions of the Cauchy problem* (1.11) *respectively associated to the initial data (u_0,u_1) and (v_0,v_1) in X_s, such that (u,u_t) and (v,v_t) are in $C([0,T],X_s)$ and such that u_t and v_t belong to $C^1([0,T],H^{s-1})$, with s a positive real number strictly greater than 4. Then there is a positive constant C such that, for all t in $[0,T]$, the following estimate holds:*

$$\|(u-v)_t(t,\cdot)\|_{L^2(\mathbb{R}^4)}+\|\nabla(u-v)(t,\cdot)\|_{L^2(\mathbb{R}^4)}\leq C\Big(\|u_1-v_1\|_{L^2(\mathbb{R}^4)}+\|\nabla(u_0-v_0)\|_{L^2(\mathbb{R}^4)}\Big)\,.$$

Proof. By straightforward computations, we readily gather that the function $w =: u - v$ solves the following Cauchy problem:

$$\left\{\begin{array}{l} w_{tt} - \displaystyle\sum_{i,j=1}^{4} a_{i,j}\big(\nabla u, u_t\big) w_{x_i x_j} - \sum_{i=1}^{4} b_i\big(\nabla u, u_t\big) w_{t x_i} = g \\ w_{|t=0} = u_0 - v_0 \\ (\partial_t w)_{|t=0} = u_1 - v_1\,, \end{array}\right. \tag{C.41}$$

where

$$g = \sum_{i,j=1}^{4} \big(a_{i,j}\big(\nabla u, u_t\big) - a_{i,j}\big(\nabla v, v_t\big)\big) v_{x_i x_j} \\ + \sum_{i=1}^{4} \big(b_i\big(\nabla u, u_t\big) - b_i\big(\nabla v, v_t\big)\big) v_{t x_i} + c\big(u, \nabla u, u_t\big) - c\big(v, \nabla v, v_t\big)\,.$$

Therefore, taking the L^2-scalar product of (C.41) with $(w_t - \frac{b}{2} \cdot \nabla w)$, we get as for the proof of Proposition C.1 the following energy inequality:

$$\|w_t(t,\cdot)\|_{L^2(\mathbb{R}^4)} + \|\nabla w(t,\cdot)\|_{L^2(\mathbb{R}^4)} \\ \leq C\Big(\|u_1 - v_1\|_{L^2(\mathbb{R}^4)} + \|\nabla(u_0 - v_0)\|_{L^2(\mathbb{R}^4)} + \int_0^t \|g(s,\cdot)\|_{L^2(\mathbb{R}^4)} ds\Big)\,.$$

As before, we have by straightforward computations

$$\|g(s,\cdot)\|_{L^2(\mathbb{R}^4)} \leq C\Big(\|w_t(t,\cdot)\|_{L^2(\mathbb{R}^4)} + \|\nabla w(t,\cdot)\|_{L^2(\mathbb{R}^4)}\Big)\,,$$

which easily achieves the proof of the continuation criterion. □

Finally the blow up criterion (1.13) results by standard arguments from the fact that if

$$\limsup_{t \nearrow T} \Big(\Big\|\frac{1}{u(t,\cdot)}\Big\|_{L^\infty} + \Big\|\frac{1}{(1 + |\nabla u|^2 - (\partial_t u)^2)(t,\cdot)}\Big\|_{L^\infty} + \sup_{|\gamma| \leq 1} \big\|\partial_x^\gamma \nabla_{t,x} u\big\|_{L^\infty}\Big) < \infty\,,$$

then the solution to the Cauchy problem (1.11) can be extended beyond T. This ends the proof of Theorem 1.1.

Appendix D. Some simple ordinary differential equations results

D.1. **Proof of Duhamel's formula** (3.17)**.** The formula results from the following lemma:

Lemma D.1. *Under the above notations, the homogeneous equation*

$$\mathcal{L} f = 0 \tag{D.1}$$

has a basis of solutions $\big\{\mathrm{e}_1, \mathrm{e}_2\big\}$ *given by:*

$$\left\{\begin{array}{l} \mathrm{e}_1(y) = (\Lambda Q)(y) \quad \textit{and} \\ \mathrm{e}_2(y) = (\Lambda Q)(y) \displaystyle\int_1^y \frac{(1 + (Q_r(r))^2)^{\frac{3}{2}}}{Q^3(r)\, r^3\, (\Lambda Q)^2(r)}\, dr \cdot \end{array}\right. \tag{D.2}$$

