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Abstract

In this paper we generalize the Lighthill-Witham-Richards model for vehicular traffic coupled with moving bottlenecks to the case of road networks. Such models can be applied to study the traffic evolution in the presence of a slow-moving vehicle, like a bus. At last, a numerical experiment is shown.

1 Introduction

In this paper we study the Lighthill-Witham-Richards (LWR) model for vehicular traffic coupled with moving bottlenecks on road networks. We recall that the LWR model was introduced in [16, 17] and gave rise to macroscopic modelling of traffic flow. A moving bottleneck models the presence of a slow vehicle, like a bus or a truck, which causes the reduction of the road capacity at its position and thus generates a moving constraint for the traffic flow. From the analytical point of view our model is the natural generalization to the case of a network of the LWR model with moving constraint on a single road developed in [8], which in turn can be considered as a generalization of the fixed in space point constraint on the flow theory, see [2, 5], [18, Chapter 6]. For completeness we mention that a 2×2 system of conservation laws coupled with a fixed in space point constraint on the flow has been studied in [1, 11, 12] in the case of a single road, while the case of a phase transition model coupled with a fixed in space point constraint on the flow has been studied in [6].

We describe the evolution of the traffic in presence of a slow-moving vehicle by the strongly coupled PDE-ODE system (1) introduced in [8, 14], where the PDE (1a) consists of a scalar conservation law which models the evolution of traffic, while the ODE (1b) describes the trajectory of the slow-moving vehicle. The study of coupled PDE-ODE systems is not new in the conservation laws framework, we refer the reader to [3, 7, 8, 9, 11, 15].

The paper is organized as follows. In the next section we consider the case of a single unidirectional road. The case of a network is then considered in Section 3. Finally, in Section 4, we compare the solutions of the standard model with that with moving constraint for the same Riemann problem.

2 A single unidirectional road

We consider a single road parametrized by the coordinate $x \in \mathbb{R}$ and assume that the vehicles move in the direction of increasing x with maximal speed $V > 0$. Let $y = y(t) \in \mathbb{R}$ be the position of the bus and $\rho = \rho(t, x) \in [0, 1]$ be the mean (normalized) density of cars at time $t \geq 0$ and position $x \in \mathbb{R}$. The resulting model is expressed by the following system

$$\partial_t \rho + \partial_x f(\rho) = 0 \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{1a}$$

$$\dot{y}(t) = \omega\big(\rho(t, y(t)+)\big) \qquad t \in \mathbb{R}_+, \tag{1b}$$

$$f\big(\rho(t, y(t))\big) - \dot{y}(t)\rho(t, y(t)) \leq \frac{\alpha}{4V}\big(V - \dot{y}(t)\big)^2 \qquad t \in \mathbb{R}_+. \tag{1c}$$

Above, the flux $f \geq 0$ is defined by $f(\rho) := \rho v(\rho)$, where $v(\rho)$ is the mean velocity of the cars. We let $v\colon [0, 1] \to \mathbb{R}_+$ be the strictly decreasing function defined by

$$v(\rho) := V(1 - \rho).$$

Clearly $f\colon [0, 1] \to \mathbb{R}_+$ is a strictly concave function such that $f(0) = f(1) = 0$,

$$f(1/2) = \max_{\rho \in [0,1]} f(\rho) \qquad \text{and} \qquad \operatorname{sign}(\rho - 1/2) f'(\rho) < 0 \ \forall \rho \in [0, 1] \setminus \{1/2\}.$$

We stress that $v(1) = 0$ and $v(0) = V$. If the bus has maximal speed $V_b \in [0, V[$, then it moves with velocity

$$\omega(\rho) = \min\{v(\rho), V_b\}.$$

As a result, the trajectory of the bus is given by the function $y : \mathbb{R}_+ \to \mathbb{R}$ satisfying (1b). Notice that if the road is sufficiently congested, then $v(\rho) < V_b$ and the speed of the bus coincides with the speed of the cars. Condition (1c) can be derived as follows. By setting $X = x - y(t)$ we obtain the bus reference frame, where the velocity of the bus is zero and the conservation law (1a) becomes

$$\partial_t \rho + \partial_X \big(f(\rho) - \dot{y}\rho\big) = 0.$$

The presence of the bus hinders the maximum flow at $X = 0$ according to the rule

$$f(\rho) - \dot{y}\rho \leq \frac{\alpha}{4V}\big(V - \dot{y}\big)^2,$$

where the constant coefficient $\alpha \in]0, 1[$ is the reduction rate of the road capacity due to the presence of the bus. Notice that a higher velocity of the bus $\dot{y}$ corresponds to a lower capacity of the road at its position and that

$$\frac{\alpha}{4V}\big(V - \dot{y}\big)^2 \in \left[\frac{\alpha}{4V}\big(V - V_b\big)^2, \frac{\alpha V}{4}\right].$$

We augment system (1) with an initial datum for the density of the form of a Heaviside function with a jump at the initial bus position, which is assumed to be $x = 0$, that is

$$\rho(0,x) = \begin{cases} \rho_\ell \text{ if } x < 0, \\ \rho_r \text{ if } x \geq 0, \end{cases} \tag{2a}$$

$$y(0) = 0, \tag{2b}$$

where $\rho_\ell, \rho_r \in [0,1]$ are fixed constants. We consider solutions of the problem (1), (2) that are self-similar, hence the bus velocity $\dot{y}(t)$ is assumed to be constant.

Let $\mathcal{RS}\colon [0,1]^2 \to \mathbf{L}^1_{loc}(\mathbb{R};[0,1])$ be the standard Riemann solver for (1a), (2a), which is described for instance in [4]. We point out that the associated self-similar weak solution $(t,x) \mapsto \mathcal{RS}(\rho_\ell,\rho_r)(x/t)$ does not always satisfy the constraint condition (1c). For this reason we define below the constrained Riemann solver $\mathcal{RS}_\alpha\colon [0,1]^2 \to \mathbf{L}^1_{loc}(\mathbb{R};[0,1])$ for the Riemann problem (1), (2), see [8]. First, see Figure 1, we need to introduce the density values $\check{\rho}_\alpha$ and $\hat{\rho}_\alpha$ defined by

$$\check{\rho}_\alpha = \min\{\rho \in [0,1]\colon f(\rho) = \rho V_b + F_\alpha\},$$
$$\hat{\rho}_\alpha = \max\{\rho \in [0,1]\colon f(\rho) = \rho V_b + F_\alpha\},$$

where $F_\alpha := \frac{\alpha}{4V}\big(V - V_b\big)^2$.

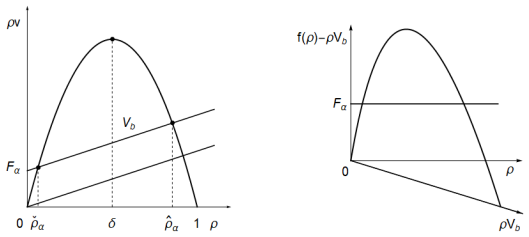

Figure 1: Fundamental diagram with constraint. Left: Fixed reference frame. Right: Bus reference frame.

Definition 2.1 *The constrained Riemann solver* $\mathcal{RS}_\alpha\colon [0,1]^2 \to \mathbf{L}^1_{loc}(\mathbb{R};[0,1])$ *is defined as follows:*

1. *If* $f\big((\mathcal{RS}(\rho_\ell,\rho_r)(V_b)\big) \leq V_b\mathcal{RS}(\rho_\ell,\rho_r)(V_b) + F_\alpha$, *then*

$$\mathcal{RS}_\alpha(\rho_\ell,\rho_r)(x/t) = \mathcal{RS}(\rho_\ell,\rho_r)(x/t) \qquad \text{and} \qquad y(t) = \omega(\rho_r)t.$$