Besides for any regular function g, the solution to the Cauchy problem

$$\left\{\begin{array}{l} \mathcal{L} f = g \\ f(0) = 0 \quad \textit{and} \quad f'(0) = 0\,, \end{array}\right. \tag{D.3}$$

writes under the following form

$$f(y)=-(\Lambda Q)(y)\int_0^y \frac{(1+(Q_r(r))^2)^{\frac32}}{Q^3(r)\,r^3\,(\Lambda Q)^2(r)}\int_0^r \frac{Q^3(s)\,s^3\,(\Lambda Q)(s)}{(1+(Q_s(s))^2)^{\frac32}}\,g(s)\,ds\,dr\,\cdot$$

Proof. Recall that it was proved page 6 that $\mathrm{e}_1:=\Lambda Q$ is a positive function on $\mathbb{R}_+$ which solves the homogeneous equation $\mathcal{L}f=0$.

In order to obtain a solution e_2 to (D.1) linearly independent with e_1, let us firstly emphasize that if we denote $f=\hat H\,\hat f$, where

$$2\frac{(\hat H)_y}{\hat H}=-\Big(\frac{3}{y}+B_1\Big)\,, \tag{D.4}$$

with B_1 defined by (1.17), then

$$\mathcal{L}f=g\Leftrightarrow\hat{\mathcal{L}}\hat f=\hat g$$

with $g=\hat H\,\hat g$ and

$$\hat{\mathcal{L}}=\partial_y^2+\hat{\mathcal{P}}\,,$$

where

$$\hat{\mathcal{P}}=B_0+\Big(\frac{3}{y}+B_1\Big)\frac{(\hat H)_y}{\hat H}+\frac{(\hat H)_{yy}}{\hat H}\,\cdot$$

Since for any two solutions $\hat f_1$ and $\hat f_2$ to the homogeneous equation

$$\hat{\mathcal{L}}\hat f=0\,, \tag{D.5}$$

the Wronskian $W(\hat f_1,\hat f_2)$ is constant, we infer that $\hat{\mathrm{e}}_2$ defined by

$$\hat{\mathrm{e}}_2(y):=\hat{\mathrm{e}}_1(y)\int_1^y\frac{ds}{\hat{\mathrm{e}}_1^2(s)}\quad\text{with}\quad\hat{\mathrm{e}}_1=\hat H^{-1}\mathrm{e}_1\,, \tag{D.6}$$

constitutes a solution to (D.5) linearly independent with $\hat{\mathrm{e}}_1$.

Since

$$B_1(y)=\frac{9\,Q_y^2}{y}-\frac{6\,Q_y}{Q}\,,$$

we get in view of (2.1)

$$B_1(y)=3\Big(\frac{Q_y}{Q}-\frac{Q_{yy}\,Q_y}{(1+Q_y^2)}\Big)\cdot$$

Therefore

$$\frac{3}{y}+B_1(y)=3\Big(\log\Big(\frac{y\,Q}{(1+Q_y^2)^{\frac12}}\Big)\Big)_y\,,$$

and thus taking account (D.4), one can choose

$$\hat H(y)=\frac{(1+(Q_y)^2(y))^{\frac34}}{(y\,Q(y))^{\frac32}}\,\cdot \tag{D.7}$$

This achieves the proof of (D.2).

To end the proof of the lemma, it remains to establish Duhamel's formula (3.17). For that purpose, let us start by noticing that since by construction $W(\hat{\mathrm{e}}_1,\hat{\mathrm{e}}_2)=1$, then $\hat f$ the solution to the inhomogeneous equation $\hat{\mathcal{L}}\hat f=\hat g$ undertakes the following form:

$$\hat f(y)=\int_0^y\big(\hat{\mathrm{e}}_1(y)\hat{\mathrm{e}}_2(s)-\hat{\mathrm{e}}_1(s)\hat{\mathrm{e}}_2(y)\big)\,\hat g(s)\,ds\,,$$

which by definition gives rise to

$$f(y)=\int_0^y \frac{\big(\mathrm{e}_1(y)\mathrm{e}_2(s)-\mathrm{e}_1(s)\mathrm{e}_2(y)\big)}{\hat H(s)^2}\, g(s)\,ds\,\cdot$$