2. *If* $f\big((\mathcal{RS}(\rho_\ell,\rho_r)(V_b)\big) > V_b\mathcal{RS}(\rho_\ell,\rho_r)(V_b) + F_\alpha$, *then*

$$\mathcal{RS}_\alpha(\rho_\ell,\rho_r)(x/t) = \begin{cases} \mathcal{RS}(\rho_\ell,\hat{\rho}_\alpha)(x/t) \text{ if } x < V_b t, \\ \mathcal{RS}(\check{\rho}_\alpha,\rho_r)(x/t) \text{ if } x \geq V_b t, \end{cases} \qquad \text{and} \qquad y(t) = V_b t.$$

Notice that if constraint condition (1c) is not satisfied by the standard weak solution $(t,x) \mapsto \mathcal{RS}(\rho_\ell,\rho_r)(x/t)$, then the weak solution $(t,x) \mapsto \mathcal{RS}_\alpha(\rho_\ell,\rho_r)(x/t)$ has a single undercompressive shock $(\hat{\rho}_\alpha,\check{\rho}_\alpha)$ moving with speed of propagation equal to V_b, according to the Rankine-Hugoniot condition.

3 Networks

In this section we introduce the LWR model with moving constraint on road networks. As in [13], we define a network as a directed graph $(\mathcal{I}, \mathcal{J})$, that is a pair consisting of a finite set $\mathcal{I}$ of unidirectional roads and a finite set $\mathcal{J}$ of junctions. For the rest of the work, if it is not stated differently, a junction is placed at $x = 0$.

Below we consider a node $J \in \mathcal{J}$ having n incoming roads $I_i =]-\infty, 0[\in \mathcal{I}$ for $i \in \mathsf{I} := \{1, \dots, n\}$ and m outgoing roads $I_j =]0, \infty[\in \mathcal{I}$ for $j \in \mathsf{J} := \{n+1, \dots, n+m\}$. Let $f_h(\rho) := V_h \rho(1-\rho)$ be the flux corresponding to the road I_h, for $h \in \mathsf{H} = \mathsf{I} \cup \mathsf{J}$. Assume that at time $t = 0$ the bus is at the junction, that is $y(0) = 0$. Let I_k, $k \in \mathsf{J}$, be the road corresponding to the route of the bus. A constrained Riemann problem at the node J is the following system of scalar conservation laws with constant initial datum on every road, augmented by the ODE for the bus trajectory and the constraint inequality:

$$\begin{aligned}
&\begin{cases} \partial_t \rho_i + \partial_x f_i(\rho_i) = 0 \\ \rho_i(0,x) = \rho_i^0 \end{cases} && (t,x) \in \mathbb{R}_+ \times I_i,\ i \in \mathsf{I}, \\
&\begin{cases} \partial_t \rho_j + \partial_x f_j(\rho_j) = 0 \\ \rho_j(0,x) = \rho_j^0 \end{cases} && (t,x) \in \mathbb{R}_+ \times I_j,\ j \in \mathsf{J}, \\
&\begin{cases} y(t) = \omega(\rho_k(t, y(t)+)), \\ y(0) = 0, \end{cases} \\
&f_k\big(\rho_k(t,y(t))\big) - \dot{y}(t)\rho_k(t,y(t)) \le \frac{\alpha_k}{4V_k}\big(V_k - \dot{y}\big)^2 \quad t \in \mathbb{R}_+,
\end{aligned} \tag{3}$$

for some α_k depending on the k-th road characteristics.

Before stating the constrained Riemann solver at the junction for (3), we define the admissible solutions.

Definition 3.1 *An admissible constrained Riemann solver at the junction $J \in \mathcal{J}$ for (3) is a map $\mathcal{RS}^J_{\alpha_k} : [0,1]^{n+m} \to [0,1]^{n+m}$ such that for any $(\rho_1^0, \dots, \rho_{n+m}^0) \in [0,1]^{n+m}$ we have that $(\bar\rho_1, \dots, \bar\rho_{n+m}) := \mathcal{RS}^J_{\alpha_k}(\rho_1^0, \dots, \rho_{n+m}^0)$ satisfies the following properties:*

- *For every $i \in \mathsf{I}$, $\mathcal{RS}_i(\rho_i^0, \bar\rho_i)$ has only waves with negative speed.*
- *For every $j \in \mathsf{J} \setminus \{k\}$, $\mathcal{RS}_j(\bar\rho_j, \rho_j^0)$ and $\mathcal{RS}_{\alpha_k}(\bar\rho_k, \rho_k^0)$ have only waves with positive speed.*
- *The mass through the junction is conserved, that is:*

$$\sum_{i=1}^{n} f_i(\bar\rho_i) = \sum_{j=n+1}^{n+m} f_j(\bar\rho_j).$$

- *$\mathcal{RS}^J_{\alpha_k}$ is consistent, that is:*

$$\mathcal{RS}^J_{\alpha_k}(\bar\rho_1, \dots, \bar\rho_{n+m}) = (\bar\rho_1, \dots, \bar\rho_{n+m}).$$

The last condition above says that the vector of traces at junction of an admissible solution is a fixed point for $\mathcal{RS}^J_{\alpha_k}$. We propose below the possible traces and their maximal fluxes. To reach this goal, we need first to define the following function.

Definition 3.2 *For any $h \in \mathsf{H}$, the function $\tau_h \colon [0,1] \to [0,1]$ is such that*

- *$f_h(\tau_h(\rho)) = f_h(\rho)$ for every $\rho \in [0,1]$;*
- *$\tau_h(\rho) \neq \rho$ for every $\rho \in [0,1] \setminus \{1/2\}$.*

The function τ_h is well defined, continuous and satisfies

$$0 \leq \rho \leq 1/2 \iff 1/2 \leq \tau_h(\rho) \leq 1, \qquad 1/2 \leq \rho \leq 1 \iff 0 \leq \tau_h(\rho) \leq 1/2.$$

In next propositions, we show the range of admissible fluxes for a given initial datum.

Proposition 1 *Let $i \in \mathsf{I}$ and ρ_i^0 be the initial datum on the incoming road I_i. The set of reachable fluxes $f_i(\bar\rho_i)$ is*

$$\Omega_i(\rho_i^0) = \begin{cases} [0, f_i(\rho_i^0)] & \text{if } \rho_i^0 \in [0,1/2], \\ [0, f_i(1/2)] & \text{if } \rho_i^0 \in \,]1/2,1]. \end{cases}$$

Proof 1 *Since the constraint does not affect an incoming road, we can apply the construction done in [13, Proposition 4.3.3]. For completeness, we consider the case $\rho_i^0 \in [0,1/2]$; the case $\rho_i^0 \in \,]1/2,1]$ is analogous. We stress that $\mathcal{RS}_i(\rho_i^0, \bar\rho_i)$ must have only waves with negative speed. If $\bar\rho_i \in \{\rho_i^0\} \cup \,]\tau_i(\rho_i^0), 1]$ then $\mathcal{RS}_i(\rho_i^0, \bar\rho_i)$ is either constant or has a single shock with negative speed. On the other hand, if $\bar\rho_i \in [0, \tau_i(\rho_i^0)] \setminus \{\rho_i^0\}$ then $\mathcal{RS}_i(\rho_i^0, \bar\rho_i)$ is either a rarefaction or a single shock, but in both cases with non negative speed, which concludes the proof.*

A direct consequence of the above proposition is the following

Corollary 1 *The maximal flow of the incoming road I_i at the junction J is*

$$\gamma_i^{\max}(\rho_i^0) = \begin{cases} f_i(\rho_i^0) & \text{if } \rho_i^0 \in [0,1/2], \\ f_i(1/2) & \text{if } \rho_i^0 \in \,]1/2,1]. \end{cases}$$

Additionally, there exists a unique $\bar\rho_i \in [0,1]$ such that the admissible solution of the Riemann problem with initial datum $(\rho_i^0, \bar\rho_i)$ consists of waves with only negative speed and the condition $f_i(\bar\rho_i) = \gamma_i^{\max}(\rho_i^0)$ holds.