In view of (D.2), we deduce that

$$\begin{aligned} f(y) &= \mathrm{e}_1(y)\int_0^y\bigg(\mathrm{e}_1(s)\int_1^s\frac{\hat H^2(s')\,ds'}{\mathrm{e}_1^2(s')}-\mathrm{e}_1(s)\int_1^y\frac{\hat H^2(s')\,ds'}{\mathrm{e}_1^2(s')}\bigg)\frac{g(s)}{\hat H(s)^2}\,ds\\ &= -\mathrm{e}_1(y)\int_0^y\frac{\mathrm{e}_1(s)\,g(s)}{\hat H(s)^2}\int_s^y\frac{\hat H^2(s')\,ds'}{\mathrm{e}_1^2(s')}\,ds\,\cdot\end{aligned}$$

Finally performing an integration by parts, we readily gather that

$$f(y)=-\mathrm{e}_1(y)\int_0^y\frac{\hat H^2(s)}{\mathrm{e}_1^2(s)}\int_0^s\frac{\mathrm{e}_1(s')\,g(s')}{\hat H(s')^2}\,ds'\,ds\,,$$

which ends the proof of the lemma by virtue of (D.7). □

D.2. **Proof of Lemma 4.3.** To prove the first item, let us for g in $\mathcal{C}^\infty(\mathbb{R}_+^*)$ look for the solution f of the inhomogeneous equation

$$\begin{cases}\widetilde{\mathcal{L}}_k f=(2z^2-1)\partial_z^2 f-(\dfrac{6}{z}+4z\nu k)\partial_z f-(\dfrac{6}{z^2}-2\,\nu k(1+\nu k))f=g\,,\\ f(\dfrac{1}{\sqrt2})=0\,,\end{cases}$$

under the form:

$$f=f^{(0)}+f^{(1)}\quad\text{with}\quad f^{(0)}(z):=\sum_{m=1}^{N+1}\alpha_m\Big(z-\frac{1}{\sqrt2}\Big)^m\,,$$

where $N:=[k\nu]+3$ and where the coefficients α_m, for $1\le m\le N+1$, are uniquely determined by the requirement that the function $\widetilde g$ defined by:

$$\widetilde g:=g-\widetilde{\mathcal{L}}_k f^{(0)}$$

verifies

$$\widetilde g^{(\ell)}\Big(\frac{1}{\sqrt2}\Big)=0,\quad\forall\,\ell\in\{0,\cdots,N\}\,. \tag{D.8}$$

Then $f^{(1)}$ has to satisfy

$$\begin{cases}\widetilde{\mathcal{L}}_k f^{(1)}=\widetilde g\,,\\ f^{(1)}(\dfrac{1}{\sqrt2})=0\,,\ f^{(1)}\in\mathcal{C}^\infty(\mathbb{R}_+^*)\,,\end{cases}$$

and can be recovered by Duhamel formula:

$$f^{(1)}(z)=\int_{\frac{1}{\sqrt2}}^z\frac{\widetilde g(s)}{2s^2-1}\,\frac{1}{\mathcal{W}(f_{k,0}^{0,+},f_{k,0}^{0,-})(s)}\big(f_{k,0}^{0,-}(z)\,f_{k,0}^{0,+}(s)-f_{k,0}^{0,+}(z)f_{k,0}^{0,-}(s)\big)\,ds,$$

where

$$\mathcal{W}(f_{k,0}^{0,+},f_{k,0}^{0,-}):=f_{k,0}^{0,+}(f_{k,0}^{0,-})_z-f_{k,0}^{0,-}(f_{k,0}^{0,+})_z$$

denotes the Wronskian of the basis $\{f_{k,0}^{0,+},f_{k,0}^{0,-}\}$ defined by (4.45). By straightforward computations, we have

$$\mathcal{W}(f_{k,0}^{0,+},f_{k,0}^{0,-})(z)=\frac{\sqrt2\,\alpha(\nu,k)\,\mathrm{sgn}(z-\frac12)|z^2-\frac12|^{\alpha(\nu,k)-1}}{z^6}\,,$$

which implies that

$$f^{(1)}(z) = \frac{1}{2\sqrt{2}\,\alpha(\nu,k)} \int_{\frac{1}{\sqrt{2}}}^{z} s^3 \widetilde{g}(s) \Big(\frac{f^{0,-}_{k,0}(z)}{|s-\frac{1}{\sqrt{2}}|^{\alpha(\nu,k)}} - \frac{f^{0,+}_{k,0}(z)}{(s+\frac{1}{\sqrt{2}})^{\alpha(\nu,k)}} \Big)\, ds \cdot \tag{D.9}$$

The uniqueness follows immediately from Remark 4.1.