Proposition 2 *Let $j \in \mathsf{J}$ and ρ_j^0 be the initial datum on the outgoing road I_j. The set of reachable fluxes $f_j(\bar\rho_j)$ is*

$$\Omega_j(\rho_j^0) = \begin{cases} \begin{cases} [0, f_j(1/2)] & \text{if } \rho_j^0 \in [0,1/2] \text{ and } j \neq k, \\ [0, f_j(\rho_j^0)] & \text{if } \rho_j^0 \in \,]1/2,1] \text{ and } j \neq k, \end{cases} \\ \begin{cases} [0, f_k(\hat\rho_{\alpha_k})] & \text{if } \rho_k^0 \in [0, \hat\rho_{\alpha_k}], \\ [0, f_k(\rho_k^0)] & \text{if } \rho_k^0 \in \,]\hat\rho_{\alpha_k}, 1]. \end{cases} \end{cases}$$

Proof 2 *The proof for $j \neq k$ is analogous to proof of Proposition 1. The only difference is that $\mathcal{RS}_j(\bar\rho_j, \rho_j^0)$ must have only waves with positive speed. Let $j = k$ and $\rho_k^0 \in [0, \hat\rho_{\alpha_k}]$. We observe that $\bar\rho_k \in [0, \check\rho_{\alpha_k}]$ can be connected with ρ_k^0 by a classical waves. For $\bar\rho_k \in \,]\check\rho_{\alpha_k}, \tau_k(\hat\rho_{\alpha_k})[\,\cup \{\hat\rho_{\alpha_k}\}$ the $\mathcal{RS}_k(\bar\rho_k, \rho_k^0)$ consists of a possibly null shock $(\bar\rho_k, \hat\rho_{\alpha_k})$, followed by a non-classical shock $(\hat\rho_{\alpha_k}, \check\rho_{\alpha_k})$ and a shock $(\check\rho_{\alpha_k}, \rho_k^0)$. Notice that $\bar\rho_k \in [\tau(\hat\rho_{\alpha_k}), \hat\rho_{\alpha_k}[$ cannot be joined with $\hat\rho_{\alpha_k}$ by a wave with positive speed. The case $j = k$ and $\rho_k^0 \in \,]\hat\rho_{\alpha_k}, 1]$ is analogous to the situation when $j \neq k$ and $\rho_j^0 \in \,]1/2,1]$.*

Corollary 2 *The maximal flow of the outgoing road I_j at the junction J is*

$$\gamma_j^{\max}(\rho_j^0) = \begin{cases} \begin{cases} f_j(1/2) & \text{if } \rho_j^0 \in [0,1/2] \text{ and } j \neq k, \\ f_j(\rho_j^0) & \text{if } \rho_j^0 \in]1/2,1] \text{ and } j \neq k, \end{cases} \\ \begin{cases} f_k(\hat{\rho}_{\alpha_k}) & \text{if } \rho_k^0 \in [0,\hat{\rho}_{\alpha_k}], \\ f_k(\rho_k^0) & \text{if } \rho_k^0 \in]\hat{\rho}_{\alpha_k},1]. \end{cases} \end{cases}$$

Additionally, there exists a unique $\bar{\rho}_j \in [0,1]$ such that the admissible solution of the Riemann problem with initial datum $(\bar{\rho}_j, \rho_j^0)$ consists of waves with only positive speed and the condition $f_j(\bar{\rho}_j) = \gamma_j^{\max}(\rho_j^0)$ holds.

For each junction we consider a traffic distribution matrix, i.e. a matrix representing the distribution of cars among the roads.

Definition 3.3 *A distribution matrix $A_{m\times n}$ for the junction $J \in \mathcal{J}$ is given by*

$$A_{m\times n} = \begin{pmatrix} \alpha_{n+1,1} & \cdots & \alpha_{n+1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n+m,1} & \cdots & \alpha_{n+m,n} \end{pmatrix},$$

where $\alpha_{j,i} \geq 0$ for every i, j and $\sum_{j=n+1}^{n+m} \alpha_{j,i} = 1$ for every i.