Now we turn our attention to the second item. Our task here is to solve uniquely (4.51) in the functional space $\mathcal{C}^\infty\big(]0,\frac{1}{\sqrt{2}}]\big)$ under Condition (4.52). Let us start with the case when $q=0$ and look for a solution f to the equation:

$$\widetilde{\mathcal{L}}_k f(z) = (\frac{1}{\sqrt{2}} - z)^\gamma h(z)\,,$$

under the form:

$$f = f^{(0)} + f^{(1)}\,,$$

with

$$f^{(0)}(z) := (\frac{1}{\sqrt{2}} - z)^{\gamma+1} \sum_{m=0}^{N} c_m (\frac{1}{\sqrt{2}} - z)^m\,,$$

where again $N=[k\nu]+3$. Due to (4.52), the coefficients c_m, for $0\leq m\leq N$, can be fixed so that

$$\widetilde{\mathcal{L}}_k f^{(1)}(z) = (\frac{1}{\sqrt{2}} - z)^\gamma \widetilde{h}(z)\,, \tag{D.10}$$

where $\widetilde{h}$ is a function in $\mathcal{C}^\infty\big(]0,\frac{1}{\sqrt{2}}]\big)$ which satisfies

$$\widetilde{h}^{(\ell)}\Big(\frac{1}{\sqrt{2}}\Big) = 0, \quad \forall\, \ell \in \{0,\cdots,N\}\,.$$

But any solution to (D.10) is under the form

$$\frac{1}{2\sqrt{2}\alpha(\nu,k)} \int_{\frac{1}{\sqrt{2}}}^{z} s^3 (\frac{1}{\sqrt{2}} - s)^\gamma\, \widetilde{h}(s) \Big(\frac{f^{0,-}_{k,0}(z)}{(\frac{1}{\sqrt{2}}-s)^{\alpha(\nu,k)}} - \frac{f^{0,+}_{k,0}(z)}{(s+\frac{1}{\sqrt{2}})^{\alpha(\nu,k)}} \Big)\, ds + a_k^+ f^{0,+}_{k,0}(z) + a_k^- f^{0,-}_{k,0}(z),$$

for some constants a_k^+ and a_k^-. Invoking the fact that we look for solutions in $\mathcal{C}^\infty\big(]0,\frac{1}{\sqrt{2}}]\big)$ vanishing at $z=\frac{1}{\sqrt{2}}$, we end up with the result in the case when $q=0$ by taking $a_k^+=a_k^-=0$.

To establish the result in the general case of any integer $q\geq 1$, we shall proceed by induction assuming that under Condition (4.52), for any integer $1\leq j\leq q-1$, the inhomogeneous equation

$$\widetilde{\mathcal{L}}_k f(z) = (\frac{1}{\sqrt{2}} - z)^\gamma \big(\log(\frac{1}{\sqrt{2}} - z)\big)^j h(z)$$

admits a unique solution f of the form:

$$f(z) = (\frac{1}{\sqrt{2}} - z)^{\gamma+1} \sum_{0\leq\ell\leq j} \big(\log(\frac{1}{\sqrt{2}} - z)\big)^\ell h_\ell(z)\,,$$

where for all $0\leq \ell\leq j$, h_ℓ is a function in $\mathcal{C}^\infty\big(]0,\frac{1}{\sqrt{2}}]\big)\cdot$ Then we look for a solution f to

$$\widetilde{\mathcal{L}}_k f(z) = (\frac{1}{\sqrt{2}} - z)^\gamma \big(\log(\frac{1}{\sqrt{2}} - z)\big)^q h(z)\,,$$

under the form:

$$f(z) = \big(\log\big(\frac{1}{\sqrt{2}} - z\big)\big)^{q}\, \widetilde{f}(z) + f^{(1)}(z)\,, \tag{D.11}$$

where

$$\widetilde{\mathcal{L}}_k \widetilde{f}(z) = \big(\frac{1}{\sqrt{2}} - z\big)^{\gamma} h(z)\,.$$

Thanks to the above computations, this implies that

$$\widetilde{f}(z) = \big(\frac{1}{\sqrt{2}} - z\big)^{\gamma+1} h_q(z)\,,$$

where h_q belongs to $\mathcal{C}^\infty\big(]0, \frac{1}{\sqrt{2}}]\big)$.