A distribution matrix $A_{m\times n}$ gives the percentage of cars from each incoming road I_i choosing the outgoing road I_j. In other words, if C is the amount of cars coming from road I_i, then $C\alpha_{j,i}$ is the amount of cars moving towards road I_j from I_i.

The construction of the admissible solution at the junction J corresponding to the initial datum $(\rho_1^0, \ldots, \rho_{n+m}^0) \in [0,1]^{n+m}$ is the following:

1. Fix a distribution matrix $A_{m\times n}$ by choosing $m \times n$ non-negative constants $\alpha_{j,i}$ such that $\sum_{j=n+1}^{n+m} \alpha_{j,i} = 1$ for every $i \in \mathsf{I}$.

2. Define the closed, convex and non-empty sets of admissible fluxes

   $$\Omega = \{(\gamma_1, \ldots, \gamma_n) \in \Omega_1 \times \ldots \times \Omega_n \colon A \cdot (\gamma_{n+1}, \ldots \gamma_{n+m})^T \in \Omega_{n+1} \times \ldots \times \Omega_{n+m}\},$$

   where $\Omega_i(\rho_i^0) = [0, \gamma_i^{\max}(\rho_i^0)]$ and $\Omega_j(\rho_j^0) = [0, \gamma_j^{\max}(\rho_j^0)]$ are respectively defined in propositions 1 and 2, see also corollaries 1 and 2.

3. Compute a vector $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n) \in \Omega$ such that

   $$\sum_{i=1}^{n} \bar{\gamma}_i = \max_{(\gamma_1,\ldots,\gamma_n)\in\Omega} \sum_{i=1}^{n} \gamma_i. \tag{4}$$

   Then by Corollary 1 there exists unique $\bar{\rho}_i \in [0,1]$ such that $f_i(\bar{\rho}_i) = \bar{\gamma}_i$.

4. Compute the vector $(\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m})$ such that

   $$\bar{\gamma}_j = \sum_{i=1}^{n} \alpha_{j,i} \bar{\gamma}_i.$$

   Then by Corollary 2 there exists unique $\bar{\rho}_j \in [0,1]$ such that $f_j(\bar{\rho}_j) = \bar{\gamma}_j$.

5. Finally, let $\mathcal{RS}^J_{\alpha_k}(\rho^0_1, \dots, \rho^0_{n+m}) = (\bar\rho_1, \dots, \bar\rho_{n+m})$.

Remark 1 *The maximization problem* (4) *may admit more than one solution. Additional assumptions are in general required to get uniqueness of the Riemann solver. This can be obtained either imposing further conditions on the distribution matrix A, see [13, Section 5.1], or introducing a priority vector as in [10].*

4 A case study

We consider a junction with two incoming ($n = 2$) and two outgoing ($m = 2$) roads. Let $V_h = 4$, namely $f_h(\rho) = 4\rho(1-\rho)$, $h \in \{1, \dots, 4\}$. Fix constant initial density $(\rho^0_1, \dots, \rho^0_4) \in [0,1]^4$, see Figure 2, center, such that

$$0 < \rho^0_1 < 1/2, \qquad 1/2 < \rho^0_2 < 1, \qquad 1/2 < \rho^0_3 < 1, \qquad 1/2 < \rho^0_4 < 1,$$
$$f(\rho^0_1) = 1/2, \qquad f(\rho^0_2) = 2/5, \qquad f(\rho^0_3) = 7/10, \qquad f(\rho^0_4) = 1/2,$$

The parameter α_3 is suitably chosen to obtain $f(\hat\rho_{\alpha_3}) = 7/20$ and we take the distribution matrix

$$A = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix}.$$

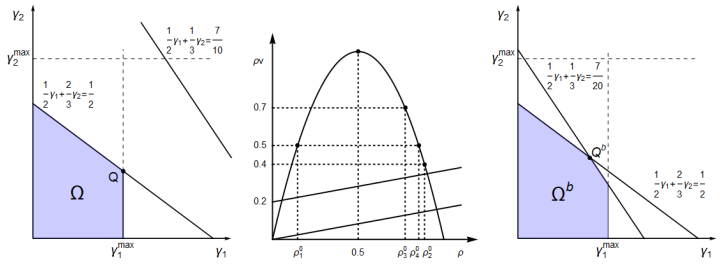

Figure 2: Left: the set Ω. Center: the fundamental diagram with initial datum. Right: the set Ω^b.

We consider two cases: the case a slow vehicle with maximal velocity $V_b = 1/6$ enters road I_3 and the case there is no slow vehicle at the junction. According to propositions 1 and 2, the sets of admissible fluxes at the junction are

$$\Omega_1 = \Omega_4 = [0, 1/2], \qquad \Omega_2 = [0,1], \qquad \Omega^b_3 = [0, 7/20], \qquad \Omega_3 = [0, 7/10],$$

where Ω^b_3 and Ω_3 are the sets of admissible fluxes on I_3 in case the bus is present or not, respectively. In the case without the bus we let

$$\Omega = \{(\gamma_1, \gamma_2) \in \Omega_1 \times \Omega_2 \colon A \cdot (\gamma_1, \gamma_2)^T \in \Omega_3 \times \Omega_4\},$$

and find that the maximal admissible flow through junction $\max_{(\gamma_1,\gamma_2)\in\Omega}(\gamma_1 + \gamma_2)$ is reached at the point $Q = (1/2, 3/8)$, see Figure 2, left, hence the solution

for the fluxes of this problem is $(\bar{\gamma}_1, \ldots, \bar{\gamma}_4) = (1/2, 3/8, 3/8, 1/2)$. In the case with the bus, we let

$$\Omega^b = \{(\gamma_1, \gamma_2) \in \Omega_1 \times \Omega_2 \colon A \cdot (\gamma_1, \gamma_2)^T \in \Omega_3^b \times \Omega_4\},$$

and find that the maximal admissible flow through the junction $\max_{(\gamma_1,\gamma_2)\in\hat{\Omega}^b}(\gamma_1 + \gamma_2)$ is reached at the point $Q^b = (2/5, 9/20)$, see Figure 2, right, therefore

$$\begin{aligned} &1/2 < \bar{\rho}_1 < 1, \quad && 1/2 < \bar{\rho}_2 < 1, \quad && \bar{\rho}_3 = \hat{\rho}_{\alpha_3}, \quad && \bar{\rho}_4 = \rho_{4,0}, \\ &f(\bar{\rho}_1) = 2/5, && f(\bar{\rho}_2) = 9/20, && f(\bar{\rho}_3) = 7/20, && f(\bar{\rho}_4) = 1/2. \end{aligned}$$

The solution of the Riemann problem at the junction is completely determined. For better understanding the solution behavior, we display in Figure 3 the two solutions at time $t = 1/5$. The blue line describes the density profile without the bus, while the red line represents the solution in the presence of the bus. We notice that a shock wave arises on road I_1, on road I_2 we observe a rarefaction wave instead of a shock wave, on road I_3 the undercompressive shock is visible in the situation with the bus. Only the solution on road I_4 is the same in both cases.

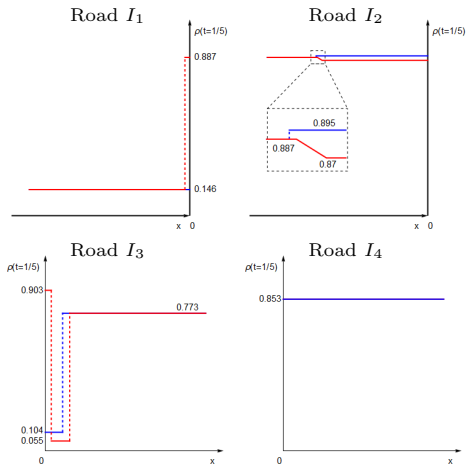

Figure 3: Blue line indicates the situation without bus and red with the bus.

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