Since in view of (D.11)

$$\widetilde{\mathcal{L}}_k f^{(1)}(z) = \big(\frac{1}{\sqrt{2}} - z\big)^{\gamma} \sum_{0\leq \ell\leq q-1} \big(\log\big(\frac{1}{\sqrt{2}} - z\big)\big)^{\ell}\, \widetilde{h}_\ell(z)\,,$$

with $\widetilde{h}_\ell \in \mathcal{C}^\infty\big(]0, \frac{1}{\sqrt{2}}]\big)$, this achieves the proof of the second item by virtue of the induction hypothesis.

Let us now establish the third item. To this end, let us for $g \in \mathcal{C}^\infty\big(]0, \frac{1}{\sqrt{2}}[\big)$ admitting an asymptotic expansion at 0 of the form:

$$g(z) = (\log z)^{\alpha_0} \sum_{\beta\geq\beta_0} g_\beta\, z^{\beta-2}\,,$$

for some integers α_0 and β_0, investigate the non homogeneous equation $\widetilde{\mathcal{L}}_k f = g$. Fixing some z_0 in $]0, \frac{1}{\sqrt{2}}[$ and invoking Duhamel's formula, we readily gather that for all z in $]0, \frac{1}{\sqrt{2}}[$, we have

$$f(z) = \frac{1}{2\sqrt{2}\alpha(\nu,k)} \int_{z_0}^{z} s^3 g(s)\Big(\frac{f^{0,-}_{k,0}(z)}{(\frac{1}{\sqrt{2}} - s)^{\alpha(\nu,k)}} - \frac{f^{0,+}_{k,0}(z)}{(s + \frac{1}{\sqrt{2}})^{\alpha(\nu,k)}}\Big) ds + a_k^+ f^{0,+}_{k,0}(z) + a_k^- f^{0,-}_{k,0}(z),$$

for some constants a_k^+ and a_k^-.

Taking into account (4.45), we infer that any solution to $\widetilde{\mathcal{L}}_k f = g$ admits for z close to 0 an asymptotic expansion of the form

$$f(z) = \sum_{\beta\geq -3} f_{0,\beta}\, z^\beta + \sum_{1\leq\alpha\leq\alpha_0} \sum_{\beta\geq\beta_0} f_{\alpha,\beta} (\log z)^\alpha z^\beta\,,$$

in the case when $\beta_0 \geq -1$, and of the type

$$f(z) = \sum_{\beta\geq \min(\beta_0,-3)} f_{0,\beta}\, z^\beta + \sum_{1\leq\alpha\leq\alpha_0} \sum_{\beta\geq\beta_0} f_{\alpha,\beta} (\log z)^\alpha z^\beta + (\log z)^{\alpha_0+1} \sum_{\beta\geq\max(\beta_0,-3)} f_{\alpha,\beta} z^\beta\,,$$

in the case when $\beta_0 \leq -2$. This completes the proof of the third item.

To end the proof of the lemma, it remains to establish the fourth item. Applying Duhamel's formula, we get for all z in $\mathcal{C}^\infty\big(]\frac{1}{\sqrt{2}}, \infty[\big)$

$$f(z) = -\frac{1}{2\sqrt{2}\alpha(\nu,k)} \int_{z}^{\infty} s^3 g(s)\Big(\frac{f^{0,-}_{k,0}(z)}{(s - \frac{1}{\sqrt{2}})^{\alpha(\nu,k)}} - \frac{f^{0,+}_{k,0}(z)}{(s + \frac{1}{\sqrt{2}})^{\alpha(\nu,k)}}\Big) ds + a_k^+ f^{0,+}_{k,0}(z) + a_k^- f^{0,-}_{k,0}(z),$$

for some constants a_k^+ and a_k^-. Since $A < \nu k$, the unique solution to (4.54) which admits an asymptotic expansion at infinity under the form:

$$f(z) = \sum_{0\le\alpha\le\alpha_0} \sum_{p\in\mathbb{N}} \hat{f}^k_{\alpha,p} (\log z)^\alpha \ z^{A-p}$$

is given by the above formula, with $a_k^+ = a_k^- = 0$. This ends the proof of the lemma.

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